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Large deviations for the largest eigenvalues and eigenvectors of spiked Gaussian random matrices

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Abstract

We consider matrices formed by a random $N \times N$ matrix drawn from the Gaussian Orthogonal Ensemble (or Gaussian Unitary Ensemble) plus a rank-one perturbation of strength θ, and focus on the largest eigenvalue, x, and the component, u, of the corresponding eigenvector in the direction associated to the rank-one perturbation. We obtain the large deviation principle governing the atypical joint fluctuations of x and u. Interestingly, for $\theta > 1$, large deviations events characterized by a small value of u, i.e. $u < 1 - 1/\theta$, are such that the second-largest eigenvalue pops out from the Wigner semi-circle and the associated eigenvector orients in the direction corresponding to the rank-one perturbation. We generalize these results to the Wishart Ensemble, and we extend them to the first n eigenvalues and the associated eigenvectors.

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1 Introduction

The large deviations theory for the spectrum of random matrix models is a very active domain of research in probability theory and theoretical physics.

A lot of works have been devoted to the statistics of the eigenvalues. In this article, we will consider the Gaussian Orthogonal ensemble (GOE) and the Gaussian Unitary Ensemble (GUE): they are sequences of self-adjoint matrices with independent real or complex centered Gaussian entries above the diagonal with covariance given by the inverse of the dimension (except on the diagonal in the real case where the covariance is two over the dimension). With such a choice, one can see that the distribution of these ensembles is invariant under conjugation by the orthogonal or unitary group, leading for instance to an explicit joint law for the eigenvalues. Following Voiculescu pioneering work on non-commutative entropy [27], G. Ben Arous and one of the authors [8] derived a large deviation principle for the distribution of the empirical measure of the eigenvalues of Gaussian ensembles in the late nineties (in physics known as Coulomb gas method [21]). The proof of the large deviation principle is based on the explicit density of the law of the eigenvalues and the speed of the large deviation principle is the square of the

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linear dimension of the random matrix. Almost twenty years later, C. Bordenave and P. Caputo [10] obtained a large deviations principle for the same empirical measure but for a Wigner matrix with heavy tails entries, in the sense that their tail decays more slowly than a Gaussian variable at infinity. Their approach is totally different as it follows from the idea that deviations are created by a few large entries: the rate then depends on the speed of decay of the tail. The large deviation principle for the spectral measure in the general sub-Gaussian case is still an open problem.

Instead of considering the deviations of the empirical measure, it is also natural to try to understand the probability of deviations of a single eigenvalue. The deviations of an eigenvalue inside the bulk is closely related to that of the empirical measure but one can seek for the probability of deviations of the extreme eigenvalues. This was achieved for Gaussian ensembles in the Appendix of [7], see also [14], where it was shown that the large deviations are on the scale of the dimension of the matrix. Again, the proof was based on the explicit joint law of the eigenvalues. The large deviations principle for the largest eigenvalue was derived in [4] for heavy tails. In the case of sharp sub-Gaussian entries, which includes Rademacher (binary) entries, it was recently proved that the large deviations of the extreme eigenvalues are the same than in the Gaussian case [18]. In the case of entries with larger (but still sub-Gaussian) Laplace's transform, large deviations estimates were shown to be governed by a smaller rate function than in the Gaussian case [5].

The probability of atypical eigenvectors has been much less studied. Again, the only large deviation result that we know concerns the Gaussian ensembles: in this case, the invariance by multiplication of the Haar measure implies that each eigenvector is uniformly distributed on the sphere. In [7], the large deviations for the empirical measure of the properly rescaled entries of an eigenvector was established. The large deviations for the supremum of the entries could also be easily derived.

In this article, we address a different question. We want to investigate the large deviations of the eigenvector in a given fixed direction. In many solvable random matrix models, eigenvectors are uniformly distributed; hence there are no meaningful atypical fluctuations or special directions to focus on. For a spiked GOE matrix, i.e. a random $N \times N$ matrix drawn from the Gaussian Orthogonal Ensemble plus a rank-one perturbation, there is instead a special direction: the one related to the perturbation. In this case an interesting phenomenon, called BBP-transition at least when Wishart matrices are concerned [6], takes place by varying the strength of the perturbation (called θ in the following). As shown in [16] and then proved rigorously in [6] the largest eigenvalue, λ_1, pops out of the semi-circle if the perturbation is strong enough. More precisely, λ_1 converges almost surely to x when the dimension goes to infinity, where x is equal to two for $\theta \leq 1$ and to $\theta + 1/\theta$ for $\theta > 1$. In the latter case, the square of the component $v_N(1)$ of the associated eigenvector in the direction associated to the perturbation converges almost surely to $u = 1 - 1/\theta^2$ [9]. In this context the question we raised before becomes meaningful, and it is natural to focus on the good rate function (GRF) that controls the *joint* atypical deviations of λ_1 and $|v_1(1)|^2$.

