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FLOWS, COALESCENCE AND NOISE. A CORRECTION

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Dr Georgii Riabov communicated to us a nonuniqueness example (see his paper [3]) of a random map which contradicts Remark 1.7 in our paper [1]: Flows, coalescence and noise. Ann. Probab. 32 (2004), no. 2, 1247–1315.

In short, with the notation of our paper, the counterexample is as follows.

COUNTEREXAMPLE TO REMARK 1.7 IN [1]. Let φ be a random variable in F (i.e., φ is a random measurable mapping on a compact metric space M) of law Q such that $M \times \Omega \ni (x, \omega) \mapsto \varphi(x, \omega) \in M$ is measurable. Suppose that Q is regular and let $\mathcal{J}$ be a regular presentation of Q. Let X be a random variable in M independent of φ. Out of φ and X, define $\psi \in F$ by $\psi(x) = \varphi(x)$ is $x \neq X$ and $\psi(x) = X$ is $x = X$. Then $M \times \Omega \ni (x, \omega) \mapsto \psi(x, \omega) \in M$ is measurable. Suppose also that the law of X has no atoms, then (reminding the definition of $\mathcal{F}$) ψ and X are independent and the law of ψ is Q. Note that $\psi(X) = X$ and (except for very special cases) we will not have that a.s. $\mathcal{J}(\psi)(X) = \psi(X) = X$.

This leads us to propose a correction to the first two sections of the paper. *In order to preserve the one-to-one correspondence between laws of stochastic flows of maps, or kernels and consistent systems of Feller semigroups, keeping the same set of notation, we slightly reinforce the definition of a stochastic flow.*

For flows of mappings the new definition is as follows.

DEFINITION 1. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space and let $\varphi = (\varphi_{s,t}, s \leq t)$ be a family of $(F, \mathcal{F})$-valued random variables such that for all $x \in M$ and all $t \in \mathbb{R}$, P-a.s. $\varphi_{t,t}(x) = x$. For $t \geq 0$, denote by Q_t the law of $\varphi_{0,t}$. The family φ is called a stochastic flow of mappings if for all $t \geq 0$, Q_t is regular and if the following properties are satisfied by φ:

(a) For all $s \leq u \leq t$, all $x \in M$ and all measurable presentation $\mathcal{J}_{t-u}$ of Q_{t-u}, P-almost surely, $\varphi_{s,t}(x) = \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi_{s,u}(x)$. (Cocycle property)
(b) For all $s \leq t$, the law of $\varphi_{s,t}$ is Q_{t-s}. (Stationarity)
(c) The flow has independent increments, that is, for all $t_1 \leq t_2 \leq \cdots \leq t_n$, the family $\{\varphi_{t_i,t_{i+1}}, 1 \leq i \leq n-1\}$ is independent.
(d) For all $f \in C(M)$ and all $s \leq t$,

$$\lim_{(u,v)\to(s,t)} \sup_{x\in M} \mathsf{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{u,v}(x))^2] = 0.$$

(e) For all $f \in C(M)$ and all $s \leq t$,

$$\lim_{d(x,y)\to 0} \mathsf{E}[(f \circ \varphi_{s,t}(x) - f \circ \varphi_{s,t}(y))^2] = 0.$$

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REMARK 2.

- The only significant difference with the previous definition is the introduction of the measurable presentation in the statement of the cocycle property. The regularity of Q_t was already a consequence of the previous definition.
- Item (a) holds for all set of measurable presentations as soon as (a) holds for one of them.
- If ψ is equal in law to a stochastic flow of mappings φ, then ψ is also a stochastic flow of mappings. Indeed, it is straightforward to check that ψ satisfies (b), (c), (d) and (e), and after having remarked that for all $x \in M$, $(\varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_3(x), \mathcal{J}_{t-u}(\varphi_2) \circ \varphi_1(x))$ is measurable, we prove that ψ satisfies (a).

