

Weak monotone rearrangement on the line*

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Abstract

Weak optimal transport has been recently introduced by Gozlan et al. The original motivation stems from the theory of geometric inequalities; further applications concern numerics of martingale optimal transport and stability in mathematical finance.

In this note we provide a complete geometric characterization of the *weak* version of the classical monotone rearrangement between measures on the real line, complementing earlier results of Alfonsi, Corbetta, and Jourdain.

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1 Introduction

Recently, there has been a growing interest in weak transport problems as introduced by Gozlan, Roberto, Samson and Tetali [14]. While the original motivation mainly stems from applications to geometric inequalities (cf. Marton [16, 15] and Talagrand [21, 22]), weak transport problems appear also in a number of further topics, including martingale transport [2, 3, 9, 5], the causal transport problem [6, 1], and stability in mathematical finance [4].

1.1 Framework and main results

Write $\Pi(\mu, \nu)$ for the set of couplings between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Starting with the seminal article of Gangbo-McCann [11] problems of the form

$$W_\theta(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} \theta(x - y) \pi(dx, dy), \quad (1.1)$$

where $\theta: \mathbb{R}^d \rightarrow \mathbb{R}$ denotes a convex function have received particular attention in optimal transport. A related problem that is considered in the weak transport literature is

$$V_\theta(\mu, \nu) := \inf_{\mu^* \leq_c \nu} W_\theta(\mu, \mu^*). \quad (1.2)$$

Here \leq_c denotes convex order, i.e. $\mu \leq_c \nu$ iff $\int \varphi d\mu \leq \int \varphi d\nu$ for all convex $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$.

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The problem (1.2), and in particular its one dimensional version, is investigated in [2, 14, 13, 19, 17, 20, 10, 12, 7, 4]. The main purpose of this note is to give a complete geometric characterization of the optimizer μ^* in one dimension.

We write $\mathcal{P}_\rho(\mathbb{R}^d)$ for the set of probability measures on \mathbb{R}^d which have finite ρ -moment.

Definition 1.1. Fix $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$. We call a function $S: \mathbb{R} \rightarrow \mathbb{R}$ admissible if it satisfies

- (i) S is increasing,
- (ii) S is 1-Lipschitz,
- (iii) $S(\mu) \leq_c \nu$.

Theorem 1.2. Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$. There exists an admissible T (μ -a.s. unique) which is maximal in the sense that $S(\mu) \leq_c T(\mu)$ for every other admissible S .

If $\theta: \mathbb{R} \rightarrow \mathbb{R}$ is convex then $\mu^* := T(\mu)$ is an optimizer of (1.2). If $\theta: \mathbb{R} \rightarrow \mathbb{R}$ is strictly convex and $V_\theta(\mu, \nu)$ is finite, $T(\mu)$ is the unique optimizer of (1.2).

We call the (μ -a.s. unique) map T in Theorem 1.2 the *weak monotone rearrangement*.

A consequence of Theorem 1.2 is that the optimizer of (1.2) does not depend on the choice of the convex function θ . We find this fact non-trivial as well as remarkable and highlight that it is not new: different independent proofs were given by Gozlan, Roberto, Samson and Tetali [14], Alfonsi, Corbetta, Jourdain [2] and Shu [20]. For $d > 1$ the optimizer of (1.2) may depend on θ , see [2, Example 2.4].

The map T can be explicitly characterized in geometric terms using the notion of irreducibility introduced in [8]: Measures $\eta, \nu \in \mathcal{P}_1(\mathbb{R})$ are in convex order iff their potential functions satisfy

$$u_\eta(y) := \int_{\mathbb{R}} |x - y| \eta(dx) \leq \int_{\mathbb{R}} |x - y| \nu(dx) =: u_\nu(y). \quad (1.3)$$

By continuity, the set U where this inequality is strict is open. Hence $U = \bigcup_n I_n$, where (I_n) is an at most countable family of disjoint open intervals; these intervals I_n are called *irreducible* with respect to (η, ν) .

Theorem 1.3. The weak monotone rearrangement T of $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ is the unique admissible map which has slope 1 on each interval $T^{-1}(I)$, where I is irreducible w.r.t. $(T(\mu), \nu)$.

We emphasize that [2, Proposition 4.3] yields that every optimiser μ^* of (1.2) is of the form $T(\mu)$ for an admissible map T and thus implies a part of Theorem 1.2. Likewise, it can be seen from [2, Proposition 4.3] that this mapping T has slope 1 on each interval $T^{-1}(I)$, I irreducible w.r.t. $(T(\mu), \nu)$. The novelty of Theorem 1.3 is that there exists only one admissible mapping with this irreducibility property.

1.2 Connection with martingale transport plans

Intuitively, the irreducible intervals of (η, ν) are the components where we need to ‘expand’ η in order to transform it into ν . In this sense Theorem 1.3 asserts that the mass of μ can *either* concentrate between μ and $T(\mu)$, or it can be expanded between $T(\mu)$ and ν (see Figure 1).

To make this precise, we recall from [14] that (1.2) can be reformulated as

$$V_\theta(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} \theta \left(x - \int_{\mathbb{R}^d} y \pi_x(dy) \right) \mu(dx), \quad (1.4)$$

where $(\pi_x)_{x \in \mathbb{R}^d}$ denotes a regular disintegration of coupling π w.r.t. its first marginal μ .

The set of optimizers of (1.4) is also straightforward to express in terms of T : Write $\Pi_M(\eta, \nu)$ for the set of *martingale couplings* (or martingale transport plans), i.e. $\pi \in$

$\Pi(\eta, \nu)$ which satisfy $\text{barycenter}(\pi_x) = x$, η -a.s. By Strassen's theorem $\Pi_M(\eta, \nu)$ is nonempty iff $\eta \leq_c \nu$. Using this notation, $\pi \in \Pi(\mu, \nu)$ is optimal iff there exists a martingale coupling $\pi^M \in \Pi_M(T(\mu), \nu)$ such that π is the concatenation of the transports described by T and π^M :

$$\pi(A \times B) = \int_A \mu(dx) \pi_{T(x)}^M(B). \tag{1.5}$$

Any $\pi^M \in \Pi_M(\eta, \nu)$ can be decomposed based on the family of irreducible intervals $(I_n)_n$: denoting $F := (\cup_n I_n)^c$ by [8, Appendix A] we have

$$\pi^M = \sum_n \pi_{|I_n \times \bar{I}_n}^M + (\text{Id}, \text{Id})(\mu|_F). \tag{1.6}$$

Plainly, (1.6) asserts that any martingale transport plan can move mass only within the individual irreducible intervals, whereas particles $x \in F$ have to stay put.

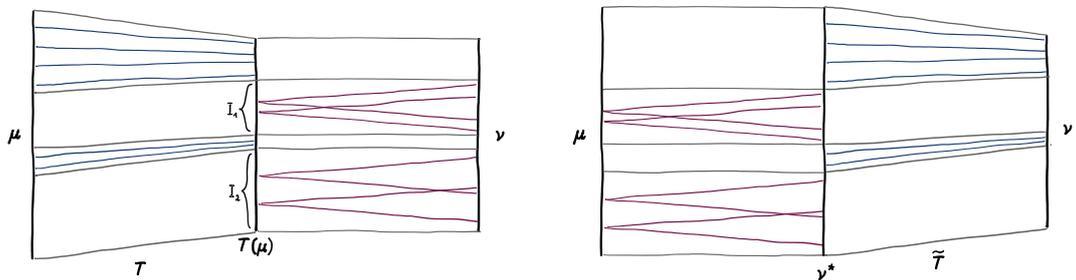


Figure 1: The solutions to (1.2)/(1.4) and (1.7), respectively. Blue lines depict contractive parts of the map, purple lines depict areas with (non trivial) martingale transport.

1.3 A reverse problem

Alfonsi, Corbetta, and Jourdain [2] proved that the same value is obtained when reversing the order of transport and convex order relaxation in (1.2), i.e.

$$V_\theta(\mu, \nu) = \inf_{\mu \leq_c \nu^*} W_\theta(\nu^*, \nu); \tag{1.7}$$

moreover they find ([2, Proposition 4.3]) a monotone mapping which is optimal between the optimizer of (1.7) and ν as well as between μ and the optimizer of (1.2).

As a counterpart to Theorems 1.2 and 1.3 we establish the following result, strengthening the connection between (1.2) and (1.7).

