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RIGHT MARKER SPEEDS OF SOLUTIONS TO THE KPP EQUATION WITH NOISE[1]

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We consider the one-dimensional KPP-equation driven by space–time white noise. We show that for all parameters above the critical value for survival, there exist stochastic wavelike solutions which travel with a deterministic positive linear speed. We further give a sufficient condition on the initial condition of a solution to attain this speed. Our approach is in the spirit of corresponding results for the nearest-neighbor contact process respectively oriented percolation. Here, the main difficulty arises from the moderate size of the parameter and the long range interaction. Stopping times and averaging techniques are used to overcome this difficulty.

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1. Introduction. The Kolmogorov–Petrovskii–Piskunov (KPP) equation (also known as the Kolmogorov or Fisher equation) with noise is given as

$$\partial_t u = \partial_{xx} u + \theta u - u^2 + u^{\frac{1}{2}}\, dW, \quad t > 0, x \in \mathbb{R}, \qquad u(0,x) = u_0(x) \geq 0, \tag{1.1}$$

where $W = W(t,x)$ is space–time white noise and $\theta > 0$ a parameter. The deterministic part of this one-dimensional stochastic partial differential equation (SPDE) is, after appropriate scaling, a case of the well-studied KPP-equation. Note that by Mueller and Tribe [17], Lemma 2.1.2, constant front-factors in the PDEs/SPDEs to be referred to, can and will be changed without comment to fit into our framework. Including the noise term, one can think of $u(t,x) = u_t(x) = u_t^{(u_0)}(x)$ as the (random) density of a population in time and space. Leaving out the term $\theta u - u^2$, the above SPDE is the density of a super-Brownian motion (cf. Perkins [19], Theorem III.4.2), the latter being the high density limit of branching particle systems. The additional term of θu models linear mass creation at rate $\theta > 0$, $-u^2$ models death due to competition or overcrowding. In [18], Mueller and Tribe obtain solutions to (1.1) as (weak) limits of *approximate* densities of occupied sites in rescaled one-dimensional long range contact processes.

Let $\mathcal{C}^+$ denote the space of nonnegative continuous functions on $\mathbb{R}$. The existence and uniqueness in law of solutions to (1.1) in the *space of nonnegative continuous functions with slower than exponential growth* $\mathcal{C}^+_{\text{tem}}$,

$$\mathcal{C}^+_{\text{tem}} = \{ f \in \mathcal{C}^+ : \|f\|_\lambda < \infty \text{ for all } \lambda > 0 \} \\ \text{with } \|f\|_\lambda = \sup_{x\in\mathbb{R}} |f(x)| e^{-\lambda|x|}, \tag{1.2}$$

is established in Tribe [21], Theorem 2.2 (see Theorem 2 below). Here, a solution to (1.1) is to be understood in the sense of a *weak solution* (see Notation 1.3 below). Denote with $\mathbb{P}_{u_0}$ the law of such a solution starting in $u_0 \in \mathcal{C}^+_{\text{tem}}$. By [21], Theorem 2.2, the map $f \mapsto \mathbb{P}_f$ on $\mathcal{C}^+_{\text{tem}}$ is continuous and the family of laws $\mathbb{P}_f$, $f \in \mathcal{C}^+_{\text{tem}}$ forms a strong Markov family. For $\nu \in \mathcal{P}(\mathcal{C}^+_{\text{tem}})$, the space of probability measures on $\mathcal{C}^+_{\text{tem}}$, denote $\mathbb{P}_\nu(A) = \int_{\mathcal{C}^+_{\text{tem}}} \mathbb{P}_f(A)\nu(df)$. Use $\mathbb{E}_{u_0}$ respectively $\mathbb{E}_\nu$ to denote respective expectations.

Let $\tau = \inf\{t \geq 0 : u(t,\cdot) \equiv 0\}$ be the *extinction-time* of the process. By [17], Theorem 1, there exists a critical value $\theta_c > 0$ such that for any initial condition $u_0 \in \mathcal{C}^+_c \backslash \{0\}$ with *compact* support and $\theta < \theta_c$, the extinction-time of u solving (1.1) is finite almost surely. For $\theta > \theta_c$, *survival*, that is, $\tau = \infty$, happens with positive probability.

The investigation of the dynamics of solutions to (1.1) is a major challenge, where the main difficulty comes from the competition term $-u^2$. Without competition, the underlying additive property facilitates the use of Laplace functionals (cf.

[19], paragraphs preceding and following Lemma II.5.9). Including competition, only subadditivity in the sense of [17], Lemma 2.1.7, or Kliem [14], Remark 2.1(i), holds, that is, for $u_0, v_0 \in \mathcal{C}^+_{\text{tem}}$ and $w_0 \equiv u_0 + v_0$ there exists a coupling of solutions $(u_t)_{t\geq 0}, (v_t)_{t\geq 0}, (w_t)_{t\geq 0}$ to (1.1) with respective initial conditions u_0, v_0, w_0 such that $w_t(x) \leq u_t(x) + v_t(x)$ for all $t \geq 0, x \in \mathbb{R}$ almost surely.

Let

$$R_0(u(t)) \equiv R_0(t) \equiv \sup\{x \in \mathbb{R} : u(t,x) > 0\} \quad \text{with } \sup \varnothing = -\infty \tag{1.3}$$

denote the right marker of a solution to (1.1) starting in $u_0 \in \mathcal{P}(\mathcal{C}^+_{\text{tem}})$. Note that $R_0(t) = -\infty$ if and only if $\tau \leq t$. Extending arguments of Iscoe [11], one can show that $R_0(u(0)) < \infty$ implies $R_0(u(t)) < \infty$ for all $t > 0$ almost surely. Indeed, the interested reader may have a look at [21], Lemma 2.1, where the crucial part of the proof is given. In combination with a Borel–Cantelli argument, this property of the right markers now follows. (Note in particular, that for $u_0 \in \mathcal{C}^+_c$, the *compact support property* holds, that is, $u_t \in \mathcal{C}^+_c$ for all $t > 0$ almost surely; also see [20].) Analogously, we denote the left marker of a solution by $L_0(u(t)) \equiv L_0(t) \equiv \inf\{x \in \mathbb{R} : u(t,x) > 0\}$ with $\inf \varnothing = -\infty$. By symmetry, $L_0(u(0)) > -\infty$ implies $L_0(u(t)) > -\infty$ for all $t > 0$ almost surely.

Using R_0 as a *(right) wavefront marker*, we look for so-called *traveling wave solutions* to (1.1), that is, solutions with the properties:

$$\begin{aligned} &\text{(i)} \quad R_0(u(t)) \in (-\infty, \infty) \quad \text{for all } t \geq 0, \\ &\text{(ii)} \quad u(t, \cdot + R_0(u(t))) \quad \text{is a stationary process in time.} \end{aligned} \tag{1.4}$$

Traveling wave solutions are of interest in models from physics, chemistry and biology (cf. Aronson and Weinberger [1]). In [21], the existence of traveling wave solutions for $\theta > \theta_c$ with nonnegative wave speed, based on solutions to (1.1) with Heavyside initial data of the form $H_0(x) \equiv 1 \wedge (-x \vee 0)$ is established. In [21], Section 4, it is established that for $\theta > \theta_c$ any traveling wave solution has an asymptotic (possibly random) *wave speed*

$$R_0(u(t))/t \to A \in [0, 2\theta^{1/2}] \quad \text{for } t \to \infty \text{ almost surely.} \tag{1.5}$$

It is further shown that for θ big enough, A is close to $2\theta^{1/2}$ with high probability. Strict positivity of A remains an open problem if θ is of moderate size. Further open problems that arise are for instance if the wave speed is deterministic or random, the dependence of the speed on the parameter θ or the distribution of the traveling wave, the uniqueness of the distribution of the traveling waves and the shape of the wavefront. In this article, we make substantial progress to resolve all of the questions relating to the wave speed.

An alternative construction of traveling wave solutions is given in [14] in case $\theta > \theta_c$. The initial Heavyside-condition H_0 is replaced by an arbitrary nonnegative continuous function $g_0 \in \mathcal{C}^+_c$ with compact support. As extinction (i.e.,

$\tau = \inf\{t \geq 0 : u_t \equiv 0\} = \inf\{t \geq 0 : \int u_t(x)\,dx = 0\} < \infty$) happens with probability $0 < \mathbb{P}_{g_0}(\tau < \infty) < 1$, we condition on nonextinction to obtain well-defined traveling wave solutions $\nu^{(g_0)}$. Note that $\nu^{(g_0)}$ denotes any subsequential limit obtained by this construction. The uniqueness of the limiting distribution remains as an open problem.

Write $\langle f, g\rangle = \int f(x)g(x)\,dx$. For $T > 0$, denote by υ_T the *left-upper measure* on $\mathcal{C}^+_{\text{tem}}$ corresponding to $\mathcal{L}(\xi_T^{(-\infty,0]\cap\mathbb{Z}})$ (here, $\mathcal{L}$ denotes "law") in the contact process setup (for details on the connection to the contact process; see Section 1.1 below; for additional motivation on upper measures, see second part of Section 1.4). One can construct υ_T (cf. [14], Remark 2.8) as the limiting distribution of $u_T^{(\zeta_N)}$ for $N \to \infty$, where $\zeta_N \in \mathcal{C}^+_{\text{tem}}$, $N \in \mathbb{N}$, $\zeta_N(x) \uparrow \infty$ for $x < 0$ and $\zeta_N(x) = 0$ for $x \geq 0$. Then

$$\int e^{-2\langle f,g\rangle} \upsilon_T(df) = \mathbb{P}\big(\langle \mathbb{1}_{(-\infty,0)}(\cdot), u_T^{(g)}\rangle = 0\big) \quad \text{for } g \in \mathcal{C}^+_{\text{tem}}. \tag{1.6}$$

Furthermore, for $u_0 \in \mathcal{C}^+_{\text{tem}}$ with $R_0(u_0) \leq 0$ and $T > 0$ arbitrarily fixed one obtains the existence of a coupling with a random continuous process $(u^{*,l}_{T+t})_{t\geq 0}$ with values in $\mathcal{C}^+_{\text{tem}}$ such that

$$u^{(u_0)}_{T+t}(x) \leq u^{*,l}_{T+t}(x) \quad \text{for all } x \in \mathbb{R}, t \geq 0 \text{ almost surely,} \tag{1.7}$$

where $\mathcal{L}((u^{*,l}_{T+t})_{t\geq 0}) = \mathbb{P}_{\upsilon_T}$ holds. Note in particular that such a coupling yields

$$R_0\big(u^{(u_0)}_{T+t}\big) \leq R_0\big(u^{*,l}_{T+t}\big) \quad \text{for all } t \geq 0 \text{ almost surely.} \tag{1.8}$$

By symmetry, analogous results hold for a *right-upper measure*, say κ_T, where we make use of the notation $L_0(f) \equiv \inf\{x \in \mathbb{R} : f(x) > 0\}$ and $u^{*,r}_{T+t}$ instead. In the Appendix (cf. (A.8)), we indicate how to modify the techniques of [14] to construct traveling wave solutions $\nu^{*,l}$ respectively $\nu^{*,r}$ from $u^{*,l}$ respectively $u^{*,r}$.

The first main result of this article is the following.

PROPOSITION 1.1. *For all $\theta > \theta_c$, the limit $B \equiv B(\theta) \equiv \lim_{t\to\infty} \mathbb{E}[R_0(u^{*,l}_t)]/t$ exists and is strictly positive. Moreover, for all $\theta_c < \underline{\theta} \leq \theta_1 \leq \theta_2 \leq \overline{\theta}$, there exists a constant $C = C(\underline{\theta}, \overline{\theta})$ such that*

$$B(\theta_2) - B(\theta_1) \geq C(\theta_2 - \theta_1). \tag{1.9}$$

We note that the strict positivity of $B(\theta)$ follows from (1.9), once $B(\theta) \geq 0$ is established for all $\theta > \theta_c$. Our approach relies on establishing the estimate (1.9) along the lines of the corresponding result for contact processes in [3], Lemma 4.2.

Recall from above that H_0 denotes Heavyside initial data of the form $H_0(x) \equiv 1 \wedge (-x \vee 0)$.

DEFINITION 1.2. Let

$$\mathcal{H} = \{ f \in \mathcal{C}^+_{\mathrm{tem}} : \exists x_0 \in \mathbb{R}, \epsilon > 0 : f(x) \geq \epsilon H_0(x - x_0) \text{ for all } x \in \mathbb{R} \} \tag{1.10}$$

and $\mathcal{H}^R = \{f \in \mathcal{H} : R_0(f) \in \mathbb{R}\}$.

Our second main result concerns the limiting speeds of several right markers. It establishes in particular the existence of at least one traveling wave with positive deterministic speed.

THEOREM 1. *Let $\theta > \theta_c$. Then*

$$R_0(u_T^{*,l})/T \to B(\theta) \quad \text{as } T \to \infty \text{ almost surely and in } \mathcal{L}^1. \tag{1.11}$$

For any traveling wave solution $\nu^{*,l}$,

$$R_0(u_T^{(\nu^{*,l})})/T \to B(\theta) \quad \text{almost surely as } T \to \infty \tag{1.12}$$

and $(0 \vee R_0(u_T^{(\nu^{*,l})}))/T \to B(\theta)$ in $\mathcal{L}^1$.

For initial conditions $\psi \in \mathcal{H}^R$,

$$R_0(u_T^{(\psi)})/T \to B(\theta) \quad \text{as } T \to \infty \text{ in probability and in } \mathcal{L}^1. \tag{1.13}$$

For any traveling wave solution $\nu^{(\psi)}$,

$$R_0(u_T^{(\nu^{(\psi)})})/T \to B(\theta) \quad \text{almost surely as } T \to \infty \tag{1.14}$$

and $(0 \vee R_0(u_T^{(\nu^{(\psi)})}))/T \to B(\theta)$ in $\mathcal{L}^1$.

Our approach is in the spirit of corresponding results for the nearest neighbor contact process respectively oriented percolation. A successful approach to prove positive wave speeds respectively survival for large parameter values θ is to employ a comparison of the system at hand to N-dependent oriented site percolation with density at least $1 - \rho$ (see, for instance, [21], Proof of Proposition 4.1(c), resp., [17], Section 2.2). Here, the main difficulty arises from the moderate size of the parameter in combination with the long range interaction and a novel approach was therefore taken. Stopping times and averaging techniques are used to overcome the above mentioned difficulties.

1.1. *Connections with the contact process setup and the significance of edge speeds.* For the process in (1.1), one has a self-duality relationship in the form

$$\mathbb{E}_{u_0}[e^{-2\langle u(t), v_0\rangle}] = \mathbb{E}_{u_0} \otimes \mathbb{E}_{v_0}[e^{-2\langle u(s), v(t-s)\rangle}] = \mathbb{E}_{v_0}[e^{-2\langle u_0, v(t)\rangle}] \tag{1.15}$$

for all $0 \leq s \leq t$ and $u_0, v_0 \in \mathcal{C}^+_{\mathrm{tem}}$, where $u(t), v(t)$ are independent solutions to (1.1) with initial condition u_0 respectively v_0 and with independent noises (cf. [14], (2.1)). Use $\mathcal{P}(E)$ to denote the space of probability measures on E. In [14], Remark 2.5, this self-duality is used to prove existence of a unique *upper invariant distribution* $\mu \in \mathcal{P}(\mathcal{C}^+_{\text{tem}})$ satisfying

$$\lim_{t\to\infty} \lim_{\psi\uparrow\infty} \mathbb{E}_\psi\left[e^{-2\langle u(T+t),\phi\rangle}\right] = \int e^{-2\langle f,\phi\rangle}\mu(df) = \mathbb{P}_\phi(\tau < \infty) \tag{1.16}$$

for all $T > 0, \phi \in \mathcal{C}^+_c$. In [9], Theorem 1, Horridge and Tribe give sufficient conditions ("uniformly distributed in space") for initial conditions to be in the domain of attraction of μ. They characterize μ by the right-hand side of (1.16) and show that it is the unique translation invariant stationary distribution satisfying $\mu(\{f : f \not\equiv 0\}) = 1$. The result and method of proof are in the spirit of Harris' convergence theorem for additive particle systems (cf. Durrett [5], Theorem 3.3).

Recall the construction of solutions to (1.1) from [18] by means of limits of densities of rescaled *long range* contact processes. When investigating solutions to the SPDE (1.1), it is only natural to anticipate and/or investigate behavior similar in spirit to the approximating systems. Indeed, [9] successfully applied the method of proof of Harris' convergence theorem for additive particle systems to prove a corresponding result in the context of SPDEs (1.1). Due to the long range interaction and the lack of a dual process, results for long range contact processes are limited. More is known for the *nearest neighbor* contact process $(\xi_t)_{t\geq 0}$ on $\mathbb{Z}$ (cf. Griffeath [7]), where the neighborhood of a site $x \in \mathbb{Z}$ is restricted to $\{x-1, x+1\}$. For the nearest neighbor contact process, a full description of the limiting law of a solution is available. The limiting law is the weighted average of the Dirac-measure on the "all-unoccupied" configuration and the upper invariant measure of the process, ν, where the weight on the former coincides with the extinction probability (see [7], Theorem 5).

In what follows, let S be the space of all subsets of $\mathbb{Z}$. By identifying the state of the process ξ_t at time t with the set of occupied sites, we can consider $(\xi_t)_{t\geq 0}$ as an S-valued process. For $A \subset \mathbb{Z}$, ξ^A_t denotes the state of the process at time t, starting with the set A as occupied sites. Let λ be the birth-parameter, the death-parameter is set to one. If we think of occupied sites as sites occupied with (exactly) one particle, then independently of each other, at rate λ a particle at site $x \in \mathbb{Z}$ attempts to give birth to a particle at a fixed neighboring site. In case this site is occupied, nothing happens, otherwise the birth is successful. (Taken together, the rate of birth is thus λ times the number of neighboring sites of x that are empty.) Furthermore, at rate 1 each, independently of each other, particles die. Set $\lambda_c = \sup\{\lambda \geq 0 : \mathbb{P}(\tau^{\{0\}} = \infty) = 0\}$, where $\tau^{\{0\}} = \inf\{t \geq 0 : \xi^{\{0\}}_t = \varnothing\}$ is the extinction time of the population starting with zero being the only occupied site at time 0. Then $\lambda_c \approx 1.6494$; see [16], page 289.

The proof of complete convergence for the *nearest neighbor* case relies in essence on the progression of the so-called edge processes $l^A_t \equiv \min\{x : x \in \xi^A_t\}$, $r^A_t \equiv \max\{x : x \in \xi^A_t\}$, $A \in S$ fixed. Due to the nearest neighbor interaction, one can easily show that

$$\xi_t^{\{0\}} = \xi_t^A \cap \big[l_t^{\{0\}}, r_t^{\{0\}}\big] = \xi_t^{(-\infty,0]\cap\mathbb{Z}} \cap \xi_t^{[0,\infty)\cap\mathbb{Z}} \quad \text{for all } 0 \in A \subset \mathbb{Z} \text{ on } \{\tau^{\{0\}} > t\} \tag{1.17}$$

(cf. [7], Theorem 3). Moreover,

$$l_t^{\{0\}} = l_t^{[0,\infty)\cap\mathbb{Z}} \quad \text{and} \quad r_t^{\{0\}} = r_t^{(-\infty,0]\cap\mathbb{Z}} \quad \text{on } \{\tau^{\{0\}} > t\}. \tag{1.18}$$

In [3], Theorem 1.4 and Section 4, respectively [4], Section 3, (8)–(9), Durrett shows for the *nearest neighbor* contact process respectively for oriented percolation in two dimensions that

$$-\lim_{t\to\infty} \frac{l_t^{[0,\infty)\cap\mathbb{Z}}}{t} = \lim_{t\to\infty} \frac{r_t^{(-\infty,0]\cap\mathbb{Z}}}{t} = \alpha \quad \text{a.s.,} \qquad \text{where } \alpha = \alpha(\lambda) \begin{cases} > 0 & \text{if } \lambda > \lambda_c, \\ < 0 & \text{if } \lambda < \lambda_c. \end{cases} \tag{1.19}$$

By (1.18), he obtains in particular $\lim_{t\to\infty} r_t^{\{0\}}/t = \alpha$ on $\{\tau^{\{0\}} = \infty\}$.

Thus, in these models, edge speeds characterize critical values. Similar features were for instance recently observed and discussed in Bessonov and Durrett [2] for planar quadratic contact processes (here, two individuals are needed to produce a new one). Under long range interaction, (1.17)–(1.18) do not hold true any longer.

For these reasons, the study of the speed of the right (and thus by symmetry left) marker (cf. (1.3)) is of independent interest and yields new insights into the dynamics of solutions to (1.1). In [14], Remark 2.8, $\mathcal{C}_{\text{tem}}^+$-valued left- and right-upper measures were derived as analogues to the law of $\xi_t^{(-\infty,0]\cap\mathbb{Z}}$, $\xi_t^{[0,\infty)\cap\mathbb{Z}}$, $t > 0$ and first rough estimates on marker speeds obtained in Section 4.

Recall (1.18). Proposition 1.1 and (1.11) are a prestep to a result in the spirit of the first case of (1.19). It remains to prove, for instance, that for $\theta < \theta_c$, $(R_0(u_T^{*,l}) \vee 0)/T$ converges (in some sense) to 0. In combination with (1.11), this would then show that the edge speeds of solutions starting in left- or right-upper measures characterize critical values. Also, it is an open question what happens to the the speed if we replace initial conditions $\psi \in \mathcal{H}^R$ in (1.13)–(1.14) by compactly supported $u_0 \in \mathcal{C}_c^+$ and a condition on survival. This is work in progress.

1.2. *A comment on the use of stopping times and averaging techniques.* Let us shortly indicate the need for additional averaging techniques and stopping times. For the contact process, ξ_t^A, $A \subset \mathbb{Z}$ models a population on $\mathbb{Z}$ with (at most) one particle of individual mass 1 located at $x \in \mathbb{Z}$ at time t if and only if $x \in \xi_t^A$. In the nearest-neighbor setup, if the right edge process r_t^A increases, it increases exactly by 1. Let τ denote a (random) time of increase in the right marker, that is, $r_\tau^A = r_{\tau-}^A + 1$. Then $\xi_\tau^A(r_\tau^A) - \xi_{\tau-}^A(r_\tau^A) = 1$. Thus an increase in the right edge yields the creation of a mass of fixed size 1 at a fixed distance of 1 to the right. Moreover, the probability for an increase of the right marker in a specific period of time can be bounded below by a positive quantity that only depends on λ. Indeed, just consider the probability that the particle at the rightmost site gives birth to a particle at its right neighboring site in the specific time period without dying out.

In our setup, we consider densities $u(t,x)$ in $\mathcal{C}_{\text{tem}}^{+}$ instead. Recall the definition of the right marker $R_0(u(t))$ from (1.3) as the right boundary of the support of the solution. As this definition does not provide any information on the shape of the right front $u(t, \cdot + R_0(u(t)))$ of the solution at time t, we have to additionally control the gain of mass to the right and its spatial distribution over time. For instance, if we gain a small amount of mass or if the gain in mass is of moderate size but distributed over a large stretch of space, the probability for another gain at the front in a specific period of time is comparatively small (this can be shown by techniques used in the proof of the finiteness of the right marker; see the comment following (1.3)). Even if we gain a large amount of mass, if its spatial distribution is strongly localized, it has a large probability of being reduced to small size in a short period of time due to competition (cf. (1.6)). As a result, we use stopping times to "wait" for times where not only the right front increases but the shape of the right front is such that another gain at the front is plausible (look ahead at the definition of $M(d_0, m_0)$ in (3.32)). By considering averages over time such as in the definition of $\alpha_T(\theta)$ in (2.1) and in particular in the definition of averages over front shapes as in (A.8) for $u_s^{*,l}$ (for the interested reader, the analogous definition for solutions with u_0 compactly supported can be found in [14], Definition 1.4), we can ensure that for a large enough fraction of time, such a shape can be found at the front of a solution with high probability.

1.3. *Notation and a basic theorem.* For the remainder, let us recall some notation and Theorem 2.2 from [21] that are often used in the present article.

NOTATION 1.3 (Notation from [21]; also see Section 1.2 of [14]).

1. Equip $\mathcal{C}_{\text{tem}}^{+}$ with the topology given by the norms $\|f\|_\lambda$ for $\lambda > 0$. Note that $d(f,g) \equiv \sum_{n\in\mathbb{N}} (1 \wedge \|f-g\|_{1/n})$ metrizes this topology and makes $\mathcal{C}_{\text{tem}}^{+}$ a Polish space. Let $(\mathcal{C}([0,\infty), \mathcal{C}_{\text{tem}}^{+}), \mathcal{U}, \mathcal{U}_t, U(t))$ be continuous path space, the canonical right continuous filtration and the coordinate variables.
2. In [21], (2.4)–(2.5), the more general equation

$$
\begin{aligned}
&\partial_t u = \partial_{xx} u + \alpha + \theta u - \beta u - \gamma u^2 + u^{\frac{1}{2}}\, dW, \quad t > 0, x \in \mathbb{R}, \\
&u(0,x) = u_0(x) \geq 0
\end{aligned}
\tag{1.20}
$$

with $\alpha(\cdot), \beta(\cdot), \gamma(\cdot) \in \mathcal{C}([0,\infty), \mathcal{C}_{\text{tem}}^{+})$ is under consideration. We may interpret α as the immigration rate, $\theta - \beta$ as the mass creation-annihilation rate and γ as the overcrowding rate.

A solution to (1.20) consists of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, an adapted white noise W and an adapted continuous $\mathcal{C}_{\text{tem}}^+$ valued process $u(t)$ such that for all $\phi \in \mathcal{C}_c^\infty$, the space of infinitely differentiable functions on $\mathbb{R}$ with compact support,

$$\begin{aligned}\langle u(t), \phi\rangle &= \langle u(0), \phi\rangle + \int_0^t \langle u(s), \phi_{xx} + (\theta - \beta(s) - \gamma(s)u(s))\phi\rangle\, ds \\ &\quad + \int_0^t \langle \alpha(s), \phi\rangle\, ds + \iint_0^t |u(s,x)|^{1/2}\phi(x)\, dW_{x,s}.\end{aligned} \tag{1.21}$$

If in addition $\mathbb{P}(u(0,x) = f(x)) = 1$, then we say the solution u starts at f.

THEOREM 2 (Theorem 2.2 of [21]).

(a) *For all* $f \in \mathcal{C}_{\text{tem}}^+$ *there is a solution to* (1.20) *started at* f.

(b) *All solutions to* (1.20) *started at* f *have the same law which we denote by* $Q^{f,\alpha,\beta,\gamma}$. *The map* $(f, \alpha, \beta, \gamma) \to Q^{f,\alpha,\beta,\gamma}$ *is continuous. The laws* $Q^{f,\alpha,\beta,\gamma}$ *for* $f \in \mathcal{C}_{\text{tem}}^+$ *form a strong Markov family.*

(c) *For* $R, T > 0$, *let* $\mathcal{U}_{R,T} = \sigma(U(t,x) : t \le T, |x| \le R)$. *Then the two laws* $Q^{f,\alpha,\beta,\gamma}$, $Q^{f,\alpha,0,0}$ *are mutually absolutely continuous on* $\mathcal{U}_{R,T}$.

Note that Tribe [21] later uses the notation $Q^f \equiv Q^{f,0,0,1}$ where we use $\mathbb{P}_f$. Also, when the parameter θ in (1.20) is not clear from the context, we write $Q^{f,\alpha,\beta,\gamma}(\theta)$.

Finally, let $\stackrel{\mathcal{D}}{=}$ denote equality in distribution. Constants may change from line to line. We drop θ if the context is clear.

1.4. *Overview of basic results obtained in* [14]. In this subsection, we give a short summary of and motivation for coupling techniques and upper measures obtained in [14] that are used in the remainder of this article.

