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Correlation structure, quadratic variations and parameter estimation for the solution to the wave equation with fractional noise

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Abstract: We compute the covariance function of the solution to the linear stochastic wave equation with fractional noise in time and white noise in space. We apply our findings to analyze the correlation structure of this Gaussian process and to study the asymptotic behavior in distribution of its spatial quadratic variation. As an application, we construct a consistent estimator for the Hurst parameter.

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1. Introduction

The stochastic wave equation driven by space-time white noise or by a Gaussian noise white in time and spatially colored has been widely studied (see e.g. [13], [9], [5], [10] and the references therein). It constitutes a recognized model for the displacement of a vibrating string under a random perturbation.

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The study of SPDEs in general (and of the stochastic wave equation in particular) driven by a fractional noise in time is more recent and it appeared as a consequence of the stochastic calculus with respect to the fractional Brownian motion (fBm in the sequel) and related processes. We refer, among others, to [1], [6], [24].

In this work, we are concerned with the analysis of the solution to the stochastic linear wave equation driven by an additive Gaussian noise which behaves as a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ with respect to the time variable and as a Wiener process in space. Our purpose is, firstly, to explicitly compute the correlation structure of the solution in space and secondly, to use it in order to obtain the asymptotic behavior in distribution of its spatial quadratic variation with application to the estimation of the Hurst parameter. We have made a similar analysis of the covariance of the solution to the wave equation with time-space white noise in the work [15]. The usual way to compute the covariance, or the mean square of its increment (in time or in space) of the solution is based (see e.g. [10], [6]) on the Fourier transform of the fundamental solution (or the Green kernel) whose expression does not depend on the dimension $d \geq 1$. But we have seen in [15] that, for $d = 1$, a direct calculation based on the formula for Green kernel of the wave equation (and not on its Fourier transform) leads to new insights and brings new information on the correlation structure of the solution. We will employ the same idea in the fractional case and we are able to obtain a closed formula for the spatial covariance of the solution. This formula is useful and gives several unknown facts concerning the correlation structure of the solution, by showing an interesting link between the solution to the wave equation and the fractional Brownian motion. The covariance formula will then be used to obtain sharp estimates for the distribution of the solution, its moments and its path regularity. Moreover, it is a crucial tool which, combinated with the techniques of the Stein-Malliavin calculus, allows to obtain the limit behavior in distribution of the centered quadratic variation in space of the solution to the wave equation. More precisely, if $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ denotes the solution of the wave equation (t being the time variable and x the space variable), we will show that the (suitably normalized) sequence

$$V_N = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \left[\frac{\left(u(t, \frac{j+1}{N}) - u(t, \frac{j}{N})\right)^2}{\mathbf{E}\left(u(t, \frac{j+1}{N}) - u(t, \frac{j}{N})\right)^2} - 1 \right]$$

converges in distribution, as $N \to \infty$ and when $H < \frac{3}{4}$, to a Gaussian random variable. We also derive the rate of convergence via the recent Stein-Malliavin theory (see [20]). The threshold $\frac{3}{4}$ also appears in the case of the fractional Brownian motion: it is well-known that the centered and suitably renormalized quadratic variation of the fractional Brownian motion satisfies a CLT for $H < \frac{3}{4}$, H being the self-similarity index of the fBm. In our case, we point out that the value $\frac{3}{4}$ does not constitute a threshold for the self-similarity index of the process u (which is self-similar in time of order $H + \frac{1}{2}$ and it is not self-similar in space) but for the self-similarity index of the noise in time. This suggests that the behavior of the noise with respect to the time variable highly affects the behavior of the spatial quadratic variation. This makes an interesting link between the theory of regularity of Gaussian stochastic processes and the estimation theory. It follows from the results in Sections 3 and 5 that the analysis of the sharp behavior of the trajectories of the solution to the fractional-white wave equations has a direct impact on many aspect of the process, including the asymptotic behavior of the Hurst index estimators.

It is also well-known that the quadratic variation constitutes a good tool to construct estimators for the self-similarity order of self-similar stochastic processes. We refer to [7], [29] and the references therein. We will assume that, for some fixed $t > 1$, the process $u(t, x)$ is observed at discrete time $u(t, \frac{i}{N}), i = 0, .., N$ and, via a standard procedure, we construct an estimator based on the spatial observation of the solution at fixed time t. We refer to Section 5.3 for the statistical interpretation of these discrete observations in the case of the vibrating string model. The behavior of the estimator is strongly related to the behavior of the sequence V_N, and hence we are able to prove that the estimator is consistent and asymptotically normal.

Our paper is organized as follows. In Section 2 we present general facts concerning the wave equation with fractional noise in time and white noise in space. In Section 3 we compute the spatial covariance of the solution and we deduce several useful facts concerning its correlation structure. Sections 4 and 5 are devoted to the analysis of the asymptotic behavior of the spatial quadratic variations of the solution, with some applications to parameter estimation. The last section contains the proof of the almost sure CLT for the quadratic variation.

2. Description of the context

In this paragraph we present some general fact concerning the wave equation, the driving Gaussian noise and the definition of the mild solution of this equation. We also present some results on the existence and basic properties of this solution. These results, although not explicitly stated in the literature, can be derived by using the lines of the proofs in Chapter 2 in [28].

We consider below the linear stochastic wave equation driven by an additive infinite-dimensional Gaussian noise W^H:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}^H(t,x), \quad t \in (0,T], \ T>0, \ x \in \mathbb{R}^d \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

Here Δ is the Laplacian on $\mathbb{R}^d$, $d \geq 1$ and $W^H = \{W_t^H(A);\ t \in [0,T],\ A \in \mathfrak{B}_b(\mathbb{R}^d)\}$ is a real valued centered Gaussian field, over a given complete filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$, whose covariance function is:

$$\mathbf{E}(W_t^H(A) W_s^H(B)) = R^H(t,s)\lambda(A \cap B), \text{ for every } A, B \in \mathfrak{B}_b(\mathbb{R}^d), \tag{2.2}$$

where λ is the d-dimensional Lebesgue measure, $\mathfrak{B}_d(\mathbb{R}^d)$ is the set of the λ-bounded Borel subsets of $\mathbb{R}^d$ and R^H is the covariance function of the fBm with Hurst parameter H, and it is given by:

$$R^H(t,s) := \frac{1}{2}\Big(t^{2H} + s^{2H} - \mid t-s \mid^{2H}\Big), \quad s,t \geq 0. \tag{2.3}$$

Throughout this paper, we consider the regular case, that is the index H is assumed to be in $(\frac{1}{2},1)$. Commonly the noise field W^H is called "the fractional-white noise", it plainly indicates that its behavior is like the fBm in time and like the Wiener process (white) in space. Thus, its spatial increments are independent while the temporal increments present a stochastic dependence and they are positively correlated when $H > \frac{1}{2}$.

2.1. The canonical Hilbert space associated to the noise field

Recall that when $H > \frac{1}{2}$, the covariance function (2.3) of fBm is a non-negative function and it can be represented in the following way:

$$R^H(t,s) = \alpha_H \int_0^t \int_0^s \mid u-v \mid^{2H-2} \mathrm{d}u\mathrm{d}v, \text{ for every } t,s \in [0,T], \tag{2.4}$$

with $\alpha_H := H(2H-1)$. A complete revue about the construction of the canonical Hilbert space $\mathcal{H}_B$ associated to the fBm can be found in [23].

Designate by ξ the set of linear combinations of the simple functions $\mathbb{1}_{\{[0,t]\times A\}}$, $t \in [0,T],\ A \in \mathfrak{B}_d(\mathbb{R}^d)$, the canonical Hilbert space $\mathcal{H}_W$ associated to the field W^H, when $H > \frac{1}{2}$, is defined as the closure of the linear space generated by ξ with respect to the inner product $\langle .,. \rangle_{\mathcal{H}_W}$ which is expressed by:

$$\begin{aligned} &\langle \mathbb{1}_{\{[0,t]\times A\}}, \mathbb{1}_{\{[0,s]\times B\}} \rangle_{\mathcal{H}_W} : \\ = \quad &\mathbf{E}(W_t^H(A) W_s^H(B)) = \alpha_H \lambda(A\cap B) \int_0^t \int_0^s \mid u-v \mid^{2H-2} \mathrm{d}u\mathrm{d}v. \end{aligned} \tag{2.5}$$

By a routine argument, as the mapping $\mathbb{1}_{\{[0,t]\times A\}} \mapsto W_t^H(A)$ defines an isometry between ξ and $Sp(W^H)$ the span of the noise W^H, and as ξ is dense in $\mathcal{H}_W$, then for every $f \in \mathcal{H}_W$, the extended mapping $f \mapsto W^H(f) := \int_0^T \int_{\mathbb{R}^d} f(t,x) W^H(\mathrm{d}t,\mathrm{d}x)$ is also an isometry between $\mathcal{H}_W$ and $Sp(W^H)$.