Theorem 1.4. *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$. Then there exists a unique \leq_c -smallest measure ν^* , $\mu \leq_c \nu^*$, which can be pushed onto ν by an increasing 1-Lipschitz mapping. Moreover we have*

1. *If $\theta: \mathbb{R} \rightarrow \mathbb{R}$ is convex then ν^* is an optimizer of (1.7). If $\theta: \mathbb{R} \rightarrow \mathbb{R}$ is strictly convex and $V_\theta(\mu, \nu)$ is finite, ν^* is the unique optimizer of (1.7).*
2. *There exists a (ν^* -unique) increasing 1-Lipschitz mapping \tilde{T} which pushes ν^* onto ν ; \tilde{T} has slope 1 on each interval I , where I is irreducible w.r.t. (μ, ν^*) .*
3. *\tilde{T} is the weak monotone rearrangement between μ and ν .*

As in the previous section, (1.7) could be interpreted as a concatenation of a martingale transport with the weak monotone rearrangement.

1.4 An auxiliary result

We close this introductory section with an auxiliary result that will be important in the proofs of our results. Since it might be of independent interest we provide the d -dimensional version. We denote the topology induced by the ρ -Wasserstein distance on the space of probability measures on \mathbb{R}^d by \mathcal{W}_ρ .

Theorem 1.5 (Stability). *Let $1 \leq \rho < \infty$, $(\mu^k)_{k \in \mathbb{N}} \in \mathcal{P}_\rho(\mathbb{R}^d)^{\mathbb{N}}$, $(\nu^k)_{k \in \mathbb{N}} \in \mathcal{P}_1(\mathbb{R}^d)^{\mathbb{N}}$ and $\theta: \mathbb{R}^d \rightarrow \mathbb{R}$ convex and such that for some constant $c > 0$ it holds*

$$\theta(x) \leq c(1 + |x|^\rho) \quad \forall x \in \mathbb{R}^d. \tag{1.8}$$

If $\mu^k \rightarrow \mu$ in \mathcal{W}_ρ and $\nu^k \rightarrow \nu$ in \mathcal{W}_1 , then $\lim_k V_\theta(\mu^k, \nu^k) = V_\theta(\mu, \nu)$. If additionally θ is strictly convex, we have that

1. $\arg \min_{\eta \leq_c \nu^k} W_\theta(\mu^k, \nu^k) \rightarrow \arg \min_{\eta \leq_c \nu} W_\theta(\mu, \nu)$ in \mathcal{W}_1 ,
2. the sequence of maps T^k , where $T^k(\mu) \leq_c \nu$ and $V_\theta(\mu, \nu^k) = W_\theta(\mu, T^k(\mu))$, converges in μ -probability to T , where $T(\mu) \leq_c \nu$ and $V_\theta(\mu, \nu) = W_\theta(\mu, T(\mu))$.

2 C-Monotonicity implies geometric characterization

In this part we prove the following

Theorem 2.1. *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ and $\theta: \mathbb{R} \rightarrow \mathbb{R}$ strictly convex. If $V_\theta(\mu, \nu)$ yields a finite value, for any optimizer $\pi \in \Pi(\mu, \nu)$ of (1.4), the map*

$$T(x) := \int_{\mathbb{R}} y \pi_x(dy),$$

is independent of the specific optimizer π . This map is further admissible in the sense of Definition 1.1 and it has slope 1 on $T^{-1}(I)$ if I is an irreducible interval for $(T(\mu), \nu)$.

To prove Theorem 2.1, we need some further properties connected to irreducibility:

Lemma 2.2. *Suppose $\{u_\mu < u_\nu\} =: (a, b)$. Then for any $\pi \in \Pi_M(\mu, \nu)$, any regular disintegration $(\pi_x)_{x \in \mathbb{R}}$ w.r.t. μ and any $c \in (a, b)$ such that $\mu((a, c)) > 0$, $\mu((c, b)) > 0$, there are $x \in (a, c]$, $y \in [c, b)$, $x \neq y$, such that the supports of π_x and π_y overlap, i.e.*

$$\text{int}(\text{co}(\text{supp}(\pi_x))) \cap \text{co}(\text{supp}(\pi_y)) \cup \text{int}(\text{co}(\text{supp}(\pi_y))) \cap \text{co}(\text{supp}(\pi_x)) \neq \emptyset. \tag{2.1}$$

Proof. To show this assertion, we assume the opposite. So there exist $c \in \{u_\mu < u_\nu\}$, $\pi \in \Pi_M(\mu, \nu)$ with fixed disintegration $(\pi_x)_{x \in \mathbb{R}}$ w.r.t. μ and

$$\mu((a, c)) > 0, \quad \mu((c, b)) > 0,$$

so that for all x, y with $a < x \leq c \leq y < b$, $x \neq y$, we have

$$\text{int}(\text{co}(\text{supp}(\pi_x))) \cap \text{co}(\text{supp}(\pi_y)) \cup \text{int}(\text{co}(\text{supp}(\pi_y))) \cap \text{co}(\text{supp}(\pi_x)) = \emptyset. \tag{2.2}$$

Since π is a martingale coupling, and by (2.2), there exists $d \in (a, b)$ with

$$\begin{aligned} \text{supp}(\pi_x) &\subseteq (-\infty, d] \quad \text{for } \mu\text{-a.e. } x < c, \\ \text{supp}(\pi_y) &\subseteq [d, \infty) \quad \text{for } \mu\text{-a.e. } y > c. \end{aligned} \tag{2.3}$$

Write d_+ for the largest and d_- for the smallest d such that (2.3) holds. Note then that $d_-, d_+ \in (a, b)$. We have either $\text{supp}(\pi_c) \subseteq [d_-, d_+]$ or $\mu(\{c\}) = 0$, which in any case implies $\mu([c \wedge d_-, c \vee d_+] \setminus \{c\}) = 0$. Thus, we infer

$$\begin{aligned} 1 &= \pi((-\infty, c) \times (-\infty, d_-) \cup \{c\} \times [d_-, d_+] \cup (c, \infty) \times (d_+, \infty)) \\ &= \pi((-\infty, d_-) \times (-\infty, d_-] \cup \{c\} \times [d_-, d_+] \cup (d_+, \infty) \times [d_+, \infty)), \end{aligned}$$

and we conclude by contradicting $u_\mu(d_-) < u_\nu(d_-)$ since

$$\begin{aligned} \int_{\mathbb{R}} |x - d_-| \mu(dx) &= \int_{(a, d_-)} |x - d_-| \mu(dx) + \int_{[d_-, b)} |x - d_-| \mu(dx) \\ &= \int_{(a, d_-)} |y - d_-| \pi_x(dy) \mu(dx) + \int_{[d_-, b)} |y - d_-| \pi_x(dy) \mu(dx) \\ &= \int_{\mathbb{R}} |x - d_-| \nu(dx). \end{aligned} \quad \square$$

Lemma 2.3. *Let $p, q \in \mathcal{P}_1(\mathbb{R})$ have overlapping supports (cf. (2.1)). Then there exists a continuous map $[0, 1] \ni \alpha \mapsto (p_\alpha, q_\alpha) \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$ such that*

$$p_\alpha + q_\alpha = p + q \quad \forall \alpha \in [0, 1], \quad (2.4)$$

and such that for some $\beta \in (0, 1)$ the functions

$$[\beta, 1] \ni \alpha \mapsto \int_{\mathbb{R}} z p_\alpha(dz), \quad [\beta, 1] \ni \alpha \mapsto \int_{\mathbb{R}} z q_\alpha(dz) \quad (2.5)$$

are strictly increasing and decreasing, respectively.