Couplings of solutions to (1.1) play an essential role in the proofs to follow. For precise statements and some additional details, see the Appendix, Section A.3 on *coupling techniques*. The first two couplings are intuitively quite simple. If we increase the initial density u_0 at time 0 or the linear mass creation parameter $\theta > 0$, then the resulting densities $u_t, t > 0$ are also (monotonically) increasing almost surely. We will refer to these couplings as a *monotonicity-coupling* respectively a *θ-coupling*. The latter lies at the heart of the definition of θ_c. The third and fourth couplings, the so-called *coupling with two independent processes* and the *immigration-coupling* state in essence that if we split the initial population density u_0 into two parts and let each of the two resulting populations evolve independently, then the sum of the respective populations dominates a population started in u_0 for all times almost surely. This is due to the additional competitive interaction between the two parts of the population in the latter nonindependent setting. It also accounts for the failure of the additive property as mentioned in the paragraph above (1.3). The last coupling, the *θ-$*$-coupling*, is an extension of the idea behind the *θ-coupling* to initial conditions "$\infty \cdot \mathbb{1}_{(-\infty,0)}$".

In [14], Remark 2.8, $\mathcal{C}^+_{\text{tem}}$-valued *left-upper measures* υ_T were derived as analogues to the laws of the contact processes $\xi_T^{(-\infty,0]\cap\mathbb{Z}}$, $T>0$. In the remainder of the article, the concepts used in this remark are often used. We thus explain them in what follows and whenever citing the remark, we will remind the reader of the following paragraphs by adding "left-upper measure" in brackets. As for the contact process, the idea is to start in "the maximal initial condition on the left of, including zero." With $\mathcal{C}^+_{\text{tem}}$ as a state space, this motivates "$u_0 = \infty \cdot \mathbb{1}_{(-\infty,0)}$". The latter is not an element of $\mathcal{C}^+_{\text{tem}}$ but can be obtained as the increasing limit of initial conditions $\zeta_N \in \mathcal{C}^+_{\text{tem}}$ such that $\zeta_N(x) \uparrow \infty$ for $x<0$ and $\zeta_N(x)=0$ for $x \geq 0$. One can then construct υ_T as the limiting distribution of $u_T^{(\zeta_N)}$ for $N \to \infty$. For intuition purposes, imagine an extension of the *monotonicity-coupling* for initial conditions $u_0 \leq v_0$ to an increasing sequence of initial conditions ζ_N, $N \in \mathbb{N}$ and define υ_T as the law of the limit of the increasing sequence $(u_T^{(\zeta_N)})_{N\in\mathbb{N}}$. The precise details of this construction can be found at the beginning of Section 2 in [14]. Note that although u_0"="$\infty \cdot \mathbb{1}_{(-\infty,0)} \notin \mathcal{C}^+_{\text{tem}}$, $\upsilon_T \in \mathcal{P}(\mathcal{C}^+_{\text{tem}})$ for all $T>0$ as the downward drift, that is, the term $(\theta-u)u$ in (1.1), immediately "brings solutions down from infinity" for u big.

Using the self-duality relationship from (1.15), one can further show for all $g \in \mathcal{C}^+_{\text{tem}}$,

$$\begin{aligned}\int e^{-2\langle f,g\rangle}\upsilon_T(df) &= \lim_{N\to\infty}\mathbb{E}_{\zeta_N}\big[e^{-2\langle u_T,g\rangle}\big] = \lim_{N\to\infty}\mathbb{E}_g\big[e^{-2\langle \zeta_N,u_T\rangle}\big] \\ &= \mathbb{P}\big(\langle \mathbb{1}_{(-\infty,0)}(\cdot), u_T^{(g)}\rangle = 0\big);\end{aligned} \tag{1.22}$$

cf. (1.6). This equality yields in particular that υ_T is independent of the sequence ζ_N.

Let us assume without loss of generality that $\zeta_1 = u_0$ for $u_0 \in \mathcal{C}^+_{\text{tem}}$ arbitrarily fixed with $R_0(u_0) \leq 0$. Let $T>0$. Again, based on the idea of *monotonicity-couplings*, one can construct a coupling of $(u_{T+t}^{(u_0)})_{t\geq 0}$ with a random continuous process $(u_{T+t}^{*,l})_{t\geq 0}$ with values in $\mathcal{C}^+_{\text{tem}}$ such that

$$u_{T+t}^{(u_0)}(x) \leq u_{T+t}^{*,l}(x) \quad \text{for all } x \in \mathbb{R}, t \geq 0 \text{ almost surely,} \tag{1.23}$$

where $\mathcal{L}((u_{T+t}^{*,l})_{t\geq 0}) = \mathbb{P}_{\upsilon_T}$ holds. Note in particular that such a coupling yields

$$R_0\big(u_{T+t}^{(u_0)}\big) \leq R_0\big(u_{T+t}^{*,l}\big) \quad \text{for all } t \geq 0 \text{ almost surely.} \tag{1.24}$$

Thanks to (1.23), $u_{T+t}^{*,l}$, $t \geq 0$ can be interpreted as a uniform upper bound on solutions to (1.1) with $R_0(u_0) \leq 0$. Here, it is important to note that the statement requires us to start in $T>0$ as the initial condition of $u_{T+t}^{*,l}$ itself is not an element of the state space. We refrain to speak of $(u^{*,l}_{T+t})_{t\geq 0}$ as "a solution" to (1.1), even when considering υ_T as a randomized initial condition. Indeed, the definition of solutions (cf. Notation 1.3) includes the specification of a white noise W, which the construction in [14] does not provide.

As an application of (1.24), bounds on $\mathbb{E}[0 \vee R_0(u_t)]$ for $t > 0$ were obtained in Section 4 of [14] that are independent of $u_0 \in \mathcal{C}^+_{\mathrm{tem}}$ satisfying $R_0(u_0) \leq 0$. Indeed, in Proposition 4.5, [14] states that for $\theta > 0$ and $0 < T \leq 1$, $\mathbb{E}[0 \vee R_0(u^{*,l}_T)] \leq C(\theta)T^{1/4}$ and in Lemma 4.6 that for $T \geq 1$, $\mathbb{E}[0 \vee R_0(u^{*,l}_T)] \leq C(\theta)T$. It then follows easily that $\mathbb{E}_{u_0}[0 \vee R_0(T)] \leq C(\theta)(T \vee T^{1/4})$ for all $T \geq 0$; cf. Corollary 4.7. These bounds are used below in Section 2.3.

By symmetry, analogous results hold for *right-upper measures* κ_T, the analogues to the laws of the contact process $\xi^{[0,\infty)\cap\mathbb{Z}}_T$, $T > 0$. Here, we make use of the notation $L_0(f) \equiv \inf\{x \in \mathbb{R} : f(x) > 0\}$ and $u^{*,r}_{T+t}$ instead.

Originally, the above ideas were used in [14], Proposition 2.2 and Corollary 2.6, to construct *upper measures* μ_T corresponding to $\mathcal{L}(\xi^{\mathbb{Z}}_T)$ satisfying

$$\int e^{-2\langle f,g\rangle}\mu_T(df) = \mathbb{P}_g(\tau \leq T) = \mathbb{P}\big(\langle \mathbb{1}_{(-\infty,\infty)}(\cdot), u^{(g)}_T\rangle = 0\big) \tag{1.25}$$

(also cf. (1.16)). They allow for a coupling $u^{(u_0)}_{T+t}(x) \leq u^*_{T+t}(x)$ for all $x \in \mathbb{R}$, $t \geq 0$ almost surely with the only restriction on u_0 being $u_0 \in \mathcal{C}^+_{\mathrm{tem}}$. Recall the two paragraphs following (1.16). In [14], Proposition 2.4, it is shown that $\mu_{T+t} \Rightarrow \mu \in \mathcal{P}(\mathcal{C}^+_{\mathrm{tem}})$ for $t \to \infty$. Here, μ is the analogue to the limit distribution of $\xi^{\mathbb{Z}}_t$ for $t \to \infty$ in the contact process setup, that is, the *upper invariant measure* ν and coincides with the *upper invariant distribution* of [9]. Indeed, take $T \to \infty$ in (1.25) to obtain the rightmost equality in (1.16), which characterizes μ.

Outline. The paper is organized as follows. Sections 2–3 are dedicated to the proof of Proposition 1.1, that is, the positivity of $B(\theta)$ for all $\theta > \theta_c$. In Sections 2.1–2.4, the groundwork is laid for the proof of Proposition 1.1, including an idea of proof for estimate (1.9) in Section 2.2. In Section 2.3, we already state the estimate that lies at the heart of the proof of Proposition 1.1; see Proposition 2.1. Its proof follows in Section 2.5. A substantial part of the proof goes into an estimate on the gain of mass at the front due to an increase in θ; see Proposition 2.17. We therefore postpone the proof of the latter to Section 3. Section 4 is dedicated to the proof of Theorem 1, that is, the convergence of the linear speed of right markers to $B(\theta)$. Essentially, in Sections 2–3 we show that $\lim_{t\to\infty}\mathbb{E}[R_0(u^{*,l}_t)]/t = B$, and in Section 4, we show that $R_0(u^{*,l}_t)/t$ also converges almost surely to B. In the Appendix, the construction of traveling wave solutions from [14] is extended to include $\nu^{*,l}$ and $\nu^{(\psi)}$ with initial conditions $\psi \in \mathcal{H}^R$ (cf. (1.10) and below). Coupling techniques that are often used are summarized for reference.

2. Preliminary results.

2.1. *The terms under investigation.* Let $\theta > \theta_c$ be arbitrarily fixed. Recall the sequence of laws $(\upsilon_T)_{T>0}$ on $\mathcal{C}_{\text{tem}}^+$ and $(u_{T+t}^{*,l})_{t\geq 0}$ for $T > 0$ fixed satisfying $\mathcal{L}(u_{T+t}^{*,l}) = \upsilon_{T+t}$ from (1.22)–(1.23). Note that $\upsilon_\cdot = \upsilon_\cdot(\theta)$ and $u_{T+\cdot}^{*,l} = u_{T+\cdot}^{*,l}(\theta)$. From [14], Corollary 4.7 and Notation 1.3–4, we conclude that the double integrals below are well defined with values in $[-\infty, \infty)$. Let

$$\alpha_T(\theta) = \alpha_T = \frac{2}{T}\int_0^{T/2} \mathbb{E}\big[R_0\big(u_{T/2+s}^{*,l}\big)\big]\,ds. \tag{2.1}$$

In fact, $\mathbb{E}[R_0(u_T^{*,l})]/T$ and α_T/T are uniformly bounded in $T \geq 1$ as we conclude from [14], Corollary 4.7, and Lemma 2.2 below. In Lemma 2.8 and Corollary 2.9 below, we will see that the limits for $T \to \infty$ exist and that the limit of the former is a constant multiple of the limit of the latter.

2.2. *Idea of proof of estimate* (1.9). In Section 2.3 to follow, we establish first that the limit

$$B = B(\theta) = \lim_{T\to\infty} \mathbb{E}\big[R_0\big(u_T^{*,l}\big)\big]/T = \frac{4}{3}\lim_{T\to\infty} \alpha_T/T \geq 0 \tag{2.2}$$

exists and is nonnegative. Here, we make use of subadditivity properties relating to $(\mathbb{E}[R_0(u_T^{*,l})])_{T\geq 1}$ to establish the existence of the limit B and use a coupling with a traveling wave solution with nonnegative linear speed as for (1.24), to establish its nonnegativity.

The proof of (1.9) then follows easily from the following.

PROPOSITION 2.1. *Let $\theta_c < \underline{\theta} < \overline{\theta}$. Then there exists $C = C(\underline{\theta}, \overline{\theta}) > 0$ and $T_0 = T_0(\underline{\theta}, \overline{\theta}) \geq 1$ such that for all $T \geq T_0$ and $\underline{\theta} \leq \theta_1 < \theta_2 \leq \overline{\theta}$,*

$$\frac{\alpha_T(\theta_2) - \alpha_T(\theta_1)}{T} \geq C(\theta_2 - \theta_1). \tag{2.3}$$

This is the main result of this section. The proof is deferred to Section 2.5. Our approach relies on establishing the estimate (2.3) along the lines of the corresponding result for contact processes in [3], Lemma 4.2. The main steps of the approach are given in what follows. For ease of understanding, let us omit at this point the necessity of considering time averages and of considering the evaluation of densities against test functions rather than the process by itself.

Step 1. Derive a uniform estimate from below on the expected increase of the right marker due to a fixed gain in mass (for $\theta_c < \underline{\theta} \leq \theta \leq \underline{\theta}$ fixed):

(1a) (cf. Lemma 2.14). With a slight abuse of notation, for $\psi, \phi \in \mathcal{C}_{\text{tem}}^+$, $R_0(\phi) \leq 0$ arbitrarily fixed,

$$\begin{aligned}&\mathbb{E}\big[R_0\big(u_{T+t}^{(``\infty\cdot\mathbb{1}_{(-\infty,0)}+\psi")}\big) - R_0\big(u_{T+t}^{(``\infty\cdot\mathbb{1}_{(-\infty,0)}")}\big)\big]\\&\quad\leq \mathbb{E}\big[R_0\big(u_{T+t}^{(\phi+\psi)}\big) - R_0\big(u_{T+t}^{(\phi)}\big)\big]\end{aligned} \tag{2.4}$$

for all $T > 0, t \geq 0$. In words, the average gain at the right front of a solution due to an increase in the initial condition by ψ can be bounded from below, uniformly over all initial conditions ϕ with $R_0(\phi) \leq 0$, by the gain we obtain from starting in the "maximal" initial condition with right marker in 0, that is, "$\infty \cdot \mathbb{1}_{(-\infty,0)}$". Recall that for $T > 0, t \geq 0$, the law $\mathcal{L}(u^{*,l}_{T+t}) = \upsilon_{T+t}$ of such a solution is a well-defined element of $\mathcal{P}(\mathcal{C}^+_{\text{tem}})$.

Intuitively, by adding ψ in the initial condition, the resulting additional population experiences death due to competition with the original population. Thus the bigger the original initial condition, the higher the resulting competition and the lower the gain in mass at the front.

(1b) (cf. Lemma 2.16). The left-hand side of (2.4) can in turn be bounded by a constant $C = C(T + t, \psi) = C(T + t, \psi, \underline{\theta}, \overline{\theta}) \geq 0$. (In *Step* (2*c*) below, it will in essence be shown that for $T + t$ appropriately chosen, $C(T + t, \psi) > 0$.)

(1c). Note that by translation invariance, the estimates in (1a), (1b) can be used at any (random stopping) time τ to obtain an estimate from below on the expected increase of the right marker in the future due to a gain in mass at time τ.

Step 2. Split $[\underline{\theta}, \overline{\theta}]$ in $O(T)$ subintervals of length $\Delta\theta = O(1/T)$. On each subinterval, say $[\theta_l, \theta_u]$, we use a θ-*coupling* (cf. Remark A.9 of the Appendix) to show: with high probability there exists a finite (random) point S in time and $\psi \in \mathcal{C}^+_{\text{tem}} \setminus \{0\}$ such that

$$(u^{*,l}_S(\theta_u) - u^{*,l}_S(\theta_l))(\cdot + R_0(u^{*,l}_S(\theta_l))) \geq \psi. \tag{2.5}$$

(2a) (cf. Proposition 2.17, (2.51)). To be more precise, show that on each subinterval $[\theta_l, \theta_u]$, (2.5) succeeds with probability of order $O(\Delta\theta) = O(1/T)$ for some $S \in [0, 1]$ and with $O(1)$ for $S \in [0, T]$ by a geometric-type series argument. How the latter is achieved will be discussed in Section 3.1 where the idea of proof is given for Proposition 2.17.

(2b). Apply first the strong Markov property at time S together with (2.5) and a *monotonicity-coupling* (cf. Remark A.8 of the Appendix), then apply a θ-*coupling* (cf. Remark A.9 of the Appendix) to get

$$\begin{aligned} &R_0(u^{*,l}_T(\theta_u)) - R_0(u^{*,l}_T(\theta_l)) \\ &\quad \geq R_0\big(u_{T-S}^{(u^{*,l}_S(\theta_l)+\psi(\cdot - R_0(u^{*,l}_S(\theta_l))))}(\theta_u)\big) - R_0\big(u_{T-S}^{(u^{*,l}_S(\theta_l))}(\theta_l)\big) \\ &\quad \geq R_0\big(u_{T-S}^{(u^{*,l}_S(\theta_l)+\psi(\cdot - R_0(u^{*,l}_S(\theta_l))))}(\theta_l)\big) - R_0\big(u_{T-S}^{(u^{*,l}_S(\theta_l))}(\theta_l)\big). \end{aligned} \tag{2.6}$$

Now refer back to *Step* 1 to bound the expectation on the right-hand side.

(2c). Show that for $S \in [0, T]$, T big enough, the bound on the expectation at the end of *Step* (1*b*) is strictly positive and independent of T. Now add things together. For each of the $O(T)$ subintervals of length $\Delta\theta$, the construction involving $S \in [0, T]$ is successful with probability of order $O(1)$. Hence, each subinterval yields a gain of $O(1)$ to the difference $\alpha_T(\theta_2) - \alpha_T(\theta_1)$ and summing up we obtain a gain of $(\theta_2 - \theta_1)O(T)$.

2.3. *Estimates on right markers.*

Note that in this subsection, for all $\theta_c < \underline{\theta} \le \theta \le \overline{\theta}$ the constants to follow only depend on θ through $\underline{\theta}, \overline{\theta}$.

LEMMA 2.2. *For all $u_0 \in \mathcal{H}$, there exists a constant $C = C(u_0) > 0$ such that*

$$\mathbb{E}_{u_0}\big[0 \vee \big(-R_0(u_t)\big)\big] \le C(1+t) \tag{2.7}$$

holds uniformly in $t \ge 0$. Moreover, there exist $C_i = C_i(u_0) > 0, i = 1, 2$ such that for all $M > 0$,

$$\mathbb{E}_{u_0}\big[-\big(R_0(u_t)/t\big)\mathbb{1}_{\{R_0(u_t)/t<-M\}}\big] \le C_1 e^{-C_2 M} \tag{2.8}$$

holds uniformly in $t \ge 1$.

PROOF. Let $u_0 \in \mathcal{H}$. Recall ϵ, x_0 from the definition of $\mathcal{H}$. Using a *monotonicity-coupling* (cf. Remark A.8) we assume without loss of generality that $u_0 = \epsilon H_0(\cdot - x_0)$. We further assume $x_0 = 0$ by the shift invariance of the dynamics.

We reason as in the proof of [21], Lemma 3.5. The author uses the wave marker $R_1(t) = \ln(\langle e^{\cdot}, u_t\rangle)$ and Heavyside initial data H_0 instead. It is shown that there exist $c = c(\theta), a = a(\theta), \delta = \delta(\theta) > 0$ such that $\mathbb{P}_{H_0}(R_1(t) \le -a - cmt) \le (1 - \delta/4)^m$ for all $t \ge 0, m \in \mathbb{N}$. We claim that this holds for H_0 replaced by $u_0 = \epsilon H_0$, $R_1(t)$ replaced by $R_0(t)$ and a replaced by 0 as well. Moreover, the constants c, δ only depend on $\epsilon, \underline{\theta}$ and $\overline{\theta}$. The main idea of proof remains the same. We give it below, so that the reader may skip the finer details.

The main idea of the proof of [21], Lemma 3.5, is to couple $u^{(H_0)}$ with a sequence of independent processes $u^{(j)}$ solving a modification of (1.1). Namely, each $u^{(j)}$ starts in a spacial shift to the left by $jr, r \ge 2, j \in \mathbb{N}$ of some continuous function $\psi_0 : \mathbb{R} \to [0, 1]$ that is symmetric and satisfies $\{x : \psi_0(x) > 0\} = (-1, 1)$. On $-jr + (r/2 - 1) \cdot (-1, 1) = (-jr - r/2 + 1, -jr + r/2 - 1)$, the solutions follow the dynamics of (1.1), outside the density is set to zero. By [17], Lemma 2.1.5, with the corresponding coupling, $u_t^{(H_0)} \ge u_t^{(j)}$ for all $j \in \mathbb{N}, t \ge 0, x \in \mathbb{R}$ almost surely. Fix $t \ge 1$. A lower bound on $R_0(u_t^{(H_0)})$ is the random $-jr - r/2 + 1$ such that $u^{(j)}$ is still alive at time t. The probability of the latter can in turn be bounded from below by the probability that a solution starting in $u_0^{(j)}$ solves (1.1), survives until time t and does not hit the boundary points $-jr \pm (r/2 - 1)$ before time t. The survival probability can be bounded from below by a constant, independent of r. The probability of hitting the boundary points can be estimated using the estimates that lie at the heart of the proof of the *compact support property* (cf. paragraph following (1.3)). For r big enough, this probability becomes small.

Let us return to our claim. Replace $a > 0$ by $a = 0$; let us reason as in the given proof with $R_1(f)$ replaced by $R_0(f)$ and ψ_0 replaced by $\psi_0' \equiv \epsilon\psi_0$ until the last set of equations. Choose $r = ct$ for $t \ge 1$ arbitrarily fixed and $r = c$

for $t \in [0, 1)$ (and thus $Q^{\psi_0}(T_0(U) \leq t) \leq Q^{\psi_0}(T_0(U) \leq 1) \leq \delta/4$ in the notation of [21]). In the last set of equations, use that for a superprocess with initial symmetric condition ψ_0' and law $\overline{\mathbb{P}}_{\psi_0'}$ (the density of the latter solves a modification of (1.1) where the competition term $-u^2$ is dropped; for an appropriate coupling, see [17], Lemma 2.1.4), there exists $\delta > 0$ small enough such that $\overline{\mathbb{P}}_{\psi_0'}(R_0(u_t) \geq 0) \geq \overline{\mathbb{P}}_{\psi_0'}(\tau > t)/2 \geq \mathbb{P}_{\psi_0'}(\langle u_t, 1\rangle \geq \delta) \geq \delta/2$ to obtain for all $t \geq 1$,

$$\mathbb{P}_{u_0}\big(R_0(t) \leq -cmt\big) \leq (1-\delta/4)^m \quad \text{for all } m \in \mathbb{N}. \tag{2.9}$$

As a result,

$$\mathbb{E}_{u_0}\big[0 \vee \big(-R_0(u_t)\big)\big] \leq ct + \sum_{m\in\mathbb{N}} (1-\delta/4)^m c(m+1)t \leq C(c,\delta)t. \tag{2.10}$$

For $t \in [0, 1)$, the different choice of r yields

$$\mathbb{P}_{u_0}\big(R_0(t) \leq -cm\big) \leq (1-\delta/4)^m \quad \text{for all } m \in \mathbb{N} \tag{2.11}$$

and

$$\mathbb{E}_{u_0}\big[0 \vee \big(-R_0(u_t)\big)\big] \leq c + \sum_{m\in\mathbb{N}} (1-\delta/4)^m c(m+1) \leq C(c,\delta) \tag{2.12}$$

instead.

By $\lfloor x \rfloor$, we denote the greatest integer that is less than or equal to $x \in \mathbb{R}$. To obtain the second claim, for $0 < M < c$ choose C_1 big enough and C_2 small enough such that $C(c,\delta) \leq C_1 e^{-C_2 c}$. For $M \geq c$ and $t \geq 1$,

$$\begin{aligned} &\mathbb{E}_{u_0}\big[-\big(R_0(u_t)/t\big)\mathbb{1}_{\{R_0(u_t)/t<-M\}}\big] \\ &\quad \leq \sum_{m=\lfloor M/c\rfloor}^{\infty} (1-\delta/4)^m c(m+1) \\ &\quad = e^{\ln(1-\delta/4)\lfloor M/c\rfloor} \sum_{m=0}^{\infty} (1-\delta/4)^m c\big(m+1+\lfloor M/c\rfloor\big) \leq C_1 e^{-C_2 M} \end{aligned} \tag{2.13}$$

for $C_1 = C_1(c,\delta)$ big enough and $C_2 = C_2(c,\delta)$ small enough. $\square$

COROLLARY 2.3. *There exists a constant $C > 0$ such that*

$$\mathbb{E}\big[0 \vee \big(-R_0\big(u_t^{*,l}\big)\big)\big] \leq C(1+t) \tag{2.14}$$

holds uniformly in $t > 0$. Moreover, there exist $C_i > 0, i = 1, 2$ such that for all $M > 0$,

$$\mathbb{E}\big[-\big(R_0\big(u_t^{*,l}\big)/t\big)\mathbb{1}_{\{R_0(u_t^{*,l})/t<-M\}}\big] \leq C_1 e^{-C_2 M} \tag{2.15}$$

holds uniformly in $t \geq 1$.

PROOF. The result follows again by domination, this time using (1.24) and $u_0 \in \mathcal{H}^R$ with $R_0(u_0) \le 0$ arbitrary. $\square$

COROLLARY 2.4. *$\mathbb{E}_{u_0}[|R_0(u_T)|]/T$, $u_0 \in \mathcal{H}^R$ and $\mathbb{E}[|R_0(u_T^{*,l})|]/T$ are uniformly bounded in $T \ge 1$ (constants may depend on u_0).*

PROOF. Combine Lemma 2.2 respectively Corollary 2.3 with [14], Lemma 4.6. $\square$

COROLLARY 2.5. *There exists a constant $C > 0$ such that $\mathbb{E}[|R_0(u_s^{*,l})|] \le C$ for all $0 < s \le 1$. Moreover, for every $u_0 \in \mathcal{H}^R$ there exists a constant $C(u_0) > 0$ such that $\mathbb{E}[|R_0(u_s)|] \le C(u_0)$ for all $0 \le s \le 1$.*

PROOF. Fix $0 < s \le 1$. Then $\mathbb{E}[0 \vee R_0(u_s^{*,l})] \le C$ follows from [14], Proposition 4.5, and $\mathbb{E}[0 \vee (-R_0(u_s^{*,l}))] \le C$ from Corollary 2.3. For $u_0 \in \mathcal{H}^R$, the bound for the positive part follows by domination and shift invariance, that is, $\mathbb{E}[0 \vee R_0(u_s)] \le |R_0(u_0)| + \mathbb{E}[0 \vee R_0(u_s^{*,l})] \le |R_0(u_0)| + C = C(u_0)$. The bound for the negative part follows from Lemma 2.2. $\square$

COROLLARY 2.6. *α_T/T is uniformly bounded in $T \ge 1$.*

REMARK 2.7. The definition of the marker R_0 together with Corollaries 2.4–2.5 yields $\inf\{t > 0 : u_t^{*,l} \equiv 0\} = +\infty$ a.s., that is, the process $u^{*,l}$ does not die out in finite time. Thus, if we consider $u^{*,l}$, we do not have to bother with conditioning on nonextinction.

The existence of the following limit will turn out to be crucial in the following chapters. Nonnegativity of the limit follows below.

LEMMA 2.8. *The limit*

$$B = B(\theta) = \lim_{T\to\infty} \mathbb{E}\big[R_0\big(u_T^{*,l}\big)\big]/T = \inf_{T\ge 1} \mathbb{E}\big[R_0\big(u_T^{*,l}\big)\big]/T \in (-\infty, \infty) \tag{2.16}$$

exists.

