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Critical percolation and the incipient infinite cluster on Galton-Watson trees

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Abstract

We consider critical percolation on Galton-Watson trees and prove quenched analogues of classical theorems of critical branching processes. We show that the probability critical percolation reaches depth n is asymptotic to a tree-dependent constant times n^{-1}. Similarly, conditioned on critical percolation reaching depth n, the number of vertices at depth n in the critical percolation cluster almost surely converges in distribution to an exponential random variable with mean depending only on the offspring distribution. The incipient infinite cluster (IIC) is constructed for a.e. Galton-Watson tree and we prove a limit law for the number of vertices in the IIC at depth n, again depending only on the offspring distribution. Provided the offspring distribution used to generate these Galton-Watson trees has all finite moments, each of these results holds almost-surely.

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1 Introduction

We consider percolation on a locally finite rooted tree T: each edge is open with probability $p \in (0,1)$, independently of all others. Let $\mathbf{0}$ denote the root of T and $\mathcal{C}_p$ be the open p-percolation cluster of the root. We may consider the *survival probability* $\theta_T(p) := \mathbf{P}[|\mathcal{C}_p| = +\infty]$ and note that θ_T is an increasing function of p. There thus exists a *critical percolation parameter* $p_c \in [0,1]$ so that $\theta_T(p) = 0$ for all $p \in [0, p_c)$ and $\theta_T(p) > 0$ for $p \in (p_c, 1]$. If T is a regular tree where each non-root vertex has degree $d+1$—i.e. each vertex has d children—then the classical theory of branching processes shows that $p_c = \frac{1}{d}$ and $\theta_T(p_c) = 0$ (see, for instance, [AN72]). Since critical percolation does not occur, we may consider the *incipient infinite cluster* (IIC), in which we condition on critical percolation reaching depth M of T and take M to infinity.

The IIC for regular trees was first constructed and considered by Kesten in [Kes86b]. In that work, along with [BK06], the primary focus was on simple random walk on the IIC for regular trees. Our focus is on three elementary quantities for random T: the probability that critical percolation reaches depth n; the number of vertices of $\mathcal{C}_p$ at depth n conditioned on percolation reaching depth n; and the number of vertices in

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the IIC at depth n. For regular trees, these questions were answered in the study of critical branching processes. In fact, these classical results apply to *annealed* critical percolation on Galton-Watson trees. If we generate a Galton-Watson tree T with progeny distribution $Z \geq 1$ with $\mathbf{E}[Z] > 1$, we may perform $p_c = 1/\mathbf{E}[Z]$ percolation at the same time as we generate T; this is known at the *annealed* process—in which we generate T and percolate simultaneously—and is equivalent to generating a Galton-Watson tree with offspring distribution $\widetilde{Z} := \mathrm{Bin}(Z, p_c)$. Since $\mathbf{E}[\widetilde{Z}] = 1$, this is a critical branching process and thus the classical theory can be used:

Theorem 1.1 ([KNS66])**.** *Suppose $\mathbf{E}[Z^2] < \infty$, and set Y_n to be the set of vertices at depth n of T connected to the root in $p_c = 1/\mathbf{E}[Z]$ percolation. Then*

(a) *The annealed probability of surviving to depth n satisfies*

$$n \cdot \mathbf{P}[|Y_n| > 0] \to \frac{2}{\mathrm{Var}\,[\widetilde{Z}]} = \frac{2\mathbf{E}[Z]^2}{\mathbf{E}[Z(Z-1)]}\,.$$

(b) *The annealed conditional distribution of $|Y_n|/n$ given $|Y_n| > 0$ converges in distribution to an exponential law with mean $\frac{\mathbf{E}[Z(Z-1)]}{2\mathbf{E}[Z]^2}$ as $n \to \infty$.*

Under the additional assumption of $\mathbf{E}[Z^3] < \infty$, parts (a) and (b) are due to Kolmogorov [Kol38] and Yaglom [Yag47] respectively; as such, they are commonly referred to as Kolmogorov's estimate and Yaglom's limit law. For a modern treatment of these classical results, see [LPP95] or [LP17, Section 12.4]. Although less widely known, Theorem 1.1 quickly gives a limit law for the size of the annealed IIC.

Corollary 1.2. *If $\mathbf{E}[Z^2] < \infty$, let C_n denote the number of vertices at depth n in the annealed incipient infinite cluster. Then C_n/n converges in distribution to the random variable with density $\lambda^2 x e^{-\lambda x}$ with $\lambda := \frac{2\mathbf{E}[Z]^2}{\mathbf{E}[Z(Z-1)]}$ on $[0, \infty)$. In other words,*

$$\lim_{n\to\infty}\left(\lim_{M\to\infty} \mathbf{P}[|Y_n|/n \in (a,b) \,|\, |Y_M| > 0]\right) = \int_a^b \lambda^2 x e^{-\lambda x}\,dx$$

for each $a < b$.

This can be easily proven from Theorem 1.1 using an argument similar to the proof of Theorem 3.11, and thus the details are omitted.

Our goal is to upgrade Theorem 1.1 and Corollary 1.2 to hold for the *quenched* process; that is, rather than generate T and perform percolation at the same time as in the annealed case, we generate T and then perform percolation on each resulting T. Before stating the quenched results, we recall some notation and facts from the theory of branching processes. If we allow $\mathbf{P}[Z = 0] > 0$ and condition on the resulting tree being infinite, we may pass to the reduced tree as in [LP17, Chapter 5.7] in which we remove all vertices that have finitely many descendants; this results in a new Galton-Watson process with some offspring distribution $\tilde{Z} \geq 1$. We therefore assume without loss of generality that $Z \geq 1$. For a Galton-Watson tree T, let Z_n denote the number of vertices at distance of n from the root; then the process $W_n = Z_n/(\mathbf{E}[Z])^n$ converges almost-surely to some random variable W.