This GRF plays an important role for the geometric properties of random high-dimensional energy landscapes, which can exhibit a number of critical points that is exponentially large in the number of dimensions, as obtained in [13, 17, 12] and rigorously proven and extended in [3, 26]. The rigorous method developed to perform those studies is based on a large dimensional version of the Kac-Rice formula [1], and is strongly related to random matrix theory, since the Hessian of the energy function at the critical points—a crucial element in the theoretical analysis—is a random matrix. In order to analyze the dynamics in those rough landscapes it is important to know not only the behavior of typical critical points, but also of atypical ones associated to index one saddles connecting minima [25]. One has therefore to study large deviations of the Hessian, i.e. one needs to condition the critical points to be of index one and to have the eigenvector associated to the negative eigenvalue oriented in the direction connecting the minima, which leads in fact to the problem discussed above.

Noise dressing and cleaning of empirical correlation matrices is another context in which the kind of large deviations addressed in this paper are relevant. In this case, a model that is often considered to interpret the data is the one of spiked Wishart random matrices, whose eigenvalue distribution consists in a Marchenko-Pastur law plus a few eigenvalues that pop out from it. Those few eigenvalues correspond to the signal buried in the noise and the associated eigenvectors play an important role in assessing the structure of the correlations, with important applications such as portfolios risk management [11]. A natural question in this context is to characterize the joint atypical fluctuations of the largest eigenvalues and associated eigenvectors that carry the signal. In this work we obtain the large deviation function that governs them.

2 Main results

We consider the matrix

$$Y = X + \theta w w^T$$

where X is taken from the GOE if $\beta = 1$ (resp. the GUE if $\beta = 2$):

$$d\mathbb{P}_N(X) = \frac{1}{Z_N} e^{-\frac{N\beta}{4}\mathrm{Tr}(XX^*)} dX$$

where dX is the Lebesgue measure and θ is a non-negative real number. w is a fixed unit vector and we may assume without loss of generality that $w = e_1 = (1, 0\cdots, 0)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of Y, with respective eigenvectors $v_1, \ldots, v_N$. The large deviations for the joint distribution of the largest eigenvalue λ_1 and the component $|v_1(1)|^2 = |\langle v_1, e_1\rangle|^2$ of the associated eigenvector along $w = e_1$ is governed by the following theorem.

Theorem 2.1. The joint law P_N of $(\lambda_1, |v_1(1)|^2)$ satisfies a large deviations principle in the scale N and good rate function I_β. In other words, for any closed set F of $\mathbb{R} \times [0,1]$

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N((\lambda_1, |v_1(1)|^2) \in F) \leq -\inf_F I_\beta \,,$$

and for any open set O of $\mathbb{R} \times [0,1]$

$$\liminf_{N\to\infty} \frac{1}{N} \log P_N((\lambda_1, |v_1(1)|^2) \in O) \geq -\inf_O I_\beta \,.$$

Moreover, I_β is a good rate function in the sense that it is non-negative and with compact level sets. I_β is infinite outside of $S = [2, +\infty) \times [0,1]$ and otherwise given by $I_\beta(x,u) = \beta(I(x,u) - \inf_S\{I\})$ where

$$I(x,u) = I(x) - \frac{1}{2}\theta x u - \frac{1}{2}\log|1-u| - \sup_{2\leq y\leq x}\{J(\sigma, \theta(1-u), y) - I(y)\} \,. \tag{2.1}$$

Here $\sigma(dx) = \sqrt{4-x^2}dx/2\pi$ is the semi-circle distribution and we denote

$$I(x) = \frac{x^2}{4} - \int \log|x-y| d\sigma(y) \,.$$

Moreover, if $G_\sigma(x) = \int (x-y)^{-1} d\sigma(y)$ denote the Cauchy transform of σ, we have for $\eta \leq G_\sigma(x)$,

$$J(\sigma, \eta, x) = \frac{1}{4}\eta^2, \tag{2.2}$$

whereas if $\eta > G_\sigma(x)$,

$$J(\sigma, \eta, x) = \frac{1}{2}\left(\eta x - 1 + \log\frac{1}{\eta} - \int \log|x-y| d\sigma(y)\right). \tag{2.3}$$

The second largest eigenvalue converges almost surely to the maximizer, y, of the variational problem (2.1) defined above.

By studying the minimizers of the rate function we have more explicit results on the behavior of the second largest eigenvalue and the component of the associated eigenvector along $w = e_1$ associated to a given large deviation of the largest eigenvalue. We in particular see that when $\theta \geq 1$, conditionally on the largest eigenvalue of Y being close to x, $|\langle v_1, e_1\rangle|^2$ converges towards $u_{\theta,x}$ below, whereas if we additionally condition by $|\langle v_1, e_1\rangle|^2$ being close to u, the second largest eigenvalue of Y pops out of the semi-circle iff $u < 1 - 1/\theta$, and is equal to $\min\{\theta(1-u) + \frac{1}{\theta(1-u)}, x\}$.