PROOF. We work with $\mathbb{P}_{\upsilon_1}$, that is randomize the initial condition according to the law of $u_1^{*,l} \in \mathcal{C}_{\mathrm{tem}}^+$. By the strong Markov property of the process, we have for arbitrary $1 \le s, t$,

$$\begin{aligned} \mathbb{E}\big[R_0\big(u_{s+t}^{*,l}\big)\big] &= \mathbb{E}\big[\mathbb{E}\big[R_0\big(u_{s+t}^{*,l}\big) \mid \mathcal{F}_t\big]\big] = \mathbb{E}\big[R_0\big(u_s^{(u_t^{*,l})}\big)\big] \\ &= \mathbb{E}\big[R_0\big(u_s^{(u_t^{*,l}(\cdot + R_0(u_t^{*,l})))}\big)\big] + \mathbb{E}\big[R_0\big(u_t^{*,l}\big)\big]. \end{aligned} \tag{2.17}$$

Use monotonicity, that is, (1.24), to further obtain

$$\mathbb{E}[R_0(u^{*,l}_{s+t})] \le \mathbb{E}[R_0(u^{*,l}_s)] + \mathbb{E}[R_0(u^{*,l}_t)]. \tag{2.18}$$

By subadditivity (cf. for instance Liggett [15], Theorem B22) and from the uniform boundedness of $\mathbb{E}[|R_0(u^{*,l}_T)|]/T$ in $T \ge 1$, we conclude

$$\lim_{T\to\infty} \mathbb{E}[R_0(u^{*,l}_T)]/T = \inf_{T>0} \mathbb{E}[R_0(u^{*,l}_T)]/T \quad \text{exists in } (-\infty, \infty). \tag{2.19}$$ □

COROLLARY 2.9. *The limit* $\lim_{T\to\infty} \alpha_T/T = \frac{3}{4}B$ *exists.*

PROOF. For all $\epsilon > 0$, there exists $T_0 \ge 1$ such that for all $T \ge T_0$,

$$\begin{aligned} \limsup_{T\to\infty} \frac{\alpha_T}{T} &= \limsup_{T\to\infty} \frac{2}{T^2} \int_0^{T/2} \mathbb{E}[R_0(u^{*,l}_{T/2+s})]\,ds \\ &\le \limsup_{T\to\infty} \frac{2}{T^2} \int_0^{T/2} (\epsilon + B)(T/2 + s)\,ds = \frac{3}{4}(\epsilon + B). \end{aligned} \tag{2.20}$$

Analogous reasoning for a lower bound concludes the proof. □

The limit is indeed nonnegative.

LEMMA 2.10. *The limit* $B = \lim_{T\to\infty} \mathbb{E}[R_0(u^{*,l}_T)]/T$ *from Lemma 2.8 is nonnegative.*

PROOF. Let $\nu \in \mathcal{P}(\mathcal{C}^+_{\text{tem}})$ be such that $\nu(\{f : R_0(f) = 0\}) = 1$ and $\mathbb{P}_\nu$ is the law of a traveling wave. For $\theta > \theta_c$, existence follows from [21], Theorem 3.8 and (3.29), and the shift invariance of the dynamics. By [21], Proposition 4.1, $R_0(u^{(\nu)}_t)/t$ converges a.s. to a (possibly random) limit $A^{(\nu)} \ge 0$. By monotonicity, that is by (1.24), we have $R_0(u^{*,l}_t)/t \ge R_0(u^{(\nu)}_t)/t$ for all $t \ge 1$ a.s., and thus $\liminf_{t\to\infty} R_0(u^{*,l}_t)/t \ge 0$ a.s.

Let $\epsilon > 0$ arbitrary. By Corollary 2.3, there exist constants $C_1, C_2 > 0$ such that for $M > 0$ satisfying $C_1 e^{-C_2 M} < \epsilon$,

$$\begin{aligned} B &\ge \limsup_{T\to\infty} \mathbb{E}[(R_0(u^{*,l}_T)/T)\mathbb{1}_{\{R_0(u^{*,l}_T)/T \ge -M\}}] - \epsilon \\ &\ge \mathbb{E}\Big[\liminf_{T\to\infty} (R_0(u^{*,l}_T)/T)\mathbb{1}_{\{R_0(u^{*,l}_T))/T \ge -M\}}\Big] - \epsilon \ge -\epsilon, \end{aligned} \tag{2.21}$$

where we applied Fatou's lemma. □

Recall the main result of this section, Proposition 2.1. As a corollary, we now obtain the strict positivity of $B(\theta)$.

COROLLARY 2.11. *For all* $\theta > \theta_c$,

$$B = B(\theta) = \lim_{T\to\infty} \mathbb{E}\big[R_0\big(u_T^{*,l}(\theta)\big)\big]/T > 0. \tag{2.22}$$

PROOF. By definition of α_T, Corollary 2.9 and Lemma 2.10, Proposition 2.1 implies that for all $\theta > \theta_c$,

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}\big[R_0\big(u_T^{*,l}(\theta)\big)\big]/T &= B(\theta) = \frac{4}{3}\lim_{T\to\infty} \alpha_T(\theta)/T \\ &> \frac{4}{3}\lim_{T\to\infty} \alpha_T\big(\theta_c + (\theta-\theta_c)/2\big)/T \geq 0. \end{aligned} \tag{2.23}$$ □

We conclude this subsection with two more results that we need for later estimates.

LEMMA 2.12. *Let* $\theta_c < \underline{\theta}$, *then there exists* $\tilde{\delta} > 0$ *such that* $\mathbb{P}(R_0(u_T^{*,l}(\theta)) \geq 0) \geq \tilde{\delta}$ *for all* $T \geq 1$ *and* $\theta \geq \underline{\theta}$.

PROOF. Recall the paragraph including (1.25). Use a *θ-$*$-coupling* (cf. Lemma A.12 of the Appendix) to see that it suffices to show the claim for $\underline{\theta}$ fixed. Note that for $T \geq 1$ arbitrarily fixed, $\mathbb{P}(R_0(u_T^{*,l}(\underline{\theta})) \geq 0) > 0$. Therefore, in the following proof by contradiction we only need to suppose to the contrary that there exists a sequence $(T_n)_{n\in\mathbb{N}}$ such that $T_n \to \infty$ for $n \to \infty$ and $\lim_{n\to\infty} \mathbb{P}(R_0(u_{T_n}^{*,l}) \geq 0) = 0$. Let $H_0(x) = 1 \wedge (-x \vee 0)$ be Heavyside initial data and set $f_1(x) = H_0(x+1) + H_0(-x-1)$. Then $f_1 \in \mathcal{C}_{\mathrm{tem}}^+$ and $\mathrm{supp}(f_1) = (-\infty, -1] \cup [1, \infty)$. Let $f_2(x) = 0 \vee (1 - |x|)$, then $f_2 \in \mathcal{C}_c^+$ with $\mathrm{supp}(f_2) = [-1, 1]$. f_1 fulfills condition [9], (6) ("nearly uniformly distributed in space") for initial conditions to be in the domain of attraction of μ, and hence [9], Theorem 1, yields $u_t^{(f_1)} \Rightarrow \mu$ for $t \to \infty$. Using a *coupling with two independent processes* (cf. Remark A.10 of the Appendix) in combination with the construction of [14], Remark 2.8(ii) *(left-upper measure)*, we construct two independent processes $(u_t^{*,l})_{t\geq 1}$ and $(u_t^{*,r})_{t\geq 1}$ such that $\mathcal{L}((u_t^{*,l})_{t\geq 1}) = \mathbb{P}_{\upsilon_1}$, $\mathcal{L}((u_t^{*,r})_{t\geq 1}) = \mathbb{P}_{\kappa_1}$ and

$$u_t^{(f_1)} \leq u_t^{*,l}(\cdot + 1) + u_t^{*,r}(\cdot - 1) \quad \text{for all } t \geq 1, x \in \mathbb{R} \text{ almost surely.} \tag{2.24}$$

By (1.16) and [17], Theorem 1 (for $\theta > \theta_c$ survival happens with positive probability),

$$\int_{\mathcal{C}_{\mathrm{tem}}^+} e^{-2\langle g, f_2\rangle} \mu(dg) = \mathbb{P}_{f_2}(\tau < \infty) < 1. \tag{2.25}$$

Note that the additional factor of 2 in the exponent results from the use of a different scaling constant in the original SPDE. We obtain by the weak convergence of

$u_t^{(f_1)}$ to μ,

$$
\begin{aligned}
1 > \lim_{n\to\infty} \int_{\mathcal{C}_{\text{tem}}^+} e^{-2\langle g, f_2\rangle} u_{T_n}^{(f_1)}(dg) &= \lim_{n\to\infty} \mathbb{E}\big[e^{-2\langle u_{T_n}^{(f_1)}, f_2\rangle}\big] \\
&\geq \lim_{n\to\infty} \mathbb{E}\big[e^{-2\langle u_{T_n}^{*,l}(\cdot+1)+u_{T_n}^{*,r}(\cdot-1), f_2\rangle}\big] \qquad (2.26)\\
&= \lim_{n\to\infty} \mathbb{E}\big[e^{-2\langle u_{T_n}^{*,l}(\cdot+1), f_2\rangle}\big]\mathbb{E}\big[e^{-2\langle u_{T_n}^{*,r}(\cdot-1), f_2\rangle}\big].
\end{aligned}
$$

The assumption $\lim_{n\to\infty} \mathbb{P}(R_0(u_{T_n}^{*,l}) \geq 0) = 0$ yields by symmetry and by the shift invariance of the dynamics, $\lim_{n\to\infty} \mathbb{P}(R_0(u_{T_n}^{*,l}(\cdot + 1) \geq -1) = 0 = \lim_{n\to\infty} \mathbb{P}(L_0(u_{T_n}^{*,r}(\cdot - 1) \leq 1) = 0$. Use a coupling $u_{T_n}^{*,r} \leq u_{T_n}^*$ with $u_{T_n}^* \Rightarrow \mu \in \mathcal{P}(\mathcal{C}_{\text{tem}}^+)$ (cf. [14], (2.34) and Proposition 2.4) to conclude by using dominated convergence that the right-hand side in (2.26) is equal to 1, a contradiction. □

LEMMA 2.13. *For all* $\theta_c < \theta \leq \overline{\theta}, T \geq 1$,

$$\mathbb{E}\big[\big(0 \vee R_0\big(u_T^{*,l}\big)\big)^2\big] \leq C(\overline{\theta})T^2. \qquad (2.27)$$

PROOF. In what follows, constants $C = C(\overline{\theta})$ may change from line to line. Note that for $a_i \geq 0, i = 1, \dots, n, n \in \mathbb{N}$, $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$.

We first show the claim for $T \in \mathbb{N}$. Let us reason as in [14], Lemma 4.2–Proposition 4.5, to show that for $T = 1$, $\mathbb{E}[(0 \vee R_0(u_1^{*,l}))^2] \leq C$. Then reason as in [14], Lemma 4.6, to show the claim for $T \in \mathbb{N}$ by induction.

Next, we extend this result to $T \geq 1$. As $\mathcal{L}(u_T^{*,l}) \in \mathcal{P}(\mathcal{C}_{\text{tem}}^+ \backslash \{0\})$ for all $T > 0$ we use [14], Remark 2.8 *(left-upper measure)* to get for $T \geq 1$ arbitrary,

$$\mathbb{E}\big[\big(0 \vee R_0\big(u_T^{*,l}\big)\big)^2\big] = \mathbb{E}\big[\big(0 \vee R_0\big(u_{T-\lfloor T\rfloor}^{(u_{\lfloor T\rfloor}^{*,l})}\big)\big)^2\big]. \qquad (2.28)$$

By [14], Remark A.1, and symmetry,

$$
\begin{aligned}
&\mathbb{E}\big[\big(0 \vee R_0\big(u_{T-\lfloor T\rfloor}^{(u_{\lfloor T\rfloor}^{*,l})}\big)\big)^2 | u_{\lfloor T\rfloor}^{*,l}\big] \\
&\quad\leq \big(0 \vee \big(R_0\big(u_{\lfloor T\rfloor}^{*,l}\big) + 2\big)\big)^2 + C \int_{0\vee(R_0(u_{\lfloor T\rfloor}^{*,l})+2)}^{\infty} 2R\big\langle e^{-\frac{(\cdot-(R-1))^2}{4(T-\lfloor T\rfloor)}}, u_{\lfloor T\rfloor}^{*,l}\big\rangle\, dR.
\end{aligned} \qquad (2.29)
$$

Take expectations and use [14], Corollaries 2.6 and 2.9, to conclude that

$$\mathbb{E}\big[\big(0 \vee R_0\big(u_{T-\lfloor T\rfloor}^{(u_{\lfloor T\rfloor}^{*,l})}\big)\big)^2\big] \leq C\lfloor T\rfloor^2 + C\int_0^\infty 2R\big\langle e^{-\frac{(\cdot-(R-1))^2}{4}}, 1\big\rangle\, dR \leq CT^2 \qquad (2.30)$$

as claimed. □

2.4. *A preliminary estimate*. The following two lemmas yield, in combination, a lower bound on the expected increase of the right front marker at time $T+t$, $T>0$, $t \geq 0$ resulting from an increase of $\psi \in \mathcal{C}_{\text{tem}}^{+}$ in the initial density of a solution to (1.1).

Recall the construction of the left-upper invariant measure υ_T and the process $(u_{T+t}^{*,l})_{t\geq 0}$ for $T>0$ fixed from Section 1.4 or [14] (cf. the corresponding construction for the upper invariant measure μ_T from [14], Proposition 2.2 and Corollary 2.6 as well as Remark 2.8 *(left-upper measure)*). For arbitrarily fixed (to be chosen later) $\psi \in \mathcal{C}_{\text{tem}}^{+}$, write

$$\Phi(x) \equiv \begin{cases} \infty & x<0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \Psi(x) \equiv \begin{cases} \infty & x<0, \\ \psi(x) & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.31}$$

In what follows, consider couplings of solutions $(u_{T+t}^{(\phi)})_{t\geq 0}$, $\phi \in \mathcal{C}_{\text{tem}}^{+}$ and $(u_{T+t}^{(\phi+\psi)})_{t\geq 0}$, $\psi \in \mathcal{C}_{\text{tem}}^{+}$ with processes $(u_{T+t}^{(\Phi)})_{t\geq 0}$ and $(u_{T+t}^{(\Psi)})_{t\geq 0}$ for $T>0$ arbitrarily fixed. Note that by a slight abuse of notation "$\Psi = \Phi + \psi$". The two latter processes are to be understood in the spirit of the construction of υ_T, that is, as in Corollary 2.6 of [14] we choose sequences $(\Psi_N)_{N\in\mathbb{N}}$ and $(\Phi_N)_{N\in\mathbb{N}}$ such that $\Psi_N \uparrow \Psi$ and $\Phi_N \uparrow \Phi$ for $N \to \infty$ to obtain $u_{T+t}^{(\Psi)}(x) \equiv\uparrow \lim_{N\to\infty} u_{T+t}^{(\Psi_N)}(x)$ and $u_{T+t}^{(\Phi)} \equiv\uparrow \lim_{N\to\infty} u_{T+t}^{(\Phi_N)}(x)$ on a common probability space.

LEMMA 2.14. *Let $\psi \in \mathcal{C}_{\text{tem}}^{+}$ arbitrarily fixed and Φ, Ψ be as above. Let $\phi \in \mathcal{C}_{\text{tem}}^{+}$ arbitrary with $R_0(\phi) \leq 0$. Then, for arbitrary $T>0, t\geq 0$, there exists a coupling of processes $(u_{T+t}^{(\Phi)})_{t\geq 0}$, $(u_{T+t}^{(\Psi)})_{t\geq 0}$ and solutions $(u_{T+t}^{(\phi)})_{t\geq 0}$, $(u_{T+t}^{(\phi+\psi)})_{t\geq 0}$ such that*

$$\mathbb{E}\big[R_0\big(u_{T+t}^{(\Psi)}\big) - R_0\big(u_{T+t}^{(\Phi)}\big)\big] \leq \mathbb{E}\big[R_0\big(u_{T+t}^{(\phi+\psi)}\big) - R_0\big(u_{T+t}^{(\phi)}\big)\big] \tag{2.32}$$

for all $t \geq 0$ almost surely. On the right-hand side, we consider a monotonicity-coupling (cf. Remark A.8) and set $R_0(u_{T+t}^{(\phi+\psi)}) - R_0(u_{T+t}^{(\phi)}) = 0$ on $\{\tau^{(\phi+\psi)} \leq T+t\}$.

REMARK 2.15. Note that the expectations on the left-hand side of (2.32) are well defined by Lemma 2.2 and Corollaries 2.4–2.5. Indeed, note that if $f_n \uparrow f$ in $\mathcal{C}_{\text{tem}}^{+}$, then $R_0(f_n) \uparrow R_0(f)$ for $n \to \infty$. Now use approximating sequences $\Psi_N, \Phi_N \in \mathcal{H}^R$ for Ψ respectively Φ from below as in [14], Remark 2.8(i) *(left-upper measure)* in combination with dominated convergence.

PROOF OF LEMMA 2.14. *Step* 1. Let ϕ, ψ as in the statement above. We first show the claim for $T=0$ and $\Phi, \Psi \in \mathcal{C}_{\text{tem}}^{+}$ satisfying $R_0(\Phi)=0$, $\Phi \geq \phi$ and $\Psi =$

$\Phi + \psi$. Consider the following coupling. Let $u_1 = u^{(\phi)}$ be a nonnegative solution to

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + (\theta - u_1)u_1 + \sqrt{u_1}\dot{W}_1, \qquad u_1(0) = \phi, \tag{2.33}$$

and v_2 be a nonnegative solution to

$$\frac{\partial v_2}{\partial t} = \Delta v_2 + (\theta - v_2 - 2u_1)v_2 + \sqrt{v_2}\dot{W}_2, \qquad v_2(0) = \Phi - \phi \tag{2.34}$$

with W_2 a white noise independent of W_1. For the construction of the latter, proceed as in Remark A.8 on *monotonicity-couplings*. Then $u_1 + v_2 \overset{\mathcal{D}}{=} u^{(\Phi)}$, that is, $u_1 + v_2$ solves (1.1) with initial condition Φ. Let v_3 be a nonnegative solution to

$$\frac{\partial v_3}{\partial t} = \Delta v_3 + \big(\theta - v_3 - 2(u_1 + v_2)\big)v_3 + \sqrt{v_3}\dot{W}_3, \qquad v_3(0) = \psi \tag{2.35}$$

with W_3 a white noise independent of W_1, W_2. Then $u_1 + v_2 + v_3 \overset{\mathcal{D}}{=} u^{(\Phi+\psi)}$ follows as above, and using that $\Psi = \Phi + \psi$,

$$\begin{aligned} R_0\big(u_t^{(\Psi)}\big) - R_0\big(u_t^{(\Phi)}\big) &\overset{\mathcal{D}}{=} R_0\big((u_1 + v_2 + v_3)_t\big) - R_0\big((u_1 + v_2)_t\big) \\ &= \big(R_0\big((v_3)_t\big) - R_0\big((u_1 + v_2)_t\big)\big) \vee 0 \end{aligned} \tag{2.36}$$

for all $t \ge 0$ a.s., where we set $R_0(u_t^{(\Psi)}) - R_0(u_t^{(\Phi)}) = 0$ on $\{\tau^{(\Psi)} \le t\}$. Finally, let d_4 be a nonnegative solution to

$$\frac{\partial d_4}{\partial t} = \Delta d_4 + 2v_2 v_3 + \big(\theta - d_4 - 2(u_1 + v_3)\big)\, d_4 + \sqrt{d_4}\dot{W}_4, \qquad d_4(0) = 0 \tag{2.37}$$

with W_4 independent of W_1, W_2, W_3 and where the term $2v_2v_3$ can be interpreted as an additional immigration term. Then $u_1 + v_3 + d_4 \overset{\mathcal{D}}{=} u^{(\phi+\psi)}$ and

$$\begin{aligned} &R_0\big(u_t^{(\phi+\psi)}\big) - R_0\big(u_t^{(\phi)}\big) \\ &\quad = R_0\big((u_1 + v_3 + d_4)_t\big) - R_0\big((u_1)_t\big) \\ &\quad = \big(R_0\big((v_3 + d_4)_t\big) - R_0\big((u_1)_t\big)\big) \vee 0 \\ &\quad \ge \big(R_0\big((v_3)_t\big) - R_0\big((u_1 + v_2)_t\big)\big) \vee 0, \end{aligned} \tag{2.38}$$

the last by the nonnegativity of the solutions d_4 and v_2.

The second part of the claim now follows from the above and (2.36). For the first part of the claim, use that $\Psi = \Phi + \psi$ and

$$\begin{gathered} u_1 \overset{\mathcal{D}}{=} u^{(\phi)}, \qquad u_1 + v_2 \overset{\mathcal{D}}{=} u^{(\Phi)}, \\ u_1 + v_2 + v_3 \overset{\mathcal{D}}{=} u^{(\Phi+\psi)}, \qquad u_1 + v_3 + d_4 \overset{\mathcal{D}}{=} u^{(\phi+\psi)} \end{gathered} \tag{2.39}$$

to obtain a coupling satisfying

$$
\begin{aligned}
u_t^{(\Psi)} - u_t^{(\Phi)} &= (u_1 + v_2 + v_3) - (u_1 + v_2) = v_3 \le v_3 + d_4 \\
&= (u_1 + v_3 + d_4) - u_1 = u_t^{(\phi+\psi)} - u_t^{(\phi)}
\end{aligned}
\tag{2.40}
$$

as claimed.

Step 2. Fix $T > 0$. Let $\Phi_N \uparrow \Phi$, Φ as in (2.31), satisfy $R_0(\Phi_N) = 0$ and $\Phi_1 \ge \phi$, $\Phi_1 \in \mathcal{H}^R$. Set $\Psi_N = \Phi_N + \psi$. By Step 1, there exists a coupling of solutions $(u_{T+t}^{(\Phi_N)})_{t\ge 0}$, $(u_{T+t}^{(\Psi_N)})_{t\ge 0}$ to (1.1) such that (2.32) holds with Φ, Ψ replaced by Φ_N, Ψ_N for $N \in \mathbb{N}$ arbitrarily fixed.

Define $u_{T+t}^{(\Phi)}(x) =\uparrow \lim_{N\to\infty} u_{T+t}^{(\Phi_N)}(x)$ and $u_{T+t}^{(\Psi)}(x) =\uparrow \lim_{N\to\infty} u_{T+t}^{(\Psi_N)}(x)$ on a common probability space (cf. [14], Remark 2.8(i)) *(left-upper measure)*. By taking limits in $N \to \infty$, the claim now follows for Φ, Ψ as well by dominated convergence (cf. Remark 2.15 above). $\square$

LEMMA 2.16. *For $t > 0$ fixed and $\psi \in \mathcal{C}_{\mathrm{tem}}^+$,*

$$
\begin{aligned}
&\mathbb{E}\big[R_0\big(u_t^{(\Psi)}\big) \vee 0 - R_0\big(u_t^{(\Phi)}\big) \vee 0\big] \\
&\quad \ge \int_0^\infty \mathbb{E}\big[\mathbb{1}_{\{-x \le L_0(u_t^{*,r}) < -x+1\}}\big(1 - e^{-2\langle \mathbb{1}_{[0,1)}\psi, u_t^{*,r}(\cdot - x)\rangle}\big)\big]\,dx
\end{aligned}
\tag{2.41}
$$

holds.

PROOF. By partial integration, for $\phi \in \mathcal{C}_{\mathrm{tem}}^+$, $t > 0$ arbitrary,

$$
\mathbb{E}_\phi\big[R_0(u_t) \vee 0\big] = \int_0^\infty \mathbb{P}_\phi\big(R_0(u_t) > x\big)\,dx. \tag{2.42}
$$

By (1.22), symmetry and by the shift invariance of the dynamics,

$$
\mathbb{P}_\phi\big(R_0(u_t) \le x\big) = \mathbb{E}\big[e^{-2\langle \phi, u_t^{*,r}(\cdot - x)\rangle}\big] = \mathbb{E}\big[e^{-2\langle \phi(\cdot + x), u_t^{*,r}\rangle}\big]. \tag{2.43}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}_\phi\big[R_0(u_t) \vee 0\big] &= \int_0^\infty \mathbb{E}\big[\big(1 - e^{-2\langle \phi, u_t^{*,r}(\cdot - x)\rangle}\big)\big]\,dx \\
&= \int_0^\infty \mathbb{E}\big[\mathbb{1}_{\{u_t^{*,r}|_{\mathrm{supp}(\phi(\cdot + x))} \not\equiv 0\}}\big(1 - e^{-2\langle \phi, u_t^{*,r}(\cdot - x)\rangle}\big)\big]\,dx.
\end{aligned}
\tag{2.44}
$$

In the following, we use Φ and $\Psi = \Phi + \psi$ as initial conditions or test functions to facilitate notation. This notation is understood as an abbreviation for taking limits of nondecreasing approximating sequences of initial conditions as explained above and using monotone convergence to obtain the respective results.

Let us reason as in Remark 2.15 to see that the following integrals are well defined. The theorem of Fubini–Tonelli yields

$$
\begin{aligned}
&\mathbb{E}\big[R_0\big(u_t^{(\Psi)}\big)\vee 0 - R_0\big(u_t^{(\Phi)}\big)\vee 0\big] \\
&\quad = \int_0^\infty \mathbb{E}\big[e^{-2\langle \Phi(\cdot+x),u_t^{*,r}\rangle}\big] - \mathbb{E}\big[e^{-2\langle \Psi(\cdot+x),u_t^{*,r}\rangle}\big]\,dx \\
&\quad = \int_0^\infty \mathbb{E}\big[e^{-2\langle \Phi(\cdot+x),u_t^{*,r}\rangle}\big(1-e^{-2\langle \psi(\cdot+x),u_t^{*,r}\rangle}\big)\big]\,dx \qquad (2.45)\\
&\quad = \int_0^\infty \mathbb{E}\big[\mathbb{1}_{\{L_0(u_t^{*,r})\geq -x\}}\big(1-e^{-2\langle \psi(\cdot+x),u_t^{*,r}\rangle}\big)\big]\,dx \\
&\quad \geq \int_0^\infty \mathbb{E}\big[\mathbb{1}_{\{-x\leq L_0(u_t^{*,r})<-x+1\}}\big(1-e^{-2\langle \mathbb{1}_{[0,1)}\psi,u_t^{*,r}(\cdot-x)\rangle}\big)\big]\,dx.
\end{aligned}
$$

This completes the proof. □

2.5. *Proof of Proposition* 2.1. Let $T\geq 1$ be arbitrarily fixed. For $\theta_c<\underline{\theta}<\overline{\theta}$ arbitrary let

$$
\delta=\frac{\overline{\theta}-\underline{\theta}}{M} \quad\text{and}\quad \theta_m=\underline{\theta}+m\frac{\delta}{T},\quad m\in\{0,1,\dots,MT\}, \qquad (2.46)
$$

where $M>0$ is arbitrarily large with $MT\in\mathbb{N}$.

For ease of notation, we only prove the case $\theta_1=\underline{\theta},\theta_2=\overline{\theta}$. Note that if we let $\delta=(\theta_2-\theta_1)/M$ instead and consider the difference $\alpha_T(\theta_2)-\alpha_T(\theta_1)$ in what follows, the proof remains unchanged.