The scalar product in $\mathcal{H}_W$ is given by

$$\begin{aligned} &\langle f,g \rangle_{\mathcal{H}_W} = \mathbf{E}(W^H(f) W^H(g)) \\ = \quad &\alpha_H \int_0^T \int_0^T \int_{\mathbb{R}^d} f(u,x) g(v,x) \mid u-v \mid^{2H-2} \mathrm{d}x\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{2.6}$$

for every $f,g \in \mathcal{H}_W$ such that $\int_0^T \int_0^T \int_{\mathbb{R}^d} |f(u,x)g(v,x)| \mid u-v \mid^{2H-2} \mathrm{d}x\mathrm{d}u\mathrm{d}v < \infty$.

2.2. The mild solution to the wave equation

We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing C^∞ test-functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions. For $f \in L^1(\mathbb{R}^d)$, we mean by $\mathcal{F}f$ the Fourier transform of f:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\mathrm{d}x, \text{ for all } \xi \in \mathbb{R}^d$$

where $\|\cdot\|$ denotes the Euclidean norm and "." the Euclidean scalar product over $\mathbb{R}^d$. An useful formula in our work is the Plancherel-identity:

$$\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x = (2\pi)^{-d}\int_{\mathbb{R}^d} (\mathcal{F}f)(\xi)\overline{\mathcal{F}(g)(\xi)}\mathrm{d}\xi, \text{ for any } f, g \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d). \tag{2.7}$$

Let G_1 be the fundamental solution (called also the Green function) of the homogeneous wave equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$. It is known that $G_1(t,\cdot)$ is a distribution in $\mathcal{S}'(\mathbb{R}^d)$ with rapid decrease. An easy way to define it is via its Fourier transform (see e.g. [26],[16]):

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}, \quad \text{for any } \xi \in \mathbb{R}^d,\ t > 0 \text{ and } d \geq 1. \tag{2.8}$$

In particular,

$$\begin{cases} G_1(t,x) &= \frac{1}{2}\mathbb{1}_{\{|x|<t\}}, \quad \text{if } d = 1 \\ G_1(t,x) &= \frac{1}{2\pi\sqrt{t^2-|x|^2}}\mathbb{1}_{\{|x|<t\}}, \quad \text{if } d = 2 \\ G_1(t,x) &= c_d\frac{\sigma_t(x)}{t}, \quad \text{if } d = 3, \end{cases} \tag{2.9}$$

with σ_t denotes the surface measure on the 3-dimensional sphere of radius t. For more details on the kernel G_1, see e.g. [13].

The solution of (2.1) in its mild formulation is a square-integrable centered field $u = \{u(t,x);\ t \in [0,T], x \in \mathbb{R}^d\}$, that is defined by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y)W^H(\mathrm{d}s, \mathrm{d}y). \tag{2.10}$$

One can refer to [8] for further lecture. We will say that the mild solution exists if the integral (2.10) is well-defined, i.e. if the integrand belongs to the space $\mathcal{H}_W$. Along this work C, C_1 and C_2 are arbitrary real constants that may change from one line to another and in such computations, they may depend or not on parameters t, s and H. From now on, we set the notation $g_{t,x}(s,y) := G_1(t-s, x-y)$, for all $(s,y) \in [0,T]\times\mathbb{R}^d$. We state the following result that gives the necessary and sufficient condition for the existence of the solution and the control of its increments in space. Its proof can be obtained by following the proofs in [1], [6] or [28].

Proposition 1. *1. The wave solution process $u = \{u(t,x);\ t \geq 0,\ x \in \mathbb{R}^d\}$ defined by (2.10) exists if and only if $d < 2H+1$.*
In this situation, for every $p \geq 2$ and $T > 0$ we have:

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \mathbf{E}\Big(\mid u(t,x) \mid^p \Big) < \infty. \tag{2.11}$$

2. Let $T, M > 0$. Fix $t \in (0,T]$, then there exist two constants $0 < C_2 < C_1$ such that:

$$C_2 \|x-y\|^{2H+1-d} \leq \mathbf{E}\Big(|u(t,x) - u(t,y)|^2 \Big) \leq C_1 \|x-y\|^{2H+1-d} \tag{2.12}$$

for every distinct $x, y \in [-M,M]^d$ and $d \in \{1,2\}$.

Proof: For the first point, we refer to the proof or Theorem 2.8 in [28] which by followed line by line (see also Corollary 2.7 in [28]). The second point follows by the same lines as the proof of Proposition 2 in [6], which presents a complete demonstration when the noise is a fractional Brownian motion with respect to the time and colored with respect to the space. For our case we take $\beta = d$, that ensures to get the desired result. $\square$

Remark 1.
- *This condition $d < 2H+1$ is fulfilled, when the spatial dimension d is 1 or 2.*
- *The bound (2.12) gives the control the spatial increment of the solution. However, in the sequel, we will restrict to the situation $d = 1$ and we will have better estimates for square mean and the spatial increment.*

Remark 2.
- *In virtue of the Kolmogorov-Centsov theorem, for a fixed $t \in (0,T]$, the solution process has a modification (still denoted by the same notation), whose sample paths $x \mapsto u(t,x)$ are almost surely Hölder continuous of exponent $\delta \in (0, H - \frac{d-1}{2})$. Whereas for $\delta \geq H - \frac{d-1}{2}$, we note the lack of Hölder continuity. This extends the case of the space-time white noise for $d = 1$, in which the spatial-wave-solution with space-time white noise is Hölder continuous of order $\delta \in (0, \frac{1}{2})$, and also it coincides with the case of heat equation with white-fractional noise.*
- *One can also show that the process $t \to u(t,x)$ is self-similar of order $H + 1 - \frac{d}{2}$, see Proposition 2.15 in [28].*

3. The spatial covariance and some consequences

Here we compute the covariance with respect to the space variable of the mild solution (2.10) in dimension $d = 1$ and we deduce some consequences on the trajectories of the process. As mentioned in the introduction, we will use use the expression of the Green kernel associated to the wave equation (2.9) instead of its Fourier transform. We are able to obtain an explicit formula that brings new information on the correlation structure of the solution $u(t,x)$ (2.10). This formula is also crucial in order to derive the results in the next section (CLT for the quadratic variation and estimation of the Hurst parameter).

3.1. The spatial covariance

From now on, assume that $d = 1$. We have the following result, which is a key results for our work.

Lemma 1. *Let $T > 0$ and fix $t \in (0, T]$. Then for every $x, y \in \mathbb{R}$ and every $H > \frac{1}{2}$, we have:*

$$\begin{aligned}\mathbf{E}\Big(u(t,x)u(t,y)\Big) &= \frac{1}{2}\left(c_H|y-x|^{2H+1} - \frac{t|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right)\mathbb{1}_{\{|y-x|<t\}} \\ &\quad + \frac{(2t-|y-x|)^{2H+1}}{8(2H+1)}\mathbb{1}_{\{t\leq|y-x|<2t\}}, \qquad (3.1)\end{aligned}$$

with $c_H := \frac{4H-1}{4(2H+1)}$.

