Electron. Commun. Probab. **24** (2019), no. 32, 1–6.
https://doi.org/10.1214/19-ECP228
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

An upper bound for the probability of visiting a distant point by a critical branching random walk in $\mathbb{Z}^4$*

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Abstract

In this paper, we study the probability of visiting a distant point $a \in \mathbb{Z}^4$ by a critical branching random walk starting at the origin. We prove that this probability is bounded by $1/(|a|^2 \log |a|)$ up to a constant factor.

Keywords: critical branching random walk; visiting probability; critical dimension.
AMS MSC 2010: 60G50; 60J80.
Submitted to ECP on December 5, 2018, final version accepted on April 8, 2019.

1 Introduction

A branching random walk is a discrete-time particle system in $\mathbb{Z}^d$ as the following. Fix a probability measure μ on $\mathbb{N}$, called offspring distribution, and another probability measure θ on $\mathbb{Z}^d$, called jump distribution. At time 0, there is a single particle at the origin $0 \in \mathbb{Z}^d$. At each time step $n \in \mathbb{N}$, every particle, say at the site $x \in \mathbb{Z}^d$, gives birth to a random number of offspring (and dies afterwards), according to μ; each of these moves independently to a site according to distribution $x + \theta$. If the mean of μ is one, we say that the branching random walk is critical.

The asymptotic behavior of the probability of visiting a distant point $a \in \mathbb{Z}^d$ by a critical branching random walk (denoted by $\mathcal{S}$) in low dimensions ($d \leq 3$) was established recently by Le Gall and Lin (Theorem 7 in [3]). Their theorem implies that (under some regularity assumption about the critical branching random walk)

$$P(\mathcal{S} \text{ visits } a) \asymp |a|^{-2} \quad \text{in } \mathbb{Z}^d \quad \text{for } d \leq 3,$$

where we write $f(a) \succeq g(a)$ ($f(a) \preceq g(a)$ respectively) if there exists a positive constant c (only depending on d, the offspring distribution μ and the jump distribution θ of the critical branching random walk) such that $f(a) \geq cg(a)$ ($f(a) \leq cg(a)$ respectively) and $f(a) \asymp g(a)$ if $f(a) \succeq g(a)$ and $f(a) \preceq g(a)$.

A simple calculation of the first and second moments gives (see e.g. Remark (2) at the end of Section 2.4 in [2])

$$P(\mathcal{S} \text{ visits } a) \asymp |a|^{2-d} \quad \text{in } \mathbb{Z}^d \quad \text{for } d \geq 5,$$

and

$$P(\mathcal{S} \text{ visits } a) \succeq 1/(|a|^2 \log |a|) \quad \text{in } \mathbb{Z}^4. \tag{1.1}$$

*The author is partially supported by ISF grant 1207/15 and ERC starting grant 676970.
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It is expected that:

$$P(\mathcal{S} \text{ visits } a) \preceq 1/(|a|^2 \log |a|) \quad \text{in } \mathbb{Z}^4. \tag{1.2}$$

In this paper, we prove (1.2) under some regularity assumption about θ- we almost assume nothing about μ, as long as μ is critical and nondegenerate i.e. μ is not the Dirac mass at 1. Let us state our main theorem.

Theorem 1.1. *Let μ be a critical probability measure on $\mathbb{N}$, which is not the Dirac mass at 1, and θ be a probability measure on $\mathbb{Z}^4$ with zero mean and finite $(4+\epsilon)$-th moment for some $\epsilon > 0$ (i.e. $\sum_{z\in\mathbb{Z}^4} \theta(z)z = 0$ and $\sum_{z\in\mathbb{Z}^4} \theta(z)|z|^{4+\epsilon} < \infty$), which is not supported on a strict subgroup of $\mathbb{Z}^4$. Write $\mathcal{S}$ for the critical branching random walk with offspring distribution μ and jump distribution θ starting at the origin. Then, there exists a positive constant C depending on μ and θ, such that, for any $a \in \mathbb{Z}^4$ with $|a|$ sufficiently large,*

$$P(\mathcal{S} \text{ visits } a) \leq C \cdot \frac{1}{|a|^2 \log |a|}. \tag{1.3}$$