Proof. Let $\alpha \in [0, 1]$ and define the inverse distribution functions by

$$s_\alpha := \inf\{x \in \mathbb{R} : F_p(x) \geq \alpha\}, \quad t_\alpha := \inf\{x \in \mathbb{R} : F_q(x) \geq \alpha\},$$

where F_p and F_q denote the cumulative distribution functions of p and q , respectively. Define two auxiliary measures

$$\tilde{p}_\alpha := p|_{(-\infty, s_\alpha)} + (\alpha - F_p(s_\alpha -))\delta_{\{s_\alpha\}}, \quad \tilde{q}_\alpha := q|_{(-\infty, t_\alpha)} + (\alpha - F_q(t_\alpha -))\delta_{\{t_\alpha\}}.$$

We define the probability measure p_α by taking the leftmost α mass of p and the leftmost $1 - \alpha$ mass of q , i.e., let p_α be given by $p_\alpha := \tilde{p}_\alpha + \tilde{q}_{1-\alpha}$. To satisfy (2.4), we set $q_\alpha := p + q - p_\alpha$, which also yields continuity of $\alpha \mapsto (p_\alpha, q_\alpha)$. Since the supports of p and q are overlapping, see (2.1), we find constants $c_1, c_2 \in \mathbb{R}$ with $c_1 > c_2$ such that

$$\alpha_1 := p([c_1, +\infty)) > 0 \text{ and } \alpha_2 := q((-\infty, c_2]) > 0.$$

Let $0 < \alpha_3 < \alpha_1 \wedge \alpha_2$, then for any $1 - \alpha_3 \leq \alpha < \alpha' \leq 1$ we have

$$\begin{aligned} \int_{\mathbb{R}} z p_{\alpha'}(dz) - \int_{\mathbb{R}} z p_\alpha(dz) &= \int_{\mathbb{R}} z (\tilde{p}_{\alpha'} - \tilde{p}_\alpha)(dz) + \int_{\mathbb{R}} z (\tilde{q}_{1-\alpha'} - \tilde{q}_{1-\alpha})(dz) \\ &\geq \int_{\mathbb{R}} c_1 (\tilde{p}_{\alpha'} - \tilde{p}_\alpha)(dz) - \int_{\mathbb{R}} c_2 (\tilde{q}_{1-\alpha} - \tilde{q}_{1-\alpha'})(dz) \\ &= (\alpha' - \alpha)(c_1 - c_2) > 0, \end{aligned}$$

and conclude that for $\beta := 1 - \alpha_3$ the maps defined in (2.5) are strictly monotone. \square

An important tool in the proof of Theorem 2.1 is C -monotonicity, a concept which was introduced for the weak optimal transport problem in [5, 12, 7]. In the current setting it takes the following form:

Definition 2.4 (C -monotonicity). *A coupling $\pi \in \Pi(\mu, \nu)$ is C -monotone (with cost $C(x, p) = \theta(x - \int y p(dy))$) if there exists a measurable set $\Gamma \subseteq \mathbb{R}$ with $\mu(\Gamma) = 1$, such that for any finite number of points x_1, \dots, x_N in Γ and probability measures $m_1, \dots, m_N \in \mathcal{P}_1(\mathbb{R})$ with $\sum_{i=1}^N m_i = \sum_{i=1}^N \pi_{x_i}$, we have*

$$\sum_{i=1}^N \theta(x_i - \int y \pi_{x_i}(dy)) \leq \sum_{i=1}^N \theta(x_i - \int y m_i(dy)).$$

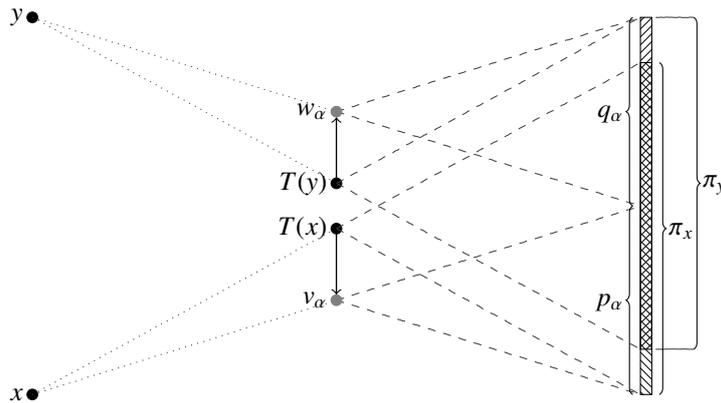


Figure 2: Sketch of usage of Lemma 2.2 and Lemma 2.3 to find contradiction to C -monotonicity of an C -optimal coupling π .

Proof of Theorem 2.1. Let π^* and $\tilde{\pi}^*$ be optimal for the weak optimal transport problem (1.4). Clearly, $\frac{\pi^* + \tilde{\pi}^*}{2}$ constitutes another optimizer of (1.4). Strict convexity of θ yields that

$$\int_{\mathbb{R}} y \pi_x^*(dy) = \int_{\mathbb{R}} y \frac{\pi^* + \tilde{\pi}^*}{2}(dy) = \int_{\mathbb{R}} y \tilde{\pi}_x^*(dy) \quad \mu\text{-a.e. } x,$$

showing that T is independent of the choice of the optimizer.

By the monotonicity principle [7, Theorem 5.2] π^* is C -monotone, therefore, there exists a set $\Gamma \subseteq \mathbb{R}$ with $\mu(\Gamma) = 1$ and such that for all $x, y \in \Gamma$, $p_1, p_2 \in \mathcal{P}_1(\mathbb{R})$ satisfying $\pi_x^* + \pi_y^* = p_1 + p_2$, we have

$$\theta\left(x - \int_{\mathbb{R}} z \pi_x^*(dz)\right) + \theta\left(y - \int_{\mathbb{R}} z \pi_y^*(dz)\right) \leq \theta\left(x - \int_{\mathbb{R}} z p_1(dz)\right) + \theta\left(y - \int_{\mathbb{R}} z p_2(dz)\right). \quad (2.6)$$

Let $T(x) = \int_{\mathbb{R}} y \pi_x^*(dy)$. If $x < y$ and $T(y) < T(x)$ we would have by strict convexity that

$$\frac{\theta(x - T(y)) - \theta(x - T(x))}{x - T(y) - (x - T(x))} < \frac{\theta(y - T(y)) - \theta(y - T(x))}{y - T(y) - (y - T(x))},$$

which then would contradict (2.6) for the choice $p_1 = \pi_y^*$ and $p_2 = \pi_x^*$. Hence T is μ -almost surely increasing. Letting $x, y \in \Gamma$, for any $\alpha \in (0, 1]$ we define the convex combination of π_x^* and π_y^* by

$$p_1^\alpha = (1 - \alpha)\pi_x^* + \alpha\pi_y^*, \quad p_2^\alpha = \alpha\pi_x^* + (1 - \alpha)\pi_y^*.$$

Due to T being a monotone map, we can assume that $x \leq y$ and $T(x) \leq T(y)$. Plugging p_1^α and p_2^α into (2.6), dividing this inequality by α , and sending $\alpha \searrow 0$, yields

$$\left(\partial_+ \theta(x - T(x)) - \partial_- \theta(y - T(y))\right)(T(x) - T(y)) \geq 0, \quad (2.7)$$

where $\partial_+ \theta$ denotes the right-hand side derivative and $\partial_- \theta$ the left-hand side derivative, respectively. By convexity of θ , (2.7) is equivalent to

$$(x - T(x) - y + T(y))(T(x) - T(y)) \geq 0.$$

Hence, $|x - y| \geq |T(x) - T(y)|$ and T is μ -a.e. 1-Lipschitz.

Further, we have that $T(\mu) \leq_c \nu$ and $\tilde{\pi} := (T, id)(\pi^*) \in \Pi_M(T(\mu), \nu)$. Since the cost only depends on x and $T(x)$, but not on the martingale part (i.e., the part which

transports mass from $T(x)$ according to $\pi_x^*(dy)$, it could happen that T is constant on some set I , i.e., $\forall x, y \in I$ we have

$$\int_{\mathbb{R}} z\pi_x^*(dz) = \int_{\mathbb{R}} z\pi_y^*(dz),$$

but π_x^* and π_y^* do not have to match necessarily. Hence, by choosing some other optimizer (the one which averages on I over all π_z^* appropriately), we may assume that π_x^* is constant on I . Therefore, we assume without loss of generality $\bar{\pi}_{T(x)} = \pi_x^*$, μ -a.e. Let $(I_k)_{k \in \mathbb{N}}$ be the intervals given by the decomposition of $(T(\mu), \nu)$ into irreducible intervals. Assume that there is an interval I_k so that on $T^{-1}(I_k)$ the map T does not have μ -a.s. slope 1. Then Lemma 2.2 provides two points \tilde{x}, \tilde{y} in $T(\Gamma \cap I_k)$ and two corresponding points $x, y \in \Gamma \cap I_k$ such that

$$x < y, \quad T(x) = \tilde{x} < \tilde{y} = T(y), \quad x - T(x) > y - T(y),$$

and the overlapping condition (2.1) holds for $\bar{\pi}_{T(x)} = \pi_x^*$, $\bar{\pi}_{T(y)} = \pi_y^*$. Lemma 2.3 allows us to define measures p_α and q_α on \mathbb{R} such that

$$\pi_x^* + \pi_y^* = \bar{\pi}_{T(x)} + \bar{\pi}_{T(y)} = p_\alpha + q_\alpha.$$