Proof: For a fixed $t \in (0, T]$, it can be shown that for every $x, y \in \mathbb{R}$, we get:

$$\begin{aligned} & \mathbf{E}\Big(u(t,x)u(t,y)\Big) \\ := {} & R(x,y) \\ = {} & \alpha_H \int_0^t \int_0^t \mathrm{d}u\mathrm{d}v|u-v|^{2H-2}\int_{\mathbb{R}} \mathrm{d}z G_1(t-u,x-z)G_1(t-v,y-z) \\ = {} & \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v|u-v|^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<t-u\}}\mathbb{1}_{\{|y-z|<t-v\}} \\ = {} & \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v|u-v|^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<u\}}\mathbb{1}_{\{|y-z|<v\}} \\ = {} & \frac{\alpha_H}{4}\int_0^t du \int_0^u \mathrm{d}v(u-v)^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<u\}}\mathbb{1}_{\{|y-z|<v\}} \\ & + \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_u^t \mathrm{d}v(v-u)^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<u\}}\mathbb{1}_{\{|y-z|<v\}} \\ := {} & R_1(x,y) + R_2(x,y). \qquad (3.2)\end{aligned}$$

We assume without loss of generality that $x \leq y$, and then we start computing the first term R_1 from above. Here two cases can be discussed: $y - x \geq u - v$ and $y - x < u - v$:

$$\begin{aligned} R_1(x,y) &:= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v(u-v)^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<u\}}\mathbb{1}_{\{|y-z|<v\}} \\ &= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v(u-v)^{2H-2}\int_{(x-u)\vee(y-v)}^{(x+u)\wedge(y+v)} \mathrm{d}z \\ &= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v(u-v)^{2H-2}\mathbb{1}_{\{u-v\leq y-x\}}\int_{(x-u)\vee(y-v)}^{(x+u)\wedge(y+v)} \mathrm{d}z \\ &\quad + \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v(u-v)^{2H-2}\mathbb{1}_{\{u-v>y-x\}}\int_{(x-u)\vee(y-v)}^{(x+u)\wedge(y+v)} \mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-v\le y-x\}} \int_{y-v}^{x+u} \mathrm{d}z \mathbb{1}_{\{x+u>y-v\}} \\
&\quad + \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-v>y-x\}} \int_{y-v}^{y+v} \mathrm{d}z \\
&= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-v\le y-x<u+v\}}(u+v-(y-x)) \\
&\quad + \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v 2v (u-v)^{2H-2}\mathbb{1}_{\{u-v>y-x\}} \\
&:= R_{1,1}(x,y) + R_{1,2}(x,y). \qquad (3.3)
\end{aligned}
$$

Separately we treat the quantities $R_{1,1}$ and $R_{1,2}$. For the first one, we make use of the change of variables $\tilde{v} = u - v$ in the integral $\mathrm{d}v$, and we infer that:

$$
\begin{aligned}
& R_{1,1}(x,y) \\
:= & \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-v\le y-x<u+v\}}(u+v-(y-x)) \\
= & \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v v^{2H-2}\mathbb{1}_{\{v\le y-x<2u-v\}}(2u-v-(y-x)) \\
= & \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^{u\wedge(y-x)\wedge(2u-(y-x))} v^{2H-2}(2u-v-(y-x)) \\
= & \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \mathbb{1}_{\{u<y-x\}} \int_0^{u\wedge(y-x)\wedge(2u-(y-x))} v^{2H-2}(2u-v-(y-x)) \\
& + \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \mathbb{1}_{\{u\ge y-x\}} \int_0^{u\wedge(y-x)\wedge(2u-(y-x))} v^{2H-2}(2u-v-(y-x)).
\end{aligned}
$$

Note that on the set $\{u < y-x\}$, it yields that: $u\wedge(y-x)\wedge(2u-(y-x)) = 2u-(y-x)$, while on the set $\{u \ge y-x\}$, it gives that: $u\wedge(y-x)\wedge(2u-(y-x)) = y-x$. Consequently this means that:

$$
\begin{aligned}
R_{1,1}(x,y) &= \frac{\alpha_H}{4}\int_0^{t\wedge(y-x)} \mathrm{d}u \mathbb{1}_{\{u>\frac{y-x}{2}\}} \int_0^{2u-(y-x)} \mathrm{d}v v^{2H-2}(2u-v-(y-x)) \\
&\quad + \frac{\alpha_H}{4}\mathbb{1}_{\{y-x<t\}} \int_{y-x}^t \mathrm{d}u \int_0^{y-x} \mathrm{d}v v^{2H-2}(2u-v-(y-x)) \\
&:= R_{1,1,A}(x,y) + R_{1,1,B}(x,y). \qquad (3.4)
\end{aligned}
$$

Looking at the quantity $R_{1,1,A}$, it can be shown that:

$$
\begin{aligned}
& R_{1,1,A}(x,y) \\
:= & \frac{\alpha_H}{4}\int_0^{t\wedge(y-x)} \mathrm{d}u \mathbb{1}_{\{u>\frac{y-x}{2}\}} \int_0^{2u-(y-x)} \mathrm{d}v v^{2H-2}(2u-v-(y-x) \\
= & \frac{\alpha_H}{4}\mathbb{1}_{\{y-x<t\}} \int_{\frac{y-x}{2}}^{y-x} \mathrm{d}u \int_0^{2u-(y-x)} \mathrm{d}v v^{2H-2}(2u-v-(y-x)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha_H}{4}\mathbb{1}_{\{y-x\geq t\}}\mathbb{1}_{\{y-x<2t\}}\int_{\frac{y-x}{2}}^{t}\mathrm{d}u\int_0^{2u-(y-x)}\mathrm{d}v v^{2H-2}(2u-v-(y-x)\\
=\;&\frac{\alpha_H}{8H(2H-1)}\mathbb{1}_{\{y-x<t\}}\int_{\frac{y-x}{2}}^{y-x}\mathrm{d}u(2u-(y-x))^{2H}\\
&+\frac{\alpha_H}{8H(2H-1)}\mathbb{1}_{\{t\leq y-x<2t\}}\int_{\frac{y-x}{2}}^{t}\mathrm{d}u(2u-(y-x))^{2H}\\
=\;&\frac{(y-x)^{2H+1}}{16(2H+1)}\mathbb{1}_{\{y-x<t\}}+\frac{(2t-(y-x))^{2H+1}}{16(2H+1)}\mathbb{1}_{\{t\leq y-x<2t\}}. \qquad (3.5)
\end{aligned}
$$

For the second term $R_{1,1,B}$, we may calculate the integral and it produces that:

$$
\begin{aligned}
&R_{1,1,B}(x,y)\\
:=\;&\frac{\alpha_H}{4}\mathbb{1}_{\{y-x<t\}}\int_{y-x}^{t}\mathrm{d}u\int_0^{y-x}\mathrm{d}v v^{2H-2}(2u-v-(y-x))\\
=\;&\frac{\alpha_H}{4}\mathbb{1}_{\{y-x<t\}}\int_{y-x}^{t}\mathrm{d}u\left[\frac{2u(y-x)^{2H-1}}{2H-1}\right.\\
&\left.-\frac{4H-1}{2H(2H-1)}(y-x)^{2H}\right]\\
=\;&\frac{\alpha_H}{4}\mathbb{1}_{\{y-x<t\}}\left[\frac{(y-x)^{2H-1}}{2H-1}(t^2-(y-x)^2)\right.\\
&\left.-\frac{4H-1}{2H(2H-1)}(y-x)^{2H}(t-(y-x))\right]\\
=\;&\left[\frac{Ht^2}{4}(y-x)^{2H-1}-\frac{(4H-1)t}{8}(y-x)^{2H}\right.\\
&\left.+\frac{2H-1}{8}(y-x)^{2H+1}\right]\mathbb{1}_{\{y-x<t\}}. \qquad (3.6)
\end{aligned}
$$

Combining (3.5) and (3.6) in (3.4), we infer that:

$$
\begin{aligned}
&R_{1,1}(x,y)\\
=\;&\left[\frac{Ht^2}{4}(y-x)^{2H-1}-\frac{(4H-1)t}{8}(y-x)^{2H}\right.\\
&\left.+\frac{8H^2-1}{16(2H+1)}(y-x)^{2H+1}\right]\mathbb{1}_{\{y-x<t\}}\\
&+\frac{(2t-(y-x))^{2H+1}}{16(2H+1)}\mathbb{1}_{\{t\leq y-x<2t\}}. \qquad (3.7)
\end{aligned}
$$