The notion of measurable flow of mappings is therefore slightly modified. We will state it after proving a preliminary proposition.

PROPOSITION 3. *Let φ be a stochastic flow of mappings, and for $t \geq 0$, let $\mathcal{J}_t$ be a measurable presentation of the law of $\varphi_{0,t}$. Then $\varphi' = (\mathcal{J}_{t-s}(\varphi_{s,t}), s \leq t)$ is a stochastic flow of mappings satisfying*:

(i) *For all $s \leq t$ and all $x \in M$, a.s. $\varphi'_{s,t}(x) = \varphi_{s,t}(x)$.*
(ii) *For all $s \leq u \leq t$ and all $x \in M$, P-almost surely, $\varphi'_{s,t}(x) = \varphi'_{u,t} \circ \varphi'_{s,u}(x)$.*

PROOF. Item (i) is a consequence of the fact that $\mathcal{J}_{t-s}$ is a measurable presentation of the law of $\varphi_{s,t}$. Then φ and φ' share the same law and φ' is a stochastic flow of mappings. Let us now prove (ii). For $s \leq u \leq t$ and $x \in M$, it holds that P-almost surely,

$$\begin{aligned}\varphi'_{s,t}(x) &= \mathcal{J}_{t-u}(\varphi'_{u,t}) \circ \varphi'_{s,u}(x)\\ &= \mathcal{J}_{t-u} \circ \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi'_{s,u}(x).\end{aligned}$$

Note that if $\mathcal{J}$ and $\mathcal{J}'$ are two measurable presentations of a regular probability measure Q and if $\mu \in \mathcal{P}(M)$, then (using Fubini's theorem),

$$\mu(dx) \otimes \mathsf{Q}(d\varphi)\text{-a.s.}, \qquad \mathcal{J}(\varphi)(x) = \mathcal{J}'(\varphi)(x).$$

Using this remark, since $\mathcal{J}_{t-u} \circ \mathcal{J}_{t-u}$ is also a measurable presentation of Q_{t-u}, it holds that P-almost surely,

$$\mathcal{J}_{t-u} \circ \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi'_{s,u}(x) = \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \varphi'_{s,u}(x) = \varphi'_{u,t} \circ \varphi'_{s,u}(x).$$

This proves (ii). □

DEFINITION 4.

- The stochastic flow of mappings φ' defined in Proposition 3 will be called a *measurable modification* of φ.
- A stochastic flow of mappings which is a measurable modification of a stochastic flow of mappings is called a measurable stochastic flow of mappings.

For stochastic flows of kernels, a similar modification can be made. We first introduce an appropriate notion of measurable presentation.

DEFINITION 5. A measurable presentation of a regular probability measure ν is a measurable mapping $\mathfrak{p} : (E, \mathcal{E}) \to (E, \mathcal{E})$ such that $(x, K) \mapsto \mathfrak{p}(K)(x)$ is measurable and such that for all $x \in M$, $\nu(dK)$-a.s. $\mathfrak{p}(K)(x) = K(x)$.

REMARK 6.

- If $\mathfrak{p}$ is a measurable presentation of a regular probability measure ν, then $(\mu, K) \mapsto \mu(\mathfrak{p}(K))$ is measurable and for all $\mu \in \mathcal{P}(M)$, $\nu(dK)$-a.s. $\mu(\mathfrak{p}(K)) = \mu K$.
- If ν is a regular probability measure on $(E, \mathcal{E})$ and if $\mathcal{J}$ is a measurable presentation of $\mathsf{Q} := \mathcal{I}^*(\nu)$, then for all $x \in M$, if $\nu(dK)$-a.s. $\delta \circ \mathcal{J} \circ \mathcal{I}(K)(x) = K(x)$. Therefore, the mapping $\mathfrak{p} = \delta \circ \mathcal{J} \circ \mathcal{I}$ is a measurable presentation of ν.
- Let $(\nu_t)_{t \geq 0}$ be a convolution semigroup on $(E, \mathcal{E})$. If K_1 and K_2 are random kernels with laws ν_s and ν_t and if $\mathfrak{p}_t$ is a measurable presentation of ν_t, then $K_1(\mathfrak{p}_t(K_2))$ is a random kernel with law ν_{s+t} (note that $(K_1, K_2) \mapsto K_1(\mathfrak{p}_t(K_2))$ is measurable).