For a graphical depiction compare with Figure 2. Hence,

$$T(x) > v_\alpha := \int_{\mathbb{R}} zp_\alpha(dz), \quad T(y) < w_\alpha := \int_{\mathbb{R}} zq_\alpha(dz),$$

are strictly monotone, continuous maps on $[\beta, 1]$. Therefore, we find $\alpha \in (\beta, 1)$ with

$$x - v_1 = x - T(x) < x - v_\alpha \leq y - w_\alpha < y - T(y) = y - w_1.$$

By strict convexity of θ we find

$$\frac{\theta(y - w_\alpha) - \theta(x - T(x))}{y - w_\alpha - x + T(x)} < \frac{\theta(y - T(y)) - \theta(x - v_\alpha)}{y - T(y) - x + v_\alpha},$$

which then yields a contradiction to C -monotonicity:

$$\begin{aligned} \theta\left(x - \int_{\mathbb{R}} z\pi_x^*(dz)\right) + \theta\left(y - \int_{\mathbb{R}} z\pi_y^*(dz)\right) &= \theta(x - T(x)) + \theta(y - T(y)) \\ &> \theta(x - v_\alpha) + \theta(y - w_\alpha) \\ &= \theta\left(x - \int_{\mathbb{R}} zp_\alpha(dz)\right) + \theta\left(y - \int_{\mathbb{R}} zq_\alpha(dz)\right). \quad \square \end{aligned}$$

3 Sufficiency of the geometric characterization

Naturally the question arises whether any map T satisfying the properties in Theorem 2.1 must be optimal. The aim of this section is to establish this:

Theorem 3.1. *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$. Then any coupling $\pi \in \Pi(\mu, \nu)$ for which $T(x) := \int_{\mathbb{R}} y\pi_x(dy)$ is admissible (in the sense of Definition 1.1) with slope 1 on each interval $T^{-1}(I)$, where I is irreducible w.r.t. $(T(\mu), \nu)$, is optimal for (1.4), i.e.,*

$$V_\theta(\mu, \nu) = \int_{\mathbb{R}} \theta(x - T(x))\mu(dx).$$

The proof is based on dual optimizers and their explicit representation. As long as T is strictly increasing, [19, Theorem 2.1] provides dual optimizers to $V_\theta(\mu, T(\mu))$. Investigating dual optimizers further, we are able to show here $V_\theta(\mu, T(\mu)) = V_\theta(\mu, \nu)$. First, Lemma 3.2 helps us to carefully approximate the increasing map T with strictly increasing maps T_ε .

Lemma 3.2. *Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing map, with $T - id$ decreasing. Then, for any $\varepsilon > 0$ there is a strictly increasing map $T_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$, with $T_\varepsilon - id$ decreasing, such that*

$$|T - T_\varepsilon|_\infty \leq \varepsilon,$$

and T is affine with slope 1 on an interval I if and only if T_ε is affine with slope 1 on I .

Proof. Since T is increasing we know that the pre-image of any point under T corresponds to an interval. Therefore, we can find at most countable many, disjoint intervals $(I_k)_{k \in \mathbb{N}}$ of finite length, where $T(I_k)$ is a singleton and T is strictly increasing on the complement, i.e., on $\bigcap_k I_k^c$. For any $\varepsilon > 0$, we define $g_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$

$$g_\varepsilon(x) := \begin{cases} \frac{\varepsilon}{\lambda(I_k)2^k} & \exists k \in \mathbb{N}: x \in I_k, \\ 0 & \text{else.} \end{cases} \quad (3.1)$$

Then the map $T_\varepsilon(x) := T(x) + \int_{-\infty}^x [g_\varepsilon(y) \wedge 1] dy$ satisfies the desired properties. \square

Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing 1-Lipschitz function. Then T induces a unique decomposition of \mathbb{R} into at most countably many maximal, closed, disjoint intervals $(I_k)_k$ and a $(G_\delta$ -set) G such that for all $k \in \mathbb{N}$ the map $T|_{I_k}$ is affine with slope 1 and $T|_G$ is properly contractive, i.e., for any two points $x, y \in G$ we have $|T(x) - T(y)| < |x - y|$. Below we call the intervals $(I_k)_k$ *irreducible* w.r.t. T .

Proof of Theorem 3.1. The convex function θ can be approximated by an increasing sequence of convex Lipschitz functions $(\theta_n)_{n \in \mathbb{N}}$, e.g.

$$\theta_n(x) := \theta(x) 1_{[-n, n]}(x) + [\theta(n) + (x - n)\partial_+\theta(n)] 1_{(n, +\infty)} + [\theta(-n) + (x + n)\partial_-\theta(-n)] 1_{(-\infty, -n)}.$$

By monotone convergence we find¹ $\sup_n V_{\theta_n}(\mu, \nu) = V_\theta(\mu, \nu)$. Indeed, if π^n optimizes $V_{\theta_n}(\mu, \nu)$ and assuming w.l.o.g. that $\pi_n \rightarrow \pi$, then

$$\begin{aligned} \lim_n \int \theta_n(x - \int y \pi_x^n(dy)) \mu(dx) &\geq \lim_n \int \theta_m(x - \int y \pi_x^n(dy)) \mu(dx) \\ &\geq \int \theta_m(x - \int y \pi_x(dy)) \mu(dx), \end{aligned}$$

by [7, Proposition 2.8]. Thus the claim follows by taking the supremum in m .

From this, it suffices to consider the case when θ is Lipschitz continuous. By Lemma 3.2 we find for any $\varepsilon > 0$ a strictly increasing map T_ε , such that $T_\varepsilon - id$ is decreasing, the decompositions of T and T_ε match (i.e., T is affine with slope 1 on an interval I iff T_ε is affine with slope 1 on I), and $|T_\varepsilon - T|_\infty \leq \varepsilon$. Then [19, Theorem 2.1] provides a convex, Lipschitz continuous function $f_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$Rf_\varepsilon(x) := \inf_{y \in \mathbb{R}} f_\varepsilon(y) + \theta(x - y) = f_\varepsilon(T_\varepsilon(x)) + \theta(x - T_\varepsilon(x)),$$

which is even affine on the parts where T_ε is affine. Write $S_\varepsilon(x) = \int_{-\infty}^x [g_\varepsilon(z) \wedge 1] dz$, where g_ε is defined as in (3.1) so $T_\varepsilon = T + S_\varepsilon$. In the following we will show that f_ε is a dual optimizer (cf. the dual problem as in [14, Theorem 2.11]) to the weak transport problem V_θ between the marginals of the coupling π^ε , where π^ε is defined as the push-forward measure of π by the function

$$(x, y) \mapsto (x, y + S_\varepsilon(x)).$$

First, we compute the barycenters of π_x^ε :

$$\int_{\mathbb{R}} y \pi_x^\varepsilon(dy) = \int_{\mathbb{R}} y + S_\varepsilon(x) \pi_x(dy) = T_\varepsilon(x),$$

¹Recall that by [13] the optimizer of the weak transport problem does not depend on the convex cost.

and $T_\varepsilon(\mu) \leq_c \text{proj}_2(\pi^\varepsilon) =: \nu_\varepsilon$. Given the sets $(I_k)_{k \in \mathbb{N}}$ and F from the decomposition of $(T(\mu), \nu)$ into irreducible intervals, we find the sets $(I_k^\varepsilon)_{k \in \mathbb{N}}$ and F_ε from the decomposition of $(T_\varepsilon(\mu), \nu_\varepsilon)$ into irreducible intervals by setting

$$I_k^\varepsilon := I_k + S_\varepsilon(x) \quad \text{for any } y \in I_k \text{ and } x \in T^{-1}(y), \quad F_\varepsilon := \bigcap_k (I_k^\varepsilon)^c.$$