A first quenched result is that of [Lyo90], which states that for a.e. supercritical Galton-Watson tree with progeny distribution Z, we have that the critical percolation probability is $p_c = 1/\mathbf{E}[Z]$; furthermore, for almost every Galton-Watson tree $\mathbf{T}$, $\theta_{\mathbf{T}}(p) = 0$ for $p \in [0, p_c]$ and $\theta_{\mathbf{T}}(p) > 0$ for $p \in (p_c, 1]$. For a fixed tree T, let $\mathbf{P}_T[\cdot]$ be the probability measure induced by performing p_c percolation on T. When T is random, this is a random variable and we may ask about the almost sure behavior of certain probabilities. Our main results are summarized in the following theorem:

Theorem 1.3. *Let $\mathbf{T}$ be a Galton-Watson tree with progeny distribution $Z \geq 1$ with $\mathbf{E}[Z] > 1$. Suppose $\mathbf{E}[Z^p] < \infty$ for each $p \geq 1$. Set $\lambda := \frac{2\mathbf{E}[Z]^2}{\mathbf{E}[Z(Z-1)]}$ and let Y_n be the set of vertices in depth n of $\mathbf{T}$ connected to the root in $p_c = 1/\mathbf{E}[Z]$ percolation. Then for a.e. $\mathbf{T}$ we have*

- (a) $n \cdot \mathbf{P_T}[|Y_n| > 0] \to W\lambda$ *a.s.*
- (b) *The conditioned variable* $(|Y_n|/n \,|\, |Y_n| > 0)$ *converges in distribution to an exponential random variable with mean* λ^{-1} *a.s.*
- (c) *Let* $\mathbf{C}_n$ *denote the number of vertices in the quenched IIC of* $\mathbf{T}$ *at depth* n*. Then* $\mathbf{C}_n/n$ *converges in distribution to the random variable with density* $\lambda^2 x e^{-\lambda x}$ *a.s.*

Note that, surprisingly, the limit laws of parts (b) and (c) of Theorem 1.3 do not depend at all on $\mathbf{T}$ itself but just on the distribution of Z. This is in sharp contrast to the case of near-critical and supercritical percolation on Galton-Watson trees, in which the behavior is dependent on the tree itself [MPR18]. One possible justification for this lack of dependence on W, for instance, is that conditioning on $|Y_n| > 0$ forces certain structure of the percolation cluster near the root; since W is mostly determined by the levels of $\mathbf{T}$ near the root, the behavior when conditioned on $|Y_n| > 0$ for large n does not depend on W. Part (a) of Proposition 3.8 corroborates this heuristic explanation.

The three parts of Theorem 1.3 are Theorems 3.3, 3.5 and 3.11 respectively. The proof of part (a) utilizes its annealed analogue, Theorem 1.1(a), along with a law of large numbers argument. Part (b) is proven by the method of moments building on the work of [MPR18]. Part (c) follows from there with a similar law of large numbers argument combined with two short facts about the structure of the percolation cluster conditioned on $|Y_n| > 0$ (this is Proposition 3.8).

Remark 1.4. Theorem 1.3 assumes that $\mathbf{E}[Z^p] < \infty$ for each $p \geq 1$, and we suspect that this condition is an artifact of the proof. Since we use the method of moments, it is natural that we require all moments of the underlying distribution to be finite. We suspect that less rigid conditions are sufficient, but this would require a different proof strategy than the method of moments, perhaps utilizing a stronger anti-concentration statement in the vein of Proposition 3.8.

2 Set-up and notation

We begin with some notation and a brief description of the probability space on which we will work. Let Z be a random variable taking values in $\{1, 2, \dots,\}$ with $\mu := \mathbf{E}[Z] > 1$ and $\mathbf{P}[Z = 0] = 0$. Define its *probability generating function* to be $\phi(z) := \sum \mathbf{P}[Z = k]z^k$. Let $\mathbf{T}$ be a random locally finite rooted tree with law equal to that of a Galton-Watson tree with progeny distribution Z and let $(\Omega_1, \mathcal{T}, \mathtt{GW})$ be the probability space on which it is defined. Since we will perform percolation on these trees, we also use variables $\{U_i\}_{i=1}^\infty$ where the U_i are i.i.d. random variables uniform on $[0, 1]$; let $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$ be the corresponding probability space. Our canonical probability space will be $(\Omega, \mathcal{F}, \mathbf{P})$ with $\Omega := \Omega_1 \times \Omega_2$, $\mathcal{F} := \mathcal{T} \otimes \mathcal{F}_2$ and $\mathbf{P} := \mathtt{GW} \times \mathbf{P}_2$. We interpret an element $\omega = (T, \omega_2) \in \Omega$ as the tree T with edge weights given by the U_i random variables. To obtain p percolation, we restrict to the subtree of edges with weight at most p. Since we are concerned with quenched probabilities, we define the measure $\mathbf{P_T}[\cdot] := \mathbf{P}[\cdot \mid \mathbf{T}] = \mathbf{P}[\cdot \mid \mathcal{T}]$. Since this is a random variable, our goal is to prove theorems $\mathtt{GW}$-a.s.

We employ the usual notation for a rooted tree T, Galton-Watson or otherwise: $\mathbf{0}$ denotes the root; T_n is the set of vertices at depth n; and $Z_n := |T_n|$. In the case of a Galton-Watson tree $\mathbf{T}$, we define $W_n := Z_n/\mu^n$ and recall that $W_n \to W$ almost surely. Furthermore, if $\mathbf{E}[Z^p] < \infty$ for some $p \in [1, \infty)$, we in fact have $W_n \to W$ in L^p [BD74, Theorems 0 and 5]. In the Galton-Watson case, define $\mathcal{T}_n := \sigma(\mathbf{T}_n)$; then $(\mathcal{T}_n)_{n=0}^\infty$ is a filtration that increases to $\mathcal{T}$. For a vertex v of T, define $T(v)$ to be the descendant tree of v and extend our notation to include $T_n(v), Z_n(v), W_n(v)$ and $W(v)$. For vertices v and w, write $v \le w$ if v is an ancestor of w.