The definition of stochastic flows of kernels is modified as follows.

DEFINITION 7. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space and let $K = (K_{s,t}, s \leq t)$ be a family of $(E, \mathcal{E})$-valued random variables such that for all $x \in M$ and all $t \in \mathbb{R}$, P-a.s. $K_{t,t}(x) = \delta_x$. For $t \geq 0$, denote by ν_t the law of $K_{0,t}$. The family K is called a stochastic flow of kernels if for all $t \geq 0$, ν_t is regular and if the following properties are satisfied by K:

(a) For all $s \leq u \leq t$, for all $x \in M$, for all $f \in C(M)$ and for all measurable presentation $\mathfrak{p}_{t-u}$ of ν_{t-u}, P-almost surely, $K_{s,t} f(x) = K_{s,u}(\mathfrak{p}_{t-u}(K_{u,t}) f)(x)$. (Cocycle property)

(b) For all $s \leq t$, the law of $K_{s,t}$ is ν_{t-s}. (Stationarity)

(c) The flow has independent increments, that is, for all $t_1 \leq t_2 \leq \cdots \leq t_n$, the family $\{K_{t_i,t_{i+1}}, 1 \leq i \leq n-1\}$ is independent.

(d) For all $f \in C(M)$ and all $s \leq t$,

$$\lim_{(u,v) \to (s,t)} \sup_{x \in M} \mathsf{E}[(K_{s,t} f(x) - K_{u,v} f(x))^2] = 0.$$

(e) For all $f \in C(M)$ and all $s \leq t$,

$$\lim_{d(x,y) \to 0} \mathsf{E}[(K_{s,t} f(x) - K_{s,t} f(y))^2] = 0.$$

REMARK 8.

- (a) holds for all set of measurable presentations as soon as (a) holds for one of them.
- If K' is equal in law to a stochastic flow of kernels K, then K' is also a stochastic flow of kernels.

PROPOSITION 9. *Let K be a stochastic flow of kernels, and for $t \geq 0$, let $\mathfrak{p}_t$ be a measurable presentation of the law of $K_{0,t}$. Then $K' = (\mathfrak{p}_{t-s}(K_{s,t}), s \leq t)$ is a stochastic flow of kernels satisfying*:

(i) *For all $s \leq t$ and $\mu \in \mathcal{P}(M)$, P-a.s. $\mu K'_{s,t}(x) = \mu K_{s,t}(x)$.*

(ii) *For all $s \leq u \leq t$ and for all $\mu \in \mathcal{P}(M)$, P-a.s. $\mu K'_{s,t} = \mu K'_{s,u} K'_{u,t}$.*

DEFINITION 10.

- The stochastic flow of kernels K' defined in the previous proposition will be called a *measurable modification* of K.
- A stochastic flow of kernels which is a measurable modification of a stochastic flow of kernels is called a measurable stochastic flow of kernels.

A new version of the paper [1], including a few more minor changes is available on arXiv [2].

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REFERENCES

[1] LE JAN, Y. and RAIMOND, O. (2004). Flows, coalescence and noise. *Ann. Probab.* **32** 1247–1315. MR2060298 https://doi.org/10.1214/009117904000000207

[2] LE JAN, Y. and RAIMOND, O. (2004). Flows, coalescence and noise. Preprint. Available at arXiv:math/0203221.

[3] RIABOV, G. V. (2018). Random dynamical systems generated by coalescing stochastic flows on $\mathbb{R}$. *Stoch. Dyn.* **18** 1850031, 24. MR3842254 https://doi.org/10.1142/S0219493718500314