Proposition 2.2. Conditionally on the largest eigenvalue of Y being close to x, $|\langle v_1, e_1\rangle|^2$ is close to the minimizer of $v \to I_\beta(x, v)$. Moreover, conditionally on the largest eigenvalue of Y being close to x and $|\langle v_1, e_1\rangle|^2$ being close to u, the second largest eigenvalue of Y converges towards the maximizer of $K_{\theta(1-u)}(y) = J(\sigma, \theta(1-u), y) - I(y)$. Moreover, we have the following characterization of these optimizers:

- *For $\theta \geq 1$ and $x > \theta + 1/\theta$:* The minimum of $I_\beta(x, .)$, for a given x, is reached at $u_{\theta,x} = 1 - \frac{\theta x - \sqrt{(\theta x)^2 - 4\theta^2}}{2\theta^2}$. The maximizer of $K_{\theta(1-u)}$ pops out of the semicircle for $u < 1 - 1/\theta$, and is equal to $y(u) = \theta(1-u) + \frac{1}{\theta(1-u)}$.
- *For $\theta \geq 1$ and $2 \leq x < \theta + 1/\theta$:* The minimum of $I_\beta(x, .)$ is reached at $u_{\theta,x}$. The maximizer of $K_{\theta(1-u)}$ pops out of the semicircle for $u < 1 - 1/\theta$, and is equal to $\inf(y(u), x)$, i.e. it increases when u decreases until reaching the value x.
- *For $\theta < 1$ and $x \geq 2$:* The minimum of the large deviation function $I_\beta(x, .)$ is taken at $u = 0$, if $u_{\theta,x}$ is not positive, or at $u_{\theta,x}$ otherwise. The latter case corresponds to large enough values of x ($x \geq \theta + 1/\theta$). The maximizer of $K_{\theta(1-u)}$ sticks to two.

Note that the component $|v_2(1)|^2$ of the eigenvector associated to the second largest eigenvalue is different from zero if and only if the associated eigenvalue is larger than two, i.e. the eigenvector v_2 orients in the direction of w when λ_2 pops out from the semi-circle. An example of the large deviation function (GRF) is shown in Fig. 1 for $\theta = 3$.

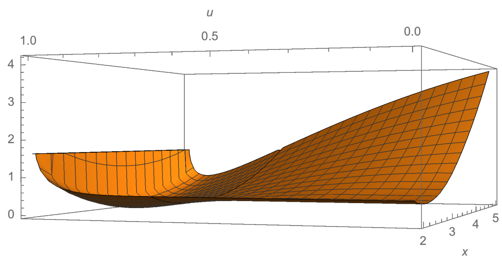

Figure 1: Large deviation function plotted for $2 < x < 5$, $0 < u < 1$ and $\theta = 3$. The global minimum is attained at $u = 1 - 1/\theta^2, x = \theta + 1/\theta$.

The previous results can be extended to the large deviations of the n largest eigenvalues $\bar{\lambda}_N^n = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the first component of their eigenvectors $\bar{u}_N^n := \left(|v_1(1)|^2, \ldots, |v_n(1)|^2\right)$. We denote by $\Delta_n = \{x_. \in [2, \infty)^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$ (resp. $U_n = \{u_. \in [0,1]^n : \sum_{i=0}^{n-1} u_i \leq 1\}$) the space where these extreme eigenvalues (resp. the first component of their eigenvectors) live.

Theorem 2.3. Let n be a fixed integer number.

The joint law of $\left(\bar{\lambda}_N^n, \bar{u}_N^n\right)$, satisfies a LDP with speed N and GRF given by $\mathcal{I}_\beta(\bar{x}, \bar{u}) = \beta(I(\bar{x}, \bar{u}) - \inf\{I\})$ where $I(\bar{x}, \bar{u})$ is infinite outside of $\Delta_n \times U_n$ and otherwise given for $\bar{x} = \{x_i\}_{0 \le i \le n-1} \in \Delta_n, \bar{u} = \{u_i\}_{0 \le i \le n-1} \in U_n$, by

$$I(\bar{x}, \bar{u}) = \sum_{i=0}^{n-1} \left(I(x_i) - \frac{1}{2}\theta x_i u_i \right) - \sup_{2 \le y \le x_{n-1}} \left\{ J\left(\sigma, \theta(1 - \sum_{i=0}^{n-1} u_i), y\right) - I(y) \right\} - \frac{1}{2} \log \left| 1 - \sum_{i=0}^{n-1} u_i \right|$$

where the functions I, J are the same than in Theorem 2.1. The $n+1$-th largest eigenvalue converges almost surely to the maximizer y of the variational problem defined above.

Moreover, we can as well extend our results to the case of Wishart matrices with covariance which is a finite dimensional perturbation of the identity. To simplify, let us assume it is one dimensional, and consider the Wishart matrix

$$W = \Sigma^{1/2} Y Y^* \Sigma^{1/2}$$

where Y is a $M \times N$ random matrix with i.i.d standard real or complex Gaussian entries with variance $1/N$. Σ is a $M \times M$ non-negative definite matrix: $\Sigma = I + \gamma e_1 e_1^*$ with e_1 a unit vector. We assume $M \le N$ and that M/N converges towards $\alpha \in (0, 1]$. We recall that the empirical measure of YY^* converges towards the so-called Marchenko-Pastur [24] law π_α with support $[(1 - \sqrt{\alpha})^2, (1 + \sqrt{\alpha})^2]$. We can study the joint large deviations of the largest eigenvalue λ_1 and the strength $|v_1(1)|^2$ of the eigenvector in the direction e_1 for W as well. We find that

Theorem 2.4. The joint law of $(\lambda_1, |v_1(1)|^2)$ satisfies a large deviations principle in the scale N and good rate function I_β^W. The function I_β^W is infinite outside of $S = [(1 + \sqrt{\alpha})^2, +\infty) \times [0, 1]$ and otherwise given by $I_\beta^W(x, u) = \frac{\beta}{2}(I^W(x, u) - \inf_S\{I^W\})$ where

$$I^W(x, u) = I^W(x) - \frac{\gamma}{(1+\gamma)} xu - \alpha \log(1 - u) - \sup_{(1+\sqrt{\alpha})^2 \le y \le x} \{2\alpha J(\pi_\alpha, \frac{\gamma(1-u)}{\alpha(1+\gamma)}, y) - I^W(y)\}$$

where $I^W(y) = y - (1 - \alpha) \log y - 2\alpha \int \log |y - t| d\pi_\alpha(t)$ is the rate function for the large deviations of the largest eigenvalue of a Gaussian Wishart matrix with covariance equal to the identity.