Focusing now on $R_{1,2}$, a modicum of calculus leads to:

$$
\begin{aligned}
&R_{1,2}(x,y)\\
:=\;&\frac{\alpha_H}{4}\int_0^t\mathrm{d}u\int_0^u\mathrm{d}v 2v(u-v)^{2H-2}\mathbb{1}_{\{u-v>y-x\}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha_H}{2}\int_0^t \mathrm{d}u \mathbb{1}_{\{u>y-x\}}\int_0^{u\wedge(u-(y-x))} \mathrm{d}v v(u-v)^{2H-2} \\
&= \frac{\alpha_H}{2}\mathbb{1}_{\{y-x<t\}}\int_{y-x}^t \mathrm{d}u \int_0^{u-(y-x)} \mathrm{d}v v(u-v)^{2H-2} \\
&= \frac{\alpha_H}{2}\mathbb{1}_{\{y-x<t\}}\int_{y-x}^t \mathrm{d}u \left[\frac{(y-x)^{2H}}{2H} - \frac{u(y-x)^{2H-1}}{2H-1} + \frac{u^{2H}}{2H(2H-1)}\right] \\
&= \frac{1}{4}\Big[(2H-1)(t-(y-x)(y-x)^{2H} \\
&\quad -2H(y-x)^{2H-1}\int_{y-x}^t \mathrm{d}u u + \int_{y-x}^t u^{2H}\Big]\mathbb{1}_{\{y-x<t\}} \\
&= \frac{1}{4}\Bigg[(2H-1)(y-x)^{2H}t - \frac{H(2H-1)(y-x)^{2H+1}}{2H+1} \\
&\quad -Ht^2(y-x)^{2H-1} + \frac{t^{2H+1}}{2H+1}\Bigg]\mathbb{1}_{\{y-x<t\}} \\
&= \frac{t^2}{4}\left(\frac{t^{2H-1}}{2H+1} - H(y-x)^{2H-1}\right)\mathbb{1}_{\{y-x<t\}} + \frac{(2H-1)(y-x)^{2H}}{4} \\
&\quad \times\left(t - \frac{H(y-x)}{2H+1}\right)\mathbb{1}_{\{y-x<t\}}. \qquad (3.8)
\end{aligned}
$$

Plugging the last two quoted results (3.7) and (3.8) in (3.3), we easily obtain that for all $x,\ y \in \mathbb{R}$:

$$
\begin{aligned}
R_1(x,y) &:= R_{1,1}(x,y) + R_{1,2}(x,y) \\
&= \frac{1}{4}\left(c_H|y-x|^{2H+1} - \frac{t|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1}\right)\mathbb{1}_{\{|y-x|<t\}} \\
&\quad + \frac{(2t-(y-x))^{2H+1}}{16(2H+1)}\mathbb{1}_{\{t\le|y-x|<2t\}}, \qquad (3.9)
\end{aligned}
$$

with $c_H := \frac{4H-1}{4(2H+1)}$.

Dealing now with the remaining part R_2 of the statement (3.2), and using the change of variables $(\tilde{u},\tilde{v}) = (t-u, t-v)$, we can show that:

$$
\begin{aligned}
&R_2(x,y) \\
&:= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u\int_u^t \mathrm{d}v(v-u)^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<u\}}\mathbb{1}_{\{|y-z|<v\}} \\
&= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u\int_0^u \mathrm{d}v(u-v)^{2H-2}\int_{\mathbb{R}} \mathrm{d}z\mathbb{1}_{\{|x-z|<t-u\}}\mathbb{1}_{\{|y-z|<t-v\}} \\
&= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u\int_0^u \mathrm{d}u(v-u)^{2H-2}\int_{(x-(t-u))\vee(y-(t-v))}^{(x+(t-u))\wedge(y+(t-v))} \mathrm{d}z \\
&= \frac{\alpha_H}{4}\int_0^t \mathrm{d}u\int_0^u \mathrm{d}v(u-v)^{2H-2}\mathbb{1}_{\{u-v\le y-x\}}\int_{y-(t-v)}^{x+(t-u)} \mathrm{d}z\mathbb{1}_{\{x+(t-u)>y-(t-v)\}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-v>y-x\}}\int_{x-(t-u)}^{x+(t-u)}\mathrm{d}z\\
= &\quad \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-(y-x)\leq v\}}\\
&\mathbb{1}_{\{x+(t-u)>y-(t-v)\}}\big(2t-(y-x)-(u+v)\big)\\
&+\frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v (u-v)^{2H-2}\mathbb{1}_{\{u-(y-x)>v\}}2(t-u)\\
= &\quad \frac{\alpha_H}{4}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v v^{2H-2}\mathbb{1}_{\{v\leq (y-x)<2t-(2u-v)\}}\big(2t-(2u-v)-(y-x)\big)\\
&+\frac{\alpha_H}{4}\int_0^t \mathrm{d}u 2(t-u)\mathbb{1}_{\{u>(y-x)\}}\int_0^{u-(y-x)}\mathrm{d}v (u-v)^{2H-2}.
\end{aligned}
$$

Proceeding by the same calculus as R_1, we find that:

$$
R_2(x,y) \quad = \quad R_{1,1}(x,y)+R_{1,2}(x,y)=R_1(x,y). \tag{3.10}
$$

Therefore, due to the expression (3.9) and (3.10), the final expression of the covariance function R can be written in the following way:

$$
\begin{aligned}
R(x,y) \quad &:= \quad R_1(x,y)+R_2(x,y)\\
&= \quad 2R_{1,1}(x,y)+2R_{1,2}(x,y)\\
&= \quad \frac{1}{2}\left(c_H|y-x|^{2H+1}-\frac{t|y-x|^{2H}}{2}+\frac{t^{2H+1}}{2H+1}\right)\mathbb{1}_{\{|y-x|<t\}}\\
&\quad\quad +\frac{(2t-|y-x|)^{2H+1}}{8(2H+1)}\mathbb{1}_{\{t\leq|y-x|<2t\}},\ \text{for all } x,\ y\in\mathbb{R},
\end{aligned}
$$

and the desired conclusion follows. □

Remark 3. *For the classical case when $H=\frac{1}{2}$, we note that for every $x,y\in\mathbb{R}$: from one hand $R_{1,2}=0$ and from another hand:*

$$
\begin{aligned}
&R_{1,1}(x,y)\\
= &\left(\frac{t^2}{8}-\frac{t}{8}|y-x|+\frac{|y-x|2}{32}\right)\mathbb{1}_{\{|y-x|<t\}}+\frac{(2t-|y-x|)^2}{32}\mathbb{1}_{\{t\leq|y-x|<2t\}}\\
= &\ \frac{(2t-|y-x|)^2}{32}\mathbb{1}_{\{|y-x|<t\}}-\frac{(2t-|y-x|)^2}{32}\mathbb{1}_{\{|y-x|<t\}}\\
&+\frac{(2t-|y-x|)^2}{32}\mathbb{1}_{\{|y-x|<2t\}}\\
= &\ \frac{1}{8}\Big(t-\frac{|y-x|}{2}\Big)^2\mathbb{1}_{\{|y-x|<2t\}},
\end{aligned}
$$

which implies that:

$$
R(x,y)=\frac{1}{4}\Big(t-\frac{|y-x|}{2}\Big)^2\mathbb{1}_{\{|y-x|<2t\}},
$$

and obviously we deduce that it coincides with the covariance's expression for the space-time white case (see [15]).

Remark 4. *Looking at the proceeding expression (3.1), we deduce that the spatial solution $(u(t,x))_{x\in\mathbb{R}}$ is a stationary process, this remains that its behavior is different from the spatial Heat-solution with white-fractional noise, which is self-similar with index $H-\frac{d-1}{2}$.*

Some results due to the expression of the spatial covariance function can be listed below in the following subsection.

3.2. Consequences of the covariance formula

3.2.1. Continuity of the covariance

From (3.1), we can infer that:

$$R(x,y) = f(|y-x|)$$

where the function f is defined for all $z \in \mathbb{R}$ in the next way:

$$f(z) = \begin{cases} \frac{1}{2}\left(c_H z^{2H+1} - \frac{t}{2}z^{2H} + \frac{t^{2H+1}}{2H+1}\right) & \text{if } z \in (-\infty, t) \\ \frac{(2t-z)^{2H+1}}{8(2H+1)} & \text{if } z \in [t, 2t) \\ 0 & \text{if } z \in [2t, +\infty). \end{cases} \tag{3.11}$$

Clearly f is a continuous function on $\mathbb{R}$, in particular at the points $z = t$ and $z = 2t$.