For percolation, recall that the critical percolation probability for GW-a.e. $\mathbf{T}$ is $p_c := 1/\mu$ and that percolation does not occur at criticality [Lyo90]. For vertices v and w with $v \le w$, let $\{v \leftrightarrow w\}$ denote the event that there is an open path from v to w in p_c percolation; let $\{v \leftrightarrow (u,w)\}$ be the event that v is connected to both u and w in p_c percolation; for a subset S of $\mathbf{T}$, let $\{v \leftrightarrow S\}$ denote the event that v is connected to some element of S in p_c percolation; lastly, let Y_n be the set of vertices in $\mathbf{T}_n$ that are connected to $\mathbf{0}$ in p_c percolation.

3 Quenched results

3.1 Moments

For $k \ge j$, let $\mathcal{C}_j(k)$ denote the set of j-compositions of k, i.e. ordered j-tuples of positive integers that sum to k. Define

$$c_{k,j} := p_c^k \sum_{a \in \mathcal{C}_j(k)} m_{a_1} m_{a_2} \cdots m_{a_j}$$

where $m_r := \mathbf{E}[\binom{Z}{r}]$. We use the following result from [MPR18]:

Theorem 3.1 ([MPR18])**.** *Define*

$$M_n^{(k)} := \mathbf{E_T}\left[\binom{|Y_n|}{k}\right] - \sum_{i=1}^{k-1} c_{k,i} \sum_{j=0}^{n-1} \mathbf{E_T}\left[\binom{|Y_j|}{i}\right] .$$

If $\mathbf{E}[Z^{2k}] < \infty$, then $M_n^{(k)}$ is a martingale with respect to the filtration $(\mathcal{T}_n)$, and there exist constants C_k and c_k so that

$$\|M_{n+1}^{(k)} - M_n^{(k)}\|_{L^2} \le C_k e^{-c_k n} .$$

While Theorem 3.1 is not stated precisely this way in [MPR18], the martingale property follows from [MPR18, Lemma 4.1], while the L^2 bound on the increments is given in [MPR18, Theorem 4.4]. This gives us the leading term of each $\mathbf{E_T}\left[|Y_n|^k\right]$.

Proposition 3.2. *For each k,*

$$\mathbf{E_T}\left[|Y_n|^k\right] n^{-(k-1)} \to k! \left(\frac{p_c^2 \phi''(1)}{2}\right)^{k-1} W$$

almost surely and in L^2.

Proof. By Theorem 3.1, $M_n^{(k)}$ is a martingale with uniformly bounded L^2 norm for each k. By the L^p martingale convergence theorem, $M_n^{(k)}$ converges in L^2 and almost surely. We now proceed by induction on k. For $k=1$, $\mathbf{E_T}[|Y_n|] = W_n$ which converges to W. Suppose that the proposition holds for all $j < k$. Then by convergence of $M_n^{(k)}$,

$$\mathbf{E_T}\left[\binom{|Y_n|}{k}\right] n^{-(k-1)} = \sum_{i=1}^{k-1} c_{k,i} n^{-(k-1)} \sum_{j=0}^{n-1} \mathbf{E_T}\left[\binom{|Y_j|}{i}\right] + o(1)$$

where the $o(1)$ term is both in L^2 and almost surely. By induction, the leading term is the contribution from $i = k-1$. Noting that $c_{k,k-1} = (k-1)p_c^2 \frac{\phi''(1)}{2}$ and the fact that $\sum_{j=0}^{n-1} j^d \sim \frac{1}{d+1} n^{d+1}$ completes the proof. $\square$

3.2 Survival probabilities

Throughout, define $\lambda := \frac{2}{p_c^2\phi''(1)}$. Our first task is to find a quenched analogue of Kolmogorov's estimate:

Theorem 3.3. *If $\mathbf{E}[Z^4] < \infty$, then*

$$n \cdot \mathbf{P_T}[|Y_n| > 0] \to W\lambda$$

almost surely.

The proof utilizes the Bonferroni inequalities. In order to control the second-order term, the variance of a sum of pairs is calculated, thereby introducing the requirement of $\mathbf{E}[Z^4] < \infty$. We begin first by proving upper and lower bounds:

Lemma 3.4. *For each n,*

$$\frac{n \cdot \mathbf{E_T}[|Y_n|]^2}{\mathbf{E_T}[|Y_n|^2]} \le n \cdot \mathbf{P_T}[|Y_n| > 0] \le \frac{2\overline{W}}{1-p_c}$$

where, $\overline{W} = \sup_n W_n$.

Proof. The lower bound is the Paley-Zygmund inequality. For the upper bound, we use [LP17, Theorem 5.24]:

$$\mathbf{P_T}[|Y_n| > 0] \le \frac{2}{\mathscr{R}(\mathbf{0} \leftrightarrow \mathbf{T}_n)}$$

where $\mathscr{R}(\mathbf{0} \leftrightarrow \mathbf{T}_n)$ is the equivalent resistance between the root and $\mathbf{T}_n$ when all of $\mathbf{T}_n$ is shorted to a single vertex and each edge branching from depth $k-1$ to k has resistance $\frac{1-p_c}{p_c^k}$. Shorting together all vertices at depth k for each k gives the lower bound

$$\mathscr{R}(\mathbf{0} \leftrightarrow \mathbf{T}_n) \ge \sum_{k=1}^{n} \frac{1-p_c}{Z_k p_c^k} = \sum_{k=1}^{n} \frac{1-p_c}{W_k} \ge (1-p_c)\frac{n}{\overline{W}}\,. \qquad \square$$

Proof of Theorem 3.3: For each fixed $m < n$, the Bonferroni inequalities imply

$$\left| n\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_n] - n \sum_{v \in \mathbf{T}_m} \mathbf{P_T}[\mathbf{0} \leftrightarrow v \leftrightarrow \mathbf{T}_n] \right| \le n \sum_{u,v \in \binom{\mathbf{T}_m}{2}} \mathbf{P_T}[\mathbf{0} \leftrightarrow (u,v) \leftrightarrow \mathbf{T}_n]\,. \tag{3.1}$$

If we can show that the right-hand side of (3.1) converges a.s. to zero for some choice of $m = m(n)$, then the survival probability is sufficiently close to a sum of i.i.d. random variables. The random variables $\mathbf{P_T}[\mathbf{0} \leftrightarrow v \leftrightarrow \mathbf{T}_n]$ are i.i.d. with mean $p_c^m \mathbf{P}[\mathbf{0} \leftrightarrow \mathbf{T}_{n-m}]$, implying that the sum is close to $W_m \mathbf{P}[\mathbf{0} \leftrightarrow \mathbf{T}_{n-m}]$. Applying the annealed result Theorem 1.1 would then complete the proof after noting that $W_m \to W$ almost surely provided $m \to \infty$. The remainder of the proof follows this sketch.