3 Strategy of the proof

We next detail the strategy of the proof for the perturbed Wigner matrix $Y = X + \theta w w^T$. The law of Y is given by

$$d\mathbb{P}_N(Y) = \frac{1}{Z_N} \exp\left\{ -\frac{\beta N}{4} \mathrm{Tr}(Y - \theta w w^T)^2 \right\} dY = \frac{1}{\tilde{Z}_N} \exp\left\{ -\frac{\beta N}{4} \mathrm{Tr}\, Y^2 + \frac{\beta N}{2} \theta \langle w, Y w \rangle \right\} dY\,.$$

Therefore, since $\langle w, Yw \rangle = \sum \lambda_i |v_i(1)|^2$ when $w = e_1$, the joint law of $(\lambda_1, |v_1(1)|^2)$ is given by

$$dP_N(x, u) = \frac{1}{\bar{Z}_N} e^{-\frac{\beta N}{4} x^2 + \frac{N\beta}{2} \theta x u} \int \prod_{i=2}^{N} |x - \lambda_i|^\beta I_N(\lambda, \theta, u) dP_{N-1}^x(\lambda) dx dB_1(u) \tag{3.1}$$

where we denoted

$$I_N(\lambda, \theta, u) = \mathbb{E}[e^{\frac{\beta N}{2} \theta \sum_{i=2}^N \lambda_i |v_i(1)|^2} || v_1(1)|^2](u) \tag{3.2}$$

and $\mathbb{E}[.||v_1(1)|^2](u)$ is the expectation on $\{|v_i(1)|, i \geq 2\}$ conditionally to $\{|v_1(1)|^2 = u\}$. $dB_1(u)$ is the distribution of $|v_1(1)|^2$. Moreover, P^x_{N-1} is the positive measure given by

$$dP^x_{N-1}(\lambda) = \frac{1}{Z^{\infty}_{N-1}} \prod_{2\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4}\sum_{i=2}^N \lambda_i^2} \prod_{i=2}^N 1_{\lambda_i \leq x} d\lambda_i \,,$$

where Z^{∞}_{N-1} is independent of x and such that P^{∞}_{N-1} is a probability measure:

$$Z^{\infty}_{N-1} = \int \prod_{2\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4}\sum_{i=2}^N \lambda_i^2} \prod_{i=2}^N d\lambda_i \,.$$

Our main goal is to estimate the density of P_N when N is large and to apply Laplace's method. We infer from concentration inequalities [22] that with $\hat{\mu}^{N-1} = \frac{1}{N-1}\sum_{i=2}^N \delta_{\lambda_i}$

$$\sup_{x>2} \limsup_{N\to\infty} \frac{1}{N} \log P^x_{N-1}\left(d(\hat{\mu}^{N-1}, \sigma) > N^{-\kappa'}\right) \leq \limsup_{N\to\infty} \frac{1}{N} \log P^{\infty}_{N-1}\left(d(\hat{\mu}^{N-1}, \sigma) > N^{-\kappa'}\right) = -\infty \,. \tag{3.3}$$

Here, d denotes the Dudley distance and κ' belongs to $(0, \frac{1}{10})$ (see [18, Lemma 6.1] for details). We deduce that for $x > 2$ (the singularity of the log can be overcome as in [7]), with probability greater than $1 - e^{-MN}$ for all $M > 0$,

$$\prod_{i=1}^{N-1} |x - \lambda_i| = e^{N \int \log|t-x| d\sigma(t) + o(N)} \,, \tag{3.4}$$

i.e. we can replace the empirical distribution $\hat{\mu}^{N-1}$ by its limit. To estimate the other terms, we first observe that since $(v_i(1))_{1\leq i\leq N}$ is uniformly distributed on the sphere with radius one, we can represent $(v_i(1))_{1\leq i\leq N}$ as

$$v_i(1) = \frac{g_i}{(\sum_{i=2}^N |g_i|^2 + |g_1|^2)^{1/2}} \tag{3.5}$$

with independent standard Gaussian variables $g_i, 1 \leq i \leq N$, which are real when $\beta = 1$ and complex when $\beta = 2$. As a consequence, a simple change of variables shows that the distribution B_1 of $|v_1(1)|^2$ is the Beta-distribution

$$dB_1(u) = C_N u^{\beta/2-1}(1-u)^{(N-1)\beta/2-1} du \tag{3.6}$$

To estimate $I_N(\lambda, \theta, u)$, we first determine the distribution of $(|v_N(1)|^2, \ldots, |v_2(1)|^2)$ conditionally to $|v_1(1)|^2$, again by using (3.5). In fact, if we fix $|v_1(1)|$ and denote $g^{N-1} = (g_2, \ldots, g_N)$, we have