In view of the structure of martingale couplings, see [8, Theorem A.4], we find that for μ -a.e. $x \in T_\varepsilon^{-1}(I_k^\varepsilon)$ we have $\text{supp}(\pi_x^\varepsilon) \subseteq I_k^\varepsilon$ and $\pi_x^\varepsilon = \delta_{T(x)+S_\varepsilon(x)}$ μ -a.e. on F_ε . Since T has slope 1 on each interval I which is irreducible w.r.t. $(T(\mu), \nu)$, we infer the same for the map T_ε and the irreducible intervals of $(T_\varepsilon(\mu), \nu_\varepsilon)$. The next computation establishes duality of the pair $(\pi^\varepsilon, f_\varepsilon)$, where we use affinity of f_ε on the irreducible components of $(T_\varepsilon(\mu), \nu_\varepsilon)$:

$$\begin{aligned} \int_{\mathbb{R}} \theta \left(x - \int_{\mathbb{R}} y \pi_x^\varepsilon(dy) \right) \mu(dx) &= \int_{\mathbb{R}} \theta(x - T_\varepsilon(x)) \mu(dx) = \int_{\mathbb{R}} Rf_\varepsilon(x) - f_\varepsilon(T_\varepsilon(x)) \mu(dx) \\ &= \int_{\mathbb{R}} Rf_\varepsilon(x) \mu(dx) - \int_{\mathbb{R}} f_\varepsilon \left(\int_{\mathbb{R}} y \pi_x^\varepsilon(dy) \right) \mu(dx) \\ &= \int_{\mathbb{R}} Rf_\varepsilon(x) \mu(dx) - \int_{\mathbb{R}} f_\varepsilon(y) \nu_\varepsilon(dy). \end{aligned}$$

This easily proves that π^ε is optimal for the optimal weak transport problem $V_\theta(\mu, \nu_\varepsilon)$. Drawing the limit for $\varepsilon \searrow 0$, we observe

$$V_\theta(\mu, \nu_\varepsilon) = \int_{\mathbb{R}} \theta(x - T_\varepsilon(x)) \mu(dx) \rightarrow \int_{\mathbb{R}} \theta(x - T(x)) \mu(dx).$$

As θ is Lipschitz, we can apply stability Theorem 1.5 and obtain optimality of π . □

4 Geometry of the weak monotone rearrangement

We can summarize Theorems 2.1 and 3.1 as follows: There exists an admissible map T with slope 1 on $T^{-1}(I)$ whenever I is an irreducible interval w.r.t. $(T(\mu), \nu)$, such that π is optimal for (1.4) iff $T(x) = \int y \pi_x(dy)$ (μ -a.s.). We now show that this map is the weak monotone rearrangement and is therefore the maximum in convex order of the set²

$$M(\mu, \nu) := \{S: \mathbb{R} \rightarrow \mathbb{R}: S \text{ is increasing and 1-Lipschitz, } S(\mu) \leq_c \nu\}.$$

Heuristically speaking, if the maximum in convex order of the set $M(\mu, \nu)$ is again given by an increasing, 1-Lipschitz map, then this map is as close as possible to a shifted identity. In turn, this is favourable when trying to find the minimum in convex order of

$$\{(id - S)(\mu): S \in M(\mu, \nu)\},$$

which gives reason to why there should exist a single optimizer to (1.4) for all convex θ .

As preparation to establishing Theorem 1.3, we prove Lemma 4.1 and Lemma 4.2.

Lemma 4.1. *Let $\mu \in \mathcal{P}_1(\mathbb{R})$, $T, S: \mathbb{R} \rightarrow \mathbb{R}$ be increasing maps with*

$$\int_{\mathbb{R}} T(x) \mu(dx) = \int_{\mathbb{R}} S(x) \mu(dx),$$

then the maximum w.r.t. the convex order³ of $T(\mu)$ and $S(\mu)$ is given by $T^(\mu)$ for T^* an increasing map. If in addition T, S are L -Lipschitz with $L > 0$, then T^* is also L -Lipschitz.*

²We abuse terminology here, meaning maximum of $\{S(\mu) : S \in M(\mu, \nu)\}$.

³The maximum in convex order of $\nu, \eta \in \mathcal{P}_1(\mathbb{R})$, denoted $\nu \vee \eta$, satisfies $\nu, \eta \leq_c \nu \vee \eta$ as well as $\nu \vee \eta \leq_c \xi$ whenever $\nu, \eta \leq_c \xi$. One can easily characterize it by the identity $u_{\nu \vee \eta}(\cdot) = u_\nu(\cdot) \vee u_\eta(\cdot)$. Similarly, the minimum of ν and η , denoted $\nu \wedge \eta$, is characterized by $u_{\nu \wedge \eta}$ being equal to the greatest convex minorant of $u_\nu \wedge u_\eta$.

Proof. The maximum of $T(\mu)$ and $S(\mu)$ w.r.t. the convex order is uniquely determined by the maximum of its potential functions, i.e. $u_{S(\mu)} \vee u_{T(\mu)} = u_{S(\mu) \vee T(\mu)}$. The right-hand side derivative of the potential function can be expressed by the cumulative distribution function, namely $\partial_+ u_\mu(x) = 2F_\mu(x) - 1$. By continuity of the potential functions, we find a minimal partition of \mathbb{R} into at most countably many disjoint intervals $((l_k, r_k])_{k \in \mathbb{N}}$, where $u_{T(\mu)} = u_{S(\mu)}$ on $\{l_k, r_k\}$, and restricted onto $(l_k, r_k]$ one of the following holds true:

(a) $u_{S(\mu)}|_{(l_k, r_k]} \leq u_{T(\mu)}|_{(l_k, r_k]}$,

(b) $u_{S(\mu)}|_{(l_k, r_k]} \geq u_{T(\mu)}|_{(l_k, r_k]}$.

Suppose w.l.o.g. (a) holds, then

$$F_{S(\mu) \vee T(\mu)}(x) = F_{T(\mu)}(x) \quad x \in (l_k, r_k]. \tag{4.1}$$

By monotonicity of the maps S and T , we can define T^* on $[\tilde{l}_k, \tilde{r}_k)$ where⁴

$$\tilde{l}_k = \sup T^{-1}(\{l_k\}) \vee \sup S^{-1}(\{l_k\}), \quad \tilde{r}_k = \sup T^{-1}(\{r_k\}) \vee \sup S^{-1}(\{r_k\}),$$

by

$$T^*(x) = \begin{cases} T(x) & x \in T^{-1}((l_k, r_k]) \cap (\tilde{l}_k, \tilde{r}_k], \\ r_k & x \in (\tilde{l}_k, \tilde{r}_k] \setminus T^{-1}((l_k, r_k]). \end{cases} \tag{4.2}$$

Hence, T^* is an increasing map and

$$\begin{aligned} F_{T^*(\mu)}(x) &= F_{T(\mu)}(x) = F_{T(\mu) \vee S(\mu)}(x) \quad x \in (l_k, r_k), \\ F_{T^*(\mu)}(r_k) &= F_{S(\mu)}(r_k) = F_{T(\mu) \vee S(\mu)}(r_k). \end{aligned}$$

Therefore, $F_{T^*(\mu)} = F_{S(\mu) \vee T(\mu)}$ which is equivalent to $S(\mu) \vee T(\mu)$ being given by the map T^* . If T and S are in addition L -Lipschitz, it follows by construction that T^* is L -Lipschitz. \square

Lemma 4.2. *Let $\eta_1 \leq_c \eta_2$ and $T_2(\eta_2) \leq_c T_1(\eta_1)$, where T_1, T_2 are increasing and 1-Lipschitz. Then $(id - T_1)(\eta_1) \leq_c (id - T_2)(\eta_2)$.*

In particular, if T and S are increasing 1-Lipschitz maps s.t. $\int T(x)\mu(dx) = \int S(x)\mu(dx)$, and we denote by R the increasing 1-Lipschitz map with $R(\mu) = S(\mu) \vee T(\mu)$, which exists by Lemma 4.1, then $(id - R)(\mu) \leq_c (id - S)(\mu) \wedge (id - T)(\mu)$.