Remark 1.2. If μ is the Dirac mass at 1, then the branching random walk is just the ordinary random walk and it is classical that (1.3) is not true and the visiting probability of a behaves like $c\|a\|^{-2}$ for some explicit positive constant c, where $\|a\| = \sqrt{a \cdot Q^{-1}a}/2$ with Q being the covariance matrix of θ.

Remark 1.3. Note that for (1.1) we need to assume that μ has finite variance. Hence if μ has finite variance in addition to the assumptions above, then: (when $|a| > 1$)

$$P(\mathcal{S} \text{ visits } a) \asymp \frac{1}{|a|^2 \log |a|}. \tag{1.4}$$

Remark 1.4. In this paper we are only interested in the case that θ is centered, i.e. the mean of θ is zero. Moreover, we need the moment assumption for θ in order to control the long jump (see the proof of (2.4)). We have not striven for the greatest generality about the assumption on θ and would like to make our proof simple.

Remark 1.5. Update: based on the result and some idea in this paper, the asymptotics of $P(\mathcal{S} \text{ visits } a)$ has been constructed in [5], under an additional and essential assumption that μ has finite variance. It is shown there (under further assumptions that μ has finite variance and that θ has finite exponential moments),

$$\lim_{a\to\infty} \|a\|^2 \log \|a\| P(\mathcal{S} \text{ visits } a + K) = \frac{1}{2\sigma^2},$$

where K is any fixed nonempty finite subset of $\mathbb{Z}^4$ and σ^2 is the variance of μ.

2 Proof of the main theorem

Before the formal proof, let us first mention the main idea. Since the branching is critical, the expectation of the number of visits to a is $G(a) = G(0, a)$ where G is the Green function of an ordinary random walk with jump distribution θ. Our assumptions about θ can guarantee $G(z) \asymp |z|^{-2}$ (see Theorem 2 in [4]). If conditionally on visiting a, the conditional expectation of the number of visits is of order $\log |a|$, then we can get (1.3). In fact, we will show that this is true with high probability.

Let us introduce some notations. Classically, a branching random walk can be regarded as a random function $\mathcal{S} : V(T) \to \mathbb{Z}^4$, where T is a random plane tree, i.e. a rooted ordered tree, and $V(T)$ is the set of all vertices of T. In our case T is a Galton-Watson tree with offspring distribution μ. Conditionally on T, we assign to every edge e of T a random variable Y_e according to θ independently. Then, $\mathcal{S}(v)$, for any $v \in V(T)$ is just the sum of the random variables Y_e over all edges e belonging to the unique simple path from the root to u in the tree (hence the root is mapped to the origin). Since we have an order $\prec$ for the children of each vertex in T, we could adopt the classical order, for all vertices, according to the so-called Depth-first search on $V(T)$ as follows. For v and v', two different vertices, let $\omega = (v_0, v_1, \dots, v_m)$ and $\omega' = (v'_0, v'_1, \dots, v'_n)$ be the unique simple paths in the tree from the root (hence $v_0 = v'_0$ is the root) to v and v' respectively. We say that v is on the left of v', if either $(v_0, v_1, \dots, v_m)$ is a subsequence of $(v'_0, v'_1, \dots, v'_n)$ or $v_i \prec v'_i$, where $i = \min\{k : v_k \neq v'_k\}$.