$$\|g\|_2^2 = \|g^{N-1}\|_2^2 + |v_1(1)|^2 \|g\|_2^2$$

so that $w_i = g_i / \|g^{N-1}\|_2, 2 \leq i \leq N$, follows the uniform law on the sphere $\mathbb{S}^{N-2}$ with radius one and

$$v_i(1) = w_i \frac{\|g^{N-1}\|_2}{\|g\|_2} = \sqrt{1 - |v_1(1)|^2} w_i$$

Observe that $(w_i)_{2\leq i\leq N}$ is independent of $|v_1(1)|^2$. Hence, we conclude that

$$I_N(\lambda, \theta, u) = \mathbb{E}_{w_i, i\geq 2}\left[e^{\frac{N}{2}\beta\theta(1-u)\sum_{i=2}^N \lambda_i |w_i|^2}\right] . \tag{3.7}$$

We next estimate this quantity, see (4.1).

4 Asymptotic of spherical integrals

Recall the definition of spherical integrals:

$$I_N(X,\eta) = \mathbb{E}_e[e^{\frac{\beta}{2}\eta N\langle e, Xe\rangle}]$$

where e is uniformly sampled on the sphere $\mathbb{S}^{N-1}$ with radius one. The asymptotics of

$$J_N(X,\eta) = \frac{1}{N}\log I_N(X,\eta)$$

were studied in [19] where the following result was proved.

Theorem 4.1. [19, Theorem 6] Let $(E_N)_{N\in\mathbb{N}}$ be a sequence of $N\times N$ real symmetric matrices if $\beta=1$ or Hermitian complex matrices if $\beta=2$ such that:

- The sequence of empirical measures $\hat{\mu}^N_{E_N}$ converges weakly to a compactly supported measure μ.
- There are two finite real numbers $C, \lambda_{\max}(E)$ such that

$$\limsup_{N\to+\infty}\|E_N\|_\infty \le C, \quad \lim_{N\to+\infty}\lambda_{\max}(E_N) = \lambda_{\max}(E).$$

For any $\eta\ge 0$,

$$\lim_{N\to+\infty} J_N(E_N,\eta) = \beta J(\mu,\eta,\lambda_{\max}(E))$$

The limit J is defined as follows. For a compactly supported probability measure $\mu\in\mathcal{P}(\mathbb{R})$ we define its Stieltjes transform G_μ by,

$$\forall z\in\mathbb{C}\setminus\mathrm{supp}(\mu),\ G_\mu(z) := \int_{\mathbb{R}}\frac{1}{z-t}d\mu(t),$$

where $\mathrm{supp}(\mu)$ denotes the support of μ. In the sequel, for any compactly supported probability measure μ, we denote by $r(\mu)$ the right edge of the support of μ. Then G_μ is a bijection from $(r(\mu),+\infty)$ to $(0, G_\mu(r(\mu))$, with

$$G_\mu(r(\mu)) = \lim_{t\downarrow r(\mu)} G_\mu(t).$$

We denote by K_μ its inverse on $(0, G_\mu(r(\mu)))$ and let $R_\mu(z) := K_\mu(z) - 1/z$ be the R-transform of μ as defined by Voiculescu in [28] (defined on $(0, G_\mu(r(\mu)))$. Then, we introduce, for any $\eta\ge 0$, and $\lambda\ge r(\mu)$,

$$J(\mu,\eta,\lambda) := \frac{1}{2}\eta v(\eta,\mu,\lambda) - \frac{1}{2}\int\log\left(1+\eta v(\eta,\mu,\lambda)-\eta y\right)d\mu(y),$$

with

$$v(\eta,\mu,\lambda) := \begin{cases} R_\mu(\eta) & \text{if } 0\le\eta\le G_\mu(\lambda),\\ \lambda-\frac{1}{\eta} & \text{if } \eta > G_\mu(\lambda).\end{cases}$$

In the case where $\mu=\sigma$ is the semi-circular law, then,

$$\forall x>2,\ G_\sigma(x) = \frac{1}{2}(x-\sqrt{x^2-4}),\ R_\sigma(x) = x,$$

from which the formulas (2.3) and (2.2) for $J(\sigma,\eta,\lambda)$ are easily deduced. From (3.7), we deduce that if the empirical measure of the λ_i converges towards μ and λ_2 converges towards y,

$$\lim_{N\to\infty}\frac{1}{N}\log I_N(\lambda,\eta,u) = \beta J(\sigma,\eta(1-u),y)\,. \tag{4.1}$$

5 Proof of Theorem 2.1

Remark that Theorem 2.1 implies the weak large deviations principle which states that for δ small enough, there exists $o(\delta)$ going to zero with δ such that

$$-I_\beta(x,u) - o(\delta) \le \liminf_{N\to\infty} \frac{1}{N} \log P_N(|\lambda_1 - x| + ||v_1(1)|^2 - u| \le \delta)$$
$$\le \limsup_{N\to\infty} \frac{1}{N} \log P(|\lambda_1 - x| + ||v_1(1)|^2 - u| \le \delta) \le -I_\beta(x,u) + o(\delta)\,.$$