Proof. By [18, Theorem 3.A.5] we have that for any two probability measures μ and ν on \mathbb{R} the convex order, i.e., $\mu \leq_c \nu$, is characterized by

$$\int_0^\alpha F_\mu^{-1}(u) du \geq \int_0^\alpha F_\nu^{-1}(u) du \quad \forall \alpha \in [0, 1], \tag{4.3}$$

with equality when $\alpha = 1$. For any increasing, continuous map $U: \mathbb{R} \rightarrow \mathbb{R}$, any measure $\mu \in \mathcal{P}(\mathbb{R})$ and $u \in (0, 1)$ we find

$$\begin{aligned} F_{U(\mu)}^{-1}(u) &= \inf\{x \in \mathbb{R}: F_{U(\mu)}(x) \geq u\} \\ &= \inf\{x \in \mathbb{R}: F_\mu(\sup U^{-1}(\{x\})) \geq u\} \\ &= \inf\{U(y): y \in \mathbb{R} \text{ and } F_\mu(y) \geq u\} \\ &= U(F_\mu^{-1}(u)). \end{aligned} \tag{4.4}$$

⁴We use the convention that the maximum/supremum of the empty set equals $-\infty$ and the preimage under T and S of $\pm\infty$ is $\pm\infty$.

Now all tools are assembled to show the assertion. Note that $(id - T_i)$, $i = 1, 2$, are increasing maps and using the equations (4.3) and (4.4), we compute for $\alpha \in [0, 1]$

$$\begin{aligned} \int_0^\alpha F_{(id-T_1)(\eta_1)}^{-1}(u)du &= \int_0^\alpha (id - T_1)(F_{\eta_1}^{-1}(u))du = \int_0^\alpha F_{\eta_1}(u) - F_{T_1(\eta_1)}^{-1}(u)du \\ &\geq \int_0^\alpha F_{\eta_2}(u) - F_{T_2(\eta_2)}^{-1}(u)du = \int_0^\alpha F_{(id-T_2)(\eta_2)}^{-1}(u)du. \end{aligned} \quad \square$$

Proof of Theorem 1.3. Existence of an admissible map T which has slope 1 on each interval $T^{-1}(I)$, where I is irreducible w.r.t. $(T(\mu), \nu)$, was already shown in Theorem 2.1. Therefore, it remains to show that the map is maximal. Denote by T the map given by Theorem 2.1 associated with an optimizer to (1.4) and some strictly convex $\theta: \mathbb{R} \rightarrow \mathbb{R}$. Let S be an arbitrary map in $M(\mu, \nu)$. Then Lemma 4.2 states that

$$(id - R)(\mu) \leq_c (id - T)(\mu),$$

where R is defined as the increasing, 1-Lipschitz map such that $R(\mu) = S(\mu) \vee T(\mu)$. Additionally to existence, strict convexity of θ ensures μ -almost sure uniqueness of T in the sense that for any optimal coupling π we have $\int_{\mathbb{R}} y\pi_x(dy) = T(x)$ μ -a.s. Thus, $R(\mu) = T(\mu)$ and $T = R$ μ -almost surely. \square

Proof of Theorem 1.2. This is a direct consequence of Theorem 2.1, which provides existence of a map with the desired geometric properties, and Theorem 1.3, which provides the equivalence between the geometric properties and maximality. \square

5 On the reverse problem of Alfonsi, Corbetta, and Jourdain

We aim to prove Theorem 1.4 pertaining the reverse problem (1.7).

Lemma 5.1. *Let $\eta_1, \eta_2 \in \mathcal{P}_1(\mathbb{R})$ with $\int_{\mathbb{R}} y\eta_1(dy) = \int_{\mathbb{R}} y\eta_2(dy)$, and $T_1, T_2: \mathbb{R} \rightarrow \mathbb{R}$ be increasing maps with*

$$T_1(\eta_1) = T_2(\eta_2) =: \nu.$$

Denote the minimum in convex order of η_1 and η_2 by η . Then there exists an increasing map T^ such that $T^*(\eta) = \nu$. If in addition, the maps are L -Lipschitz with $L > 0$, then the same holds true for T^* .*

Proof. Suppose there exist increasing maps T_i and measures η_i , $i = 1, 2$, such that $T_1(\eta_1) = \nu = T_2(\eta_2)$. Then the potential function of the minimum η of η_1 and η_2 w.r.t. the convex order is given by the convex hull of u_{η_1} and u_{η_2} . The potential function u_η completely specifies the cumulative distribution function through $\partial_+ u_\eta = 2F_\eta - 1$. Thus, we can find a partition of \mathbb{R} into countably many, disjoint intervals $I_k = [a_k, b_k] \cap \mathbb{R}$. For $k \in \mathbb{N}$ we find $i \neq j \in \{1, 2\}$ with $u_\eta(a_k) = u_{\eta_i}(a_k)$, $u_\eta(b_k) = u_{\eta_j}(b_k)$ such that one of the following holds

- (a) $u_\eta(x) = u_{\eta_i}(x)$ on I_k ,
- (b) $F_\eta(a_k) = F_\eta(b_k-) = F_{\eta_j}(b_k-)$ and $u_\eta(x) < u_{\eta_1}(x) \wedge u_{\eta_2}(x)$ on (a_k, b_k) .

According to this decomposition, we can define an increasing map T^* via

$$T^*(x) = \begin{cases} T_i(x) & x \in I_k, \text{ (a) holds,} \\ T_i(a_k) & x \in I_k, \text{ (b) holds.} \end{cases}$$

Note that T^* is L -Lipschitz if T_i , $i = 1, 2$, are L -Lipschitz. Let $y \in \mathbb{R}$, due to the continuity of the maps T_1 and T_2 , we can find points $x_1, x_2 \in \mathbb{R}$ with $p = F_\nu(y)$, $x_1 = F_{\eta_1}^{-1}(p)$ $x_2 =$

$F_{\eta_2}^{-1}(p)$. Assume that $i = 1, j = 2$ with $x_1 \in I_k$. If (a) holds, then we have $F_\eta(x) = F_{\eta_1}(x)$ for all $x \in I_k$. Now presume that (b) holds, then $F_{\eta_2}(b_k-) = F_\eta(a_k) \leq F_{\eta_1}(x_1)$. Then

$$\begin{aligned} F_{\eta_2}(b_k) = p &\implies x_2 = b_k \text{ and } F_\eta(b_k) = p, \\ F_{\eta_2}(b_k) < p &\implies \exists l \in \mathbb{N}: x_2 \in I_l \text{ s.t. } i = 2 \text{ and (a) holds.} \end{aligned}$$

Hence, by monotonicity of the map T^* we conclude $F_{T^*(\eta)} = F_\nu$. □

Proof of Theorem 1.4. Define ν^* via the weak monotone rearrangement T between μ and ν , as μ on the contraction parts of T and accordingly shifted ν on the affine (irreducible) intervals such that $\tilde{T}(\nu^*) = \nu$. Let $\eta \geq_c \mu$. Then for any strictly convex $\theta: \mathbb{R} \rightarrow \mathbb{R}$ and coupling $\pi^2 \in \Pi(\eta, \nu)$ we have $\int_{\mathbb{R} \times \mathbb{R}} \theta(y - z) \pi^2(dy, dz) \geq \int_{\mathbb{R}} \theta(y - \int_{\mathbb{R}} z \pi_y^2(dz)) \eta(dy)$, with equality iff π^2 is actually given by a map. Hence, if the optimizer π^2 of $W_\theta(\eta, \nu)$ is not given by a map and $\pi^1 \in \Pi_M(\mu, \eta)$, we have

$$W_\theta(\eta, \nu) > \int_{\mathbb{R}} \theta(x - \int_{\mathbb{R}} \int_{\mathbb{R}} z \pi_y^2(dz) \pi_x^1(dy)) \mu(dx) \geq V_\theta(\mu, \nu).$$

Thus, by the structure of the weak monotone rearrangement, we deduce optimality of ν^* for Problem (1.7). To show uniqueness of (1.7), assume that η attains the minimum of (1.7) and the optimizer of $W_\theta(\eta, \nu)$ is given by the map R . For any martingale coupling $\pi^1 \in \Pi_M(\mu, \eta)$ we define a map by

$$L(x) := \int_{\mathbb{R}} R(y) \pi_x^1(dy).$$

Then by optimality $L(\mu) = T(\mu)$ and, in particular,

$$\int_{\mathbb{R}} \theta(y - R(y)) \eta(dy) = \int_{\mathbb{R}} \theta(x - L(x)) \mu(dx) = \int_{\mathbb{R}} \theta(x - T(x)) \mu(dx),$$

which shows $L = T$ μ -almost surely. By strict convexity, we have

$$y - R(y) = x - L(x) = x - T(x) \quad \pi^1\text{-a.s.}$$

Since π^1 was arbitrary in $\Pi_M(\mu, \eta)$ we get that R is affine with slope 1 on I , whenever I is an irreducible interval w.r.t. (μ, η) . Therefore η and ν^* restricted to I coincide. Hence, $\eta = \nu^*$.