We proceed to observe that $\theta_0=\underline{\theta},\theta_{MT}=\overline{\theta}$ and that we therefore rewrite

$$
\begin{aligned}
&\alpha_T(\overline{\theta})-\alpha_T(\underline{\theta}) \\
&\quad = \sum_{m=1}^{MT}\big\{\alpha_T(\theta_m)-\alpha_T(\theta_{m-1})\big\} \qquad (2.47)\\
&\quad = \sum_{m=1}^{MT}\frac{2}{T}\int_0^{T/2}\big\{\mathbb{E}\big[R_0\big(u_{T/2+s}^{*,l}(\theta_m)\big)\big]-\mathbb{E}\big[R_0\big(u_{T/2+s}^{*,l}(\theta_{m-1})\big]\big\}\,ds.
\end{aligned}
$$

Let $\xi>0$ arbitrary and $S=S(\omega,m)$, $m\in\mathbb{N}$ with $\xi\leq S\leq T/2-\xi$ be random stopping times to be made more precise later on. Then, by the strong Markov property of the processes involved,

$$
\begin{aligned}
\alpha_T(\overline{\theta})-\alpha_T(\underline{\theta}) = \sum_{m=1}^{MT}\frac{2}{T}\int_0^{T/2}&\mathbb{E}\big[\mathbb{E}_{u_S^{*,l}(\theta_m)}\big[R_0\big(u_{T/2-S+s}(\theta_m)\big)\big] \\
&-\mathbb{E}_{u_S^{*,l}(\theta_{m-1})}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big]\big]\,ds.
\end{aligned} \qquad (2.48)
$$

The expectations are well defined by Corollaries 2.4–2.5. Using a θ-*coupling* (cf. Remark A.9), we bound (2.48) from below by

$$\sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \mathbb{E}\big[\mathbb{E}_{u_S^{*,l}(\theta_m)}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big] - \mathbb{E}_{u_S^{*,l}(\theta_{m-1})}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big]\big]\, ds. \tag{2.49}$$

A shift in space, using the shift invariance of the dynamics, further allows to rewrite this to

$$\sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \mathbb{E}\big[\mathbb{E}_{u_S^{*,l}(\theta_m)(\cdot+R_0(u_S^{*,l}(\theta_{m-1})))}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big] - \mathbb{E}_{u_S^{*,l}(\theta_{m-1})(\cdot+R_0(u_S^{*,l}(\theta_{m-1})))}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big]\big]\, ds. \tag{2.50}$$

For $S \geq \xi > 0$, use a θ-$*$-*coupling* (cf. Lemma A.12) to obtain

$$0 \leq \Delta_S^{*,l}(\theta_{m-1}, \theta_m) \equiv u_S^{*,l}(\theta_m)\big(\cdot + R_0\big(u_S^{*,l}(\theta_{m-1})\big)\big) - u_S^{*,l}(\theta_{m-1})\big(\cdot + R_0\big(u_S^{*,l}(\theta_{m-1})\big)\big) \in \mathcal{C}_{\text{tem}}^+. \tag{2.51}$$

Hence, we use the strong Markov property of the family of laws $\mathbb{P}_f$, $f \in \mathcal{C}_{\text{tem}}^+$ to apply Lemma 2.14, using that $T/2 - S \geq \xi > 0$ and $S \geq \xi > 0$, to see that

$$\begin{aligned}
&\alpha_T(\overline{\theta}) - \alpha_T(\underline{\theta}) \\
&\quad\geq \sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \mathbb{E}\big[\mathbb{E}\big[\mathbb{E}_{u_S^{*,l}(\theta_m)(\cdot+R_0(u_S^{*,l}(\theta_{m-1})))}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big] \\
&\qquad - \mathbb{E}_{u_S^{*,l}(\theta_{m-1})(\cdot+R_0(u_S^{*,l}(\theta_{m-1})))}\big[R_0\big(u_{T/2-S+s}(\theta_{m-1})\big)\big] | \mathcal{F}_S\big]\big]\, ds \\
&\quad\geq \sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \mathbb{E}\big[\mathbb{E}\big[R_0\big(u_{T/2-S+s}^{(\Phi+\Delta_S^{*,l}(\theta_{m-1},\theta_m))}(\theta_{m-1})\big)\big] \\
&\qquad - \mathbb{E}\big[R_0\big(u_{T/2-S+s}^{(\Phi)}(\theta_{m-1})\big)\big]\big]\, ds \\
&\quad\geq \sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \mathbb{E}\big[R_0\big(u_{T/2-S+s}^{(\Phi+\Delta_S^{*,l}(\theta_{m-1},\theta_m))}(\theta_{m-1})\big) \vee 0 \\
&\qquad - R_0\big(u_{T/2-S+s}^{(\Phi)}(\theta_{m-1})\big) \vee 0\big]\, ds.
\end{aligned} \tag{2.52}$$

With the help of Lemma 2.16, we further bound this from below by

$$\sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \int_0^{\infty} \mathbb{E}\big[\mathbb{1}_{\{-x \leq L_0(u_{T/2-S+s}^{*,r}(\theta_{m-1})) < -x+1\}} \times \big(1 - e^{-2\langle \mathbb{1}_{[0,1)} \Delta_S^{*,l}(\theta_{m-1},\theta_m), (u_{T/2-S+s}^{*,r}(\theta_{m-1}))(\cdot - x)\rangle}\big)\big]\, dx\, ds. \tag{2.53}$$

For $d_0, m_0 > 0$, let

$$\tilde{M}(d_0, m_0) = \bigl\{ f \in \mathcal{C}^+_{\text{tem}} : \text{ there exist } 0 \le l_0 < r_0 \le 1/2, |r_0 - l_0| = d_0 \\ \text{such that } f \ge m_0 \mathbb{1}_{[l_0, r_0]} \bigr\}. \tag{2.54}$$

Fix $\epsilon > 0$ arbitrary and let $d_0 = d_0(\epsilon), m_0 = m_0(\epsilon) > 0$ as in Corollary A.5, where we note that instead of considering right markers we now consider left markers. We obtain as a further lower bound to the above

$$\begin{aligned} &\sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \int_0^\infty \mathbb{E}[\mathbb{1}_{\{u^{*,r}_{T/2-S+s}(\theta_{m-1})(\cdot + L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1}))) \in \tilde{M}(d_0, m_0)\}} \\ &\quad \times \mathbb{1}_{\{-x \le L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1})) < -x+1\}} \\ &\quad \times \bigl(1 - e^{-2\langle \mathbb{1}_{[0,1)} \Delta^{*,l}_S(\theta_{m-1}, \theta_m), (u^{*,r}_{T/2-S+s}(\theta_{m-1}))(\cdot - x)\rangle}\bigr)] \, dx \, ds \end{aligned} \tag{2.55}$$

for all $T \ge 1$. We next make use of the following crucial observation. Recall that $\theta_m - \theta_{m-1} = \delta / T$ for $m \in \{1, \ldots, MT\}$ with $\delta = (\overline{\theta} - \underline{\theta})/M$.

PROPOSITION 2.17. *For all $\xi > 0$ and $\varphi \in \mathcal{C}^+_{\text{tem}}$ with $L_0(\varphi) \in (0, 1)$, there exist $T_0 > 0$ big enough and $\rho, C_0, C_1 > 0$ small enough; all constants only dependent on $\xi, \underline{\theta}, \overline{\theta}, \varphi$, such that for all $T \ge T_0$ and $m \in \{0, 1, \ldots, MT\}$, $M \in \mathbb{N}$ there exist stopping times $\xi \le S = S(m, \varphi) \le T/2 - \xi$ such that*

$$\mathbb{P}\bigl(\langle \Delta^{*,l}_{S(m,\varphi)}(\theta_{m-1}, \theta_m), \varphi\rangle \ge \rho\bigr) \ge C_0\bigl(1 - \exp(-C_1 \delta)\bigr). \tag{2.56}$$

The proof of the proposition follows in Section 3 below. First, we complete the proof of Proposition 2.1. We obtain as a lower bound to the term in (2.55) with $\varphi = m_0 \mathbb{1}_{[1/2, 1/2 + d_0/2]}$,

$$\begin{aligned} &C_0\bigl(1 - e^{-C_1 \delta}\bigr)\bigl(1 - e^{-2\rho}\bigr) \\ &\quad \times \sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \int_0^\infty \mathbb{E}[\mathbb{1}_{\{u^{*,r}_{T/2-S+s}(\theta_{m-1})(\cdot + L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1}))) \in \tilde{M}(d_0, m_0)\}} \\ &\quad \times \mathbb{1}_{\{-x \le L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1})) < -x+1\}} \\ &\quad \times \mathbb{1}_{\{\mathbb{1}_{[0,1)}(\cdot)(u^{*,r}_{T/2-S+s}(\theta_{m-1}))(\cdot - x) \ge m_0 \mathbb{1}_{[1/2, 1/2 + d_0/2]}(\cdot)\}}] \, dx \, ds \end{aligned} \tag{2.57}$$

for all $T \ge T_0$. By definition of $\tilde{M}(d_0, m_0)$, using the theorem of Fubini–Tonelli, this is bounded from below by

$$\begin{aligned} &C_0\bigl(1 - e^{-C_1 \delta}\bigr)\bigl(1 - e^{-2\rho}\bigr) \frac{d_0}{2} \\ &\quad \times \sum_{m=1}^{MT} \frac{2}{T} \int_0^{T/2} \mathbb{E}[\mathbb{1}_{\{u^{*,r}_{T/2-S+s}(\theta_{m-1})(\cdot + L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1}))) \in \tilde{M}(d_0, m_0)\}} \\ &\quad \times \mathbb{1}_{\{L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1})) < 0\}}] \, ds. \end{aligned} \tag{2.58}$$

By symmetry and Lemma 2.12, we have for T big enough,

$$\frac{2}{T}\int_0^{T/2}\mathbb{E}[\mathbb{1}_{\{L_0(u^{*,r}_{T/2-S+s}(\theta_{m-1}))<0\}}]\,ds \geq \tilde{\delta}/2 > 0 \tag{2.59}$$

with $\tilde{\delta}$ as in Lemma 2.12. Recall the definition of $(\nu_T^{*,l}(\theta))$ from (A.8). We conclude using Corollary A.5 and symmetry that

$$\alpha_T(\overline{\theta}) - \alpha_T(\underline{\theta}) \geq C_0\big(1-e^{-C_1\delta}\big)\big(1-e^{-2\rho}\big)\frac{d_0}{2}\sum_{m=1}^{MT}(\tilde{\delta}/2-\epsilon) \tag{2.60}$$

for all $T \geq T_0$. Choose ϵ small enough and recall that $\delta = (\overline{\theta}-\underline{\theta})/M$, that is $M = (\overline{\theta}-\underline{\theta})/\delta$ to conclude that

$$\frac{\alpha_T(\overline{\theta}) - \alpha_T(\underline{\theta})}{T} \geq C_0\big(1-e^{-2\rho}\big)\frac{d_0}{2}(\tilde{\delta}/2-\epsilon)\big(1-e^{-C_1(\overline{\theta}-\underline{\theta})/M}\big)M. \tag{2.61}$$

Let $M \to \infty$ to obtain $C_0(1-e^{-2\rho})\frac{d_0}{2}(\tilde{\delta}/2-\epsilon)(\overline{\theta}-\underline{\theta})$ as a lower bound to the left-hand side. This completes the proof.

2.6. *Proof of Proposition* 1.1. Lemma 2.8 yields the existence of the limit $B = B(\theta)$. Its positivity follows from Corollary 2.11. Combine Proposition 2.1 and Corollary 2.9 to obtain (1.9) by taking $T \to \infty$. This concludes the proof.

3. Proof of Proposition 2.17. In this section, we prove Proposition 2.17. We start out by giving the main idea of the proof.

3.1. *Idea of proof.* Let $T_1, T_2 > 0$, $m \in \mathbb{N}$ and $\xi > 0$ be arbitrarily fixed. Let θ_m as in (2.46) and suppose that $\theta_c < \underline{\theta} \leq \theta_{m-1} < \theta_m \leq \overline{\theta}$. For $t \in [\xi, T/2-\xi-T_1-T_2]$ fixed, on a time-interval of length T_1+T_2, we look for a (random) point in time $S = S(m) \in [t, t+T_1+T_2]$ such that

$$\mathbb{1}_{[0,1)}\Delta_S^{*,l}(\theta_{m-1},\theta_m) \geq \rho\mathbb{1}_{[0,1)}, \tag{3.1}$$

where

$$\begin{aligned}\Delta_S^{*,l}(\theta_{m-1},\theta_m) \equiv{}& u_S^{*,l}(\theta_m)\big(\cdot + R_0\big(u_S^{*,l}(\theta_{m-1})\big)\big)\\ &- u_S^{*,l}(\theta_{m-1})\big(\cdot + R_0\big(u_S^{*,l}(\theta_{m-1})\big)\big)\end{aligned} \tag{3.2}$$

as in (2.51).

We investigate the difference between the solutions $u^{*,l}(\theta_m)$ and $u^{*,l}(\theta_{m-1})$ over time with the goal of finding S such that (3.1) holds. For t fixed as above, condition on $\mathcal{F}_t$. Aside from the shift in space, by monotonicity, the difference on the time-interval $[t, t+T_1+T_2]$ is greater or equal to the difference of solutions $u^{*,l}_{t+\cdot}(\theta_m)$ and $u^{*,l}_{t+\cdot}(\theta_{m-1})$ with common initial condition $u_t^{*,l}(\theta_{m-1})$ at time t.

First step: Start out with density $u_t^{*,l}(\theta_{m-1})$. Use a time-interval of length T_1 to gain additional mass-density v_{T_1} of height of order $O(\epsilon)$, $\epsilon \equiv \theta_m - \theta_{m-1}$ on the support of $u_t^{*,l}(\theta_{m-1})$ with probability of order $O(1)$. This amount is due to an immigration term of order $\epsilon u_{t+s}^{*,l}(\theta_{m-1})$, $s \in [0, T_1]$ in a θ*-coupling* (cf. Remark A.9). For T_1 not too big, the mass created, v_s, $s \in [0, T_1]$ remains small and immigration dominates the annihilation term of order v_s.

Second step: Use a *monotonicity-coupling* (cf. Remark A.8) to compare the original solution $u_{t+T_1+\cdot}^{*,l}(\theta_{m-1})$ with parameter θ_{m-1} for a time-interval of length T_2 with a solution with the same parameter θ_{m-1} but with mass v_{T_1} (cf. *first step*) added to the initial condition (at time $t + T_1$). With probability of order $O(\epsilon)$, the mass v_{T_1} gets a constant distance and an amount of mass $O(1)$ in front of the original solution $u_{t+T_1+\cdot}^{*,l}(\theta_{m-1})$ after a time-period of length T_2. To be more precise, we use this time-period of length T_2 twofold. First, we show that the mass stays "ahead" with probability of order $O(\epsilon)$ and second, that if it stays "ahead," then it has acquired a size of order $O(1)$ at the front.

We now give the *mathematical framework* for the coupling-techniques mentioned above. Let $T_1 > 0$ be arbitrarily fixed and $\theta_c < \underline{\theta} \le \theta_1 < \theta_2 \le \overline{\theta} < \infty$ with $\epsilon \equiv \theta_2 - \theta_1 > 0$. For $u_0 \in \mathcal{P}(\mathcal{C}_{\mathrm{tem}}^+)$ fixed, use a θ*-coupling* (cf. Remark A.9) to construct solutions $u_s^{(1)}(x) = u_s(\theta_1)(x), u_s^{(2)}(x) = u_s(\theta_2)(x), 0 \le s \le T_1$ to (1.1) such that $u_s^{(1)}(x) \le u_s^{(2)}(x)$ for all $s \in [0, T_1], x \in \mathbb{R}$ a.s. solve

$$\begin{aligned} &u_s(\theta_2)(x) = u_s(\theta_1)(x) + v_s(x) \\ &\quad \text{with } v_s(x) \ge 0 \text{ for all } s \in [0, T_1], x \in \mathbb{R} \text{ a.s.} \end{aligned} \tag{3.3}$$

and v as in (A.23), that is, *conditional on* $\sigma(u_s(\theta_1)) : 0 \le s \le T_1)$, v has distribution $Q^{0,(\theta_2-\theta_1)u(\theta_1),2u(\theta_1),1}(\theta_2)$ (cf. (1.20) and Theorem 2) on $[0, T_1]$.

Let $T_2 > 0$ be arbitrarily fixed. Extend the above coupling to include a process $(w_s)_{s\in[0,T_1+T_2]}$ such that

$$u_s(\theta_1)(x) \le w_s(x) \le u_s(\theta_2)(x) \quad \text{for all } s \in [0, T_1 + T_2], x \in \mathbb{R} \text{ a.s.} \tag{3.4}$$

as follows. Set

$$\begin{aligned} w_s(x) &\equiv u_s(\theta_1)(x) + v_s(x) \\ &\equiv \begin{cases} u_s^{(u_0)}(\theta_2)(x) = u_s^{(u_0)}(\theta_1)(x) + v_s(x) & \text{for } 0 \le s \le T_1, \\ u_{s-T_1}^{(w_{T_1})}(\theta_1)(x) = u_{s-T_1}^{(u_{T_1}^{(u_0)}(\theta_1)+v_{T_1})}(\theta_1)(x) & \text{for } s \ge T_1. \end{cases} \end{aligned} \tag{3.5}$$

That is, *conditional on* $\mathcal{F}_{T_1}$, $w_{T_1+\cdot}$ has distribution $Q^{w_{T_1},0,0,1}(\theta_1)$. Indeed, to construct the coupling for the case $s > T_1$, condition on $\mathcal{F}_{T_1}$ and use a combination of a *monotonicity-coupling* (cf. Remark A.8) and a θ*-coupling* (cf. Remark A.9). To be more precise, use a *monotonicity-coupling* based on two independent white noises W_1, W_2 to construct

$$\begin{aligned} u^{(u_0)}_{T_1+r}(\theta_1)(x) &= u_r^{(u_{T_1}(\theta_1))}(\theta_1)(x) \leq u_r^{(w_{T_1})}(\theta_1)(x) \\ &\equiv u_r^{(u_{T_1}(\theta_1))}(\theta_1)(x) + v_{T_1+r}(x) \end{aligned} \tag{3.6}$$

for all $r \geq 0, x \in \mathbb{R}$ almost surely, with $v_{T_1+\cdot}$ solving (A.20). Then use a θ*-coupling* to obtain

$$u_r^{(w_{T_1})}(\theta_1)(x) \leq u_r^{(w_{T_1})}(\theta_2)(x) \equiv u_r^{(w_{T_1})}(\theta_1)(x) + \hat{v}_{T_1+r}(x) \tag{3.7}$$

for all $r \geq 0, x \in \mathbb{R}$ almost surely, where the difference process $\hat{v}$ solves (A.23) with a white noise W_3 independent of W_1, W_2 from above. As a result,

$$\begin{aligned} u^{(u_0)}_{T_1+r}(\theta_1)(x) &\leq u^{(u_0)}_{T_1+r}(\theta_1)(x) + v_{T_1+r}(x) = w_{T_1+r}(x) \\ &\leq u^{(u_0)}_{T_1+r}(\theta_1)(x) + v_{T_1+r}(x) + \hat{v}_{T_1+r}(x) \\ &= u_r^{(w_{T_1})}(\theta_2)(x) = u_r^{(u^{(u_0)}_{T_1}(\theta_2))}(\theta_2)(x) = u^{(u_0)}_{T_1+r}(\theta_2)(x) \end{aligned} \tag{3.8}$$

holds indeed true.

3.2. *A first estimate*. The following estimate is fundamental in the *first step* of the construction. Recall (3.3) and thus compare the following SPDE with (A.23) from the θ*-coupling* which quantifies the gain in density due to an increase in θ.

Let

$$\begin{aligned} \Upsilon = \bigl\{ f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \|f\|_\lambda = \sup\{|f(x)| \exp(-\lambda|x|) : x \in \mathbb{R}\} < \infty \\ \text{for some } \lambda < 0 \bigr\} \end{aligned} \tag{3.9}$$

be the set of continuous functions with exponential decay. For existence and uniqueness of solutions to all of the SPDEs mentioned in the proof below, see Theorem 2. Also let

$$\tilde{\Upsilon} \equiv \left\{ \psi \in \mathcal{C}^{1,2} \text{ and } \sup_{t\in[0,T]} |\psi_t(\cdot)| \wedge \left| \frac{\partial \psi_t(\cdot)}{\partial t} \right| \wedge |\Delta \psi_t(\cdot)| \in \Upsilon \right\}. \tag{3.10}$$

LEMMA 3.1. *Let $T > 0, \zeta \in \mathcal{C}([0,T], \mathcal{C}^+_{\mathrm{tem}}) \backslash \{0\}$, W a white noise and $\epsilon > 0$ be arbitrarily fixed. Let $v = v(\epsilon, \theta, \zeta)$ be a solution to*

$$\frac{\partial v}{\partial t} = \Delta v + \epsilon\zeta + (\theta - v - 2\zeta)v + \sqrt{v}\dot{W}, \qquad v(0) = 0, \quad t \in [0,T]. \tag{3.11}$$

For $g \in \Upsilon, g \geq 0, g \not\equiv 0$ fixed,

$$\begin{aligned} 0 < c_-(\zeta, g, \theta, T) &= \liminf_{\epsilon \downarrow 0^+} \frac{\mathbb{E}[1 - e^{-2\langle v_T, g\rangle}]}{\epsilon} \\ &\leq \limsup_{\epsilon \downarrow 0^+} \frac{\mathbb{E}[1 - e^{-2\langle v_T, g\rangle}]}{\epsilon} = c_+(\zeta, g, \theta, T) < \infty \end{aligned} \tag{3.12}$$

holds true.

PROOF. *The lower bound.* Fix ζ, g, θ and T as above. Let

$$I(\epsilon) = \mathbb{E}\big[1 - e^{-2\langle v_T, g\rangle}\big]. \tag{3.13}$$

Subsequently, dominate $v = v(\epsilon)$ by the sum of two independent solutions (cf. the construction of the *coupling with two independent processes* in Remark A.10 below) satisfying

$$\begin{aligned} \frac{\partial v^{(i)}}{\partial t} &= \Delta v^{(i)} + \frac{\epsilon}{2}\zeta + \big(\theta - v^{(i)} - 2\zeta\big)v^{(i)} + \sqrt{v^{(i)}}\dot{W}_i, \\ v^{(i)}(0) &= 0, \quad i = 1, 2, t \geq 0 \end{aligned} \tag{3.14}$$

such that $v(t, x) \leq v^{(1)}(t, x) + v^{(2)}(t, x)$ for all $t \geq 0, x \in \mathbb{R}$ a.s. Note that $v^{(i)} = v^{(i)}(\epsilon) \stackrel{\mathcal{D}}{=} v(\epsilon/2), i = 1, 2$. We obtain by the independence and the identical distribution of the two nonnegative solutions, for all $\epsilon, T > 0$,

$$\begin{aligned} I(\epsilon) &\leq \mathbb{E}\big[1 - e^{-2\langle v_T^{(1)} + v_T^{(2)}, g\rangle}\big] \\ &= \mathbb{E}\big[1 - e^{-2\langle v_T^{(1)}, g\rangle}\big]\mathbb{E}\big[1 + e^{-2\langle v_T^{(2)}, g\rangle}\big] \leq 2I(\epsilon/2) \\ &\iff \left(\frac{I(\epsilon)}{\epsilon} \leq \frac{I(\epsilon/2)}{\epsilon/2}\right). \end{aligned} \tag{3.15}$$

By Theorem 2(b), $I(\epsilon)$ is continuous in ϵ. Hence, to establish the lower bound, it is enough to show that there exists $\epsilon_0 > 0$ such that $I(\epsilon) > 0$ for all $\epsilon \in [\epsilon_0, 2\epsilon_0]$. Indeed, by the continuity of I and (3.15), it then follows that

$$\inf_{\epsilon \in (0, 2\epsilon_0]} \frac{I(\epsilon)}{\epsilon} \geq \inf_{\epsilon \in [\epsilon_0, 2\epsilon_0]} \frac{I(\epsilon)}{\epsilon} > 0. \tag{3.16}$$

By reasoning as for an *immigration-coupling* (cf. Remark A.11 of the Appendix), it follows that $I(\epsilon)$ is monotonically increasing in ϵ. It is therefore enough to find $\epsilon_0 = \epsilon_0(T) > 0$ such that $I(\epsilon_0) > 0$. By definition of v, this holds true for arbitrary $T > 0$. Indeed, use for instance Theorem 2(c) to see that with $\mathbb{P}(v)$ denoting the distribution of v, $\mathbb{P}(v) = Q^{0,\epsilon\zeta,2\zeta,1}$ and $Q^{0,\epsilon\zeta,0,0}$ are mutually absolutely continuous on $\mathcal{U}_{R,T}$ (recall the notation from Theorem 2) for $R, T > 0$ arbitrarily fixed. Here, the law $Q^{0,\epsilon\zeta,0,0}$ is the law of the solution to

$$\frac{\partial w}{\partial t} = \Delta w + \epsilon\zeta + \theta w + \sqrt{w}\dot{W}, \qquad w(0) = 0, \quad t \geq 0. \tag{3.17}$$

The latter is a superprocess with immigration, and thus satisfies $\mathbb{P}(\langle w_T, g\rangle > 0) > 0$.

The upper bound. We now derive the upper bound in (3.12). Couple a solution v of (3.11) with a solution V of

$$\frac{\partial V}{\partial t} = \Delta V + \epsilon\zeta + \theta V + \sqrt{V}\dot{W}_3, \qquad V(0) = 0, \quad t \geq 0, \tag{3.18}$$

such that $v(t,x) \le V(t,x)$ for all $t \ge 0, x \in \mathbb{R}$ a.s. Here, W_3 is an appropriate white noise and we use techniques as in Section A.3 of the Appendix. See the beginning of [21], Section 2, for the theoretical background of what follows. We obtain for test functions $\psi(t,x) = \psi_t(x), t \ge 0, x \in \mathbb{R}$ satisfying $\psi \in \tilde{\Upsilon}$ and for $t \ge 0$ arbitrary, by an application of Itô's formula,

$$\begin{aligned} & e^{-2\langle V_t, \psi_t\rangle} \\ & = 1 + M_t \\ & \quad - 2\int_0^t e^{-2\langle V_s,\psi_s\rangle}\left\{\epsilon\langle \zeta_s, \psi_s\rangle + \left\langle V_s, \frac{\partial \psi_s}{\partial s} + \Delta\psi_s + \theta\psi_s - \frac{2}{2}\psi_s^2\right\rangle\right\} ds, \end{aligned} \tag{3.19}$$

where $(M_t)_{t\ge 0}$ is a local martingale with quadratic variation

$$\langle M.\rangle_t = 4\iint_0^t e^{-4\langle V_s,\psi_s\rangle} V_s(x)\psi_s^2(x)\,dx\,ds. \tag{3.20}$$

For $T > 0$ fixed and $0 \le s \le T$, choose $\psi(s,z) \equiv \Psi(T-s,z)$, where $\Psi(s,x) = \Psi_s(x)$ is the unique nonnegative solution to the partial differential equation (PDE)

$$\frac{\partial \Psi_s}{\partial s} = \Delta\Psi_s + \theta\Psi_s - \Psi_s^2, \qquad \Psi_0 = g, \quad 0 \le s \le T \tag{3.21}$$

(cf. Iscoe [10], Theorem A of the appendix, with $A\psi = \Delta\psi + \theta\psi$, $g(x) = x^2$ and $\mathcal{D}(A) = \{f \in \mathcal{C}^2(\mathbb{R},\mathbb{R}) : f, \Delta f \in \Upsilon)$. Then $\psi \in \tilde{\Upsilon}$ and we obtain for $0 \le t \le T$,

$$e^{-2\langle V_t,\psi_t\rangle} = 1 + M_t - 2\epsilon\int_0^t e^{-2\langle V_s,\psi_s\rangle}\langle \zeta_s, \psi_s\rangle\,ds. \tag{3.22}$$

Note that the integral on the right-hand side is finite as $\zeta \in \mathcal{C}([0,\infty), \mathcal{C}_{\text{tem}}^+)$ and $\sup_{s\in[0,T]} |\psi_s(\cdot)| \in \Upsilon$. Let $t = T$. In case $(M_t)_{t\in[0,T]}$ is a martingale, take expectations to conclude

$$\mathbb{E}\big[1 - e^{-2\langle V_T, g\rangle}\big] = 2\epsilon\int_0^T \mathbb{E}\big[e^{-2\langle V_s,\psi_s\rangle}\langle \zeta_s,\psi_s\rangle\big]\,ds. \tag{3.23}$$

In case $(M_t)_{t\in[0,T]}$ is only a local martingale, take a sequence of increasing stopping times $\tau_n \uparrow T$ such that $(M_{t\wedge\tau_n})_{t\in[0,T]}$ is a martingale for each $n \in \mathbb{N}$ fixed. Take expectations and subsequently use dominated convergence to obtain the same conclusion.