Set $m = \lceil n^{1/4} \rceil$; we then bound the second moment

$$\begin{aligned}
&\mathbf{E}\left[\left(\sum_{u,v \in \binom{\mathbf{T}_m}{2}} \mathbf{P_T}[\mathbf{0} \leftrightarrow (u,v) \leftrightarrow \mathbf{T}_n]\right)^2\right] \\
&\quad = \mathbf{E}\left[\left(\sum_{u,v \in \binom{\mathbf{T}_m}{2}} \mathbf{P_T}[\mathbf{0} \leftrightarrow (u,v)]\mathbf{P_T}[u \leftrightarrow \mathbf{T}_n]\mathbf{P_T}[v \leftrightarrow \mathbf{T}_n]\right)^2\right] \\
&\quad = \mathbf{E}\left[\mathbf{E}\left[\left(\sum_{u,v \in \binom{\mathbf{T}_m}{2}} \mathbf{P_T}[\mathbf{0} \leftrightarrow (u,v)]\mathbf{P_T}[u \leftrightarrow \mathbf{T}_n]\mathbf{P_T}[v \leftrightarrow \mathbf{T}_n]\right)^2 \,\middle|\, \mathcal{T}_m\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left[\mathbf{E}\left[\left(\sum_{u,v\in\binom{\mathbf{T}_m}{2}} \mathbf{P}_{\mathbf{T}}[\mathbf{0}\leftrightarrow(u,v)]\mathbf{P}_{\mathbf{T}}[u\leftrightarrow\mathbf{T}_n]\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n]\right)^2 \,\middle|\, \mathcal{T}_m\right]^{(1/2)\cdot 2}\right] \\
&\le \mathbf{E}\left[\left(\sum_{u,v\in\binom{\mathbf{T}_m}{2}} \mathbf{P}_{\mathbf{T}}[\mathbf{0}\leftrightarrow(u,v)]\, \|\mathbf{P}_{\mathbf{T}}[u\leftrightarrow\mathbf{T}_n]\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n]\|_{L^2}\right)^2\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by the triangle inequality} \\
&\le \left(\frac{2}{1-p_c}\right)^4 \mathbf{E}[\overline{W}^2]^2\cdot(n-m)^{-4}\mathbf{E}\left[\binom{|Y_m|}{2}^2\right] \qquad \text{by Lemma 3.4} \\
&\le Cm^2n^{-4} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Theorem 3.1}\,.
\end{aligned}
$$

Multiplying by n, the second moment of the right-hand side of (3.1) is bounded above by $Cm^2n^{-2} = O(n^{-3/2})$ which is summable in n. By Chebyshev's Inequality together with the Borel-Cantelli Lemma, the right-hand side of (3.1) converges to zero almost surely. This implies

$$
n\mathbf{P}_{\mathbf{T}}[\mathbf{0}\leftrightarrow\mathbf{T}_n] = n\sum_{v\in\mathbf{T}_m}\mathbf{P}_{\mathbf{T}}[\mathbf{0}\leftrightarrow v\leftrightarrow\mathbf{T}_n] + o(1) = \sum_{v\in\mathbf{T}_m}\frac{n\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n]}{\mu^m} + o(1)\,. \tag{3.2}
$$

We want to show that the right-hand side of (3.2) converges to $W\lambda$, so we first calculate

$$
\begin{aligned}
&\mathrm{Var}\left[\sum_{v\in\mathbf{T}_m}\frac{n\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n] - n\mathbf{P}[\mathbf{0}\leftrightarrow\mathbf{T}_{n-m}]}{\mu^m}\right] \\
&\qquad = \mathbf{E}\left[\mathrm{Var}\left[\sum_{v\in\mathbf{T}_m}\frac{n\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n] - n\mathbf{P}[\mathbf{0}\leftrightarrow\mathbf{T}_{n-m}]}{\mu^m}\,\middle|\,\mathcal{T}_m\right]\right] \\
&\qquad = \mathbf{E}\left[\frac{1}{\mu^{2m}}\sum_{v\in\mathbf{T}_m}\mathrm{Var}\left[n\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n]\right]\right] \\
&\qquad \le \frac{C}{\mu^m}
\end{aligned}
$$

where the last inequality is via Lemma 3.4. Since this is summable in n, Chebyshev's Inequality and the Borel-Cantelli Lemma again imply

$$
\sum_{v\in\mathbf{T}_m}\frac{n\mathbf{P}_{\mathbf{T}}[v\leftrightarrow\mathbf{T}_n]}{\mu^m} = \sum_{v\in\mathbf{T}_m}\frac{n\mathbf{P}[\mathbf{0}\leftrightarrow\mathbf{T}_{n-m}]}{\mu^m} + o(1) = W_m(n\cdot\mathbf{P}[\mathbf{0}\leftrightarrow\mathbf{T}_{n-m}]) + o(1)\,.
$$

Taking $n\to\infty$ and utilizing Theorem 1.1 together with (3.2) completes the proof. ☐

3.3 Conditioned survival

Theorem 3.5. *Suppose $\mathbf{E}[Z^p] < \infty$ for all $p\ge 1$. Then the conditional variable $(|Y_n|/n \,|\, |Y_n|>0)$ converges in distribution to an exponential random variable with mean λ^{-1} for* GW*-almost every $\mathbf{T}$.*

By conditional random variable $(|Y_n|/n \,|\, |Y_n|>0)$, we mean the random variable with law $\mathbf{P}_{\mathbf{T}}[|Y_n|/n\in\cdot \,|\, |Y_n|>0]$.