We finally show that ν^* is minimal in the convex order as stated. By Lemma 5.1, we can assume $\mu \leq_c \eta \leq_c \nu^*$ and that η can be pushed forward onto ν via an increasing 1-Lipschitz map S . It follows by Lemma 4.2 that $(id - S)(\eta) \leq_c (id - \tilde{T})(\nu^*)$, so

$$V_\theta(\mu, \nu) \leq \int \theta(x - S(x)) \eta(dx) \leq \int \theta(x - \tilde{T}(x)) \nu^*(dx) = V_\theta(\mu, \nu),$$

and by the uniqueness obtained above we deduce $\eta = \nu^*$. □

6 Stability of barycentric weak transport problems in multiple dimensions

The final part of the article is concerned with stability of the weak optimal transport problem under barycentric costs, see (1.4). Unlike in the rest of the article we work here on \mathbb{R}^d . The final aim is to prove Theorem 1.5.

One surprising aspect of this result is that we only require $\nu^k \rightarrow \nu$ in \mathcal{W}_1 and not necessarily in \mathcal{W}_ρ . This relates to the conditional expectation in (1.4) being ‘inside of θ .’ We first prove an illuminating intermediate result:

Proposition 6.1. *Let $1 \leq \rho < \infty$, $(\mu^k)_{k \in \mathbb{N}} \in \mathcal{P}_\rho(\mathbb{R}^d)^{\mathbb{N}}$, $(\nu^k)_{k \in \mathbb{N}} \in \mathcal{P}_\rho(\mathbb{R}^d)^{\mathbb{N}}$ and $\theta: \mathbb{R}^d \rightarrow \mathbb{R}$ convex and satisfying the growth condition (1.8). Suppose that $\mu^k \rightarrow \mu$ and $\nu^k \rightarrow \nu$ in \mathcal{W}_ρ , and that $\eta \leq_c \nu$. Then there exist $\eta^k \leq_c \nu^k$ such that*

- (i) $\eta^k \rightarrow \eta$ in \mathcal{W}_ρ ,
- (ii) $\lim_k W_\theta(\mu^k, \eta^k) = W_\theta(\mu, \eta)$.

Proof. It is well-known that (i) together with the stated convergence of the μ^k 's implies (ii), so we proceed to prove the former. Let π^k be an optimal coupling attaining $W_\rho(\nu, \nu^k)$. Let M be any martingale coupling with first marginal η and second marginal ν , the existence of which is guaranteed by the assumption $\eta \leq_c \nu$ together with Strassen's theorem. We convene on the notation $M(dx, dy)$ and $\pi(dy, dz)$, and define the measure

$$P(dx, dy, dz) = M_x(dx)\pi_y^k(dz)\nu(dy).$$

This measure has η, ν and ν^k as first, second and third marginals. We next define $R^k(x)$ as the conditional expectation under P of the third variable given the first one, namely

$$R^k(x) = \int \int z \pi_y^k(dz)M_x(dy).$$

Next we introduce $\eta^k := R^k(\eta)$ so by definition $\eta^k \leq_c \nu^k$. Finally

$$\begin{aligned} W_\rho(\eta, \eta^k)^\rho &\leq \int |x - R^k(x)|^\rho \eta(dx) = \int \left| \int y M_x(dy) - \int \left(\int z \pi_y^k(dz) \right) M_x(dy) \right|^\rho \eta(dx) \\ &\leq \int |y - \int z \pi_y^k(dz)|^\rho \nu(dy) \leq \int \int |y - z|^\rho \pi^k(dy, dz) = W_\rho(\nu, \nu^k)^\rho, \end{aligned} \tag{6.1}$$

by the martingale property and two applications of Jensen's inequality. The desired conclusion follows. \square

Remark 6.2. In the context of the previous proposition, if η is supported in finitely many atoms, then the condition that $\nu^k \rightarrow \nu$ in \mathcal{W}_ρ can be relaxed to convergence in \mathcal{W}_1 . To wit, if $\eta = \sum_{i=1}^\ell \alpha_i \delta_{x^i}$, one can take $\rho = 1$ in (6.1) and prove

$$\forall i \leq \ell : |x^i - R^k(x^i)| \leq (\min\{\alpha_j\})^{-1} W_1(\nu, \nu^k),$$

so taking ρ -power and integrating w.r.t. η we get $W_\rho(\eta, \eta^k)^\rho \leq K W_1(\nu, \nu^k)^\rho \rightarrow 0$.

The previous remark shows that we need to reduce to the finite-support setting. We carry to this in the next two lemmas:

Lemma 6.3. *Let $\eta \in \mathcal{P}_1(\mathbb{R}^d)$. Then for any $\varepsilon > 0$ there is a compactly supported, sub-probability measure $\tilde{\eta}$ with*

$$\tilde{\eta} \leq \eta, \quad \tilde{\eta}(\mathbb{R}^d) \geq 1 - \varepsilon, \quad \int_{\mathbb{R}^d} z \eta(dz) = \frac{1}{\tilde{\eta}(\mathbb{R}^d)} \int_{\mathbb{R}^d} z \tilde{\eta}(dz). \tag{6.2}$$

Proof. We first partition \mathbb{R}^d into countably many, disjoint d -dimensional cubes $(Q_k^\delta)_{k \in \mathbb{N}}$ of length $\delta > 0$. Define an approximation η^δ of η by

$$\eta^\delta := \sum_{k \in \mathbb{N}} \delta_{z_k^\delta} \eta(Q_k^\delta), \quad z_k^\delta := \begin{cases} \frac{1}{\eta(Q_k^\delta)} \int_{Q_k^\delta} z \eta(dz) & \eta(Q_k^\delta) > 0, \\ 0 & \text{else.} \end{cases}$$

Note that $\eta^\delta \leq_c \eta$ and $\eta^\delta \rightarrow \eta$ in \mathcal{W}_1 when $\delta \searrow 0$. If there exists an approximation η^δ such that the assertion holds, then it is straightforward to construct the corresponding measure for η , which in turn satisfies the assertion with respect to η . W.l.o.g., we may assume that

$$\left\{ \sum_{i=1}^{2d} \alpha_i v_i : (\alpha_i)_{i=1}^{2d} \in \mathbb{R}_+^{2d}, v_1, \dots, v_{2d} \in \left\{ x - \bar{z} \in \mathbb{R}^d : x \in \text{supp}(\eta) \right\} \right\} = \mathbb{R}^d,$$

where \bar{z} denotes the barycenter of η . Then we can find $\delta > 0$ such that

$$\left\{ \sum_{i=1}^{2d} \alpha_i v_i : (\alpha_i)_{i=1}^{2d} \in \mathbb{R}_+^{2d}, v_1, \dots, v_{2d} \in \left\{ x - \bar{z} \in \mathbb{R}^d : x \in \text{supp}(\eta^\delta) \right\} \right\} = \mathbb{R}^d.$$

Let $z_{n_1}^\delta, \dots, z_{n_{2d}}^\delta$ span \mathbb{R}^d in the sense above and

$$\eta^\delta(z_{n_j}^\delta) = \eta(Q_{n_j}^\delta) > 0 \quad j = 1, \dots, 2d.$$

For any $\varepsilon > 0$ there is a $\tilde{\varepsilon} \in (0, \varepsilon)$ such that

$$\left\{ \sum_{i=1}^{2d} \alpha_i z_{n_i} : (\alpha_i)_{i=1}^{2d} \in \mathbb{R}_+^{2d}, \sum_{i=1}^{2d} \alpha_i < \varepsilon \right\} \supseteq B_{\tilde{\varepsilon}}(0).$$