3.2.2. Explicit distribution of the spatial solution

Another interesting consequence is the second moment of the spatial solution, which is for a fixed $t > 0$ and for every $x \in \mathbb{R}$ equal to:

$$R(x,x) := \mathbf{E}\Big(u(t,x)^2\Big) = \frac{t^{2H+1}}{2(2H+1)}. \tag{3.12}$$

Recall that the solution process $(u(t,x))_{x\in\mathbb{R}}$ is a centered Gaussian one, hence for fixed $t > 0$, we may make out that:

$$u(t,x) \quad \sim \quad \frac{t^{H+\frac{1}{2}}}{\sqrt{2(2H+1)}} Z, \text{ with } Z \sim \mathcal{N}(0,1). \tag{3.13}$$

In particular, the p-moment is:

$$\mathbf{E}\Big(|u(t,x)|^p\Big) \quad = \quad \left(\frac{t^{H+\frac{1}{2}}}{\sqrt{2(2H+1)}}\right)^p \mathbf{E}\big(|Z|^p\big). \tag{3.14}$$

3.2.3. Sharp estimates of the spatial increment

Looking again at the mean square of the spatial increment of the solution, we can find sharp estimates according to the covariance function's expression. In fact we retrieve the result established in Proposition 1, point 2, which is based on using the Fourier transform of the Green kernel and on mimicking the proof evoked in [6], it shows that for $d = 1$:

$$C_2 | \, x - y \, |^{2H} \leq \mathbf{E}\Big(\, |u(t,x) - u(t,y)|^2 \, \Big) \leq C_1 | \, x - y \, |^{2H} \tag{3.15}$$

for every distinct $x, y \in [-M, M]$.

Yet, according to the covariance (3.1) we can get this result and even more precise bounds with explicit expressions for the constants appearing in the result. Indeed, fix $t > 0$ and for every $x, y \in \mathbb{R}$, we have:

$$\begin{aligned} & \mathbf{E}\Big(|u(t,x) - u(t,y)|^2\Big) \\ = \; & 2\Big(R(x,x) - R(x,y)\Big) \\ = \; & \frac{t^{2H+1}}{2H+1} - \left(c_H |y-x|^{2H+1} - \frac{t|y-x|^{2H}}{2} + \frac{t^{2H+1}}{2H+1} \right) \mathbb{1}_{\{|y-x|<t\}} \\ & - \frac{(2t - |y-x|)^{2H+1}}{4(2H+1)} \mathbb{1}_{\{t \leq |y-x| < 2t\}}. \end{aligned} \tag{3.16}$$

For the first situation when $|y - x| < t$, we get:

$$\begin{aligned} \mathbf{E}\Big(\, |u(t,x) - u(t,y)|^2 \, \Big) &= \frac{t}{2}|y-x|^{2H} - c_H |y-x|^{2H+1} \\ &= |y-x|^{2H} \left(\frac{t}{2} - \frac{4H-1}{4(2H+1)} |y-x| \right). \end{aligned} \tag{3.17}$$

Which obviously leads to:

$$\mathbf{E}\Big(|u(t,x) - u(t,y)|^2\Big) \geq \frac{3t}{4(2H+1)} |y-x|^{2H} \tag{3.18}$$

and

$$\begin{aligned} \mathbf{E}\Big(|u(t,x) - u(t,y)|^2\Big) &\leq |y-x|^{2H} \left(\frac{t}{2} + \frac{4H-1}{4(2H+1)} |y-x| \right) \\ &\leq |y-x|^{2H} \left(\frac{t}{2} + \frac{4H-1}{4(2H+1)} t \right) \\ &= \frac{(8H+1)t}{4(2H+1)} |y-x|^{2H}. \end{aligned}$$

For the second situation when $t \leq |y - x| < 2t$, it implies that:

$$\mathbf{E}\Big(\, |u(t,x) - u(t,y)|^2 \, \Big) = \frac{1}{2H+1} \left(t^{2H+1} - \frac{1}{4}\Big(2t - |y-x|\Big)^{2H+1} \right)$$

$$= \frac{t^{2H+1}}{2H+1}\left(1-\frac{1}{4}\Big(2-\frac{|y-x|}{t}\Big)^{2H+1}\right). \quad (3.19)$$

Since in this case we have $1 \le \frac{|y-x|}{t} < 2$, so it yields that:

$$\frac{3}{4} \le 1-\frac{1}{4}\Big(2-\frac{|y-x|}{t}\Big)^{2H+1} < 1. \quad (3.20)$$

Consequently we deduce that:

$$\begin{aligned} \mathbf{E}\Big(|u(t,x)-u(t,y)|^2 \Big) &< \frac{t^{2H+1}}{2H+1} \\ &\le \frac{t}{2H+1}|y-x|^{2H} \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} \mathbf{E}\Big(|u(t,x)-u(t,y)|^2 \Big) &\ge \frac{3t^{2H+1}}{4(2H+1)} \\ &\ge \frac{3t}{2^{2H+2}(2H+1)}|y-x|^{2H}. \end{aligned} \quad (3.22)$$

Actually, in this second case and by the same reasoning we can obtain more regularity for the mean square spatial increment, for instance:

$$C_2|y-x|^{2H+1} \le \mathbf{E}\Big(|u(t,x)-u(t,y)|^2 \Big) \le C_1|y-x|^{2H+1}. \quad (3.23)$$

3.2.4. Spatial modulus of continuity

From the above estimates on the spatial increment of the solution, we can deduce useful information on the modulus of continuity of the process $(u(t,x))_{x\in\mathbb{R}}$ given by (2.10) with respect to its space variable.

Let us fist recall some general definitions. Let f be an increasing function in $\mathbb{R}_+$ such that $\lim_{x\to 0^+} f(x) = 0$. Let $(Y_t)_{t\in I}$ be a stochastic process with index set $I \subset \mathbb{R}$ and let ρ be a metric on I. We say that the function f is an almost sure uniform modulus of continuity on (I,ρ), if there exists an almost-surely positive random variable α_0 such that for $\alpha < \alpha_0$ one has

$$\sup_{s,t\in T;\rho(s,t)<\alpha} |Y_t - Y_s| \le f(\alpha).$$

For (sub)Gaussian processes, there is a wide theory on the modulus of continuity in terms of the covariance structure of the process and the magnitude of its increment (see, among others, [11], [12], [18], [27], [30]). In particular, if Y is a Gaussian process such that

$$\mathbf{E}\big(|Y_t - Y_s|\big)^2 \le G(|s-t|) \quad (3.24)$$

for all $s, t \in I$, where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $G(0) = 0$, then

$$f(\varepsilon) = \int_0^\varepsilon (\log r)^{\frac{1}{2}} \mathrm{d}G(r)$$

is an almost sure uniform modulus of continuity for Y on I with respect to the Euclidean metric.

From the estimates (3.19) and (3.20) in Section 3.2.3, we can immediately get the modulus of continuity of the Gaussian process $(u(t,x))_{x\in\mathbb{R}}$. Indeed, fix $t > 0$. By the bound (3.19) and (3.20), we easily deduce that

$$\mathbf{E}\big(|u(t,x) - u(t,y)|\big)^2 \le C|x-y|^{2H}$$

for every $x, y \in [-M, M]$ with $M > t$ and $C > 0$ a non-random constant (depending on t and H). Consequently (3.24) is satisfied for $(u(t,x))_{x\in\mathbb{R}}$ and the function

$$f(r) = \int_0^r (\log x)^{\frac{1}{2}} d(x^H) \sim (\log r)^{\frac{1}{2}} r^H$$

will be an almost sure uniform modulus of continuity for $(u(t,x))_{x\in\mathbb{R}}$ on any interval $[-M, M]$ with $M > t$.

Remark 5. *There is an interesting impact on the fractional noise in time of the spatial regularity of the solution. Indeed, the almost sure modulus of continuity of the solution coincides with the modulus of continuity of the fractional Brownian motion. A similar phenomenon can be observed in the next section in the study of the central limit theorem for the spatial quadratic variations of u.*

3.2.5. Relation with the fractional Brownian motion

Another interesting property resulting from the formula (3.1) is that the spatial increments of the solution are related to the increments of the fractional Brownian motion. Indeed, assume t is fixed and x, y are such that $|y - x| < t$ (this will be the case in the next section). In this case, it follows from (3.1) that

$$\mathbf{E}\left((u(t,x+1) - u(t,x))(u(t,y+1) - u(t,y))\right) = \frac{t}{2}\varphi_H(|y-x|) - c_H \varphi_{H+\frac{1}{2}}(|y-x|)$$

where

$$\varphi_H(k) := \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right), \quad k \in \mathbb{R}. \tag{3.25}$$

That means that the increment of $u(t,x)$ with t is fixed and x, y such that $|y - x| < t$ is the same in distribution as the increment of B^H and $B^{H+\frac{1}{2}}$ where B^H and $B^{H+\frac{1}{2}}$ are two independent fractional Brownian motions with Hurst indices H and $H + \frac{1}{2}$ respectively (if the second fBm would exist).