Proof. The proof is via the method of moments. In particular, since the moment generating function of an exponential random variable has a positive radius of convergence, its distribution is uniquely determined by its moments. Thus, any sequence of random variables with each moment converging to the moment of an exponential random variable must converge in distribution to that exponential random variable [Bil95, Theorems 30.1 and 30.2].

Let X_n be a random variable with distribution $(|Y_n|/n \,|\, |Y_n| > 0)$. It is sufficient to show $\mathbf{E_T}[X_n^k] \to k!\lambda^{-k}$ GW-a.s. since $k!\lambda^{-k}$ is the kth moment of an exponential random variable. Proposition 3.2 and Theorem 3.3 imply

$$\begin{aligned}\mathbf{E_T}[X_n^k] &= \frac{\mathbf{E_T}[|Y_n|^k]}{n^k \mathbf{P_T}[|Y_n| > 0]} \\ &= \frac{\mathbf{E_T}[|Y_n|^k}{n^{k-1}} \cdot \frac{1}{n \cdot \mathbf{P_T}[|Y_n| > 0]} \\ &\to k! W \lambda^{-(k-1)} \cdot \frac{1}{\lambda W} \\ &= k!\lambda^{-k}\,. \end{aligned}$$ □

More can be said about the structure of the open percolation cluster of the root conditioned on $\mathbf{0} \leftrightarrow \mathbf{T}_n$, but we require two general, more or less standard lemmas first.

Lemma 3.6. *For any events A and B with $\mathbf{P}[B] \neq 0$,*

$$|\mathbf{P}[A \,|\, B] - \mathbf{P}[A]| \leq \mathbf{P}[B^c]\,.$$

Proof. Expand

$$\mathbf{P}[A] = \mathbf{P}[A \,|\, B](1 - \mathbf{P}[B^c]) + \mathbf{P}[A \,|\, B^c]\mathbf{P}[B^c]$$

and solve

$$\mathbf{P}[A] - \mathbf{P}[A \,|\, B] = (\mathbf{P}[A \,|\, B^c] - \mathbf{P}[A \,|\, B])\mathbf{P}[B^c]\,.$$

Taking absolute values and bounding $|\mathbf{P}[A \,|\, B^c] - \mathbf{P}[A \,|\, B]| \leq 1$ completes the proof. □

Lemma 3.7. *Let X_k be i.i.d. centered random variables with $\mathbf{E}[|X_1|^p] < \infty$ for some $p \in [2, \infty)$. Then there exists a constant C_p so that*

$$\mathbf{P}\left[\left|\sum_{k=1}^{n} \frac{X_k}{n}\right| > t\right] \leq C_p t^{-p} n^{-p/2} + 2\exp\left(-\frac{nt^2}{\mathrm{Var}\,[X_1]}\right)$$

for all $t > 0$.

Proof. This is a straightforward application of [Che09, Theorem 2.1] which states that for independent random variables M_i with $\mathbf{E}[M_i] = 0$ and $\mathbf{E}[|M_i|^p] < \infty$ for some $p > 2$ we have

$$\mathbf{P}\left[\left|\sum_{i=1}^{n} M_i\right| \geq t\right] \leq C_p t^{-p} \max\left(r_{n,p}(t), (r_{n,2}(t))^{p/2}\right) + \exp\left(-\frac{t^2}{16 b_n}\right)$$

where $r_{n,u}(t) = \sum_{i=1}^n \mathbf{E}(|M_i|^u \mathbf{1}_{|M_i| \geq 3b_n/t})$, $b_n = \sum_{i=1}^n \mathbf{E}[M_i^2]$ and C_p is a positive constant. Setting $M_i = X_i/n$ completes the proof. □

For a fixed tree and $m < n$, define $B_m(n)$ to be the event that $\mathbf{0} \leftrightarrow \mathbf{T}_n$ through precisely one vertex at depth m.

Proposition 3.8. *Suppose $\mathbf{E}[Z^p] < \infty$ for all $p \geq 1$. There exists an $N = N(\mathbf{T})$ with $N < \infty$ almost surely so that for all $n \geq N$, we have*

(a) $\mathbf{P_T}[B_m(n)^c \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] < Cn^{-1/4}$ *for* $m = m(n) := \lceil \frac{\log n}{4 \log \mu} \rceil$

(*b*) $\max_{v\in\mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] = O(n^{-1/8})$

for some constant $C > 0$.

Proof. Note first that for the choice of m as in part (a), we have $\frac{1}{2\mu}Wn^{1/4} \leq Z_m \leq 2\mu W n^{1/4}$ for sufficiently large n.

(a) Using Theorem 3.3 and Lemma 3.4, we bound

$$\begin{aligned}
\mathbf{P_T}[B_m(n)^c \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] &\leq \frac{\left(\sum_{v\in T_m} \mathbf{P_T}[v \leftrightarrow \mathbf{T}_n]\right)^2}{\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_n]} \\
&\leq \left(\frac{2}{1-p_c}\right)^2 \left(\frac{\sum_{v\in\mathbf{T}_m} \overline{W}(v)}{Z_m}\right)^2 \frac{Z_m^2}{(n-m)^2 \mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_n]} \\
&\leq C \left(\frac{\sum_{v\in\mathbf{T}_m} \overline{W}(v)}{Z_m}\right)^2 W n^{-1/2} \qquad (3.3)
\end{aligned}$$

for n sufficiently large, and some choice of $C > 0$ depending on the distribution of Z. Applying Lemma 3.7 for $p = 9$ gives

$$\mathbf{P}\left[\left|\frac{\sum_{v\in\mathbf{T}_m} \overline{W}(v)}{Z_m} - \mathbf{E}[\overline{W}]\right| > n^{1/8}\right] \leq C_9 n^{-9/8} + 2\exp\left(-n^{1/4}/\mathrm{Var}\,[\overline{W}]\right)$$

where we use the trivial bound of $1 \leq Z_m$. Since this is summable in n, the Borel-Cantelli Lemma implies that this event only occurs finitely often. In particular, this means that for sufficiently large n

$$\mathbf{P_T}[B_m(n)^c \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] \leq CWn^{-1/4} \qquad (3.4)$$

for some constant $C > 0$ depending only on the distribution of Z.