For any branching random walk sample $\mathcal{S} : V(T) \to \mathbb{Z}^4$ that visits a, $V_a := \{v \in V(T) : \mathcal{S}(v) = a\}$ is not empty. Let u be the leftmost point in V_a and $(v_0, v_1, \dots, v_k)$ be the unique simple path in T from the root to u. Then $(\mathcal{S}(v_0), \mathcal{S}(v_1), \dots, \mathcal{S}(v_k))$ is a path in $\mathbb{Z}^4$ from the origin to a. We denote this path by $\tilde{\gamma}(\mathcal{S})$. Let N be the number of visits to a. For any γ, a path from the origin to a, define $p(\gamma) = P(N > 0, \tilde{\gamma}(\mathcal{S}) = \gamma)$ and $e(\gamma) = E(N|N > 0, \tilde{\gamma}(\mathcal{S}) = \gamma)$. Note that $N > 0$ iff $\mathcal{S}$ visits a. For any path $\gamma = (z_0, \dots, z_n)$ in $\mathbb{Z}^4$, define $g(\gamma) = \sum_{i=0}^n G(z_i, a) = \sum_{i=0}^n G(a - z_i)$. Let $\mathcal{G} = 1\{\mathcal{S} \text{ visits } a\} \cdot g(\tilde{\gamma}(\mathcal{S}))$. The following lemmas are key ingredients for our main theorem.

Lemma 2.1. *For any γ, a path from the origin to a such that $p(\gamma) > 0$, we have:*

$$e(\gamma) \geq g(\gamma) \sum_{i \geq 2} \mu(i). \tag{2.1}$$

Lemma 2.2. *There exists positive constants c, c_1, c_2, such that for all $a \in \mathbb{Z}^4$ with $|a|$ sufficiently large, we have*

$$P(0 < \mathcal{G} \leq c_1 \log |a|) \leq c_2/|a|^{2+c}. \tag{2.2}$$

We postpone proofs of these two lemmas and start the proof of Theorem 1.1. Since μ is critical, we have:

$$EN = G(0, a) \asymp |a|^{-2}.$$

By Lemma 2.1, we have:

$$\begin{aligned} |a|^{-2} \asymp EN &\geq P(\mathcal{G} \geq c_1 \log |a|) E(N|\mathcal{G} \geq c_1 \log |a|) \\ &\geq P(\mathcal{G} \geq c_1 \log |a|)(\sum_{i \geq 2} \mu(i)) c_1 \log |a| \\ &\succeq P(\mathcal{G} \geq c_1 \log |a|) \log |a|. \end{aligned}$$

Note that since μ is critical and nondegenerate, $\sum_{i \geq 2} \mu(i) > 0$. Therefore:

$$P(\mathcal{G} \geq c_1 \log |a|) \preceq 1/(|a|^2 \log |a|).$$

Then we have:

$$\begin{aligned} P(\mathcal{S} \text{ visits } a) = P(\mathcal{G} > 0)& \\ &= P(0 < \mathcal{G} < c_1 \log |a|) + P(\mathcal{G} \geq c_1 \log |a|) \\ &\preceq 1/|a|^{2+c} + 1/(|a|^2 \log |a|) \\ &\preceq 1/(|a|^2 \log |a|). \end{aligned}$$

Proof of Lemma 2.1. Fix a $\gamma = (z_0, z_1, \dots, z_k)$ such that $p(\gamma) > 0$. For any branching random walk sample $\mathcal{S}$ such that $\tilde{\gamma}(\mathcal{S}) = \gamma$, write a_i (b_i respectively) for the number of siblings of z_i on the left of z_i (on the right respectively), for $i = 1, \dots, k$. From the tree structure, one can easily see that, for any $l_1, \ldots, l_k, m_1, \ldots, m_k \in \mathbb{N}$, we have

$$P(N > 0, \tilde{\gamma}(\mathcal{S}) = \gamma; a_i = l_i, b_i = m_i, \text{ for } i = 1, \ldots, k) \\ = s(\gamma) \prod_{i=1}^{k} \left(\mu(l_i + m_i + 1)(q(a - z_{i-1}))^{l_i} \right), \quad (2.3)$$

where $s(\gamma)$ is the probability weight for the random walk with jump distribution θ, i.e., $s(\gamma) = \prod_{i=1}^{k} \theta(z_i - z_{i-1})$ and $q(z)$ is the probability that the branching random walk avoids z conditioned on the initial particle having only one child.