For example, if $t > 2$ the process $(u(t,x+1) - u(t,x))_{x\in[0,1]}$ has the same distribution as $\left((B^H(x+1) - B^H(x)) + B^{H+\frac{1}{2}}(x+1) - B^{H+\frac{1}{2}}(x)\right)_{x\in[0,1]}$.

Since the behavior of our sequence $(V_N)_{N\geq 1}$(defined below in(4.1)) relies on how big is the correlation of its spatial increments, we will estimate its $L^2(\Omega)$-norm by using essentially the earlier expression (3.1) and some elements of Malliavin calculus.

4. The spatial quadratic variation

Fix $d = 1$ and the time $t \in (0, T]$, $T > 0$. Take an equidistant spatial partition of the unit rectangle $[0, 1]$ such that for every $N \geq 1$ and for every $j = 0, \ldots, N$, we designate by $x_j = \frac{j}{N}$. The centered renormalized quadratic variation statistic over the unit interval $[0, 1]$, can be defined in the following way:

$$V_N = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \left[\frac{\left(u(t, \frac{j+1}{N}) - u(t, \frac{j}{N})\right)^2}{\mathbf{E}\left(u(t, \frac{j+1}{N}) - u(t, \frac{j}{N})\right)^2} - 1 \right]. \tag{4.1}$$

Our purpose is to find the asymptotic behavior of the renormalized partial sum V_N as N goes to infinity.

To prove this, a convenient device that we will use the recent Stein-Malliavin theory, see [20]. To this end, we need to introduce the basic tools of the Malliavin calculus.

4.1. Malliavin calculus

We assume that the reader has a basic knowledge of such notions from stochastic analysis. Here, we shall only recall some elementary facts; our main reference on this realm is [22]. Consider $\mathcal{H}$ a real separable infinite-dimensional Hilbert space with its associated inner product $\langle ., . \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$, for every $\varphi, \psi \in \mathcal{H}$. Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{q!}}\| \cdot \|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by:

$$H_q(x) = \frac{(-1)^q}{q!} \exp\left(\frac{x^2}{2}\right) \frac{\mathrm{d}^q}{\mathrm{d}x^q} \left(\exp\left(-\frac{x^2}{2}\right)\right), \; x \in \mathbb{R}. \tag{4.2}$$

The isometry of multiple integrals can be written as: for p, $q \geq 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbf{E}\Big(I_p(f)I_q(g)\Big) = \begin{cases} q!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

It also holds that:

$$I_q(f) = I_q\big(\tilde{f}\big),$$

where $\tilde{f}$ denotes the canonical symmetrization of f and it is defined by:

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}),$$

in which the sum runs over all permutations σ of $\{1, \ldots, q\}$.

We recall that any square-integrable random variable F, which is measurable with respect to the σ-algebra generated by B, can be expanded into an orthogonal sum of multiple stochastic integrals:

$$F = \mathbf{E}\,(F) + \sum_{q=1}^{\infty} I_q(f_q), \tag{4.4}$$

where the series converges in $L^2(\Omega)$-sense and the kernels f_q, belonging to $\mathcal{H}^{\odot q}$, are uniquely determined by F.

We denote by D the Malliavin derivative operator that acts on cylindrical random variables of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\varphi_i \in \mathcal{H}$. This derivative is an element of $L^2(\Omega, \mathcal{H})$ and it is defined as:

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}\left(\mathcal{H}\right)$ into $\mathbb{D}^{\alpha-1,p}\left(\mathcal{H}\right)$.

We will need the general formula for calculating products of Wiener chaos integrals of any orders $p, q \geq 1$, so, for any symmetric integrands $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, it is:

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{4.5}$$

In the particular case when $\mathcal{H} = L^2([0,T])$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$, which is defined by:

$$\begin{aligned} &(f \otimes_r g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \textstyle\int_{[0,T]^r} \mathrm{d}u_1 \ldots \mathrm{d}u_r f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r), \end{aligned} \tag{4.6}$$

for every $f \in L^2([0,T]^p)$, $g \in L^2([0,T]^q)$ and $r = 1, \ldots, p \wedge q$.

4.2. Renormalization of V_N

To renormalize V_N we need to understand the behavior of $\mathbf{E}\big(V_N^2\big)$ as $N \to \infty$. To this end, we need a careful analysis of the auto-correlation $\mathbf{E}\Big((u(t,x_{i+1}) - u(t,x_i))(u(t,x_{j+1}) - u(t,x_j))\Big)$ with $x_i = \frac{i}{N}$ for $i = 0, 1, .., N$.

Applying the expression (3.1), we clearly obtain for every $0 \le i, j \le N$:

$$\begin{aligned}
&\Delta R(i,j) \\
:= \;& \mathbf{E}\Big((u(t,x_{i+1}) - u(t,x_i))(u(t,x_{j+1}) - u(t,x_j))\Big) \\
= \;& \frac{1}{2}\Bigg[2\Big(c_H \frac{|i-j|^{2H+1}}{N^{2H+1}} - \frac{t|i-j|^{2H}}{2N^{2H}} + \frac{t^{2H+1}}{2H+1}\Big)\mathbb{1}_{\{|i-j|<Nt\}} \\
& -\Big(c_H \frac{|i-j+1|^{2H+1}}{N^{2H+1}} - \frac{t|i-j+1|^{2H}}{2N^{2H}} + \frac{t^{2H+1}}{2H+1}\Big)\mathbb{1}_{\{|i-j+1|<tN\}} \\
& - \Big(c_H \frac{|i-j-1|^{2H+1}}{N^{2H+1}} - \frac{t|i-j-1|^{2H}}{2N^{2H}} + \frac{t^{2H+1}}{2H+1}\Big)\mathbb{1}_{\{|i-j-1|<tN\}}\Bigg] \\
& -\frac{1}{8(2H+1)}\Bigg[\Big(2t - \frac{|i-j+1|}{N}\Big)^{2H+1}\mathbb{1}_{\{tN \le |i-j+1| < 2tN\}} \\
& -2\Big(2t - \frac{|i-j|}{N}\Big)^{2H+1}\mathbb{1}_{\{tN \le |i-j| < 2tN\}} \\
& +\Big(2t - \frac{|i-j-1|}{N}\Big)^{2H+1}\mathbb{1}_{\{tN \le |i-j-1| < 2tN\}}\Bigg]. \qquad (4.7)
\end{aligned}$$

According to the value of the temporal index t, the former covariance function (4.7) can be written in the following way:

• If $t \in (1, T]$, it yields that $|i-j+1| < tN$ and we get:

$$\Delta R(i,j) = k_1 \varphi_H(\frac{i-j}{N}) + k_2 \varphi_{H+\frac{1}{2}}(\frac{i-j}{N}), \qquad (4.8)$$

where φ_H is given by (3.25) and

$$k_1 = \frac{t}{2}, \quad k_2 = -c_H. \qquad (4.9)$$

Recall that $\varphi_H(k) \sim_{k\to\infty} 2H(2H-1)|k|^{2H-2}$, with the symbol "$\sim$" means that both sides have the same limit as k goes to ∞.

As mentioned in the subsection 3.2.5, this means that, when the time is large enough, the increment of the spatial wave solution behaves as the sum of the increments generated by the fBm B^H modulo a constant and the fBm $B^{H+\frac{1}{2}}$ modulo a constant, with $B^{H+\frac{1}{2}}$ independent by B^H.

• If $t \in (0, 1]$, several situations can be evoked according to the values of tN and also of $2tN$, that is why it seems to be quiet hard to estimate the $L^2(\Omega)$-norm of the increments by using our approach basing on the expression of the covariance function (4.7).