(b) Applying Lemma 3.6 to the measure $\mathbf{P_T}[\cdot \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n]$ and recalling $B_m(n) \subseteq \mathbf{0} \leftrightarrow \mathbf{T}_n$,

$$\Big|\mathbf{P_T}[v \in Y_n \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] - \mathbf{P_T}[v \in Y_n \,|\, B_m(n)]\Big| \leq \mathbf{P_T}\left[B_m(n)^c \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n\right]$$

which is $O(n^{-1/4})$ by part (a). It is thus sufficient to bound $\mathbf{P_T}[v \in Y_n \,|\, B_m(n)]$. For a vertex $v \in \mathbf{T}_n$ and $m < n$, let $P_m(v)$ be the ancestor of v in $\mathbf{T}_m$. We then have

$$\mathbf{P_T}[v \in Y_n \,|\, B_m(n)] \leq \mathbf{P_T}[\mathbf{0} \leftrightarrow P_m(v) \leftrightarrow \mathbf{T}_n \,|\, B_m(n)]\,.$$

Conditioned on $B_m(n)$, there exists a unique vertex $w \in \mathbf{T}_m$ so that $\mathbf{0} \leftrightarrow w \leftrightarrow \mathbf{T}_n$; this vertex w is chosen with probability bounded above by

$$\begin{aligned}
&\mathbf{P_T}[\mathbf{0} \leftrightarrow w \leftrightarrow \mathbf{T}_n \,|\, B_m(n)] \\
&\quad\leq \frac{\mathbf{P_T}[\mathbf{0} \leftrightarrow w \leftrightarrow \mathbf{T}_n]}{\sum_{u\in\mathbf{T}_m} \mathbf{P_T}[\mathbf{0} \leftrightarrow u \leftrightarrow \mathbf{T}_n] - \sum_{(u_1,u_2)\in\binom{\mathbf{T}_m}{2}} \mathbf{P_T}[\mathbf{0} \leftrightarrow (u_1,u_2) \leftrightarrow \mathbf{T}_n]} \\
&\quad\leq \frac{\mathbf{P_T}[w \leftrightarrow \mathbf{T}_n]}{\sum_{u\in\mathbf{T}_m} \mathbf{P_T}[u \leftrightarrow \mathbf{T}_n] - \left(\sum_{u\in\mathbf{T}_m} \mathbf{P_T}[u \leftrightarrow \mathbf{T}_n]\right)^2} \\
&\quad\leq \frac{c(n-m)^{-1}\overline{W}(w)}{(1+o(1))\sum_{u\in\mathbf{T}_m} \mathbf{P_T}[u \leftrightarrow \mathbf{T}_n]} \qquad (3.5)
\end{aligned}$$

where the latter inequality is by applying the bound of Lemma 3.4 to the numerator and arguing as in (3.3) to almost-surely bound the denominator. In particular, the $o(1)$ term is uniform in w.

We want to take the maximum over all possible $w \in \mathbf{T}_m$, and note that for any $\alpha > 0$,

$$\begin{aligned}\mathbf{P}\left[\max_{w\in\mathbf{T}_m} \overline{W}(w) > n^\alpha\right] &= \mathbf{E}\left[\mathbf{P}\left[\max_{w\in\mathbf{T}_m} \overline{W}(w) > n^\alpha \,\middle|\, \mathcal{T}_m\right]\right] \\ &\le \mathbf{E}[Z_m]\mathbf{P}[\overline{W} > n^\alpha] \\ &\le \mu^m \cdot \frac{\mathbf{E}[\overline{W}^{2/\alpha}]}{n^2} \\ &= O(n^{-7/4})\end{aligned}$$

which is summable, implying that for any fixed $\alpha > 0$, we eventually have $\max_{w\in\mathbf{T}_m} \overline{W}(w) \le n^\alpha$. It merely remains to bound the denominator of (3.5).

Note that by Proposition 3.2, the lower bound given in Lemma 3.4 converges almost surely to $\frac{W\lambda}{2}$ as $n \to \infty$. In particular, this means that if we set

$$p_n := \mathbf{P}\left[\frac{W\lambda}{4} \le n\mathbf{P}_\mathbf{T}[|Y_n| > 0]\right],$$

then $p_n \to 1$. By Hoeffding's inequality together with Borel-Cantelli, the number of vertices $u \in \mathbf{T}_m$ for which we have

$$\frac{W(u)\lambda}{4} \le (n-m)\mathbf{P}_\mathbf{T}[u \leftrightarrow \mathbf{T}_n]$$

is almost surely at least $1/2$ of $\mathbf{T}_m$ for n sufficiently large. This gives

$$(n-m)\sum_{u\in\mathbf{T}_m} \mathbf{P}_\mathbf{T}[u \leftrightarrow \mathbf{T}_n] \ge \frac{\lambda}{4}\sum_{u\in\mathbf{T}_m} W(u)\mathbf{1}_{W(u)\lambda/4\le(n-m)\mathbf{P}_\mathbf{T}[u\leftrightarrow\mathbf{T}_n]} = \Omega(Z_m)\,.$$

Recalling that $Z_m = \Theta(Wn^{-1/4})$ and plugging the above into (3.5) completes the proof. □

3.4 Incipient infinite cluster

As in [Kes86a], we sketch a proof of the construction of the IIC. For an infinite tree T, define $T[n]$ to be the finite subtree of T obtained by restricting to vertices of depth at most n.