Indeed, the weak large deviations principle is simply the restriction of the full large deviations principle to small balls. To recover the full large deviations principle from its weak version, it is enough to show that the probability is exponentially tight in the sense that deviations mostly occur in a compact set. We refer the reader to [2, Theorem D.4, Corollary D.6] or [15]. The latter is easy to check since $|v_1(1)|^2$ lives in a compact set and

$$|\lambda_1| \le \|X\|_\infty + \theta$$

where it is known that $\|X\|_\infty \le M$ with probability greater than $1 - e^{-NcM^2}$ [7]. Hence, we only need to prove the weak large deviations principle, that is estimate the probability that $(\lambda_1, |v_1(1)|^2)$ is close to some (x,u). To this end, we estimate the density of $P_N(x,u)$, and according to (3.1), we estimate in the limit $N\to\infty$

$$\frac{1}{N} \log \frac{\int \prod_{i=2}^N |x-\lambda_i|^\beta I_N(\lambda,\theta,u) dP_{N-1}^x(\lambda)}{\int I_N(\lambda,\theta,u) dP_{N-1}^x(\lambda)} = \beta \int \log|x-y| d\sigma(y) + o(1)$$

where we used that (3.3) also holds under P_{N-1}^x. Since $I_N(\lambda,\theta,u)$ depends continuously on λ_2 [23], and that λ_2 satisfies a LDP under P_{N-1}^x [7] with good rate function which is infinite above x and below 2, and otherwise given by $\beta(I(y) - \inf I)$ we deduce by Varadhan's Lemma [15, Theorem 4.3.1] and (4.1) that

$$\lim_{N\to\infty} \frac{1}{N} \log \int \prod_{i=2}^N |x-\lambda_i|^\beta I_N(\lambda,\theta,u) dP_{N-1}^x(\lambda)$$
$$= \beta \int \log|x-y| d\sigma(y) + \sup_{2\le y\le x} \{\beta J(\sigma, \theta(1-u), y) - \beta(I(y) - \inf I)\} \quad (5.1)$$

Therefore, plugging (3.4), (3.6) and the above estimate (5.1) in (3.1), we deduce that the density of the joint law of $(\lambda_1, |v_1(1)|^2)$ is approximately given by

$$\frac{dP_N(x,u)}{dxdu} = \frac{1}{Z_N'} e^{-N\beta\left(I(x) - \frac{1}{2}\theta xu - \sup_{2\le y\le x}\{J(\sigma,\theta(1-u),y) - I(y) + \inf I\} + o(1)\right)} (1-u)^{\beta(N-1)/2-1} u^{\frac{\beta}{2}-1}$$

where $o(1)$ goes to zero when N goes to infinity. The final result follows by Laplace's method (and the continuity of the limiting spherical integral [23]).

6 Proof of Theorem 2.3

The law of Y is given by

$$d\mathbb{P}_N(Y) = \frac{1}{Z_N} \exp\left\{-\frac{\beta N}{4} \mathrm{Tr}(Y - \theta vv^T)^2\right\} dY = \frac{1}{\tilde{Z}_N} \exp\left\{-\frac{\beta N}{4} \mathrm{Tr}(Y)^2 + \frac{\beta N}{2} \theta \langle v, Yv\rangle\right\} dY$$

and therefore, since $\langle v, Yv\rangle = \sum \lambda_i |v_i(1)|^2$, the joint law of $\{\lambda_i, |v_i(1)|^2\}$ for $i = 1, \ldots, n$ is given, for $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{u} = (u_1, \ldots, u_n)$, by

$$dP_N^n(\bar{x}, \bar{u}) = \frac{1}{\bar{Z}_N} e^{-\frac{\beta N}{4}\sum_{i=1}^n x_i^2 + \frac{\beta N}{2}\theta \sum_{i=1}^n x_i u_i} I_N^n(\bar{x}, \theta, \bar{u}) \prod_i dx_i dB_1(\bar{u})$$

where $I_N^n(x,\theta,u)$ equals

$$\int \prod_{i=1}^{n}\prod_{k=n+1}^{N}|x_i-\lambda_k|^\beta \prod_{i<j}^{n-1}|x_i-x_j|^\beta \mathbb{E}[e^{\frac{N}{2}\beta\theta\sum_{i=n+1}^{N}\lambda_i|v_i(1)|^2}||v_i|^2, i\le n](\bar u)dP_{N-n}^{\bar x}(\lambda)\,.$$

Here we have denoted

- $\mathbb{E}[.||v_i|^2, i\le n](\bar u)$ the expectation on $|v_i(1)|^2, i>n$ conditionally to $\{|v_i(1)|^2 = u_i, 0\le i\le n-1\}$.
- If $\bar x=\{x_1\ge x_2\cdots\ge x_n\}$,

  $$dP_{N-n}^{\bar x}(\lambda)=\frac{1}{Z_{N-1}^\infty}\prod_{n+1\le i<j\le N}|\lambda_i-\lambda_j|^\beta e^{-\frac{\beta N}{4}\sum_{i=n+1}^{N}\lambda_i^2}\prod_{i=n+1}^{N}1_{\lambda_i\le x_n}d\lambda_i\,,$$

  if

  $$Z_{N-n}^\infty=\int\prod_{n+1\le i<j\le N}|\lambda_i-\lambda_j|^\beta e^{-\frac{\beta N}{4}\sum_{i=n+1}^{N}\lambda_i^2}\prod_{i=n+1}^{N}1_{\lambda_i\le x_n}d\lambda_i\,.$$

- $dB_1(\bar u)$ the distribution of $\{|v_i(1)|^2, 1\le i\le n\}$.