For the rest of the work we adopt that the time $t \in (1, T]$. Due to the Subsection 4.1, we denote by $\mathcal{H}$ the canonical Hilbert space associated to the Gaussian solution process $(u(t,x))_{x\in[0,1]}$. This Hilbert space is defined as the closure of the set ξ of indicator functions $\mathbb{1}_{[0,x]},\ x>0$, with respect to the inner product:

$$\langle \mathbb{1}_{[0,x]}, \mathbb{1}_{[0,y]}\rangle_{\mathcal{H}} = \mathbf{E}\Big(u(t,x)u(t,y)\Big), \text{ for a fixed } t\in(1,T].$$

The Gaussian space generated by $(u(t,x))_{x\in[0,1]}$, $t\in(1,T]$, can be identified with an isonormal Gaussian process of the type $(X(h))_{h\in\mathcal{H}}$. We also designate by $I_q,\ q\geq 1$ the multiple Wiener-integral with respect to the Gaussian process $(u(t,x))_{x\in[0,1]}$, so the increment $u(t,y)-u(t,x)$ can be expressed as $I_1(\mathbb{1}_{[x,y]})$, for every $x<y$.

Define now the next sequence:

$$F_N = \frac{V_N}{\sqrt{v_N}}, \quad \text{with } v_N = \mathbf{E}(V_N^2),\ N\geq 1. \tag{4.10}$$

Using again (4.7) when $t\in(1,T]$, it implies that:

$$\mathbf{E}\Big(u(t,x_{j+1})-u(t,x_j)\Big)^2 = \frac{k_1}{N^{2H}} + \frac{k_2}{N^{2H+1}}, \tag{4.11}$$

and consequently, for $N\geq 1$, F_N can be re-written as:

$$\begin{aligned}
F_N &= \frac{1}{\sqrt{v_N N}}\left[\frac{1}{\frac{k_1}{N^{2H}}+\frac{k_2}{N^{2H+1}}}\right] \\
&\sum_{j=0}^{N-1}\left[\left(u(t,\frac{j+1}{N})-u(t,\frac{j}{N})\right)^2 - \mathbf{E}\left(u(t,\frac{j+1}{N})-u(t,\frac{j}{N})\right)^2\right] \\
&= \frac{1}{\sqrt{v_N N}}\left[\frac{1}{\frac{k_1}{N^{2H}}+\frac{k_2}{N^{2H+1}}}\right] I_2\left(\sum_{j=0}^{N-1}\mathbb{1}_{[x_j,x_{j+1}]}^{\otimes 2}\right) \\
&= \frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1N+k_2)} I_2\left(\sum_{j=0}^{N-1}\mathbb{1}_{[x_j,x_{j+1}]}^{\otimes 2}\right).
\end{aligned} \tag{4.12}$$

The next lemma shows that the deterministic sequence $v_N>0$ converges, as N goes to ∞, to a strictly positive constant. It will play no role in the limit behavior of $(F_N)_{N\geq 1}$, it is just used to normalize and to guarantee that $\mathbf{E}({F_N}^2)=1$.

Lemma 2. *Assume $H\in(\frac{1}{2},\frac{3}{4})$ and $t>1$. Then, there exists a constant $C_0\in(0,\infty)$ such that:*

$$v_N \to_{N\to\infty} C_0,$$

with $C_0 := 2\sum_{k\in\mathbb{Z}}(\varphi_H(k))^2$.

Proof: Thanks to (4.8), we have:

$$v_N = \frac{2N^{4H+1}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}$$

$$
\begin{aligned}
& \left[\mathbf{E}\left(\left(u(t,\frac{i+1}{N})-u(t,\frac{i}{N})\right)\left(u(t,\frac{j+1}{N})-u(t,\frac{j}{N})\right)\right)\right]^2 \\
= \; & \frac{2N^{4H+1}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1} \\
& \left[k_1\frac{\varphi_H(i-j)}{N^{2H}}+k_2\frac{\varphi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}}\right]^2 \\
= \; & 2k_1^2\frac{N^{4H+1}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\left(\frac{\varphi_H(i-j)}{N^{2H}}\right)^2+ \\
& 4k_1k_2\frac{N^{4H+1}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\frac{\varphi_H(i-j)}{N^{2H}}\frac{\varphi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}} \\
& +2k_2^2\frac{N^{4H+1}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\left(\frac{\varphi_{H+\frac{1}{2}}(i-j)}{N^{2H+1}}\right)^2 \\
:= \; & T_1+T_2+T_3. \qquad (4.13)
\end{aligned}
$$

It turns out to check that the first term T_1 is the dominant one, and it converges to a finite limit, while the other terms are negligible. So, by using the dominated convergence theorem, we get that:

$$
\begin{aligned}
T_1 \;:=\; & 2k_1^2\frac{N^{4H+1}}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\left(\frac{\varphi_H(i-j)}{N^{2H}}\right)^2 \\
= \; & \frac{2k_1^2N^2}{(k_1N+k_2)^2}\sum_{k\in\mathbb{Z}}\left(\varphi_H(k)\right)^2\left(1-\frac{|k|}{N}\right)\mathbb{1}_{\{|k|<N\}} \\
& \to_{N\to\infty} 2\sum_{k\in\mathbb{Z}}\left(\varphi_H(k)\right)^2 = C_0. \qquad (4.14)
\end{aligned}
$$

Recall that the function $\varphi_H(k)$ behaves, for k large enough, as $H(2H-2)|k|^{2H-2}$, which yields that the series $\sum_{k\in\mathbb{Z}}\left(\varphi_H(k)\right)^2$ is finite only if $H<\frac{3}{4}$. Also, as $\varphi_H(0)=1$, it is obvious that $C_0\in(0,\infty)$.

Moving to the second term T_2, we obtain:

$$
\begin{aligned}
T_2 \;=\; & \frac{4k_1k_2}{(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\varphi_H(i-j)\varphi_{H+\frac{1}{2}}(i-j) \\
\leq \; & \frac{4k_1k_2N}{(k_1N+k_2)^2}\sum_{k=0}^{N-1}\left|\varphi_H(k)\varphi_{H+\frac{1}{2}}(k)\right| \\
\leq \; & \frac{C}{N}\sum_{k=1}^{N}k^{4H-3}\to_{N\to\infty}0, \qquad (4.15)
\end{aligned}
$$

which is true due to the Cesàro-Lemma and the fact that $H<\frac{3}{4}$.

Similarly as for the second term, we deduce that:

$$\begin{aligned} T_3 &= \frac{2k_2^2}{N(k_1N+k_2)^2}\sum_{i,j=0}^{N-1}\Big(\varphi_{H+\frac{1}{2}}(i-j)\Big)^2 \\ &\leq \frac{2k_2^2}{(k_1N+k_2)^2}\sum_{k=0}^{N-1}\Big(\varphi_{H+\frac{1}{2}}(k)\Big)^2 \\ &\leq \frac{C}{N^2}\sum_{k=1}^{N}k^{4H-2} \\ &\leq CN^{4H-3}\rightarrow_{N\rightarrow\infty} 0. \end{aligned} \tag{4.16}$$

By putting together (4.14), (4.15) and (4.16), the proof is completed. □

Remark 6. *The appearance of the threshold $H=\frac{3}{4}$ is interesting. Note that H represents the self-similarity index of the noise (in time) and not of the solution. This is related to the observation noticed in the Section 3.2.5: the main part of V_N comes from the fBm B^H. A similar phenomenon has been noticed in [25].*

5. Central Limit Theorem and rate of convergence

In order to verify the CLT for the sequence $(V_N)_{N\geq 1}$, our main tool will be the Theorem 5.2.6 in [20], which provides a description of the normal approximation of multiple stochastic integrals, by the aid of explicit bounds of the well-known distances d (Kolmogorov, Total Variation, Wasserstein). This result is based on the classical Berry-Esseen inequality.

Theorem 1. *Fix $q\geq 1$. Let $(F_N)_{N\geq 1}=(I_q(f_N))_{N\geq 1}$ with $f_N\in\mathcal{H}^{\odot q}$, be a sequence of random variables belonging to the qth Wiener chaos such that:*

$$\mathbf{E}(F_N^2)\longrightarrow_{N\rightarrow\infty}\sigma^2.$$

Then, F_N converges in law to $Z\sim\mathcal{N}(0,1)$ if and only if

$$\|DF_N\|^2_{\mathcal{H}}\longrightarrow_{N\rightarrow\infty}q\sigma^2.$$

Furthermore,

$$d\left(F_N;\mathcal{N}(0,1)\right)\leq C\sqrt{\mathbf{Var}\Big(\frac{1}{q}\|DF_N\|^2_{\mathcal{H}}\Big)}.$$

5.1. Main result

Our first main result is summarized in the following lemma.