Lemma 3.9. *Suppose $\mathbf{E}[Z^4] < \infty$; for a subtree t of $\mathbf{T}[n]$, we have*

$$\lim_{M\to\infty} \mathbf{P}_\mathbf{T}[\mathcal{C}_{p_c}[n] = t \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_M] = \frac{\sum_{v\in t_n} W(v)}{W}\mathbf{P}_\mathbf{T}[\mathcal{C}_{p_c}[n] = t]$$

almost surely for each tree t.

The random measure $\mu_\mathbf{T}$ on subtrees of $\mathbf{T}$ with marginals

$$\mu_\mathbf{T}\big|_{\mathcal{T}_n}[t] := \frac{\sum_{v\in t_n} W(v)}{W}\mathbf{P}_\mathbf{T}[\mathcal{C}_{p_c}[n] = t]$$

has a unique extension to a probability measure on rooted infinite trees `GW` *almost surely. The IIC is thus the random subtree of $\mathbf{T}$ with law $\mu_\mathbf{T}$.*

Proof. Since each $\mathbf{T}$ has countably many vertices, Theorem 3.3 assures that $n\mathbf{P}_\mathbf{T}[v \leftrightarrow \mathbf{T}_{n+|v|}] = \lambda W(v)$ for each vertex v of $\mathbf{T}$ a.s. When all of these limits hold, we then have

$$\mathbf{P}_\mathbf{T}[\mathcal{C}_{p_c}[n] = t \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_M] = \frac{\mathbf{P}_\mathbf{T}[\mathcal{C}_{p_c}[n] = t, \mathbf{0} \leftrightarrow \mathbf{T}_M]}{\mathbf{P}_\mathbf{T}[\mathbf{0} \leftrightarrow \mathbf{T}_M]}$$

$$= \mathbf{P_T}[\mathcal{C}_{p_c}[n] = t] \left(\frac{\sum_{v \in t_n} \mathbf{P_T}[v \leftrightarrow \mathbf{T}_M] + O(|t_n|^2 M^{-2})}{\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_M]} \right)$$
$$\xrightarrow{M\to\infty} \mathbf{P_T}[\mathcal{C}_{p_c}[n] = t] \frac{\sum_{v \in t_n} W(v)}{W}$$

for each t. To show that the measure $\mu_{\mathbf{T}}$ can be extended, we note that its marginals are consistent, as can be seen via the recurrence $W(v) = p_c \sum_w W(w)$ where the sum is over all children of v. Applying the Kolmogorov extension theorem [Dur10, Theorem 2.1.14] completes the proof. □

It is easy to show that the law of the IIC can in fact be generated by conditioning on $p > p_c$ percolation to survive and then taking $p \to p_c^+$:

Corollary 3.10. *For a subtree t of $\mathbf{T}[n]$, we have*

$$\lim_{p \to p_c^+} \mathbf{P_T}[\mathcal{C}_p[n] = t \,|\, |\mathcal{C}_p| = \infty] = \frac{\sum_{v \in t_n} W(v)}{W} \mathbf{P_T}[\mathcal{C}_{p_c}[n] = t]$$

almost surely.

Proof. As shown in [MPR18], we have

$$\lim_{p \to p_c} \frac{\mathbf{P_T}[|\mathcal{C}_p| = \infty]}{p - p_c} = KW$$

almost-surely for some constant K depending only on the offspring distribution. The Corollary follows from Bayes' theorem in the same manner as Lemma 3.9. □

In light of Lemma 3.9, it is natural to guess that the number of vertices in the IIC at depth n will asymptotically be the size-biased version of $(|Y_n| \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n)$: the sum $\sum_{v \in t_n} W(v)$ will be relatively close to $|t_n| W$, therefore biasing each choice of t by a factor of $|t_n|$. In order to make this argument rigorous, we will invoke Proposition 3.8 which shows that no single vertex has high probably of surviving conditionally. Throughout, we use the notation $n(a, b) = (na, nb)$ for $a < b$ and $\mathbf{C}$ to denote the IIC.

Theorem 3.11. *Suppose $\mathbf{E}[Z^p] < \infty$ for each $1 \le p < \infty$. Then for each $0 \le a < b$,*

$$\lim_{n \to \infty} \mathbf{P_T}[\mathbf{C}_n \in n(a, b)] = \int_a^b \lambda^2 x e^{-\lambda x}\, dx$$

almost surely. In fact, $\mathbf{C}_n / n$ converges in distribution to the random variable with density $\lambda^2 x e^{-\lambda x}$ for GW*-almost every $\mathbf{T}$.*

Proof. To see that convergence in distribution follows from the almost sure limit, apply the almost sure limit to each interval (a, b) with $a, b \in \mathbb{Q}$; since there are only countably many such intervals, there exists a set of full GW measure on which these limits simultaneously exist for each rational interval, thereby implying convergence in distribution [Dur10, Theorem 3.2.5].

We have

$$\mathbf{P_T}[\mathbf{C}_n \in n(a, b)] = \lim_{M \to \infty} \mathbf{P_T}[Y_n \in n(a, b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_{n+M}] \,.$$

For a fixed n, write

$$\begin{aligned} &\mathbf{P_T}[|Y_n| \in n(a, b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_{n+M}] \\ &\quad = \frac{\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_{n+M} \,|\, |Y_n| \in n(a, b)] \cdot \mathbf{P_T}[|Y_n| \in n(a, b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] \cdot \mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_n]}{\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_{n+M}]} \,. \end{aligned} \tag{3.6}$$

We then calculate

$$\begin{aligned}
&\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_{n+M} \,|\, |Y_n| \in n(a,b)] \\
&\quad = \sum_S \mathbf{P_T}[Y_n = S \,|\, |Y_n| \in n(a,b)] \mathbf{P_T}[S \leftrightarrow \mathbf{T}_{n+M}] \\
&\quad = \sum_S \mathbf{P_T}[Y_n = S \,|\, |Y_n| \in n(a,b)] \sum_{v \in S} \mathbf{P_T}[v \leftrightarrow \mathbf{T}_{n+M}] + O(M^{-2}) \\
&\quad = \sum_{v \in \mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)] \mathbf{P_T}[v \leftrightarrow \mathbf{T}_{n+M}] + O(M^{-2})\,.
\end{aligned}$$