The coupling of v and V yields

$$\begin{aligned} \mathbb{E}\big[1 - e^{-2\langle v_T, g\rangle}\big] &\le \mathbb{E}\big[1 - e^{-2\langle V_T, g\rangle}\big] \\ &= 2\epsilon\int_0^T \mathbb{E}\big[e^{-2\langle V_s,\psi_s\rangle}\langle \zeta_s,\psi_s\rangle\big]\,ds \\ &\le 2\epsilon\int_0^T \langle \zeta_s, \Psi_{T-s}\rangle\,ds. \end{aligned} \tag{3.24}$$

It thus remains to show that $\int_0^T \langle \zeta_s, \Psi_{T-s}\rangle\,ds < \infty$. The latter follows from the assumption $\zeta \in \mathcal{C}([0,\infty), \mathcal{C}_{\text{tem}}^+)$ and as $(\psi_s)_{s\in[0,T]} = (\Psi_{T-s})_{s\in[0,T]}$ satisfies (3.10). □

3.3. *Increase of the right marker.* We now follow the strategy as outlined in Section 3.1. We start by investigating the increase of the right marker of a solution due to an increase in θ.

Let $f \in \mathcal{C}_{\text{tem}}^{+}$ with $R_0(f) < \infty$ and $\mathbb{P}_f(\tau = \infty) = 1$. Recall the notation from Section 3.1, in particular the definition of $u, v, w = u^{(f)}(\theta_1) + v$ with $u_0 = f$ from (3.5). In what follows, write $u_\cdot(x) = u_\cdot^{(f)}(\theta_1)(x)$ and set $\mathcal{F}_T^u = \sigma(u_t : 0 \le t \le T)$ for $T > 0$ arbitrary. Note that in the proofs to follow we will often only write $\mathbb{E}$ or $\mathbb{P}$ when the context is clear. In the main statements, the indices are kept however. This will allow us to avoid changes in indexing when using duality relations.

LEMMA 3.2. *Let $T_1, T_2 > 0$ be arbitrarily fixed and $\theta_c < \underline{\theta} \le \theta_1 < \theta_2 \le \overline{\theta}$. For all $\delta' > 0$, there exists $\eta_1 = \eta_1(\delta', T_1, T_2, \underline{\theta}, \overline{\theta}) > 0$ small enough such that*

$$\int_{\mathcal{C}_{\text{tem}}^{+}} \mathbb{P}_f\big(\mathbb{E}_f\big[0 \vee \big(R_0(w_{T_1+T_2}) \wedge 1\big) - 0 \vee \big(R_0(u_{T_1+T_2}) \wedge 1\big) | \mathcal{F}_{T_1}^u\big] \ge \eta_1(\theta_2 - \theta_1)\big)\big(\nu_T^{*,l}(\theta_1)\big)(df) \ge 1 - 2\delta' \tag{3.25}$$

for all $T > 1$.

REMARK 3.3. Note that $\mathbb{P}_{\nu_T^{*,l}(\theta_1)}(\tau = \infty) = 1$ by (A.8) and Remark 2.7.

PROOF OF LEMMA 3.2. Recall $Q^{f,\alpha,\beta,\gamma} \equiv Q_u^{f,\alpha,\beta,\gamma}(\theta)$ from Notation 1.3 and Theorem 2. With regards to the conditional expectation $\mathbb{E}[0 \vee (R_0(w_{T_1+T_2}) \wedge 1) - 0 \vee (R_0(u_{T_1+T_2}) \wedge 1)|\mathcal{F}_{T_1}^u]$ to follow, recall that for the coupling from (3.5), the difference $(0 \vee (R_0(w_t) \wedge 1)) - (0 \vee (R_0(u_t) \wedge 1))$ is nonnegative for all $t \ge 0$ almost surely. Moreover, *conditional on* $\mathcal{F}_{T_1}^u$, v has law $Q^{0,(\theta_2-\theta_1)u,2u,1}(\theta_2) \equiv Q_v^{0,(\theta_2-\theta_1)u,2u,1}(\theta_2)$ on $[0, T_1]$ and *conditional on* $\mathcal{F}_{T_1}$, w has law $Q^{u_{T_1}+v_{T_1},0,0,1}(\theta_1) \equiv Q_w^{u_{T_1}+v_{T_1},0,0,1}(\theta_1)$ on $[T_1, T_1 + T_2]$. Thus, as the laws $Q^{f,\alpha,\beta,\gamma}$ for $f \in \mathcal{C}_{\text{tem}}^{+}$ form a strong Markov family by Theorem 2(b), and by (3.4),

$$\begin{aligned} &\mathbb{E}\big[0 \vee \big(R_0(w_{T_1+T_2}) \wedge 1\big) - 0 \vee \big(R_0(u_{T_1+T_2}) \wedge 1\big)|\mathcal{F}_{T_1}^u\big] \\ &\quad = Q_v^{0,(\theta_2-\theta_1)u,2u,1}(\theta_2)\big[Q_w^{u_{T_1}+v_{T_1},0,0,1}(\theta_1)\big[0 \vee \big(R_0(u_{T_2}) \wedge 1\big)\big]\big] \\ &\qquad - Q_u^{u_{T_1},0,0,1}(\theta_1)\big[0 \vee \big(R_0(u_{T_2}) \wedge 1\big)\big] \quad \text{a.s.} \end{aligned} \tag{3.26}$$

Note that by Remark 3.3, u survives almost surely. Recall (2.42)–(2.44) to rewrite

$$\begin{aligned} &\mathbb{E}\big[0 \vee \big(R_0(w_{T_1+T_2}) \wedge 1\big) - 0 \vee \big(R_0(u_{T_1+T_2}) \wedge 1\big)|\mathcal{F}_{T_1}^u\big] \\ &\quad = \int_0^1 \mathbb{P}\big(R_0(w_{T_1+T_2}) > x|\mathcal{F}_{T_1}^u\big)\, dx \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^1 \mathbb{P}\big(R_0(u_{T_1+T_2}) > x | \mathcal{F}^u_{T_1}\big)\,dx \\
&= \int_0^1 \mathbb{E}\big[1 - e^{-2\langle w_{T_1}(\cdot+x), u^{*,r}_{T_2}(\theta_1)\rangle} | \mathcal{F}^u_{T_1}\big]\,dx \\
&\quad - \int_0^1 \mathbb{E}\big[1 - e^{-2\langle u_{T_1}(\cdot+x), u^{*,r}_{T_2}(\theta_1)\rangle} | \mathcal{F}^u_{T_1}\big]\,dx \\
&= \int_0^1 \mathbb{E}\big[e^{-2\langle u_{T_1}(\cdot+x), u^{*,r}_{T_2}(\theta_1)\rangle}\big(1 - e^{-2\langle v_{T_1}(\cdot+x), u^{*,r}_{T_2}(\theta_1)\rangle}\big) | \mathcal{F}^u_{T_1}\big]\,dx.
\end{aligned} \tag{3.27}
$$

Use a θ-$*$-*coupling* (cf. Lemma A.12) to conclude that this is bounded below by

$$
\begin{aligned}
&\int_0^1 \mathbb{E}\big[e^{-2\langle u_{T_1}(\cdot+x), u^{*,r}_{T_2}(\overline{\theta})\rangle}\big(1 - e^{-2\langle v_{T_1}(\cdot+x), u^{*,r}_{T_2}(\underline{\theta})\rangle}\big) | \mathcal{F}^u_{T_1}\big]\,dx \\
&\quad = \int_0^1 \mathbb{E}\big[e^{-2\langle u_{T_1}(\cdot+x), u^{*,r}_{T_2}(\overline{\theta})\rangle} \\
&\qquad \times \mathbb{E}\big[1 - e^{-2\langle v_{T_1}(\cdot+x), u^{*,r}_{T_2}(\underline{\theta})\rangle} | \sigma\big(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1}\big)\big] | \mathcal{F}^u_{T_1}\big]\,dx,
\end{aligned} \tag{3.28}
$$

where we let $\mathcal{F}^{u^{*,r}} = \sigma(u^{*,r}_t(\underline{\theta}), u^{*,r}_t(\overline{\theta}) : t \geq 0)$.

Let $\epsilon = \theta_2 - \theta_1$. We now randomize the initial condition. Recall $\nu^{*,l}_T(\theta_1)$ as defined in (A.8) of the Appendix. Let $\eta_1, T > 0$ be arbitrarily fixed. The quantity we are interested in is

$$
\begin{aligned}
I_1 \equiv \int_{\mathcal{C}^+_{\mathrm{tem}}} &\mathbb{P}_f\big(\mathbb{E}\big[0 \vee \big(R_0(w_{T_1+T_2}) \wedge 1\big) \\
&- 0 \vee \big(R_0(u_{T_1+T_2}) \wedge 1\big) | \mathcal{F}^u_{T_1}\big] \geq \eta_1 \epsilon\big)\big(\nu^{*,l}_T(\theta_1)\big)(df).
\end{aligned} \tag{3.29}
$$

We get with the help of (3.27)–(3.28) as a lower bound to (3.29),

$$
\begin{aligned}
&\int_{\mathcal{C}^+_{\mathrm{tem}}} \mathbb{P}_f\bigg(\int_0^1 \mathbb{E}\big[e^{-2\langle u_{T_1}(\cdot+x), u^{*,r}_{T_2}(\overline{\theta})\rangle} \\
&\qquad \times \mathbb{E}\big[1 - e^{-2\langle v_{T_1}(\cdot+x), u^{*,r}_{T_2}(\underline{\theta})\rangle} | \sigma\big(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1}\big)\big] | \mathcal{F}^u_{T_1}\big]\,dx \\
&\quad \geq \eta_1 \epsilon\bigg)\big(\nu^{*,l}_T(\theta_1)\big)(df).
\end{aligned} \tag{3.30}
$$

Here, we note that $(v_t)_{t \in [0,T_1]}$ solves (A.23) or (3.11) with $\theta = \theta_2$ and $(\zeta_t)_{t \in [0,T_1]} = (u_t)_{t \in [0,T_1]} = (u^{(f)}_t)(\theta_1)_{t \in [0,T_1]}$ and f drawn according to $\nu^{*,l}_T(\theta_1)$. By (A.11) and Corollary A.5, for every $\delta' > 0$ there exist a compact set $K_{\delta'} \subset \mathcal{C}^+_{\mathrm{tem}}$ and $d_0 = d_0(\delta'), m_0 = m_0(\delta') > 0$ such that

$$
\inf_{\underline{\theta} \leq \theta \leq \overline{\theta}} \big(\nu^{*,l}_T(\theta)\big)\big(K_{\delta'} \cap M(d_0, m_0)\big) \geq 1 - \delta' \quad \text{for all } T > 1 \tag{3.31}
$$

with

$$M(d_0, m_0) \equiv \{f \in \mathcal{C}^+_{\text{tem}} : \text{ there exist } -1/2 \le l_0 < r_0 \le 0, \\ |r_0 - l_0| = d_0 \\ \text{such that } f \ge m_0 \mathbb{1}_{[l_0, r_0]}\} \tag{3.32}$$

for $d_0, m_0 > 0$. Observe first that $M(d_0, m_0) \cap K_{\delta'}$ is compact in $\mathcal{C}^+_{\text{tem}}$. Indeed, use that if $(f_n)_n \subset M(d_0, m_0) \cap K_{\delta'}$, then there exists a subsequence $(f_{n_k})_k \subset M(d_0, m_0)$ that converges to a limit in $K_{\delta'}$. Let $x_{n_k} = l_{n_k} + d_0/2$ such that $f_{n_k} \ge m_0 \mathbb{1}_{[x_{n_k} - d_0/2, x_{n_k} + d_0/2]}$ and $l_{n_k} \le x_{n_k} \le r_{n_k}$, $|r_{n_k} - l_{n_k}| = d_0$, $l_{n_k}, r_{n_k} \in [-1/2, 0]$. By the compactness of $[-1/2, 0]$, there exists a subsequence $x_{n_{k_l}} \to x_0 \in [-1/2, 0]$ for $l \to \infty$ and as a result, $f_{n_{k_l}}$ converges to a limit in $M(d_0, m_0) \cap K_{\delta'}$.

Conditional on $\sigma(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1})$, we now apply the lower bound of Lemma 3.1 for some $0 \not\equiv g = g_x \in \Upsilon$, $0 \le g \le u^{*,r}_{T_2}(\underline{\theta})(\cdot - x)$. Recall that $u_t = u_t(\theta_1)$. From below (3.16), it follows that it is enough to show for $\epsilon_0 > 0$ arbitrarily fixed that

$$\begin{aligned} \inf_{\theta_1 \in [\underline{\theta}, \overline{\theta}]} \int_{K_{\delta'} \cap M(d_0, m_0)} & \mathbb{P}_f\biggl(\int_0^1 \mathbb{E}\bigl[e^{-2\langle u_{T_1}(\cdot + x), u^{*,r}_{T_2}(\overline{\theta})\rangle} \\ & \quad \times \mathbb{E}\bigl[1 - e^{-2\langle v_{T_1}(\epsilon_0, \theta_2, (u_t)_{t \in [0, T_1]}), g_x\rangle} | \sigma(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1})\bigr] | \mathcal{F}^u_{T_1}\bigr] \, dx \\ & \quad \ge \eta_1 \epsilon_0\biggr) (\nu^{*,l}_T(\theta_1))(df) \\ & \ge 1 - 2\delta' \end{aligned} \tag{3.33}$$

for η_1 small enough and $\theta_2 = \theta_1 + \epsilon_0$. The left-hand side in the above can be bounded from below by

$$\begin{aligned} (1 - \delta') \inf_{\theta \in [\underline{\theta}, \overline{\theta}]} \inf_{f \in K_{\delta'} \cap M(d_0, m_0)} & \mathbb{P}_f\biggl(\int_0^1 \mathbb{E}\bigl[e^{-2\langle u_{T_1}(\cdot + x), u^{*,r}_{T_2}(\overline{\theta})\rangle} \\ \times \mathbb{E}\bigl[1 - e^{-2\langle v_{T_1}(\epsilon_0, \theta_2, (u_t)_{t \in [0, T_1]}), g_x\rangle} & | \sigma(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1})\bigr] | \mathcal{F}^u_{T_1}\bigr] \, dx \ge \eta_1 \epsilon_0\biggr), \end{aligned} \tag{3.34}$$

where we used (3.31).

The map $(f, \alpha, \beta, 1) \mapsto Q^{f, \alpha, \beta, 1}$ is continuous by Theorem 2(b). Hence, the law $\mathbb{P}_f(\theta_1) = Q^{f, 0, 0, 1}(\theta_1) = Q^{f, 0, (\overline{\theta} - \theta_1), 1}(\overline{\theta})$ of u is continuous in f and θ_1. Furthermore, by the continuous mapping theorem, the law of $v(\epsilon_0, \theta_2, u)$, that is $Q^{0, \epsilon_0 u, 2u, 1}(\theta_1 + \epsilon_0) = Q^{0, \epsilon_0 u, 2u + (\overline{\theta} - \theta_1), 1}(\overline{\theta} + \epsilon_0)$ is also continuous in f and θ_1. As $[\underline{\theta}, \overline{\theta}]$ is a compact interval and $M(d_0, m_0) \cap K_{\delta'}$ is compact in $\mathcal{C}^+_{\text{tem}}$, the infimum is attained for some $\theta' \in [\underline{\theta}, \overline{\theta}]$, $f' \in M(d_0, m_0) \cap K_{\delta'}$. Let θ', f' be arbitrarily fixed. The innermost expectation is nonzero almost surely by reasoning as in (3.17) of the proof of the lower bound in Lemma 3.1. Let $x \in [0, 1]$ be arbitrarily fixed. Then

(1.22) and symmetry yield for $u_t = u_t^{(f')}(\theta')$,

$$
\begin{aligned}
&\mathbb{E}[e^{-2\langle u_{T_1}(\cdot+x), u_{T_2}^{*,r}(\overline{\theta})\rangle} | \mathcal{F}_{T_1}^u] \\
&\quad = \mathbb{P}(\langle \mathbb{1}_{(0,\infty)}(\cdot), u_{T_2}^{(u_{T_1}(\cdot+x))}(\overline{\theta})\rangle = 0 | \mathcal{F}_{T_1}^u) \qquad (3.35)\\
&\quad = \mathbb{P}(R_0(u_{T_2}^{(u_{T_1}(\cdot+x))}(\overline{\theta})) \le 0 | \mathcal{F}_{T_1}^u).
\end{aligned}
$$

The latter is nonzero almost surely. Thus, using dominated convergence, we can choose $\eta_1 > 0$ small enough such that $I_1 \ge (1-\delta')^2 \ge 1 - 2\delta'$. □

COROLLARY 3.4. *Let $T_1, T_2 > 0$ be arbitrarily fixed and $\theta_c < \underline{\theta} \le \theta_1 < \theta_2 \le \overline{\theta}$. For all $\delta' > 0$, there exists $\eta_1 = \eta_1(\delta', T_1, T_2, \underline{\theta}, \overline{\theta}) > 0$ small enough such that*

$$
\begin{aligned}
&\int_{\mathcal{C}_{\mathrm{tem}}^+} \mathbb{P}_f(\mathbb{P}_f(R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2}) | \mathcal{F}_{T_1}^u) \qquad (3.36)\\
&\qquad \ge \eta_1(\theta_2 - \theta_1))(\nu_T^{*,l}(\theta_1))(df) \ge 1 - 2\delta'
\end{aligned}
$$

for all $T > 1$.

PROOF. Use that $\mathbb{1}_{\{X>Y\}} \ge 0 \vee (X \wedge 1) - 0 \vee (Y \wedge 1)$ for $X \ge Y$. □

LEMMA 3.5. *Let $T_1, T_2 > 0$ be arbitrarily fixed and $\theta_c < \underline{\theta} \le \theta_1 < \theta_2 \le \overline{\theta}$. For all $\delta' > 0$, there exists $\eta_2 = \eta_2(\delta', T_1, T_2, \underline{\theta}, \overline{\theta}) > 0$ big enough such that*

$$
\begin{aligned}
&\int_{\mathcal{C}_{\mathrm{tem}}^+} \mathbb{P}_f(\mathbb{P}_f(R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2}) | \mathcal{F}_{T_1}^u) \qquad (3.37)\\
&\qquad \le \eta_2(\theta_2 - \theta_1))(\nu_T^{*,l}(\theta_1))(df) \ge 1 - 4\delta'
\end{aligned}
$$

for all $T > 1$.

PROOF. First, note that $w = u + v$ as in (3.5), and thus

$$
\mathbb{P}(R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2}) | \mathcal{F}_{T_1}^u) = \mathbb{P}(R_0(v_{T_1+T_2}) > R_0(u_{T_1+T_2}) | \mathcal{F}_{T_1}^u). \qquad (3.38)
$$

Recall the construction of $v_{T_1+r}, r \in [0, T_2]$ by means of a *monotonicity-coupling* (cf. Remark A.8) from (3.6). Extend this coupling as follows:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \Delta u + (\theta_1 - u)u + \sqrt{u}\dot{W}_1, \qquad u(T_1) = u_{T_1}^{(u_0)}(\theta_1),\\
&\frac{\partial v}{\partial t} = \Delta v + (\theta_1 - v - 2u)v + \sqrt{v}\dot{W}_2, \qquad (3.39)\\
&v(T_1) = v_{T_1} = w_{T_1} - u_{T_1}^{(u_0)}(\theta_1),\\
&\frac{\partial d}{\partial t} = \Delta d + 2uv + (\theta_1 - d - 2v)d + \sqrt{d}\dot{W}_3, \qquad d(T_1) = 0,
\end{aligned}
$$

$t \geq T_1$, with $W_i, i = 1, 2, 3$ independent white noises. Then $U \equiv v + d$ solves, conditional on $\mathcal{F}^u_{T_1+T_2}$,

$$\frac{\partial U}{\partial t} = \Delta U + (\theta_1 - U)U + \sqrt{d}\dot{W}_4, \qquad U(T_1) = v_{T_1}, \quad t \geq T_1 \tag{3.40}$$

for some white noise W_4 independent of W_1. By construction, $v_{T_1+t}(x) \leq U_{T_1+t}(x)$ for all $x \in \mathbb{R}, t \in [0, T_2]$ almost surely and the law of U only depends on $\mathcal{F}^u_{T_1}$ through the initial condition.

Now reason similar to (3.27)–(3.28) to obtain

$$\begin{aligned}
&\mathbb{P}\big(R_0(v_{T_1+T_2}) > R_0(u_{T_1+T_2})|\mathcal{F}^u_{T_1}\big)\\
&\quad \leq \mathbb{P}\big(R_0(U_{T_1+T_2}) > R_0(u_{T_1+T_2})|\mathcal{F}^u_{T_1}\big)\\
&\quad = \mathbb{P}\big(\mathbb{P}\big(R_0(U_{T_1+T_2}) > R_0(u_{T_1+T_2})|\mathcal{F}^u_{T_1+T_2}\big)|\mathcal{F}^u_{T_1}\big)\\
&\quad = \mathbb{E}\big[\mathbb{E}\big[1 - e^{-2\langle U_{T_1}(\cdot + R_0(u_{T_1+T_2})), u^{*,r}_{T_2}(\theta_1)\rangle}|\mathcal{F}^u_{T_1+T_2}\big]|\mathcal{F}^u_{T_1}\big]\\
&\quad = \mathbb{E}\big[1 - e^{-2\langle v_{T_1}(\cdot + R_0(u_{T_1+T_2})), u^{*,r}_{T_2}(\theta_1)\rangle}|\mathcal{F}^u_{T_1}\big]\\
&\quad \leq \mathbb{E}\big[1 - e^{-2\langle v_{T_1}(\cdot + R_0(u_{T_1+T_2})), u^{*,r}_{T_2}(\overline{\theta})\rangle}|\mathcal{F}^u_{T_1}\big]\\
&\quad = \mathbb{E}\big[\mathbb{E}\big[1 - e^{-2\langle v_{T_1}, (u^{*,r}_{T_2}(\overline{\theta}))(\cdot - R_0(u_{T_1+T_2}))\rangle}|\sigma\big(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1}\big)\big]|\mathcal{F}^u_{T_1}\big].
\end{aligned} \tag{3.41}$$

In the third equality, we used that $U(T_1) = v_{T_1}$.

Use Lemma A.2 to obtain that for all $\theta \in [\underline{\theta}, \overline{\theta}]$, $T_1, T_2 > 0$, $A > 0$, $T \geq 1$,

$$\mathbb{P}_{\nu^{*,l}_T(\theta)}\big(\big|R_0(u_{T_1+T_2})\big| \geq A\big) \leq C(\underline{\theta}, \overline{\theta}, T_1 + T_2)/A. \tag{3.42}$$

By (A.11), it follows that for every $\delta' > 0$ there exist $A_{\delta'} > 0$ big enough and a compact set $K_{\delta'} \subset \mathcal{C}^+_{\text{tem}}$ such that

$$\inf_{\underline{\theta} \leq \theta \leq \overline{\theta}} \big(\nu^{*,l}_T(\theta)\big)\big(K_{\delta'} \cap \big\{\big|R_0(u_{T_1+T_2})\big| < A_{\delta'}\big\}\big) \geq 1 - \delta' \quad \text{for all } T > 1. \tag{3.43}$$

For $I_2 \equiv \int_{\mathcal{C}^+_{\text{tem}}} \mathbb{P}_f(\mathbb{P}(R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})|\mathcal{F}^u_{T_1}) \leq \eta_2(\theta_2 - \theta_1))(\nu^{*,l}_T(\theta_1))(df)$ as in (3.37), we obtain

$$\begin{aligned}
I_2 \geq \int_{K_{\delta'}} \mathbb{P}_f\Big(&\sup_{a \in [-A_{\delta'}, A_{\delta'}]} \mathbb{E}\big[\mathbb{E}\big[1 - e^{-2\langle v_{T_1}, (u^{*,r}_{T_2}(\overline{\theta}))(\cdot - a)\rangle}|\sigma\big(\mathcal{F}^{u^{*,r}}, \mathcal{F}^u_{T_1}\big)\big]|\mathcal{F}^u_{T_1}\big]\\
&\leq \eta_2(\theta_2 - \theta_1)\Big)\big(\nu^{*,l}_T(\theta_1)\big)(df) - \delta'.
\end{aligned} \tag{3.44}$$

By symmetry, Corollaries 2.4–2.5 and the Markov inequality, for $T_2 > 0$ fixed we can choose $l > 0$ big enough such that

$$\mathbb{P}\big(\big|L_0\big(u^{*,r}_{T_2}(\overline{\theta})\big)\big| \geq l\big) \leq \delta'. \tag{3.45}$$

Recall (3.4)–(3.5) and use monotonicity to conclude that for $T_1 > 0$ fixed, for all $r > 0, T > 1$,

$$\mathbb{P}_{\nu_T^{*,l}(\theta_1)}\big(0 \vee R_0(v_{T_1}) \geq r\big) \leq \mathbb{P}_{\nu_T^{*,l}(\theta_1)}\big(0 \vee R_0\big(u_{T_1}(\theta_2)\big) \geq r\big) \leq \frac{C(\underline{\theta}, \overline{\theta}, T_1)}{r}. \tag{3.46}$$

Thus, for $\delta' > 0$ fixed, we can pick $l, r > 0$ big enough such that

$$\begin{aligned} I_2 \geq \int_{K_{\delta'}} \mathbb{P}_f\Big(& \sup_{a \in [-A_{\delta'}, A_{\delta'}]} \mathbb{E}\big[\mathbb{E}\big[1 - e^{-2\langle v_{T_1}, (\mathbb{1}_{[-l,r]} u_{T_2}^{*,r}(\overline{\theta}))(\cdot - a)\rangle} \big| \\ & \sigma\big(\mathcal{F}^{u^{*,r}}, \mathcal{F}_{T_1}^u\big)\big] \big| \mathcal{F}_{T_1}^u\big] \\ & \leq \eta_2(\theta_2 - \theta_1)\Big)\big(\nu_T^{*,l}(\theta_1)\big)(df) - 3\delta'. \end{aligned} \tag{3.47}$$

Now reason as from above (3.33) to below (3.34), this time using (3.24) from the proof of the upper bound from Lemma 3.1, to obtain the claim. □

Recall the following observation for the coupling from (3.5) from the beginning of the proof of Lemma 3.2. Conditional on $\mathcal{F}_{T_1}^u$, v has law $Q_v^{0,(\theta_2-\theta_1)u,2u,1}(\theta_2)$ on $[0, T_1]$ and conditional on $\mathcal{F}_{T_1}$, w has law $Q_w^{u_{T_1}+v_{T_1},0,0,1}(\theta_1)$ on $[T_1, T_1 + T_2]$. Finally, recall that $w = u + v$.

LEMMA 3.6. *Let $T_1, T_2 > 0$ be arbitrarily fixed, $\theta_c < \underline{\theta} \leq \theta_1 < \theta_2 \leq \overline{\theta}$. For all $\delta' > 0$, there exists $\eta_3 = \eta_3(\delta', T_1, T_2, \underline{\theta}, \overline{\theta}) > 0$ small enough such that*

$$\begin{aligned} \int_{\mathcal{C}_{\text{tem}}^+} & \mathbb{E}_f\big[\mathbb{1}\big(\mathbb{P}_f\big(\{R_0(w_{T_1+T_2}) - R_0(u_{T_1+T_2}) \geq \eta_3\} \big| \mathcal{F}_{T_1}^u\big) \\ & \geq \eta_3^2 \mathbb{P}_f\big(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\} \big| \mathcal{F}_{T_1}^u\big)\big)\big]\big(\nu_T^{*,l}(\theta_1)\big)(df) \\ & \geq 1 - 6\delta' \end{aligned} \tag{3.48}$$

for all $T > 1$.