The analysis of the expressions above allows to extend straightforwardly Theorem 2.1 to 2.3. This follows from four remarks:

1. The term $\prod_{i<j}^{n-1}|x_i-x_j|$ does not lead to any contribution to the GRF as long as the x_is are distinct.
2. The spherical integral $\mathbb{E}[e^{\frac{N}{2}\beta\theta\sum_{i=n+1}^{N}\lambda_i|v_i(1)|^2}||v_i(1)|^2, i\le n]$ is performed on $(v_i(1), i\ge n+1)$ that are uniform on the sphere $\mathbb{S}^{N-n-1}$ with radius given by

   $$\sum_{i=n+1}^{N}|v_i(1)|^2=1-\sum_{i=1}^{n}u_i$$

   As a consequence, one can write

   $$v_i(1)=w_i\sqrt{1-\sum_{i=1}^{n}u_i}$$

   where $(w_i)_{1\le i\le N-n}$ is uniform on the sphere $\mathbb{S}^{N-n-1}$ with radius one, independent from $\bar u$. Therefore the spherical integral is the same one evaluated for Theorem 2.1 with u replaced by $\sum_{i=1}^{n}u_i$.
3. Conditionally to $\bar u=(|v_1|^2,\dots,|v_n|^2)$, $(v_{n+1},\dots,v_N)$ follows the uniform law on the sphere of radius $\sqrt{1-\sum u_i}$. Therefore, the contribution to the total GRF due to $dB_1(\bar u)$ is the same one of Theorem 2.1 with u replaced by $\sum_{i=1}^{n}u_i$.
4. The above implies by Laplace method the weak large deviations principle at any strictly ordered sequence of points $x_1>\cdots>x_n$, that is

   $$\limsup_{\delta\to0}\limsup_{N\to\infty}\frac{1}{N}\log P_N(\cap_{1\le i\le n}\{|\lambda_i-x_i|+||v_i(1)|^2-u_i|\le\delta\})=-I(\bar x,\bar u)\,,$$

   and the same when the limsup are replaced by liminf. We deduce the same case for x_i ordered and eventually equal by taking approximating sequences $\underline{x}_i^\delta$ and $\bar x_i^\delta$ which are strictly ordered and such that

   $$\begin{aligned}&P_N(\cap_{1\le i\le n}\{|\lambda_i-\underline{x}_i^\delta|+||v_i(1)|^2-u_i|\le\delta/2\})\\&\quad\le P_N(\cap_{1\le i\le n}\{|\lambda_i-x_i|+||v_i(1)|^2-u_i|\le\delta\})\\&\qquad\le P_N(\cap_{1\le i\le n}\{|\lambda_i-\bar x_i^\delta|+||v_i(1)|^2-u_i|\le2\delta\})\end{aligned}$$

By the previous bounds we deduce that

$$-I(\underline{x}^{\delta},\bar{u})\leq\limsup_{N\to\infty}\frac{1}{N}\log P_N(\cap_{1\leq i\leq n}\{|\lambda_i-x_i|+||v_i(1)|^2-u_i|\leq\delta\})\leq-\inf_{(\bar{y},\bar{u})\in B^{\delta}}I(\bar{y},\bar{u})$$

where $B^{\delta}=\cap_{1\leq i\leq n}\{|y_i-\bar{x}_i^{\delta}|+||v_i(1)|^2-u_i|\leq 2\delta\}$. The continuity of I in the x_i allows to conclude by letting δ going to zero.

7 Study of the rate function and proof of Proposition 2.2

We can give a more explicit formula of the rate function by noticing that the supremum

$$H(x,\theta(1-u))=\sup_{2\leq y\leq x}\{J(\sigma,\theta(1-u),y)-\frac{y^2}{4}+\int\log|y-t|d\sigma(t)\}$$

was already studied in [23]. In the notations of [23], we are maximizing $-\frac{1}{2}F^2_{\theta(1-u)}(y)$ on $y\in[2,x]$. According to [23, Section 3.2] of this paper we find that

- If $\theta(1-u)\geq 1$, the maximum is achieved at

  $$y(u)=\theta(1-u)+\frac{1}{\theta(1-u)}\,,$$

  or at x, if x is smaller than $y(u)$, and

  $$H(x,\theta(1-u))=-\frac{1}{2}F^2_{\theta(1-u)}(\inf(y(u),x)).$$

- If $\theta(1-u)\leq 1$, we are optimizing a decreasing function and therefore the maximum is taken at 2 and

  $$H(x,\theta(1-u))=J(\sigma,\theta(1-u),2)+C'=\frac{\theta^2(1-u)^2}{4}+C'$$

  with $C'=-1+\int\log|2-x|d\sigma(x)\,.$

Note that these two cases correspond to different asymptotic behaviors of λ_{N-1} when λ_N goes to x and $|v_N(1)|^2$ goes to u: if $\theta(1-u)$ is larger than 1, λ_{N-1} goes to $\inf(y(u),x)$, and otherwise to 2.