For a fixed n, we take $M \to \infty$ and utilize Theorem 3.3 to get

$$\lim_{M\to\infty} \frac{\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_{n+M} \,|\, |Y_n| \in n(a,b)]}{\mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_{n+M}]} = \frac{1}{W} \sum_{v \in \mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)] \cdot W(v)\,. \tag{3.7}$$

We plug this into (3.6) to get the limit

$$\begin{aligned}
&\lim_{M\to\infty} \mathbf{P_T}[|Y_n| \in n(a,b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_{n+M}] \\
&\quad = \left(\sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]}{n} \cdot W(v) \right) \\
&\qquad \times \left(\mathbf{P_T}[|Y_n| \in n(a,b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] \right) \left(\frac{n \cdot \mathbf{P_T}[\mathbf{0} \leftrightarrow \mathbf{T}_n]}{W} \right)\,.
\end{aligned}$$

Theorems 3.3 and 3.5 show that the latter two factors above have almost sure limits $\int_a^b \lambda e^{-\lambda x}\,dx$ and λ as $n \to \infty$, leaving only the first term. We note that

$$\begin{aligned}
\mathbf{E}\left[\sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]}{n} \cdot W(v) \,\middle|\, \mathcal{T}_n \right] &= \sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]}{n} \\
&= \mathbf{E_T}\left[\frac{|Y_n|}{n} \,\middle|\, |Y_n| \in n(a,b) \right] \\
&= \frac{\mathbf{E_T}\left[\frac{|Y_n|}{n} \cdot \mathbf{1}_{|Y_n|/n \in (a,b)} \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n \right]}{\mathbf{P_T}\left[\frac{|Y_n|}{n} \in (a,b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n \right]} \\
&\to \frac{\int_a^b \lambda x e^{-\lambda x}\,dx}{\int_a^b \lambda e^{-\lambda x}\,dx}
\end{aligned}$$

where the limit is by the continuous mapping theorem [Dur10, Theorem 3.2.4] and Theorem 3.5. It's thus sufficient to show that

$$\left| \sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]}{n} \cdot (W(v) - 1) \right| \xrightarrow{n\to\infty} 0 \tag{3.8}$$

almost surely.

Our strategy is to use a conditional version of the Borel-Cantelli Lemma together with Chebyshev's inequality. We bound the conditional variance

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]}{n} \cdot (W(v) - 1) \,\middle|\, \mathcal{T}_n \right] \\
&\qquad = \mathrm{Var}\,(W) \sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]^2}{n^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \mathrm{Var}\,(W) \max_{v \in \mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)] \sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]}{n^2} \\
&\leq \mathrm{Var}\,(W) \max_{v \in \mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)] \cdot \frac{\mathbf{E}[Y_n \,|\, |Y_n| \in n(a,b)]}{n^2} \\
&\leq \mathrm{Var}\,(W) \cdot \frac{b}{n} \cdot \max_{v \in \mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)]\,. \qquad (3.9)
\end{aligned}$$

We want to show that this is summable, and thus look to bound the $\max$ term. Applying Lemma 3.6 to the measure $\mathbf{P_T}[\cdot \,|\, |Y_n| \in n(a,b)]$ gives

$$\begin{aligned}
|\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)] - &\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b), B_m(n)]| \\
&\leq \mathbf{P_T}[B_m(n)^c \,|\, |Y_m| \in n(a,b)] \\
&\leq \frac{\mathbf{P_T}[B_m(n)^c \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n]}{\mathbf{P_T}[|Y_n| \in n(a,b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n]} \\
&= O(n^{-1/4}) \qquad (3.10)
\end{aligned}$$

by Proposition 3.8 and Theorem 3.5. Similarly,

$$\begin{aligned}
\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b), B_m(n)] &= \frac{\mathbf{P_T}[v \in Y_n, |Y_n| \in n(a,b), B_m(n)]}{\mathbf{P_T}[|Y_n| \in n(a,b), B_m(n)]} \\
&\leq \frac{\mathbf{P_T}[v \in Y_n, B_m(n)]}{\mathbf{P_T}[|Y_n| \in n(a,b), B_m(n)]} \\
&= \frac{\mathbf{P_T}[v \in Y_n \,|\, B_m(n)]}{\mathbf{P_T}[|Y_n| \in n(a,b) \,|\, B_m(n)]}\,. \qquad (3.11)
\end{aligned}$$

Using Lemma 3.6 once again expands the denominator

$$\Big|\mathbf{P_T}[|Y_n| \in n(a,b) \,|\, B_m(n)] - \mathbf{P_T}[|Y_n| \in n(a,b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n]\Big| \leq \mathbf{P_T}[B_m(n)^c \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] \leq Cn^{-1/4}$$

by Proposition 3.8. Plugging into (3.11) gives the upper bound

$$\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b), B_m(n)] \leq \frac{\mathbf{P_T}[v \in Y_n \,|\, B_m(n)]}{\mathbf{P_T}[|Y_n| \in n(a,b) \,|\, \mathbf{0} \leftrightarrow \mathbf{T}_n] - Cn^{-1/4}}\,. \qquad (3.12)$$

Combining (3.10), (3.12) and Proposition 3.8 bounds

$$\max_{v \in \mathbf{T}_n} \mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in n(a,b)] = O(n^{-1/8})\,.$$

Thus, by (3.9), the conditional variance is almost surely summable. For any fixed $\delta > 0$, Chebyshev's inequality then implies

$$\mathbf{P}\left[\left|\sum_{v \in \mathbf{T}_n} \frac{\mathbf{P_T}[v \in Y_n \,|\, |Y_n| \in (a,b)]}{n} \cdot (W(v) - 1)\right| > \delta \,\middle|\, \mathcal{T}_n\right]$$

is summable almost surely. Applying a conditional Borel-Cantelli Lemma (e.g. [Che78]) shows that (3.8) holds almost surely. □

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