REMARK 3.7. Note that by (3.36), $\mathbb{P}(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\} | \mathcal{F}_{T_1}^u) > 0$ almost surely under $\nu_T^{*,l}(\theta_1)$.

PROOF OF LEMMA 3.6. Let $X \geq 0$ be a random variable on some probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$. Then (cf. [9], Proof of Lemma 3)

$$\tilde{\mathbb{P}}\big(X > \tilde{\mathbb{E}}[X]/2\big) \geq \big(\tilde{\mathbb{E}}[X]\big)^2 / \big(4\tilde{\mathbb{E}}[X^2]\big). \tag{3.49}$$

In the coupling from above, let

$$0 \leq X \equiv 0 \vee \big(R_0(w_{T_1+T_2}) \wedge 1\big) - 0 \vee \big(R_0(u_{T_1+T_2}) \wedge 1\big) \leq 1. \tag{3.50}$$

In what follows, we make use of regular conditional distributions. For $\mathbb{P}(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1}) > 0$, set

$$\tilde{\mathbb{P}}(\{\cdot\}) = \frac{\mathbb{P}(\{\cdot\} \cap \{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1})}{\mathbb{P}(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1})}. \tag{3.51}$$

Now apply (3.49) to get for $\eta_3 \le \tilde{\mathbb{E}}[X]/2$,

$$\begin{aligned}
&\tilde{\mathbb{P}}(\{R_0(w_{T_1+T_2}) - R_0(u_{T_1+T_2}) \ge \eta_3\}) \\
&\quad \ge \tilde{\mathbb{P}}(X \ge \eta_3) \ge \tilde{\mathbb{P}}(X > \tilde{\mathbb{E}}[X]/2) \ge (\tilde{\mathbb{E}}[X])^2/(4\tilde{\mathbb{E}}[X^2]) \\
&\quad \ge (\tilde{\mathbb{E}}[X])^2/4 \ge \eta_3^2.
\end{aligned} \tag{3.52}$$

Therefore, it suffices to show that there exists $\eta_3 > 0$ small enough such that

$$\int_{\mathcal{C}^+_{\text{tem}}} \mathbb{E}_f[\mathbb{1}_{\{\eta_3 \le \tilde{\mathbb{E}}[X]/2\}}](\nu_T^{*,l}(\theta_1))(df) \ge 1 - 6\delta'. \tag{3.53}$$

By (3.25) and (3.37), we have for $\eta_3 = \eta_1/(2\eta_2)$,

$$\begin{aligned}
&\int_{\mathcal{C}^+_{\text{tem}}} \mathbb{E}_f[\mathbb{1}(\eta_3 \le \tilde{\mathbb{E}}[X]/2)](\nu_T^{*,l}(\theta_1))(df) \\
&= \int_{\mathcal{C}^+_{\text{tem}}} \mathbb{E}_f[\mathbb{1}(\mathbb{E}[0 \vee (R_0(w_{T_1+T_2}) \wedge 1) - 0 \vee (R_0(u_{T_1+T_2}) \wedge 1)|\mathcal{F}^u_{T_1}] \\
&\qquad \ge 2\eta_3 \mathbb{P}(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1}))](\nu_T^{*,l}(\theta_1))(df) \\
&\ge \int_{\mathcal{C}^+_{\text{tem}}} \mathbb{E}_f[\mathbb{1}(\mathbb{E}[0 \vee (R_0(w_{T_1+T_2}) \wedge 1) - 0 \vee (R_0(u_{T_1+T_2}) \wedge 1)|\mathcal{F}^u_{T_1}] \\
&\qquad \ge \eta_1(\theta_2 - \theta_1))](\nu_T^{*,l}(\theta_1))(df) \\
&\qquad - \int_{\mathcal{C}^+_{\text{tem}}} \mathbb{E}_f\Bigg[\mathbb{1}\Bigg(\mathbb{P}(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1}) \\
&\qquad\qquad \ge \frac{\eta_1(\theta_2 - \theta_1)}{2\eta_3}\Bigg)\Bigg](\nu_T^{*,l}(\theta_1))(df) \\
&\ge 1 - 2\delta' - 4\delta'.
\end{aligned} \tag{3.54}$$

□

LEMMA 3.8. *Let $\varphi \in \mathcal{C}^+_{\text{tem}}$ with $L_0(\varphi) \in (0, 1)$ be arbitrarily fixed. Further, let $T_1, T_2 > 0$ be arbitrarily fixed and $\theta_c < \underline{\theta} \le \theta_1 < \theta_2 \le \overline{\theta}$. For all $\delta' > 0$, there exists $\eta_4 = \eta_4(\varphi, \delta', T_1, T_2, \underline{\theta}, \overline{\theta}) > 0$ small enough such that*

$$\begin{aligned}
&\int_{\mathcal{C}^+_{\text{tem}}} \mathbb{E}_f[\mathbb{1}(\mathbb{P}_f(\{\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle \ge \eta_4\}|\mathcal{F}^u_{T_1}) \\
&\quad \ge (1 - e^{-2\eta_4})^2
\end{aligned} \tag{3.55}$$

$$\times \mathbb{P}_f(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1}))](\nu^{*,l}_T(\theta_1))(df)$$
$$\geq 1-\delta'$$

for all $T > 1$.

PROOF. Let $\varphi \in \mathcal{C}^+_{\text{tem}}$ with $L_0(\varphi) \in (0,1)$ and $\eta_4 > 0$ be arbitrarily fixed. Note that in what follows we condition first on $\mathcal{F}^u_{T_1+T_2}$ rather than on $\mathcal{F}^u_{T_1}$. Next, rewrite

$$\begin{aligned} &\mathbb{P}(\{\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle \geq \eta_4\}|\mathcal{F}^u_{T_1+T_2}) \\ &\quad = \mathbb{P}(\{1-e^{-2\langle v_{T_1+T_2}(\cdot+R_0(u_{T_1+T_2})),\varphi(\cdot)\rangle} \geq 1-e^{-2\eta_4}\}|\mathcal{F}^u_{T_1+T_2}). \end{aligned} \tag{3.56}$$

Recall from (3.6) that by means of a *monotonicity-coupling* (cf. Remark A.8), $u_{T_1+t}(x) = u_t^{(u_{T_1}(\theta_1))}(\theta_1)(x) \leq w_{T_1+t}(x) = u_t^{(w_{T_1})}(\theta_1)(x) = u_{T_1+t}(x) + v_{T_1+t}(x)$ for $0 \leq t \leq T_2$ with v solving (cf. (A.20))

$$\begin{aligned} &\frac{\partial v}{\partial t} = \Delta v + (\theta_1 - v - 2u)v + \sqrt{v}\dot{W}_2, \\ &v(T_1) = w_{T_1} - u_{T_1}, \quad T_1 \leq t \leq T_1 + T_2. \end{aligned} \tag{3.57}$$

By Corollary A.1, we have

$$\begin{aligned} &\mathbb{E}[1-e^{-2\langle v_{T_1+T_2}(\cdot+R_0(u_{T_1+T_2})),\varphi(\cdot)\rangle}|\mathcal{F}^u_{T_1+T_2}] \\ &\quad = \mathbb{E}[1-e^{-2\langle v_{T_1+T_2},\varphi(\cdot-R_0(u_{T_1+T_2}))\rangle}|\mathcal{F}^u_{T_1+T_2}] \\ &\quad = \mathbb{E}[1-e^{-2\langle v_{T_1},z_{T_1+T_2}\rangle}|\mathcal{F}^u_{T_1+T_2}], \end{aligned} \tag{3.58}$$

where z solves

$$\begin{aligned} &\frac{\partial z}{\partial t} = \Delta z + (\theta - z - 2u_{2T_1+T_2-\cdot})z + \sqrt{z}\dot{W}_3, \\ &z(T_1) = \varphi(\cdot - R_0(u_{T_1+T_2})), \quad T_1 \leq t \leq T_1 + T_2, \end{aligned} \tag{3.59}$$

where W_2, W_3 are independent white noises. That is, conditional on $\mathcal{F}^u_{T_1+T_2}$, $(z_t)_{T_1 \leq t \leq T_1+T_2}$ has law $\mathbb{P}(z) \equiv Q^{z(T_1),0,2u_{2T_1+T_2-\cdot},1}$. By Theorem 2(c), $\mathbb{P}(z)$ and $Q^{z(T_1),0,0,0}$ are mutually absolutely continuous on $\mathcal{U}_{R,T}$ for $R, T > 0$ arbitrarily fixed. The latter is the law of a superprocess with nonzero initial condition, and thus is nonzero with positive probability at time $T_1 + T_2$. Similarly, v_{T_1} is nonzero with positive probability.

Now reason as in (3.28)–(3.34) with the following modifications. Use a θ-*coupling* (cf. Remark A.9) for z to obtain (3.28). Then investigate

$$\mathbb{E}[\mathbb{E}[1-e^{-2\langle v_{T_1},z_{T_1+T_2}(\underline{\theta})\rangle}|\sigma(\mathcal{F}^z,\mathcal{F}^u_{T_1+T_2})]\mathcal{F}^u_{T_1}] \tag{3.60}$$

instead of the (outer) conditional expectation in (3.30). Only apply the lower bound of Lemma 3.1 in case $z_{T_1+T_2} \not\equiv 0$. This way we obtain a result in the spirit of

(3.25). Here, we do not require $z_{T_1+T_2}$ or v_{T_1} to be nonzero a.s. as we do not have to multiply the (inner) conditional expectation from (3.60) with a front factor as in (3.30).

Note in particular, that the final statement is phrased in terms of conditioning on $\mathcal{F}^u_{T_1}$. Analogous reasoning to the proof of Lemma 3.6, using that $L_0(\varphi) \in (0,1)$, and thus $\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle = 0$ if $R_0(w_{T_1+T_2}) \le R_0(u_{T_1+T_2})$, completes the proof. $\square$

LEMMA 3.9. *Let $\varphi \in \mathcal{C}^+_{\text{tem}}$ with $L_0(\varphi) \in (0,1)$ and $T_1, T_2 > 0$ be arbitrarily fixed and $\theta_c < \underline{\theta} \le \theta_1 < \theta_2 \le \overline{\theta}$. For all $\delta'' > 0$, there exists $\eta_5 = \eta_5(\varphi, \delta'', T_1, T_2, \underline{\theta}, \overline{\theta}) > 0$ small enough such that*

$$\begin{aligned}\int_{\mathcal{C}^+_{\text{tem}}} &\mathbb{P}_f\big(\mathbb{P}_f\big(\{\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle \ge \eta_5\}|\mathcal{F}^u_{T_1}\big)\\ &\quad\ge \eta_5(\theta_2-\theta_1)\big)\big(\nu^{*,l}_T(\theta_1)\big)(df)\\ &\ge 1-\delta''\end{aligned} \tag{3.61}$$

for all $T > 1$.

PROOF. By Corollary 3.4, with $\nu^{*,l}_T(\theta_1)(df)$-measure of at least $1-2\delta'$,

$$\mathbb{P}_f\big(\mathbb{P}_f\big(R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})|\mathcal{F}^u_{T_1}\big) \ge \eta_1(\theta_2-\theta_1)\big) \ge 1-2\delta'. \tag{3.62}$$

Hence, with $\nu^{*,l}_T(\theta_1)(df)\mathbb{P}_f(d\omega)$-measure of at least $(1-2\delta')^2 \ge 1-4\delta'$,

$$\mathbb{P}_f\big(R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})|\mathcal{F}^u_{T_1}\big)(\omega) \ge \eta_1(\theta_2-\theta_1). \tag{3.63}$$

Also, by Lemma 3.8, with $\nu^{*,l}_T(\theta_1)(df)$-measure of at least $1-\delta'$,

$$\begin{aligned}&\mathbb{E}_f\big[\mathbb{1}\big(\mathbb{P}_f\big(\{\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle \ge \eta_4\}|\mathcal{F}^u_{T_1}\big)\\ &\quad\ge \big(1-e^{-2\eta_4}\big)^2\mathbb{P}_f\big(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1}\big)\big)\big] \ge 1-\delta'.\end{aligned} \tag{3.64}$$

Hence, with $\nu^{*,l}_T(\theta_1)(df)\mathbb{P}_f(d\omega)$-measure of at least $(1-\delta')^2 \ge 1-2\delta'$,

$$\begin{aligned}&\mathbb{P}_f\big(\{\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle \ge \eta_4\}|\mathcal{F}^u_{T_1}\big)(\omega)\\ &\quad\ge \big(1-e^{-2\eta_4}\big)^2\mathbb{P}_f\big(\{R_0(w_{T_1+T_2}) > R_0(u_{T_1+T_2})\}|\mathcal{F}^u_{T_1}\big)(\omega).\end{aligned} \tag{3.65}$$

Together with (3.63), this yields that with $\nu^{*,l}_T(\theta_1)(df)\mathbb{P}_f(d\omega)$-measure of at least $1-6\delta'$,

$$\begin{aligned}&\mathbb{P}_f\big(\{\langle v_{T_1+T_2}(\cdot + R_0(u_{T_1+T_2})), \varphi(\cdot)\rangle \ge \eta_4\}|\mathcal{F}^u_{T_1}\big)(\omega)\\ &\quad\ge \big(1-e^{-2\eta_4}\big)^2\eta_1(\theta_2-\theta_1).\end{aligned} \tag{3.66}$$

The claim now follows. $\square$

PROOF OF PROPOSITION 2.17. Let $\varphi \in \mathcal{C}^+_{\mathrm{tem}}$ with $L_0(\varphi) \in (0,1)$, $T_1, T_2, \xi > 0$ and θ_{m-1}, θ_m be arbitrarily fixed. For ease of notation, write θ_1, θ_2 instead of θ_{m-1}, θ_m and set $\epsilon = \theta_2 - \theta_1$. By Lemma 3.9 and the definition of $\nu_T^{*,l}(\theta_1)$ (cf. (A.8)), for all $\delta'' > 0$ there exists $\eta_6 > 0$ small enough and $T_0 > 0$ big enough, all constants only dependent on $\varphi, \delta'', T_1, T_2, \underline{\theta}, \overline{\theta}$, such that

$$\frac{1}{T}\int_0^T \mathbb{P}\big(\big\{\mathbb{P}_{u_s^{*,l}(\cdot + R_0(s))}\big(\big\langle v_{T_1+T_2}\big(\cdot + R_0(u_{T_1+T_2})\big), \varphi(\cdot)\big\rangle \geq \eta_6\big) \\ \geq \eta_6(\theta_2 - \theta_1)\big\}\big)\, ds \geq 1 - \delta'' \tag{3.67}$$

for all $T \geq T_0$ and $\theta_c < \underline{\theta} \leq \theta_1 < \theta_2 \leq \overline{\theta}$. Hence, using Fubini–Tonelli's theorem, there exists a set Ω' with $\mathbb{P}(\Omega') \geq 1 - \delta''$, such that for all $\omega \in \Omega'$,

$$\frac{1}{T}\int_0^T \mathbb{1}\big(\mathbb{P}_{u_s^{*,l}(\cdot + R_0(s))}\big(\big\langle v_{T_1+T_2}\big(\cdot + R_0(u_{T_1+T_2})\big), \varphi(\cdot)\big\rangle \geq \eta_6\big) \geq \eta_6\epsilon\big)\, ds \\ \geq 1 - \delta''. \tag{3.68}$$

For all $\omega \in \Omega'$, there exists

$$s_1 = s_1(\omega) \equiv \inf\big\{s \geq \xi : \mathbb{P}_{u_s^{*,l}(\cdot + R_0(s))}\big(\big\langle v_{T_1+T_2}\big(\cdot + R_0(u_{T_1+T_2})\big), \varphi(\cdot)\big\rangle \geq \eta_6\big) \\ \geq \eta_6\epsilon\big\} \tag{3.69}$$

satisfying $s_1 \leq T/2 - \xi - T_1 - T_2$. In case $\langle v_{T_1+T_2}^{(u_{s_1}^{*,l}(\cdot + R_0(s_1)))}(\cdot + R_0(u_{T_1+T_2}^{(u_{s_1}^{*,l}(\cdot + R_0(s_1)))})), \varphi(\cdot)\rangle \geq \eta_6$, set $S = s_1 + T_1 + T_2$ and call this a success. In case of no success, by (3.68), there exists

$$s_2 = s_2(\omega) \equiv \inf\big\{s \geq s_1 + T_1 + T_2 : \\ \mathbb{P}_{u_s^{*,l}(\cdot + R_0(s))}\big(\big\langle v_{T_1+T_2}\big(\cdot + R_0(u_{T_1+T_2})\big), \varphi(\cdot)\big\rangle \geq \eta_6\big) \geq \eta_6\epsilon\big\} \tag{3.70}$$

satisfying $s_2 \leq T/2 - \xi - T_1 - T_2$. Continue as above with s_2 instead of s_1. By choosing $C > 0$ small enough, we can repeat this procedure $\lceil CT \rceil$ times. If the above procedure fails, which can only happen if $\omega \notin \Omega'$ or $\omega \in \Omega'$ but there was no success in $\lceil CT \rceil$ trials, set $S = T/2 - \xi$.

Note that for $\eta, \delta'' \in (0,1)$ arbitrarily fixed, $\max_{x \in [0,\delta'']}(x + (1-x)\eta) = \delta'' + (1-\delta'')\eta$. As a result, using the strong Markov property of the family of laws $\mathbb{P}_f$, $f \in \mathcal{C}^+_{\mathrm{tem}}$, we get

$$\mathbb{P}\big(\nexists S : \xi \leq S \leq T/2 - \xi : \big\langle \Delta_S^{*,l}(\theta_1, \theta_2), \varphi\big\rangle \geq \eta_6\big) \\ \leq \delta'' + \big(1 - \delta''\big)(1 - \eta_6\epsilon)^{\lceil CT \rceil}. \tag{3.71}$$

Recall from (2.46) that $\epsilon = \delta/T$ to conclude that

$$\mathbb{P}\big(\exists S : \xi \leq S \leq T/2 - \xi : \big\langle \Delta_S^{*,l}(\theta_1, \theta_2), \varphi\big\rangle \geq \eta_6\big) \\ \geq \big(1 - \delta''\big)\big(1 - (1 - \eta_6\delta/T)^{\lceil CT \rceil}\big). \tag{3.72}$$

For $T \to \infty$, this bound approaches $(1-\delta'')(1-\exp(-C\eta_6\delta)) > 0$. This completes the proof of the claim. □

4. The speed of the right marker. Note the construction of traveling wave solutions from Theorem 3 or Remark A.7 of the Appendix. Let $\nu_T^{*,l} \in \mathcal{P}(\mathcal{C}_{\mathrm{tem}}^+)$ be given as in (A.8) and denote any arbitrary subsequential limit of the tight set $\{\nu_T^{*,l} : T \ge 1\}$ by $\nu = \nu^{*,l}$ in what follows. This limit yields a traveling wave solution to (1.1). By Proposition A.6, $\nu^{*,l}(\{f : R_0(f) = 0\}) = 1$ and $\mathbb{P}_{\nu^{*,l}}(u(t) \not\equiv 0) = 1$ for all $t \ge 0$. Denote with $\nu^{(u_0)}$ any subsequential limit that is obtained as in Remark A.7 for $u_0 \in \mathcal{H}$ with analogous properties.

Recall from [21], Proposition 4.1, that for $\theta > \theta_c$ and $(u_t^{(\nu)})_{t\ge 0}$ a traveling wave solution to (1.1),

$$R_0(u_t^{(\nu)})/t \to A^{(\nu)} = A^{(\nu)}(\omega) \in [0, 2\theta^{1/2}] \quad \text{almost surely as } t \to \infty \tag{4.1}$$

holds. This convergence also holds in $\mathcal{L}^1$ if we replace $R_0(u(t))$ by $0 \vee R_0(u(t))$ as we see below.

In this section, we show that the limiting speed of the dominating right marker $R_0(u_t^{*,l})$ and that of any traveling wave solution $\nu^{*,l}$ coincide. Moreover, the speed is deterministic, namely it equals $B = B(\theta)$ from Lemma 2.8. We extend this result to right front markers of solutions to (1.1) with initial conditions ψ satisfying $\psi \in \mathcal{H}^R$, where the convergence is now in probability and $\mathcal{L}^1$. For the right front marker of a corresponding traveling wave solution, we obtain almost sure convergence to $A^{(\nu^{(\psi)})} = B$.

LEMMA 4.1. *Let $\theta > \theta_c$. Then, for any $(u_t^{(\nu)})_{t\ge 0}$ a traveling wave solution to* (1.1),

$$(0 \vee R_0(u_t^{(\nu)}))/t \to A^{(\nu)} = A^{(\nu)}(\omega) \in [0, 2\theta^{1/2}] \quad \text{as } t \to \infty \text{ in } \mathcal{L}^1. \tag{4.2}$$

Moreover, $\mathbb{E}[A^{(\nu)}] \le B$.

PROOF. By (4.1), for all $N \in \mathbb{N}$, $((0 \vee R_0(u_t^{(\nu)})) \wedge (Nt))/t \to A^{(\nu)} \wedge N$ almost surely for $t \to \infty$. By dominated convergence,

$$((0 \vee R_0(u_t^{(\nu)})) \wedge (Nt))/t \to A^{(\nu)} \wedge N \quad \text{as } t \to \infty \text{ in } \mathcal{L}^1. \tag{4.3}$$

By (1.24) and Lemma 2.13, for $m > 0$ arbitrary,

$$\mathbb{P}(R_0(u_t^{(\nu)})/t \ge m) \le Cm^{-2} \tag{4.4}$$

uniformly in $t \ge 1$. In combination with (4.1), this gives $\mathbb{P}(A^{(\nu)} \ge 2m) \le Cm^{-2}$. Thus $A^{(\nu)} \in \mathcal{L}^1$. For $N \in \mathbb{N}$ arbitrary, we get

$$0 \le \mathbb{E}[0 \vee (R_0(u_t^{(\nu)})/t)] - \mathbb{E}[(0 \vee (R_0(u_t^{(\nu)})/t)) \wedge N] \le \int_N^\infty Cm^{-2}\, dm \tag{4.5}$$

and similarly, $|\mathbb{E}[A^{(\nu)} \wedge N] - \mathbb{E}[A^{(\nu)}]| \le 2C/N$. Hence,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\big[\big|0 \vee \big(R_0(u_t^{(\nu)})/t\big) - A^{(\nu)}\big|\big] \\
&\quad \le \lim_{t\to\infty} \mathbb{E}\big[\big|\big(0 \vee \big(R_0(u_t^{(\nu)})/t\big)\big) \wedge N - A^{(\nu)} \wedge N\big|\big] + 3C/N \qquad (4.6)\\
&\quad = 3C/N
\end{aligned}$$

and the first claim follows after taking $N \to \infty$.

Moreover, for all $N \in \mathbb{N}$, using once more (1.24) and the $\mathcal{L}^1$-convergence of the first claim,

$$\begin{aligned}
\mathbb{E}\big[A^{(\nu)}\big] &\le \mathbb{E}\big[A^{(\nu)} \wedge N\big] + 2C/N \\
&= \lim_{T\to\infty} \mathbb{E}\big[\big(0 \vee \big(R_0(u_T^{(\nu)})/T\big)\big) \wedge N\big] + 2C/N \qquad (4.7)\\
&\le \lim_{T\to\infty} \mathbb{E}\big[0 \vee R_0(u_T^{*,l})\big]/T + 2C/N \le B + 2C/N.
\end{aligned}$$

Take $N \to \infty$ to conclude that $\mathbb{E}[A^{(\nu)}] \le B$. □

Recall from Lemma 2.8 and Corollary 2.11 that

$$\lim_{T\to\infty} \mathbb{E}\big[R_0(u_T^{*,l})\big]/T = \inf_{T\ge 1} \mathbb{E}\big[R_0(u_T^{*,l})\big]/T \equiv B \in (0,\infty). \qquad (4.8)$$

Then

$$B = \lim_{T\to\infty} \mathbb{E}\big[0 \vee R_0(u_T^{*,l})\big]/T \qquad (4.9)$$

holds as well. Indeed, let $(u_t^{(\nu)})_{t\ge 0}$ be an arbitrary traveling wave solution with $R_0(\nu) = 0$ almost surely. By Corollary 2.3 and (1.24), (4.1), for $M \in \mathbb{N}$ arbitrary,

$$\begin{aligned}
&\lim_{T\to\infty} \mathbb{E}\big[0 \vee \big(-R_0(u_T^{*,l})\big)\big]/T \\
&\quad \le \lim_{T\to\infty} \mathbb{E}\big[\big(0 \vee \big(-R_0(u_T^{*,l})\big)\big) \wedge (MT)\big]/T + C_1 e^{-C_2 M} \\
&\quad \le \lim_{T\to\infty} \mathbb{E}\big[\big(0 \vee \big(-R_0(u_T^{(\nu)})\big)\big) \wedge (MT)\big]/T + C_1 e^{-C_2 M} \qquad (4.10)\\
&\quad = \mathbb{E}\big[0 \vee \big(-A^{(\nu)} \wedge M\big)\big] + C_1 e^{-C_2 M} \\
&\quad = C_1 e^{-C_2 M}.
\end{aligned}$$

Take $M \to \infty$ and the claim follows.

LEMMA 4.2. *Let $\theta > \theta_c$. With the notation from above, $\mathbb{E}[A^{(\nu^{*,l})}] = B$ holds true. Moreover, if $\psi \in \mathcal{H}^R$ satisfies $R_0(u_T^{(\psi)})/T \to B$ for $T \to \infty$ in $\mathcal{L}^1$, then $\mathbb{E}[A^{(\nu^{(\psi)})}] = B$ holds as well.*

REMARK 4.3. Note that we will show at the end of this section (Corollary 4.7) that every $\psi \in \mathcal{H}$ indeed satisfies the above assumption.

PROOF OF LEMMA 4.2. Let $\nu = \nu^{*,l}$ or $\nu = \nu^{(\psi)}$ with ψ as above. By Lemma 4.1, $\mathbb{E}[A^{(\nu)}] \le B$. It remains to show that $\mathbb{E}[A^{(\nu)}] \ge B$. In what follows, we provide a proof in case $\nu = \nu^{*,l}$. The proof in case $\nu = \nu^{(\psi)}$ is analogous except for the changes indicated below.