Besides, there exists a compact set $K \subseteq \mathbb{R}^d$ such that

$$\eta^\delta(K^c) < \tilde{\varepsilon}, \quad \left| \bar{z} - \int_K z \eta^\delta(dz) \right| < \tilde{\varepsilon}$$

and $z_{n_1}^\delta, \dots, z_{n_{2d}}^\delta \in K$. Therefore, we find $(\tilde{\alpha}_i)_{i=1}^{2d} \in \mathbb{R}_+^{2d}$ with

$$\bar{z} - \int_K z \eta^\delta(dz) = \sum_{i=1}^{2d} \tilde{\alpha}_i z_{n_i}, \quad \sum_{i=1}^{2d} \tilde{\alpha}_i < \varepsilon.$$

If ε is chosen smaller than $\eta^\delta(z_{n_i}^\delta)$ for all $i = 1, \dots, 2d$, we can define $\tilde{\eta}^\delta$ via

$$\tilde{\eta}^\delta := \eta^\delta \upharpoonright_K - \sum_{i=1}^{2d} \tilde{\alpha}_i \delta_{z_{n_i}}. \quad \square$$

Lemma 6.4. *Let $\mu, \eta \in \mathcal{P}_\rho(\mathbb{R}^d)$ and $\theta: \mathbb{R}^d \rightarrow \mathbb{R}$ convex satisfying the growth condition (1.8). Then there exists a sequence $(\eta^k)_{k \in \mathbb{N}}$ of finitely supported measures with $\eta^k \leq_c \eta$, $\eta^k \rightarrow \eta$ in \mathcal{W}_ρ and $W_\theta(\mu, \eta^k) \rightarrow W_\theta(\mu, \eta)$.*

Proof. For any $\varepsilon > 0$ we find a compact set $K_\varepsilon \subseteq \text{supp}(\eta)$ such that $\int_{K_\varepsilon^c} |y|^\rho \eta(dy) < \varepsilon$. For any $\delta > 0$, the set K_ε can be covered by finitely many, disjoint sets $(A_i^{\varepsilon, \delta})_{i=1}^{N_{\varepsilon, \delta}}$ with diameter smaller than δ and $K_\varepsilon = \bigcup_i A_i^{\varepsilon, \delta}$. Define the measure $\eta^{\varepsilon, \delta}$ by

$$\eta^{\varepsilon, \delta} = \delta_{z_\varepsilon} \eta(K_\varepsilon^c) + \sum_{i=1}^{N_{\varepsilon, \delta}} \delta_{z_i^{\varepsilon, \delta}} \eta(A_i^{\varepsilon, \delta}),$$

where the points z_ε and $(z_i^{\varepsilon, \delta})_{i=1}^{N_{\varepsilon, \delta}}$ are given by

$$z_\varepsilon := \frac{1}{\eta(K_\varepsilon^c)} \int_{K_\varepsilon^c} y \eta(dy), \quad z_i^{\varepsilon, \delta} := \frac{1}{\eta(A_i^{\varepsilon, \delta})} \int_{A_i^{\varepsilon, \delta}} y \eta(dy).$$

By construction, there exists a martingale coupling between $\eta^{\varepsilon, \delta}$ and η , thus, $\eta^{\varepsilon, \delta} \leq_c \eta$. Note that θ restricted to K_ε is Lipschitz continuous. Then drawing the limit $\delta \rightarrow 0$ yields

$$\sum_{i=1}^{N_{\varepsilon, \delta}} \delta_{z_i^{\varepsilon, \delta}} \eta(A_i^{\varepsilon, \delta}) \rightarrow \eta \upharpoonright_{K_\varepsilon} \text{ in } \mathcal{W}_\rho,$$

and by convexity, $|z_\varepsilon|^\rho \eta(K_\varepsilon^c) \leq \int_{K_\varepsilon^c} |y|^\rho \eta(dy)$. Choosing $\delta(\varepsilon)$ sufficiently small, we have

$$\mathcal{W}_\rho(\eta, \eta^{\varepsilon, \delta(\varepsilon)}) \leq 2\varepsilon \text{ and } \eta^{\varepsilon, \delta(\varepsilon)} \rightarrow \eta \text{ in } \mathcal{W}_\rho.$$

By the growth condition (1.8) we obtain $W_\theta(\mu, \eta^{\varepsilon, \delta(\varepsilon)}) \rightarrow W_\theta(\mu, \eta)$. □

We can now prove a version of Proposition 6.1 under weaker assumptions:

Lemma 6.5. *Let $(\nu^k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{P}_1(\mathbb{R}^d)$ and let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence $\mathcal{P}_\rho(\mathbb{R}^d)$ with $\nu^k \rightarrow \nu$ in \mathcal{W}_1 , $\mu^k \rightarrow \mu$ in \mathcal{W}_ρ , where $\rho \geq 1$, and let $\theta: \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex functions satisfying the growth constraint (1.8). Then for any $\eta \leq_c \nu$ we find a sequence of $\eta^k \leq_c \nu^k$ such that $W_\theta(\mu^k, \eta^k) \rightarrow W_\theta(\mu, \eta)$ and $\eta^k \rightarrow \eta$ in \mathcal{W}_1 .*

Proof. W.l.o.g. assume that θ is positive. By Lemma 6.3, we can find for any $\varepsilon > 0$ a compactly supported $\hat{\eta} = \tilde{\eta} + (1 - \tilde{\eta}(\mathbb{R}^d))\delta_{\bar{z}} \in \mathcal{P}_\rho(\mathbb{R}^d)$ with $\hat{\eta} \leq_c \eta$, $\mathcal{W}_1(\hat{\eta}, \eta) < \varepsilon$ and

$$W_\theta(\mu, \hat{\eta}) \leq W_\theta(\mu, \eta) + c \int_{\mathbb{R}^d} 1 + |x - \bar{z}|^\rho \mu(dx) < \infty,$$

where $\bar{z} := \int_{\mathbb{R}^d} y \eta(dy)$. Using stability of classical optimal transport, see [23, Theorem 5.20], we may assume that $|W_\theta(\mu, \hat{\eta}) - W_\theta(\mu, \eta)| < \varepsilon$. By Lemma 6.4 we may reduce to the case of finitely supported $\hat{\eta}$. We conclude the proof with Remark 6.2. □

Finally we can give the pending proof of Theorem 1.5:

Proof of Theorem 1.5. Lower-semicontinuity of the map $(\mu, \nu) \mapsto V_\theta(\mu, \nu)$ follows from [7, Theorem 1.3]. By [7, Lemma 6.1] we have

$$V_\theta(\mu, \nu) = \inf_{\eta \leq_c \nu} W_\theta(\mu, \eta), \quad (6.3)$$

where the infimum is attained for a measure $\eta \leq_c \nu$. By Lemma 6.5 we find a sequence $\eta^k \leq_c \nu^k$, so that again using [7, Lemma 6.1] we have

$$V_\theta(\mu, \nu) = W_\theta(\mu, \eta) = \lim_k W_\theta(\mu^k, \eta^k) \geq \limsup_k V_\theta(\mu^k, \nu^k).$$

If θ is strictly convex, the infimum in (6.3) is attained by a unique probability measure $\eta^k \leq_c \nu^k$,⁵ which in turn is the push-forward of μ^k under a μ^k -uniquely defined map T^k . Moreover, the W_θ -optimal transport plan $\pi^k \in \Pi(\mu^k, \nu^k)$ is uniquely determined by $\mu^k(dx)\delta_{T^k(x)}(dy)$: Suppose the contrary and let $T'(x) := \int_{\mathbb{R}^d} y\pi_x^k(dy)$, then $T'(\mu^k) \leq_c \nu^k$ and we find the contradiction

$$V_\theta(\mu^k, \nu^k) \leq \int_{\mathbb{R}^d} \theta(x - T'(x))\mu^k(dx) < \int_{\mathbb{R}^d \times \mathbb{R}^d} \theta(x - y)\pi^k(dx, dy) = V_\theta(\mu^k, \nu^k).$$

Hence, by convergence of the values of V_θ and tightness of $(\eta_k)_{k \in \mathbb{N}}$, we deduce the convergence of the η^k to the optimal $\eta \leq_c \nu$ in \mathcal{W}_1 . Suppose that $\mu^k = \mu$ for all $k \in \mathbb{N}$, then due to the uniqueness of the optimal transport maps T between T and $T(\mu)$, we can apply Theorem [23, Corollary 5.23] and obtain convergence of the transport maps T^k to T . \square

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⁵The uniqueness of η^k , T^k and π^k was already shown in [2, Theorem 2.1] for $|\cdot|^\rho$, $\rho > 1$

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