Conditionally on the event in (2.3), the expectation of N is:

$$G(0) + \sum_{i=1}^{k} m_i G(a - z_{i-1}).$$

Recall that $g(\gamma) = \sum_{i=0}^{k} G(a - z_i) = G(0) + \sum_{i=1}^{k} G(a - z_{i-1})$. Thus it suffices to show:

$$E(b_i | N > 0, \tilde{\gamma}(\mathcal{S}) = \gamma) \geq \sum_{i \geq 2} \mu(i).$$

A straight computation using (2.3) gives:

$$\begin{aligned} E(b_i | N > 0, \tilde{\gamma}(\mathcal{S}) = \gamma) &= \frac{\sum_{l \geq 0, m \geq 0} m\mu(l + m + 1)(q(a - z_{i-1}))^l}{\sum_{l \geq 0, m \geq 0} \mu(l + m + 1)(q(a - z_{i-1}))^l} \\ &\geq \frac{\sum_{l = 0, m \geq 1} 1 \cdot \mu(l + m + 1)}{\sum_{l \geq 0, m \geq 0} \mu(l + m + 1)} \\ &= \frac{\sum_{m \geq 1} \mu(m + 1)}{\sum_{j \geq 1} j\mu(j)} \\ &= \frac{\sum_{i \geq 2} \mu(i)}{1} \\ &= \sum_{i \geq 2} \mu(i). \end{aligned}$$

□

Proof of Lemma 2.2. A straight calculation using (2.3) gives:

$$\begin{aligned} p(\gamma) &= s(\gamma) \prod_{i=1}^{k} (\sum_{l_i \geq 0, m_i \geq 0} \mu(l_i + m_i + 1)(q(a - z_{i-1}))^{l_i}) \\ &\leq s(\gamma) \prod_{i=1}^{k} (\sum_{l_i \geq 0, m_i \geq 0} \mu(l_i + m_i + 1)) \\ &= s(\gamma) \prod_{i=1}^{k} (\sum_{j \geq 1} j\mu(j)) \\ &= s(\gamma). \end{aligned}$$

Hence, we have:

$$\begin{aligned} P(0 < \mathcal{G} \leq c_1 \log |a|) &= \sum_{\gamma: 0 \to a, g(\gamma) \leq c_1 \log |a|} p(\gamma) \\ &\leq \sum_{\gamma: 0 \to a, g(\gamma) \leq c_1 \log |a|} s(\gamma). \end{aligned}$$

Then Lemma 2.2 is implied by the following proposition. □

Proposition 2.3. *There exist c, c_1, c_2 such that for $a \in \mathbb{Z}^4$ with $|a|$ sufficiently large,*

$$P(\tau_a < \infty, \sum_{i=0}^{\tau_a} G(S_i) \leq c_1 \log |a|) \leq c_2 |a|^{-(2+c)},$$

where $(S_i)_{i\in\mathbb{N}}$ is the ordinary random walk starting from 0 with jump distribution θ and τ_a is the hitting time for a.

Note that we actually deduce Lemma 2.2 by applying the previous proposition to the random walk with jump distribution $-\theta$.

This proposition is an adjusted version of Lemma 10.1.2 (a) in [1]. It is assumed there that θ has finite support which is stronger than our case, though its conclusion is also stronger than ours. Following the argument there, we present a proof here.

Proof of Proposition 2.3. It suffices to show:

$$P(\sum_{i=0}^{\tau_n} G(S_i) \leq c_1 \log n) \leq c_2 n^{-(2+c)}, \tag{2.4}$$

where $\tau_n = \min\{k \geq 0 : |S_k| \geq n\}$.