We can therefore study the optimizer in u of I_β for a given x.

- For $\theta(1-u)\geq 1$, the contribution to the GRF that depends on u and that we have to minimize reads:

  $$-\frac{1}{2}\theta(x-\inf(y(u),x))u-\frac{1}{2}\int\log|\inf(y(u),x)-t|d\sigma(t)+\frac{\inf(y(u),x)^2}{4}$$

  which is independent of u if $x\leq y(u)$. If $x\geq y(u)$ the total derivative of this expression is simply

  $$-\frac{1}{2}\theta(x-\inf(y(u),x))\leq 0$$

  since the partial derivative with respect to $y(u)$ is zero ($y(u)$ is an extremum). As a consequence, the expression above is a decreasing function of u in the entire available range of u.

- for $\theta(1-u)\leq 1$, the contribution to the GRF that depends on u and that we have to minimize reads:

  $$-\frac{1}{2}\theta xu-\frac{\theta^2(1-u)^2}{4}-\frac{1}{2}\log|1-u|$$

We find that the supremum is taken for $\theta > 1$ at $u_{\theta,x}$ given by

$$1 - u_{\theta,x} = \frac{\theta x - \sqrt{(\theta x)^2 - 4\theta^2}}{2\theta^2} = \frac{x - \sqrt{x^2-4}}{2\theta}\,.$$

$\theta(1-u_{\theta,x})$ belongs to $[0,1]$ for all $x \geq 2$, hence u_θ is in $[1-\frac{1}{\theta},1]$. For $\theta < 1$ and for all $x \geq 2$, the supremum is taken at $u_{\theta,x}$ if it is positive or at $u=0$ otherwise.

Collecting all previous results proves Proposition 2.2.

8 The case of Wishart matrices

In the case of $M\times M$ Wishart matrix $W = \Sigma^{1/2}YY^*\Sigma^{1/2}$, we can diagonalize the matrix $W = UD(\lambda)U^*$ where $D(\lambda)$ is a diagonal matrix with entries given by the eigenvalues and U the matrix of the eigenvectors. We start from the law of $L = YY^*$ which reads [20]:

$$d\mathbb{P}_M^W(L) = \frac{1}{Z_M^L} e^{-\frac{\beta}{2}N\mathrm{Tr}(L)}\det(L)^{\frac{\beta}{2}(N-M+1)-1}dL$$

where dL denotes the Lebesgue measure on the entries of L. By a standard change of variables we obtain the joint law of the eigenvalues of W and U:

$$d\mathbb{P}_M^W(\lambda,U) = \frac{1}{Z_M^W}\prod_{1\leq i<j\leq M}|\lambda_i-\lambda_j|^\beta e^{-\frac{\beta N}{2}\mathrm{Tr}(UD(\lambda)U^*\Sigma^{-1})}\prod_{1\leq i\leq M}\lambda_i^{\frac{\beta}{2}(N-M+1)-1}1_{\lambda_i\geq 0}d\lambda_i dU$$

where dU is the Haar measure on the unitary group $U(M)$. Writing that $\Sigma^{-1} = I - \frac{\gamma}{\gamma+1}ee^*$ we deduce that

$$d\mathbb{P}_M^W(\lambda,U) = \frac{1}{Z_N^W}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-\frac{N\beta}{2}\sum\lambda_i + \frac{N\beta\gamma}{2(1+\gamma)}\langle e,UDU^*e\rangle}\prod_{1\leq i\leq N}\lambda_i^{\frac{\beta}{2}(N-M+1)-1}1_{\lambda_i\geq 0}d\lambda_i dU$$

and we conclude that the joint law P_M^W of the maximum eigenvalue λ_1 and $|\langle e,u_1\rangle|^2$ is the measure on $\mathbb{R}^+\times[0,1]$ given by

$$\frac{dP_M^W(x,u)}{dxdB_M(u)} = \frac{1}{\bar{Z}_M^W}e^{-\frac{\beta N}{2}x + \frac{N\beta\gamma}{2(1+\gamma)}xu}x^{\frac{\beta}{2}(N-M+1)-1}\int\prod_{2\leq i\leq M}|\lambda_i - x|^\beta I_M(\lambda,\frac{\gamma}{1+\gamma}\frac{N}{M},u)d\mathbb{P}_{M-1}^{W,x}(\lambda)$$

where $I_M(\lambda,\theta,u)$ is the spherical integral as defined in (3.2) and

$$d\mathbb{P}_{M-1}^{W,x} = \frac{1}{Z_N^{W,\infty}}\prod_{2\leq i<j\leq M}|\lambda_i-\lambda_j|^\beta\prod_{2\leq i\leq M}1_{0\leq\lambda_i\leq x}e^{-\frac{N\beta}{2}\lambda_i}\lambda_i^{\frac{\beta}{2}(N-M+1)-1}d\lambda_i$$

where the constant $Z_N^{W,\infty}$ is chosen so that $\mathbb{P}_{M-1}^{W,\infty}$ is a probability measure. B_M is the Beta-distribution of (3.6) (with dimension M instead of N). The rest of the proof is similar to the Wigner case.

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