Fix $T \ge 1$ arbitrary. Review the definitions and comments in [14], (5.18)–(5.20). Note in particular that for fixed $N \in \mathbb{N}$ and $m_0 > 0$, $R^N_{m_0}$ is a continuous function on $\mathcal{C}^+_{\mathrm{tem}}$ with $|R^N_{m_0}(f)| \le N$ and $R^{m_0,N}(f) \le R^N_{m_0}(f) \le R_0(f)$ on $\{R_0(f) \ge -N\}$ and $R^N_{m_0}(f) = -N$ on $\{R_0(f) < -N\}$. Hence, for $m_0 = m_0(T)$, $N = N(T)$, by Lemma 4.1,

$$\mathbb{E}[A^{(\nu^{*,l})}] = \lim_{T\to\infty} \mathbb{E}_{\nu^{*,l}}[0 \vee R_0(u_T)]/T \ge \lim_{T\to\infty} \mathbb{E}_{\nu^{*,l}}[0 \vee R^N_{m_0}(u_T)]/T. \tag{4.11}$$

By the definition of tightness and the continuity of $f \mapsto \mathbb{P}_f$, (A.8) yields for $\nu^{*,l}_{T_n} \Rightarrow \nu^{*,l}$ $(n \to \infty)$,

$$\begin{aligned}
&\mathbb{E}_{\nu^{*,l}}[0 \vee R^N_{m_0}(u_T)] \\
&\quad = \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \mathbb{E}[0 \vee R^N_{m_0}(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s)))]\,ds \\
&\quad \ge \liminf_{n\to\infty} \biggl\{ \frac{1}{T_n} \int_0^{T_n} \mathbb{E}[R_0(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s)))]\,ds \\
&\qquad - \frac{1}{T_n} \int_0^{T_n} \mathbb{E}[0 \vee \{R_0(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s))) \\
&\qquad\quad - R^N_{m_0}(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s)))\}]\,ds \biggr\} \\
&\quad \ge \liminf_{n\to\infty} \biggl\{ \frac{1}{T_n} \int_0^{T_n} \mathbb{E}[R_0(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s)))]\,ds \biggr\} \\
&\qquad - \limsup_{n\to\infty} \biggl\{ \frac{1}{T_n} \int_0^{T_n} \mathbb{E}[0 \vee \{R_0(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s))) \\
&\qquad\quad - R^N_{m_0}(u^{*,l}_{s+T}(\cdot + R_0(u^{*,l}_s)))\}]\,ds \biggr\} \\
&\quad \equiv I_1(T) - E_1(T, m_0(T), N(T)) \\
&\quad = I_1(T) - E_1(T).
\end{aligned} \tag{4.12}$$

We obtain for I_1, using Corollaries 2.4–2.5, that

$$I_1(T) = \liminf_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \mathbb{E}[R_0(u^{*,l}_{s+T}) - R_0(u^{*,l}_s)]\,ds$$

$$
\begin{aligned}
&= \liminf_{n\to\infty} \frac{1}{T_n}\left(\int_{T_n}^{T_n+T} \mathbb{E}[R_0(u_s^{*,l})]\,ds - \int_0^T \mathbb{E}[R_0(u_s^{*,l})]\,ds\right) \qquad (4.13)\\
&= TB
\end{aligned}
$$

by (4.8) respectively the assumption $R_0(u_T^{(\psi)})/T \to B$ for $T \to \infty$ in $\mathcal{L}^1$ for $\psi \in \mathcal{H}^R$. It therefore remains to show that $\limsup_{T\to\infty} E_1(T)/T = 0$.

For $\epsilon > 0$ arbitrarily fixed, recall the definition of $R^{m_0,N}(f)$ from [14], (5.18). Also recall from above that $|R_{m_0}^N(f)| \le N$ and $R^{m_0,N}(f) \le R_{m_0}^N(f) \le R_0(f)$ on $\{R_0(f) \ge -N\}$ and $R_{m_0}^N(f) = -N$ on $\{R_0(f) < -N\}$ to obtain that

$$
\begin{aligned}
E_1(T) &\le \limsup_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \mathbb{E}[2R_0(u_{s+T}^{*,l}(\cdot + R_0(u_s^{*,l})))\mathbb{1}_{\{R_0(u_{s+T}^{*,l}(\cdot+R_0(u_s^{*,l})))>N\}}]\,ds\\
&\quad + \limsup_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \mathbb{E}[2N\mathbb{1}_{\{\langle u_{s+T}^{*,l}(\cdot+R_0(u_{s+T}^{*,l})),\mathbb{1}_{[-\epsilon T,\infty)}\rangle<m_0\}}]\,ds + \epsilon T \qquad (4.14)\\
&= 2\limsup_{n\to\infty} \mathbb{E}_{\nu_{T_n}^{*,l}}[R_0(u_T)\mathbb{1}_{\{R_0(u_T)>N\}}]\\
&\quad + 2N\limsup_{n\to\infty} \mathbb{P}_{\nu_{T_n}^{*,l}}(\langle u_T(\cdot + R_0(T)), \mathbb{1}_{[-\epsilon T,\infty)}\rangle < m_0) + \epsilon T.
\end{aligned}
$$

By (1.24) and Lemma 2.13, we choose $N > CT/\epsilon$ such that

$$
\begin{aligned}
\mathbb{E}_{\nu_{T_n}^{*,l}}[R_0(u_T)\mathbb{1}_{\{R_0(u_T)>N\}}] &\le \mathbb{E}[R_0(u_T^{*,l})\mathbb{1}_{\{R_0(u_T^{*,l})>N\}}]\\
&\le N^{-1}\mathbb{E}[(0\vee R_0(u_T^{*,l}))^2] \qquad (4.15)\\
&\le N^{-1}CT^2 < \epsilon T
\end{aligned}
$$

for all $n \in \mathbb{N}$. Finally, by Lemma A.4 with $a = \epsilon T/2$, we choose $b = b(T)$ and $\tilde{m} = m_0(T)$ small enough such that

$$
\mathbb{P}_{\nu_{T_n}^{*,l}}(\langle u_T(\cdot + R_0(T)), \mathbb{1}_{[-\epsilon T,\infty)}\rangle < m_0) \le \frac{\epsilon T}{2N(T)} \qquad (4.16)
$$

for all $n \in \mathbb{N}$. Thus, $E_1(T)/T \le 4\epsilon$ for all $T \ge 1$ and the claim follows after taking $\epsilon \downarrow 0^+$. □

PROPOSITION 4.4. *Let $\theta > \theta_c$. Then $A^{(\nu^{*,l})} \equiv \mathbb{E}[A^{(\nu^{*,l})}] = B$ almost surely and*

$$
R_0(u_T^{*,l})/T \to B \quad \text{almost surely as } T \to \infty. \qquad (4.17)
$$

In particular, $A^{(\nu)} \le B$ almost surely for all $A^{(\nu)}$ as in (4.1).

PROOF. The last claim follows immediately from (1.24).

Fix $T_0 > 0$. By (1.24) and as $\nu^{*,l}(\{f : R_0(f) = 0\}) = 1$ by Proposition A.6, there exists a coupling such that

$$R_0\big(u_{T_0+t}^{(\nu^{*,l})}\big) \le R_0\big(u_{T_0+t}^{*,l}\big) \quad \text{for all } t \ge 0 \text{ almost surely.} \tag{4.18}$$

By Corollaries 2.4–2.5, $\mathbb{E}[|R_0(u_{T_0}^{*,l})|] \le C(T_0)$, and thus, using Lemma 4.1 and (4.1),

$$\begin{aligned} A^{(\nu^{*,l})} &= \lim_{T\to\infty} 0 \vee \big(R_0\big(u_T^{(\nu^{*,l})}\big)/T\big) = \lim_{T\to\infty} R_0\big(u_T^{(\nu^{*,l})}\big)/T \\ &\le \liminf_{T\to\infty} R_0\big(u_T^{*,l}\big)/T \quad \text{a.s.,} \end{aligned} \tag{4.19}$$

where the left equality also holds in $\mathcal{L}^1$.

Note that by reasoning as in the proof of [21], Proposition 4.1(a), for $T > 0$ fixed, once we bound $\limsup_{n\to\infty} R_0(u_{T_0+nT}^{*,l})/nT$, the same bound holds for $\limsup_{t\to\infty} R_0(u_t^{*,l})/t$ almost surely. We therefore fix $T > 0$ and rewrite

$$\begin{aligned} &\frac{1}{nT} R_0\big(u_{T_0+nT}^{*,l}\big) - \frac{1}{nT} R_0\big(u_{T_0}^{*,l}\big) \\ &\quad = \frac{1}{nT} \sum_{i=1}^{n} \big(R_0\big(u_{T_0+iT}^{*,l}\big) - R_0\big(u_{T_0+(i-1)T}^{*,l}\big)\big) \\ &\quad = \frac{1}{nT} \sum_{i=1}^{n} R_0\big(u_T^{(u_{T_0+(i-1)T}^{*,l}(\cdot + R_0(u_{T_0+(i-1)T}^{*,l})))}\big). \end{aligned} \tag{4.20}$$

Fix $i \in \mathbb{N}$. By (1.24), there exists a coupling such that

$$R_0\big(u_T^{(u_{T_0+(i-1)T}^{*,l}(\cdot + R_0(u_{T_0+(i-1)T}^{*,l})))}\big) \le R_0\big(u_T^{*,l}(i)\big) \quad \text{almost surely,} \tag{4.21}$$

where $\mathcal{L}(u_T^{*,l}(i)) = \mathcal{L}(u_T^{*,l})$ for all $i \in \mathbb{N}$. By construction, the $\mathcal{L}(u_T^{*,l}(i))$, $i \in \mathbb{N}$ are independent. Indeed, we show this by induction. Let $u_{T_0+(i-1)T}^{*,l}$ be given. Then $\zeta_1 \ge u_{T_0+(i-1)T}^{*,l}(\cdot + R_0(u_{T_0+(i-1)T}^{*,l}))$ is chosen in the construction. Nevertheless, as $\zeta_N(x) \uparrow \infty$ for $x < 0$ and $\zeta_N(x) = 0$ for $x \ge 0$, the law of $u_T^{*,l}(i)$ conditional on $u_{T_0+(i-1)T}^{*,l}$ remains $\mathcal{L}(u_T^{*,l})$. Thus

$$\frac{1}{nT} R_0\big(u_{T_0+nT}^{*,l}\big) - \frac{1}{nT} R_0\big(u_{T_0}^{*,l}\big) \le \frac{1}{nT} \sum_{i=1}^{n} R_0\big(u_T^{*,l}(i)\big), \tag{4.22}$$

where $(R_0(u_T^{*,l}(i)))_{i\in\mathbb{N}}$ is an i.i.d. sequence of real valued random variables with $R_0(u_T^{*,l}(1)) \overset{\mathcal{D}}{=} R_0(u_T^{*,l})$. By Corollaries 2.4–2.5, $R_0(u_T^{*,l}) \in \mathcal{L}^1$.

By the ergodic theorem (cf. for instance Klenke [13], Theorems 20.14, 20.16 and Example 20.12),

$$\frac{1}{nT}\sum_{i=1}^{n} R_0(u_T^{*,l}(i)) \to \mathbb{E}[R_0(u_T^{*,l})]/T \tag{4.23}$$

almost surely and in $\mathcal{L}^1$ for $n \to \infty$.

As a result,

$$\limsup_{n\to\infty} \frac{R_0(u_{T_0+nT}^{*,l})}{nT} \le \limsup_{n\to\infty} \frac{R_0(u_{T_0}^{*,l})}{nT} + \frac{\mathbb{E}[R_0(u_T^{*,l})]}{T} = \frac{\mathbb{E}[R_0(u_T^{*,l})]}{T} \tag{4.24}$$

for all $T > 0$. Take $T \to \infty$ to conclude that

$$\begin{aligned} A^{(\nu^{*,l})} &\le \liminf_{T\to\infty} \frac{R_0(u_T^{*,l})}{T} \le \limsup_{T\to\infty} \frac{R_0(u_T^{*,l})}{T} \\ &\le \limsup_{T\to\infty} \frac{\mathbb{E}[R_0(u_T^{*,l})]}{T} = B = \mathbb{E}[A^{(\nu^{*,l})}] \end{aligned} \tag{4.25}$$

and, therefore, $\lim_{T\to\infty} R_0(u_T^{*,l})/T = B$ almost surely. □

COROLLARY 4.5. *Let $\theta > \theta_c$. Then*

$$R_0(u_T^{(\nu^{*,l})})/T \to B \quad \text{almost surely as } T \to \infty \tag{4.26}$$

and

$$R_0(u_T^{*,l})/T \to B \quad \text{in } \mathcal{L}^1 \text{ as } T \to \infty. \tag{4.27}$$

PROOF. The first claim follows from (4.1) and Proposition 4.4.

By Proposition 4.4, we have $(R_0(u_T^{*,l})/T \wedge N) \vee (-N) \to B$ for $T \to \infty$ almost surely for all $N \in \mathbb{N}$, $N > B$ fixed. As these random variables are bounded, dominated convergence implies convergence in $\mathcal{L}^1$. We conclude that for all $N \in \mathbb{N}$, $N > B$,

$$\begin{aligned} &\limsup_{T\to\infty} \mathbb{E}[|(R_0(u_T^{*,l})/T) - B|] \\ &\quad \le 2 \limsup_{T\to\infty} \mathbb{E}[|(R_0(u_T^{*,l})/T)| \mathbb{1}_{\{|R_0(u_T^{*,l})/T|>N\}}]. \end{aligned} \tag{4.28}$$

The second claim now follows from the bounds on the positive part from Lemma 2.13 and on the negative part from (2.15) for $N \to \infty$. □

Finally, we consider initial conditions $\psi \in \mathcal{H}$ with $\mathcal{H}$ as in (1.10).

LEMMA 4.6. *For initial conditions $\psi \in \mathcal{H}^R$, $R_0(u_T^{(\psi)})/T \stackrel{\mathcal{D}}{\to} B$ as $T \to \infty$.*

PROOF. Without loss of generality, we assume that $\psi(x) = \epsilon H_0(x - x_0)$ for some $x_0 \in \mathbb{R}, \epsilon > 0$. Indeed, by definition of $\mathcal{H}^R$, for every $\psi \in \mathcal{H}$ there exist $x_0 \in \mathbb{R}, \epsilon > 0$ such that $\psi \geq \epsilon H_0(\cdot - x_0)$ and $R_0(\psi) \in \mathbb{R}$. Let us reason as in [14], Remark 2.8(ii) *(left-upper measure)* to construct a coupling such that for $T_0 > 0$ arbitrarily fixed, $u^{(\epsilon H_0(\cdot - x_0))}(t, x) \leq u^{(\psi)}(t, x) \leq u^{*,l}(t, x - R_0(\psi))$ for all $t \geq T_0, x \in \mathbb{R}$ almost surely. Then

$$R_0\big(u_t^{(\epsilon H_0(\cdot - x_0))}\big)/t \leq R_0\big(u_t^{(\psi)}\big)/t \leq \big(R_0\big(u_t^{*,l}\big) + R_0(\psi)\big)/t \quad \text{for all } t \geq T_0 \text{ almost surely} \tag{4.29}$$

and by Proposition 4.4,

$$\lim_{t \to \infty} \big(R_0\big(u_t^{*,l}\big) + R_0(\psi)\big)/t = B \quad \text{almost surely.} \tag{4.30}$$

Hence, $R_0(u_t^{(\epsilon H_0(\cdot - x_0))})/t \overset{\mathcal{D}}{\to} B$ implies $R_0(u_t^{(\psi)})/t \overset{\mathcal{D}}{\to} B$. By the shift invariance of the dynamics, assume further that $x_0 = 1$.

Let $c \in \mathbb{R}$ be arbitrary. For $c > B$, $\lim_{T \to \infty} \mathbb{P}(R_0(u_T^{(\psi)}) \geq cT) = 0$ follows from (4.29)–(4.30). Note that $\psi(x) = \epsilon H_0(x - 1) \geq \epsilon \mathbb{1}_{(-\infty,0]}(x)$ for all $x \in \mathbb{R}$. By (1.22), symmetry and by the shift invariance of the dynamics, for all $T > 0$,

$$\begin{aligned} \mathbb{P}\big(R_0\big(u_T^{(\psi)}\big) > cT\big) &= 1 - \mathbb{E}\big[e^{-2\langle \psi, u_T^{*,r}(\cdot - cT)\rangle}\big] \\ &\geq 1 - \mathbb{E}\big[e^{-2\epsilon\langle \mathbb{1}_{(-\infty,0]}, u_T^{*,r}(\cdot - cT)\rangle}\big] \\ &\geq 1 - \mathbb{P}\big(\big\langle \mathbb{1}_{(-\infty,0]}, u_T^{*,r}(\cdot - cT)\big\rangle < N\big) - e^{-2\epsilon N} \end{aligned} \tag{4.31}$$

for all $N \in \mathbb{N}$. Suppose $c < B$. Let $\delta > 0$ and choose N big enough such that $e^{-2\epsilon N} < \delta$. As we will show below, for N fixed,

$$\lim_{T \to \infty} \mathbb{P}\big(\big\langle \mathbb{1}_{(-\infty,0]}, u_T^{*,r}(\cdot - cT)\big\rangle < N\big) = 0 \tag{4.32}$$

and, therefore, $\mathbb{P}(R_0(u_T^{(\psi)}) > cT) \geq 1 - 2\delta$ for all T big enough. Then $\lim_{T \to \infty} \mathbb{P}(R_0(u_T^{(\psi)}) \geq cT) = \mathbb{1}_{(-\infty,B)}(c)$ for $c \neq B$ arbitrary follows.

It thus remains to show (4.32) for $0 < c < B$ and N arbitrarily fixed. Let $\Delta = (B - c)/2, 0 < \Delta < B$. By symmetry, $L_0(u_T^{*,r})/T \to -B$ almost surely. A *coupling with two independent processes* (cf. Remark A.10) at time T yields

$$\begin{aligned} &\mathbb{P}\big(L_0\big(u_{T+1}^{*,r}\big) \geq (-B + \Delta)T\big) \\ &\geq \mathbb{E}\big[\mathbb{1}_{\{\langle \mathbb{1}_{(-\infty,-cT]}, u_T^{*,r}(\cdot)\rangle < N\}} \mathbb{P}_{\mathbb{1}_{(-\infty,-cT]} u_T^{*,r}}(\tau \leq 1) \\ &\qquad \times \mathbb{P}_{\mathbb{1}_{[-cT,\infty)} u_T^{*,r}}\big(L_0(u_1) \geq (-B + \Delta)T\big)\big]. \end{aligned} \tag{4.33}$$

By [14], (2.11), on $\{\langle \mathbb{1}_{(-\infty,-cT]}, u_T^{*,r}(\cdot)\rangle < N\}$, $\mathbb{P}_{\mathbb{1}_{(-\infty,-cT]}u_T^{*,r}}(\tau \leq 1) \geq \exp(\frac{-2\theta N}{1-e^{-\theta}})$. Note that $-B+\Delta = -c-\Delta$ to further get by symmetry and domination, $\mathbb{P}_{\mathbb{1}_{[-cT,\infty)}u_T^{*,r}}(L_0(u_1) \geq (-B+\Delta)T) \geq \mathbb{P}(R_0(u_1^{*,l}) \leq \Delta T)$. Hence,

$$\begin{aligned}
&\mathbb{P}\big(L_0\big(u_{T+1}^{*,r}\big) \geq (-B+\Delta)T\big) \\
&\quad \geq \exp\left(\frac{-2\theta N}{1-e^{-\theta}}\right)\mathbb{P}\big(\big\langle \mathbb{1}_{(-\infty,-cT]}, u_T^{*,r}(\cdot)\big\rangle < N\big)\mathbb{P}\big(R_0\big(u_1^{*,r}\big) \leq \Delta T\big).
\end{aligned} \tag{4.34}$$

As $\lim_{T\to\infty}\mathbb{P}(L_0(u_{T+1}^{*,r}) \geq (-B+\Delta)T) = 0$ and $\lim_{T\to\infty}\mathbb{P}(R_0(u_1^{*,r}) \leq \Delta T) = 1$ by [14], Lemma 4.6, and the Markov inequality, (4.32) follows. □

COROLLARY 4.7. *For initial conditions $\psi \in \mathcal{H}^R$, $R_0(u_T^{(\psi)})/T \to B$ for $T \to \infty$ in probability and in $\mathcal{L}^1$.*

PROOF. As the limit is deterministic, the convergence in distribution (cf. Lemma 4.6) implies convergence in probability (cf. Grimmett and Stirzaker [8], Theorem 7.2.(4)(a)). Use (2.8) for the negative part and Lemma 2.13 and domination for the positive part of $R_0(u_T^{(\psi)})$ to see that the family $\{R_0(u_T^{(\psi)})/T, T \geq 1\}$ is bounded in $\mathcal{L}^2$, and thus uniformly integrable (cf. [13], Corollary 6.21). By [13], Theorem 6.25 (and Definition 6.2), the convergence in $\mathcal{L}^1$ now follows from the convergence in probability of $R_0(u_T^{(\psi)})/T$ in combination with the uniform integrability of this sequence. □

COROLLARY 4.8. *For initial conditions $\psi \in \mathcal{H}^R$,*

$$R_0\big(u_T^{(\nu^{(\psi)})}\big)/T \to B \quad \textit{almost surely as } T \to \infty \tag{4.35}$$

and $(0 \vee R_0(u_T^{(\nu^{(\psi)})}))/T \to B$ *in* $\mathcal{L}^1$.

PROOF. By (4.1), Lemma 4.2 and Corollary 4.7,

$$R_0\big(u_T^{(\nu^{(\psi)})}\big)/T \to A^{(\nu^{(\psi)})} \quad \text{almost surely as } T \to \infty \tag{4.36}$$

with $\mathbb{E}[A^{(\nu^{(\psi)})}] = B$. By Proposition 4.4, $A^{(\nu^{(\psi)})} \leq B$ almost surely. Hence, $A^{(\nu^{(\psi)})} = B$ almost surely and the first claim follows. The second claim follows by Lemma 4.1. □

4.1. *Proof of Theorem* 1. The first claim and (1.12) follow from Proposition 4.4 and Corollary 4.5. Lemma 4.1 yields the $\mathcal{L}^1$-convergence of the positive part of the right-hand side of (1.12). The third claim follows from Corollary 4.7. The fourth and last claim follow from Corollary 4.8. This concludes the proof.

APPENDIX

A.1. Duality. A *self-duality relation* in the form (1.15) holds for solutions to (1.1). For solutions with additional annihilation due to competition with a deterministic process β (see (A.1) below), a *duality relation* is obtained analogously. Such solutions appear for instance in the context of *monotonicity-couplings*; see (A.20). For existence and uniqueness of solutions to (A.1) or (A.2), see Theorem 2.

COROLLARY A.1. *Let $\theta > 0, T > 0, \beta \in \mathcal{C}([0,T], \mathcal{C}^+_{\text{tem}})$ arbitrarily fixed. Let v, z be independent solutions to*

$$\frac{\partial v}{\partial t} = \Delta v + (\theta - v - \beta)v + \sqrt{v}\dot{W}_1, \qquad v(0) = v_0, \tag{A.1}$$

respectively,

$$\frac{\partial z}{\partial t} = \Delta z + (\theta - z - \beta_{T-\cdot})z + \sqrt{z}\dot{W}_2, \qquad z(0) = z_0 \tag{A.2}$$

for $0 \le t \le T$ with $v_0, z_0 \in \mathcal{C}^+_{\text{tem}}$ and W_1, W_2 independent white noises. Then we have for $0 \le s \le T$,

$$\mathbb{E}[e^{-2\langle v(T), z(0)\rangle}] = \mathbb{E}[e^{-2\langle v(s), z(T-s)\rangle}] = \mathbb{E}[e^{-2\langle v(0), z(T)\rangle}]. \tag{A.3}$$

PROOF. Let us reason as in [9], Section 1.2. Let

$$H(f,g) = 2e^{-2\langle f,g\rangle}(\langle f^2, g\rangle + \langle f, g^2\rangle - \langle f, \Delta g\rangle - \theta\langle f, g\rangle). \tag{A.4}$$

Integration by parts yields $H(f,g) = H(g,f)$. The additional factor of 2 in the exponent results from the use of different scaling constants in the original SPDEs. Then

$$e^{-2\langle v_t, g\rangle} - \int_0^t H(v_s, g)\,ds - 2\int_0^t e^{-2\langle v_s, g\rangle}\langle \beta_s v_s, g\rangle\,ds \tag{A.5}$$

is a local martingale as well as

$$e^{-2\langle z_t, f\rangle} - \int_0^t H(z_s, f)\,ds - 2\int_0^t e^{-2\langle z_s, f\rangle}\langle \beta_{T-s} z_s, f\rangle\,ds. \tag{A.6}$$

As

$$\begin{aligned} &\frac{d}{ds}\mathbb{E}[e^{-2\langle v_s, z_{T-s}\rangle}] \\ &\quad = \mathbb{E}[(H(v_s, z_{T-s}) + 2e^{-2\langle v_s, z_{T-s}\rangle}\langle \beta_s v_s, z_{T-s}\rangle) \\ &\qquad - (H(z_{T-s}, v_s) + 2e^{-2\langle z_{T-s}, v_s\rangle}\langle \beta_s z_{T-s}, v_s\rangle)] = 0, \end{aligned} \tag{A.7}$$

the duality relation follows. $\square$

A.2. Traveling waves for the right-upper invariant measure. We extend the construction of traveling wave solutions from solutions with compactly supported initial conditions (cf. [14]) respectively Heavyside initial data (cf. [21]) to the right-upper invariant measure case. As in [14], the right marker is used to center the waves. Recall the set $\mathcal{H}^R$ from (1.10). The constructions extend to initial conditions $u_0 \in \mathcal{H}^R$.

Let $\theta_c < \underline{\theta} \le \theta \le \overline{\theta}$ and $\nu_T^{*,l} = \nu_T^{*,l}(\theta) \in \mathcal{P}(\mathcal{C}_{\mathrm{tem}}^+)$ be given by

$$\nu_T^{*,l}(A) \equiv T^{-1} \int_0^T \mathbb{P}\bigl(u_s^{*,l}(\cdot + R_0(s)) \in A\bigr)\, ds, \tag{A.8}$$

with $\mathcal{L}(u_s^{*,l}) = \upsilon_s, s > 0$ as in (1.22)–(1.23). Note that by Corollaries 2.4–2.5, $R_0(u_s^{*,l})$ is almost surely finite, and thus $\nu_T^{*,l}$ is well defined. Then the analogues of the following results of [14] hold, where constants only depend on θ through $\underline{\theta}$ and $\overline{\theta}$. Here, it is important to note that the tightness-result of Lemma A.3 from below is uniform in $\underline{\theta} \le \theta \le \overline{\theta}$ as well. Note that in the proof of Lemma A.2 we use Corollaries 2.4–2.5 in place of [14], Lemma 4.8. Also note that the constants in [21], Lemmas 3.2–3.4, hold uniform in $\underline{\theta} \le \theta \le \overline{\theta}$, which is easily be deduced using that $p_t^\theta(x) = e^{\theta t} p_t(x)$. Finally, note that the restriction to $N \in \mathbb{N}$ in [21], Lemma 3.7, and [14], Chapter 5, was only due to the fact that the sequence $\{\nu_T : T \in \mathbb{N}\}$ was under consideration rather than the set $\{\nu_T : T \ge 1\}$. As we integrate from 1 to T in the proof of [21], Lemma 3.7, and [14], (5.2), part of the statements below are only valid for $T > 1$.

LEMMA A.2 (Analogue to [14], Lemma 4.9). *If $t > 0$, then there exists $C(\underline{\theta}, \overline{\theta}, t)$ such that for all $a > 0$, $0 < s \le t$ and $T \ge 1$,*

$$\mathbb{P}_{\nu_T^{*,l}}\bigl(|R_0(s)| \ge a\bigr) \le C(\underline{\theta}, \overline{\theta}, t)/a. \tag{A.9}$$

In particular, for $0 < t \le 1$,

$$\mathbb{P}_{\nu_T^{*,l}}\bigl(|R_0(s)| \ge a\bigr) \le C(\underline{\theta}, \overline{\theta}) t^{1/4}/a \tag{A.10}$$

holds.