Choose $\alpha < \beta < c \in (0, 0.1)$, such that

$$(4+\epsilon)(1-\beta) - 2(1-\alpha) > 2 + c. \tag{2.5}$$

Let A be the event that $|X_i| \leq M \doteq \lfloor n^{1-\beta} \rfloor$ for $i = 1, 2, \ldots, T \doteq 2\lfloor n^{2(1-\alpha)} \rfloor$ (where $X_i = S_i - S_{i-1}$). Write $X'_i = X_i 1\{|X_i| \leq M\}$ and $S'_i = X'_1 + \cdots + X'_i$. Note that on A, $S_i = S'_i$ for $i = 1, \ldots, T$ and

$$P(A^c) \preceq n^{2(1-\alpha)} P(|X_1| \geq M) \leq n^{2(1-\alpha)} \frac{E|X_1|^{4+\epsilon}}{M^{4+\epsilon}} \overset{(2.5)}{\preceq} n^{-(2+c)}.$$

Write $l = \max\{i \in \mathbb{N} : 4^i \leq n^{1-\beta}\}$ and $L = \max\{i \in \mathbb{N} : 4^i \leq n^{1-\alpha}\}$. Define ξ_i for $i = l, l+1, \ldots, L$, inductively by $\xi_l = 0$, $\xi_{i+1} = (\xi_i + (4^{i+1})^2) \wedge \min\{k \in \mathbb{N} : |S'_k| \geq 4^{i+1}\}$, where we write $x \wedge y = \min\{x, y\}$. Note that $\xi_L \leq (4^L)^2 (1 + \frac{1}{16} + \left(\frac{1}{16}\right)^2 + \ldots) \leq 2(4^L)^2 \leq 2\lfloor n^{2(1-\alpha)} \rfloor$ and $L - l \asymp (\beta - \alpha) \log n$.

On the other hand, since $G(x) \succeq (|x|+1)^{-2}$, we have

$$\sum_{i=\xi_k}^{\xi_{k+1}-1} G(S'_i) \succeq (\xi_{k+1} - \xi_k)(4^{k+1})^{-2}.$$

It is not difficult to see that for every $b \in (0, 0.1)$ we could find some $t \in (0, 0.1)$ such that

$$sup_{k,n\in\mathbb{N}, x\in\mathbb{Z}^4 : l \leq k \leq L-1, |x| \leq 2\cdot 4^k} P(\xi_{k+1} - \xi_k \leq t(4^{k+1})^2 | S'_{\xi_k} = x) < b.$$

For example, one could first show, using Kolmogorov's maximal inequality, the corresponding result when the role of X'_i is replaced by X_i and then note that on A, $S_i = S'_i$ and that $P(A^c)$ is very small.

Write I_k for $1\{\xi_{k+1} - \xi_k \leq t(4^{k+1})^2\}$. Then we have

$$P(I_{k+1} = 1 | S'_0, \ldots, S'_{\xi_k}) < b.$$

Therefore, $J \doteq \sum_{k=l}^{L-1} I_k$ is stochastically bounded by a binomial random variable with parameters $L - l$ and b. By choosing b small enough and standard estimates for binomial random variables, one could get

$$P(J \geq \frac{L-l}{2}) \leq (2\sqrt{b(1-b)})^{L-l} \preceq n^{-3}.$$

On the event $\{J < \frac{L-l}{2}\}$, we have

$$\sum_{i=\xi_l}^{\xi_L - 1} G(S_i') \geq \frac{L-l}{2} t \asymp (\beta - \alpha) t \log n.$$

Noting that $P(A^c) \preceq n^{-(2+c)}$ and on A, $\sum_{i=0}^{\tau_n} G(S_i) \geq \sum_{i=\xi_l}^{\xi_L - 1} G(S_i')$, we finish the proof. $\square$

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Acknowledgments. This work was done when the author was a PhD student in the University of British Columbia. The author would like to thank his advisor, Professor Omer Angel for inspiring discussions.