LEMMA A.3 (Extension of [14], Lemma 5.1). *Let $\theta_c < \underline{\theta} \le \theta \le \overline{\theta}$ be arbitrarily fixed. Then the set $\{\nu_T^{*,l}(\theta) : T \ge 1\}$ is tight. In particular, for every $\epsilon > 0$ there exists a compact set $K_\epsilon = K_\epsilon(\underline{\theta}, \overline{\theta}) \subset \mathcal{C}_{\mathrm{tem}}^+$ such that*

$$\inf_{\underline{\theta} \le \theta \le \overline{\theta}} \bigl(\nu_T^{*,l}(\theta)\bigr)(K_\epsilon) \ge 1 - \epsilon \quad \textit{for all } T > 1. \tag{A.11}$$

PROOF. Let $\lambda > 0$ arbitrary. Then there exist $C < \infty, \gamma, \delta > 0, \mu < \lambda$ and $A > 0$, such that

$$\begin{aligned} &\inf_{\underline{\theta} \le \theta \le \overline{\theta}} \bigl(\nu_T^{*,l}(\theta)\bigr)\bigl(K(C, \delta, \gamma, \mu) \cap \{f \in \mathcal{C}_{\mathrm{tem}}^+ : \langle f, \phi_1 \rangle \le A\}\bigr) \ge 1 - \epsilon \\ &\qquad \text{for all } T > 1, \end{aligned} \tag{A.12}$$

where $\phi_1(x) \equiv \exp(-|x|)$ and

$$K(C,\delta,\gamma,\mu) \equiv \{f \in \mathcal{C}_{\mathrm{tem}}^+ : |f(x) - f(x')| \le C|x - x'|^{\gamma} e^{\mu|x|} \text{ for all } |x - x'| \le \delta\}. \tag{A.13}$$

Indeed, a look at the proof of [14], Lemma 5.1, shows that the sets under consideration, namely $K(C,\delta,\gamma,\overline{\mu})$ and $\{f : \langle f, \phi_1\rangle \le N\}$ are independent of θ. The bounds are derived from previous statements, where constants only depend on θ through $\underline{\theta}$ and $\overline{\theta}$.

Recall that $K \subset \mathcal{C}_{\mathrm{tem}}^+$ is (relatively) compact if and only if it is (relatively) compact in $\mathcal{C}_{\lambda}^+$ for all $\lambda > 0$ and that $K(C,\delta,\gamma,\mu) \cap \{f \in \mathcal{C}_{\mathrm{tem}}^+ : \langle f, \phi_1\rangle \le A\} \equiv K(\epsilon,\lambda)$ satisfying (A.12) is compact in $\mathcal{C}_{\lambda}^+$ (cf. [21], above (1.2)). Now set

$$K_\epsilon = \overline{\bigcap_{n\in\mathbb{N}} K(\epsilon 2^{-n}, 1/n)} \tag{A.14}$$

to conclude the proof of the claim. □

LEMMA A.4 (Analogue to [14], Lemma 5.2). *Let $t \ge 0$ and $a, \tilde{m} > 0$, $0 < b \le 1$ be arbitrarily fixed. Then*

$$\begin{aligned} &\mathbb{P}_{\nu_T^{*,l}}(\langle u_t(\cdot + R_0(u_t)), \mathbb{1}_{(-2a,\infty)}(\cdot)\rangle < \tilde{m}) \\ &\quad \le \left(\left(1 - \frac{C_1(\underline{\theta},\overline{\theta})b^{1/4}}{a}\right) \vee 0\right)^{-1} \\ &\qquad \times \left\{\frac{T+t}{T}\frac{C_2(\underline{\theta},\overline{\theta})b^{1/4}}{a} + \left(1 - e^{-2\overline{\theta}\frac{\tilde{m}}{1-e^{-\underline{\theta}b}}}\right)\right\} \end{aligned} \tag{A.15}$$

for all $T \ge 1$.

Recall from (3.32) for $d_0, m_0 > 0$ the definition of

$$M(d_0,m_0) = \{f \in \mathcal{C}_{\mathrm{tem}}^+ : \text{there exist } -1/2 \le l_0 < r_0 \le 0, |r_0 - l_0| = d_0 \text{ such that } f \ge m_0 \mathbb{1}_{[l_0,r_0]}\}. \tag{A.16}$$

COROLLARY A.5. *Let $\epsilon > 0$ arbitrary. Then there exist $d_0 = d_0(\epsilon), m_0 = m_0(\epsilon) > 0$ such that*

$$\inf_{\underline{\theta}\le\theta\le\overline{\theta}} (\nu_T^{*,l}(\theta))(M(d_0,m_0)) \ge 1 - \epsilon \tag{A.17}$$

for all $T > 1$.

PROOF. In Lemma A.4, choose $t = 0$ and $a = 1/4$. Then choose b small enough and then $\tilde{m}$ small enough to obtain

$$\inf_{\underline{\theta}\le\theta\le\overline{\theta}} (\nu_T^{*,l}(\theta))(\{f : \langle f, \mathbb{1}_{(-1/2,0]}(\cdot)\rangle \ge \tilde{m}\}) \ge 1 - \epsilon/2 \quad \text{for all } T \ge 1. \tag{A.18}$$

By (A.12)–(A.13) for $\lambda = 1$, there exist $C < \infty, \gamma, \delta > 0, \mu < 1$ and $A > 0$, such that

$$\inf_{\underline{\theta} \le \theta \le \bar{\theta}} (\nu_T^{*,l}(\theta))(K(C,\delta,\gamma,\mu) \cap \{f \in \mathcal{C}_{\text{tem}}^+ : \langle f, \phi_1 \rangle \le A\}) \ge 1 - \epsilon/2 \quad \text{for all } T > 1. \tag{A.19}$$

For deterministic $f_0 \in \mathcal{C}_{\text{tem}}^+$, note that if $f_0 \in \{\mathcal{C}_{\text{tem}}^+ : \langle f, \mathbb{1}_{[-1/2,0]} \rangle \ge \tilde{m}\} \cap \{f \in \mathcal{C}_{\text{tem}}^+ : |f(x) - f(x')| \le C|x - x'|^\gamma \text{ for all } x, x' \in [-1, 0], |x - x'| \le \delta\}$, then there exists $x_0 \in [-1/2, 0]$ such that $f(x_0) \ge 2\tilde{m}$. Now use the Hölder-γ-continuity of f_0 around x_0 to obtain the existence of $d_0 > 0$ such that there exist $l_0 \le x_0 \le r_0, l_0, r_0 \in [-1/2, 0], |r_0 - l_0| = d_0$ and $0 < m_0 < 2\tilde{m}$ such that $f(x) \ge m_0$ for all $l_0 \le x \le r_0$. $\square$

PROPOSITION A.6 (Analogue to [14], Proposition 1.7). *Let $\nu_{T_n}^{*,l}$ be a subsequence that converges to $\nu^{*,l}$. Then $\nu^{*,l}(\{f : R_0(f) = 0\}) = 1$ and $\mathbb{P}_{\nu^{*,l}}(u(t) \not\equiv 0) = 1$ for all $t \ge 0$.*

THEOREM 3 (Analogue to [14], Theorem 1.6). *Every subsequential limit of the tight set $\{\nu_T^{*,l} : T \ge 1\}$ yields a traveling wave solution to equation* (1.1).

REMARK A.7. Recall the set $\mathcal{H}^R$ from (1.10). The constructions and statements from above extend to initial conditions $u_0 \in \mathcal{H}^R$. Here, we use Lemma 2.2 instead of (2.14).

A.3. Coupling techniques. In what follows, we shortly introduce the main coupling techniques and ideas that are used in this article. We start with the *monotonicity-coupling* from [14], Remark 2.1(i).

REMARK A.8 (Monotonicity-coupling). Let $0 < \theta$ and $u_i \in \mathcal{C}_{\text{tem}}^+, i = 1, 2$ with $u_1(x) \le u_2(x)$ for all $x \in \mathbb{R}$. Then there exists a coupling of solutions $u^{(i)}, i = 1, 2$ to (1.1) with initial conditions $u_i, i = 1, 2$ such that $u^{(1)}(t, x) \le u^{(2)}(t, x)$ for all $t \ge 0, x \in \mathbb{R}$ almost surely. For intuition purposes, compare the construction of [17], Lemma 2.1.7. The main idea is to write

$$\begin{aligned} \frac{\partial u^{(1)}}{\partial t} &= \Delta u^{(1)} + (\theta - u^{(1)})u^{(1)} + \sqrt{u^{(1)}}\dot{W}_1, \qquad u^{(1)}(0) = u_1, \\ \frac{\partial v}{\partial t} &= \Delta v + (\theta - v - 2u^{(1)})v + \sqrt{v}\dot{W}_2, \qquad v(0) = u_2 - u_1, \end{aligned} \tag{A.20}$$

where W_1, W_2 are independent white noises and $u^{(2)} \equiv u^{(1)} + v$ with $v(t, x) \ge 0$ for all $t \ge 0, x \in \mathbb{R}$ almost surely. v is constructed (conditional on $u^{(1)}$) as a process with annihilation due to competition with $u^{(1)}$. Now recall [14], (1.8), to note that

$$
\begin{aligned}
&\left\langle \iint_0^{\cdot} \left|u^{(1)}(s,x)\right|^{1/2}\phi(x)\,dW_1(x,s) + \left|v(s,x)\right|^{1/2}\phi(x)\,dW_2(x,s)\right\rangle_t \\
&\quad = \iint_0^t \left(u^{(1)}(s,x)+v(s,x)\right)\phi^2(x)\,dx\,ds \\
&\quad = \left\langle \iint_0^{\cdot} \left|u^{(1)}(s,x)+v(s,x)\right|^{1/2}\phi(x)\,dW(x,s)\right\rangle_t \\
&\quad = \left\langle \iint_0^{\cdot} \left|u^{(2)}(s,x)\right|^{1/2}\phi(x)\,dW(x,s)\right\rangle_t
\end{aligned}
\tag{A.21}
$$

for W a white noise appropriately chosen.

In this article, we call a *θ-coupling* a coupling in the spirit of [17], Lemma 2.1.6. To be more precise, use the techniques of [14], (2.2)–(2.4), to show the following.

REMARK A.9 (θ-coupling). Let $0 < \theta_1 < \theta_2$. Let $u_0 \in \mathcal{C}_{\text{tem}}^+$. Then there exists a coupling of solutions $u^{(i)}, i = 1, 2$ to (1.1) with common initial condition u_0 but different parameters θ_1 respectively θ_2 such that $u^{(1)}(t,x) \le u^{(2)}(t,x)$ for all $t \ge 0, x \in \mathbb{R}$ almost surely. The main idea is to write

$$
\begin{aligned}
\frac{\partial u^{(1)}}{\partial t} &= \Delta u^{(1)} + \left(\theta_1 - u^{(1)}\right)u^{(1)} + \sqrt{u^{(1)}}\,\dot{W}_1, \qquad u^{(1)}(0) = u_0, \\
\frac{\partial v}{\partial t} &= \Delta v + (\theta_2 - \theta_1)u^{(1)} + \left(\theta_2 - v - 2u^{(1)}\right)v + \sqrt{v}\,\dot{W}_2, \qquad v(0) = 0,
\end{aligned}
\tag{A.22}
$$

where W_1, W_2 are independent white noises and $u^{(2)} \equiv u^{(1)} + v$ with $v(t,x) \ge 0$ for all $t \ge 0, x \in \mathbb{R}$ almost surely. v is constructed (conditional on $u^{(1)}$) as a process with annihilation due to competition with $u^{(1)}$ and an immigration-term $(\theta_2 - \theta_1)u^{(1)}$.

In what follows, we call a *coupling with two independent processes* a coupling in the spirit of [17], Lemma 2.1.7. To be more precise, use the techniques of [14], (2.2)–(2.4), to show the following.

REMARK A.10 (Coupling with two independent processes). Let $0 < \theta$. Let $u_1, u_2 \in \mathcal{C}_{\text{tem}}^+$ and $u_0 \equiv u_1 + u_2$. Then there exists a coupling of solutions $u^{(i)}, i = 0, 1, 2$ to (1.1) with initial conditions $u_i, i = 0, 1, 2$ such that $u^{(1)}$ and $u^{(2)}$ are independent and $u^{(0)}(t,x) \le u^{(1)}(t,x) + u^{(2)}(t,x)$ for all $t \ge 0, x \in \mathbb{R}$ almost surely. The main idea is to write

$$
\frac{\partial u^{(1)}}{\partial t} = \Delta u^{(1)} + \left(\theta - u^{(1)}\right)u^{(1)} + \sqrt{u^{(1)}}\,\dot{W}_1, \qquad u^{(1)}(0) = u_1,
$$

$$\text{(A.23)}\qquad \frac{\partial v}{\partial t} = \Delta v + \big(\theta - v - 2u^{(1)}\big)v + \sqrt{v}\dot{W}_2, \qquad v(0) = u_2,$$
$$\frac{\partial u^{(2)}}{\partial t} = \Delta u^{(2)} + \big(\theta - u^{(2)}\big)u^{(2)} + \sqrt{u^{(2)}}\dot{W}_2, \qquad u^{(2)}(0) = u_2,$$

where W_1, W_2 are independent white noises and $u^{(0)} \equiv u^{(1)} + v$ with $v(t,x) \leq u^{(2)}(t,x)$ for all $t \geq 0, x \in \mathbb{R}$ almost surely. v is constructed (conditional on $u^{(1)}$) as a process with annihilation due to competition with $u^{(1)}$ contrary to $u^{(2)}$, where no annihilation takes place. The independence of $u^{(1)}$ and $u^{(2)}$ follows from the independence of the white noises W_1, W_2.

An *immigration-coupling* is constructed similar to a θ*-coupling* (cf. Remark A.9), where the immigration-term only depends on an outside source.

REMARK A.11 (Immigration-coupling). Let $\alpha_1, \alpha_2 - \alpha_1 \in \mathcal{C}([0,\infty), \mathcal{C}^+_{\text{tem}})$. Let $u_0 \in \mathcal{C}^+_{\text{tem}}$. Then there exists a coupling of solutions $u^{(i)}, i = 1, 2$ solving

$$\text{(A.24)}\qquad \begin{aligned} &\frac{\partial u^{(i)}}{\partial t} = \Delta u^{(i)} + \alpha_i + \big(\theta - u^{(i)}\big)u^{(i)} + \sqrt{u^{(i)}}\dot{W}_i, \\ &u^{(i)}(0) = u_0, \quad i = 1, 2 \end{aligned}$$

with W_1, W_2 two independent white noises, such that $u^{(1)}(t,x) \leq u^{(2)}(t,x)$ for all $t \geq 0, x \in \mathbb{R}$ almost surely. The main idea is to write $u^{(2)} \equiv u^{(1)} + v$ with $v \geq 0$ satisfying

$$\text{(A.25)}\qquad \frac{\partial v}{\partial t} = \Delta v + (\alpha_2 - \alpha_1) + \big(\theta - v - 2u^{(1)}\big)v + \sqrt{v}\dot{W}_2, \qquad v(0) = 0.$$

v is constructed (conditional on $u^{(1)}$) as a process with annihilation due to competition with $u^{(1)}$ and an immigration-term $\alpha_2 - \alpha_1$.

Note that conditional on $u^{(1)} \in \mathcal{C}([0,\infty), \mathcal{C}^+_{\text{tem}})$ all the processes $(v(t))_{t\geq 0}$ fit into the framework of (1.20) and are as such nonnegative. The final coupling we present is of a different flavor. It is based on the approximation of solutions to (1.1) with initial conditions $u_0 \in \mathcal{C}^+_{\text{tem}}$ by means of densities of rescaled long-range contact processes; see [18], Theorem 1, for the convergence result. Note that the parameter θ_c in [18] denotes an arbitrary $\theta > 0$ and does not have any relation to the critical parameter θ_c of the present article.

We use the construction of an approximating particle system $(\xi^n_t(f_0))_{t\geq 0}$ for $n \in \mathbb{N}$ resulting in a solution to (1.1) with initial condition $f_0 \in \mathcal{C}^+_{\text{tem}}$ from [18]. The dynamics are modeled by means of i.i.d. Poisson processes given at the beginning of Section 2. Their rates depend in a monotone way on the parameter θ. Initial conditions f_0 get approximated by approximate densities $A_c(\xi^n_0(f_0)), n \in \mathbb{N}$, compare the definition preceding Theorem 1. The approximate densities $(A_c(\xi_t^n(f_0))_{t\geq 0}$ converge to a solution to (1.1) with initial condition f_0.

For the next lemma, recall the definition of $\upsilon_T = \upsilon_T(\theta)$ from [14], Remark 2.8 *(left-upper measure)*. Note in particular the use of the nondecreasing sequence $\zeta_N \in \mathcal{C}_{\mathrm{tem}}^+$, $N \in \mathbb{N}$.

LEMMA A.12 (θ-$*$-coupling). *Let $0 < \theta_1 < \theta_2$ and $T > 0$ be arbitrarily fixed. There exists a coupling of two processes $(u_{T+t}^{*,l}(\theta_i))_{t\geq 0}$, $i = 1, 2$ such that $\mathcal{L}((u_{T+t}^{*,l}(\theta_i))_{t\geq 0}) = \mathbb{P}_{\upsilon_T(\theta_i)}$, $i = 1, 2$. Moreover,*

$$(A.26) \qquad \big(u_{T+t}^{*,l}(\theta_1)\big)(x) \leq \big(u_{T+t}^{*,l}(\theta_2)\big)(x) \quad \textit{for all } x \in \mathbb{R}, t \geq 0 \textit{ a.s.}$$

This result also holds for a finite number of $0 < \theta_1 < \cdots < \theta_m$, $m \in \mathbb{N}$.

This coupling relies on two properties of the processes involved. First, we use the monotonicity of the respective solutions resulting from $\theta_1 < \theta_2$ for each initial condition ζ_N, $N \in \mathbb{N}$; second, for θ_i fixed, we use the construction of $(u_{T+t}^{*,l}(\theta_i))_{t\geq 0}$ by means of a nondecreasing sequence $(u_{T+t}^{(\zeta_N)}(\theta_i))_{t\geq 0}$, $N \in \mathbb{N}$ as in [14], Remark 2.8 *(left-upper measure)*. Unfortunately, we could not make the constructions from the above work to integrate these two steps into one. Thus we had to make use of the approximation by discrete particle systems, where at least the motivation for the veracity of the above result should be easily accessible to the reader.

PROOF OF LEMMA A.12. The *dynamics* of the nth approximation for θ_i, $i = 1, 2$ use the same set of i.i.d. Poisson processes for death events. For birth events, consider i.i.d. Poisson processes

$$(A.27) \qquad \begin{aligned} &\big(P_t(x, y) : x, y \in n^{-2}\mathbb{Z}, x \text{ neighbor of } y\big) \text{ with rate } \big(2c_1 n^{3/2}\big)^{-1}(n + \theta_1), \\ &\big(Q_t(x, y) : x, y \in n^{-2}\mathbb{Z}, x \text{ neighbor of } y\big) \text{ with rate } \big(2c_1 n^{3/2}\big)^{-1}(\theta_2 - \theta_1), \end{aligned}$$

where $c_1(n) \to 1$ as $n \to \infty$. For the θ_1-system, at a jump of $P_t(x, y)$, if the site x is occupied, there is a birth and the site y, if vacant, becomes occupied (cf. beginning of [18], Section 2). In our coupling, for the θ_2-system, at a jump of $P_t(x, y)$ or $Q_t(x, y)$ the same holds. Note that $(P_t(x, y) + Q_t(x, y) : x, y \in n^{-2}\mathbb{Z}, x \text{ neighbor of } y)$ is a family of i.i.d. Poisson processes with rate $(2c_1 n^{3/2})^{-1}(n + \theta_2)$. As a result, given the same initial configurations, the θ_2-system dominates the θ_1-system.

Additionally, we construct a set of *initial conditions* $(\xi_0^n(\zeta_N) : N \in \mathbb{N})$ of the nth approximating particle systems as follows below. They are the same for the θ_1- and θ_2-system. After linear interpolation in space, $A_c(\xi_0^n(\zeta_N))$ converges in $\mathcal{C}_{\mathrm{tem}}^+$ to ζ_N for $n \to \infty$ for all $N \in \mathbb{N}$ and (use that the sequence $(\zeta_N)_{N\in\mathbb{N}}$ is nondecreasing), $\xi_0^n(\zeta_{N_1}) \leq \xi_0^n(\zeta_{N_2})$ for $N_1 \leq N_2$. By [18], Theorem 1, the approximating densities $(A_c(\xi_t^n(\zeta_N))(\theta_i))_{t\geq 0}, i=1,2$ converge in distribution for $n\to\infty$ to *continuous* solutions $(u_t^{(\zeta_N)}(\theta_i))_{t\geq 0}, i=1,2$ of (1.1) with initial conditions $\zeta_N\in\mathcal{C}^+_{\mathrm{tem}}$. By construction,

$$A_c\big(\xi_t^n(\zeta_{N_1})\big)(\theta_i)\leq A_c\big(\xi_t^n(\zeta_{N_2})\big)(\theta_j) \tag{A.28}$$

for all $t\geq 0$, $N_1\leq N_2$, $\theta_i\leq\theta_j$, $i,j\in\{1,2\}$.

For $n\in\mathbb{N}$ fixed, use the following coupling to obtain nth-approximations $(\xi_0^n(\zeta_N):N\in\mathbb{N})$ for a family of initial conditions $(\zeta_N)_{N\in\mathbb{N}}$ as in [14], Remark 2.8 *(left-upper measure)*. Assume without loss of generality that for all $N\in\mathbb{N}$, $\zeta_N\in\mathcal{C}^+_{\mathrm{tem}}$ is a bounded continuously differentiable function with bounded first derivatives. To construct the initial conditions of the nth approximating particle system recall that each site $z\in n^{-2}\mathbb{Z}$ has $2c_1n^{3/2}$ neighbors (including z) and for $f_0\in\mathcal{C}^+_{\mathrm{tem}}$, $A_c(\xi_0^n(f_0))(z)=(2c_1n^{1/2})^{-1}\sum_{y\text{ neighbor of }z}(\xi_0^n(f_0))(y)$. For $z\in n^{-2}\mathbb{Z}$, let

$$\begin{aligned}&\big(\xi_0^n(f_0)\big)(z)\\&\quad=\begin{cases}1 & \exists k\in\mathbb{Z}:z\in\left\{\dfrac{k\cdot 2c_1n^{3/2}}{n^2},\dfrac{k\cdot 2c_1n^{3/2}+1}{n^2},\dots,\right.\\ & \qquad\left.\dfrac{k\cdot 2c_1n^{3/2}+\lfloor 2c_1n^{1/2}f_0(kc_1n^{-1/2})\rfloor}{n^2}\right\},\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{A.29}$$

Then $A_c(\xi_0^n(\zeta_N))$, after linear interpolation in space, converges in $\mathcal{C}^+_{\mathrm{tem}}$ to ζ_N for $n\to\infty$ for all $N\in\mathbb{N}$ and by construction, (A.28) is fulfilled.

Assume without loss of generality that $\tilde{d}$ is such that $(\mathcal{D}([0,\infty),\mathcal{C}^+_{\mathrm{tem}}),\tilde{d})$ is a Polish space (recall d from below [14], (1.6); cf. Ethier and Kurtz [6], Theorem III.5.6). We have

$$\begin{aligned}&\big(A_c\big(\xi_\cdot^n(\zeta_N)\big)(\theta_1),A_c\big(\xi_\cdot^n(\zeta_N)\big)(\theta_2),A_c\big(\xi_\cdot^n(\zeta_N)\big)(\theta_2)-A_c\big(\xi_\cdot^n(\zeta_N)\big)(\theta_1),\\&\qquad A_c\big(\xi_\cdot^n(\zeta_{N+1})\big)(\theta_2)-A_c\big(\xi_\cdot^n(\zeta_N)\big)(\theta_2),A_c\big(\xi_\cdot^n(\zeta_{N+1})\big)(\theta_1)\\&\qquad -A_c\big(\xi_\cdot^n(\zeta_N)\big)(\theta_1)\big)_{N\in\mathbb{N}}\\&\quad\equiv\big(A_n(N,\theta_1),A_n(N,\theta_2),A_n(N,\Delta\theta),A_n(\Delta N,\theta_1),A_n(\Delta N,\theta_2)\big)_{N\in\mathbb{N}}\\&\quad\equiv A_n\in X\equiv\big(\big(\mathcal{D}\big([0,\infty),\mathcal{C}^+_{\mathrm{tem}}\big)\big)^5\big)^{\mathbb{N}}.\end{aligned} \tag{A.30}$$

If we equip X with the metric $\rho(\bar{f},\bar{g})\equiv\sum_{i\in\mathbb{N}}2^{-i}(\sum_{j=1,2,3,4,5}\tilde{d}(f_{i1},g_{i1}))\wedge 1$ where $\bar{f}=(f_{ij})_{i\in\mathbb{N},j=1,2,3,4,5}$, $\bar{g}=(g_{ij})_{i\in\mathbb{N},j=1,2,3,4,5}\in X$, $f_{ij},g_{ij}\in\mathcal{D}([0,\infty),\mathcal{C}^+_{\mathrm{tem}})$, then X is a Polish space as well. Let us reason as in Jacod and Shiryaev [12], Corollary VI.3.33, to see that the convergence in distribution of the sequences $(A_c(\xi_\cdot^n(\zeta_N))(\theta_j))_{n\in\mathbb{N}}$ for $j\in\{1,2\}$, $N\in\mathbb{N}$ fixed and $n\to\infty$ to a *continuous* (in t) limit implies the convergence of

$$\big(A_n(N,\theta_1),A_n(N,\theta_2),A_n(N,\Delta\theta_1),A_n(\Delta N,\theta_1),A_n(\Delta N,\theta_2)\big)_{n\in\mathbb{N}} \tag{A.31}$$

to

$$\begin{aligned}(u_\cdot^{(\zeta_N)}(\theta_1), u_\cdot^{(\zeta_N)}(\theta_2), u_\cdot^{(\zeta_N)}(\theta_2) - u_\cdot^{(\zeta_N)}(\theta_1), u_\cdot^{(\zeta_{N+1})}(\theta_1) \\ - u_\cdot^{(\zeta_N)}(\theta_1), u_\cdot^{(\zeta_{N+1})}(\theta_2) - u_\cdot^{(\zeta_N)}(\theta_2)).\end{aligned} \tag{A.32}$$

By the definition of $\rho(\bar{f}, \bar{g})$, we can choose a subsequence such that $(A_{n_k})_{k\in\mathbb{N}}$ converges in X. Note in particular that the marginal distributions of every subsequential limit are given by their respective one-dimensional limits.

Fix this convergent subsequence. Now apply Skorokhod's theorem (cf. [6], Theorem III.1.8) to obtain that after possibly changing to another probability space, this convergence becomes almost sure convergence, that is,

$$\begin{aligned}&(A_{n_k}(N,\theta_1), A_{n_k}(N,\theta_2), A_{n_k}(N,\Delta\theta_1), A_{n_k}(\Delta N,\theta_1), A_{n_k}(\Delta N,\theta_2))_{N\in\mathbb{N}} \\ &\quad \to ((u_\cdot^{(\zeta_N)}(\theta_1), u_\cdot^{(\zeta_N)}(\theta_2), u_\cdot^{(\zeta_N)}(\theta_2) - u_\cdot^{(\zeta_N)}(\theta_1), \\ &\qquad u_\cdot^{(\zeta_{N+1})}(\theta_1) - u_\cdot^{(\zeta_N)}(\theta_1), u_\cdot^{(\zeta_{N+1})}(\theta_2) - u_\cdot^{(\zeta_N)}(\theta_2))_{N\in\mathbb{N}} \in X \\ &\quad \text{a.s. for } k\to\infty,\end{aligned} \tag{A.33}$$

where $u^{(\zeta_N)}(\theta_i)$ solves (1.1) with initial condition ζ_N and parameter θ_i and

$$u^{(\zeta_{N_1})}(\theta_{j_1}) \le u^{(\zeta_{N_2})}(\theta_{j_2}) \quad \text{for all } N_1 \le N_2, j_1 \le j_2 \text{ a.s.} \tag{A.34}$$

by (A.33) in combination with the definition of X.

Fix $T > 0$. Let $u_{T+t}^{*,l}(\theta_i) = \uparrow \lim_{N\to\infty} u_{T+t}^{(\zeta_N)}(\theta_i)$, $i = 1, 2$, $t \ge 0$. Let us reason as in [14], Corollary 2.6, to conclude that $\mathcal{L}((u_{T+t}^{*,l}(\theta_i))_{t\ge 0} = \mathbb{P}_{\upsilon_T}$. From (A.34), (A.26) follows. □

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