Lemma 3. *Fix $t\in(1,T]$ and assume that $H\in(\frac{1}{2},\frac{3}{4})$, the sequence of random variables $(F_N)_{N\geq 1}$ given by (4.12) converges in distribution, as $N\rightarrow\infty$, to* *the standard normal law $\mathcal{N}(0,1)$. Moreover, for $N \geq 3$, there exists a constant $C > 0$ (depending only on t and H) such that:*

$$d\left(F_N;\mathcal{N}(0,1)\right) \leq C \begin{cases} \frac{1}{\sqrt{N}} & \textit{if } H \in (\frac{1}{2},\frac{5}{8}) \\ \sqrt{\frac{\log^3 N}{N}} & \textit{if } H = \frac{5}{8} \\ N^{4H-3} & \textit{if } H \in (\frac{5}{8},\frac{3}{4}). \end{cases} \tag{5.1}$$

Proof: We apply Theorem 1. Since for all $N \geq 1$:

$$F_N = I_2(f_N), \text{ with } f_N = \frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1 N + k_2)} \sum_{j=0}^{N-1} \mathbb{1}_{[x_j,x_{j+1}]}^{\otimes 2} \in \mathcal{H}^{\odot 2},$$
$$\text{and } \mathbf{E}(F_N)^2 = 1,$$

it is a enough to estimate the quantity:

$$\mathbf{Var}\Big(\frac{1}{2}\|DF_N\|_{\mathcal{H}}^2\Big).$$

From Subsection 4.1, we get:

$$\begin{aligned} &\|DF_N\|_{\mathcal{H}}^2 - \mathbf{E}\Big(\|DF_N\|_{\mathcal{H}}^2\Big) \\ = \ &\frac{4N^{4H+1}}{v_N(k_1N+k_2)^2} \sum_{j,k=0}^{N-1} \langle \mathbb{1}_{[x_j,x_{j+1}]}, \mathbb{1}_{[x_k,x_{k+1}]}\rangle_{\mathcal{H}} I_2\left(\mathbb{1}_{[x_j,x_{j+1}]} \otimes \mathbb{1}_{[x_k,x_{k+1}]}\right), \end{aligned}$$

which obviously gives:

$$\begin{aligned} &\mathbf{Var}\Big(\frac{1}{2}\|DF_N\|_{\mathcal{H}}^2\Big) \qquad (5.2) \\ = \ &\frac{1}{4}\mathbf{E}\Big(\|DF_N\|^2 - \mathbf{E}(\|DF_N\|^2)\Big)^2 \\ = \ &\frac{4N^{8H+2}}{v_N^2(k_1N+k_2)^4} \sum_{i,j,k,l=0}^{N-1} \langle \mathbb{1}_{[x_j,x_{j+1}]}, \mathbb{1}_{[x_k,x_{k+1}]}\rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_i,x_{i+1}]}, \mathbb{1}_{[x_l,x_{l+1}]}\rangle_{\mathcal{H}} \\ &\times \langle \mathbb{1}_{[x_j,x_{j+1}]}, \mathbb{1}_{[x_i,x_{i+1}]}\rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_k,x_{k+1}]}, \mathbb{1}_{[x_l,x_{l+1}]}\rangle_{\mathcal{H}} \\ = \ &\frac{4N^{8H+2}}{v_N^2(k_1N+k_2)^4} \sum_{i,j,k,l=0}^{N-1} \left[k_1 \frac{\varphi_H(j-k)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-k)}{N^{2H+1}}\right] \\ &\times \left[k_1 \frac{\varphi_H(i-l)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(i-l)}{N^{2H+1}}\right] \left[k_1 \frac{\varphi_H(j-i)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-i)}{N^{2H+1}}\right] \\ &\times \left[k_1 \frac{\varphi_H(k-l)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(k-l)}{N^{2H+1}}\right] \\ := \ &\sum_{i=1}^{5} P_i. \qquad (5.3) \end{aligned}$$

We designate by P_1 the product of four elements that are the functions φ_H. As well, P_2 contains all the terms with the form of a product of only one function φ_H and three functions $\varphi_{H+\frac{1}{2}}$. P_3 is composed of all the terms which are the product of two functions φ_H and two functions $\varphi_{H+\frac{1}{2}}$. P_4 contains all the terms which are written as a product of three functions φ_H and one function $\varphi_{H+\frac{1}{2}}$, and finally P_5 is the product of four functions $\varphi_{H+\frac{1}{2}}$.

The biggest term in (5.2) is P_1, straightforward calculations show that all the other terms are negligible. Typically referred to the proof of the Theorem 7.3.1 in [20], we set, for all $N \geq 1$, the functions:

$$\varphi_{H,N}(k) = |\varphi_H(k)|\mathbb{1}_{\{|k|\leq N-1\}} \text{ and } \varphi_{H+\frac{1}{2},N}(k) = |\varphi_{H+\frac{1}{2}}(k)|\mathbb{1}_{\{|k|\leq N-1\}}$$

for every $k \in \mathbb{Z}$. Also, for two sequences $(u(n), n \in \mathbb{Z})$ and $(v(n), n \in \mathbb{Z})$, we define their convolution by

$$(u * v)(j) = \sum_{n\in\mathbb{Z}} u(n)v(j-n).$$

We will need the Young's inequality

$$\|u * v\|_{\ell^s(\mathbb{Z})} \leq \|u\|_{\ell^p(\mathbb{Z})}\|v\|_{\ell^q(\mathbb{Z})} \tag{5.4}$$

if $s, p, q \geq 1$ with $\frac{1}{s} + 1 = \frac{1}{p} + \frac{1}{q}$.

We can write:

$$\begin{aligned}
P_1 &= \frac{4k_1^4 N^{8H+2}}{v_N^2(k_1N+k_2)^4} \sum_{i,j,k,l=0}^{N-1} \frac{\varphi_H(j-k)\varphi_H(i-l)\varphi_H(j-i)\varphi_H(k-l)}{N^{8H}} \\
&\leq \frac{4k_1^4 N^2}{v_N^2(k_1N+k_2)^4} \\
&\quad \sum_{i,k=0}^{N-1}\sum_{j,l\in\mathbb{Z}} \varphi_{H,N}(j-k)\varphi_{H,N}(i-l)\varphi_{H,N}(j-i)\varphi_{H,N}(k-l) \\
&\leq \frac{C}{N^2}\sum_{i,k=0}^{N-1} \left(\varphi_{H,N} * \varphi_{H,N}\right)^2(k-i) \\
&= \frac{C}{N}\sum_{k\in\mathbb{Z}} \left(\varphi_{H,N} * \varphi_{H,N}\right)^2(k) \\
&= \frac{C}{N}\|\varphi_{H,N} * \varphi_{H,N}\|^2_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

By the aid of Young's inequality (5.4) for $s = 2$ and $p = q = \frac{4}{3}$, we obtain:

$$\begin{aligned}
P_1 &\leq \frac{C}{N}\|\varphi_{H,N}\|^4_{\ell^{\frac{4}{3}}(\mathbb{Z})} \\
&= \frac{C}{N}\Big(\sum_{k=1-N}^{N-1} |\varphi_H(k)|^{\frac{4}{3}}\Big)^3.
\end{aligned}$$

As the function φ_H behaves like $C|k|^{2H-2}$ when $|k|$ goes to ∞, it yields that:

$$\sum_{k=1-N}^{N-1} |\varphi_H(k)|^{\frac{4}{3}} = \begin{cases} O(1) & \text{if } H \in (0, \frac{5}{8}) \\ O(\log N) & \text{if } H = \frac{5}{8} \\ O(N^{\frac{8H-5}{3}}) & \text{if } H \in (\frac{5}{8}, 1). \end{cases} \tag{5.5}$$

Hence, recall that $v_n \in (0, \infty)$ only for $H \in (\frac{1}{2}, \frac{3}{4})$, we deduce the following result:

$$P_1 \le C \begin{cases} \frac{1}{N} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log N)^3}{N} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \tag{5.6}$$

The same reasoning will be used for the other terms, giving:

$$\begin{aligned} P_2 &= \frac{4k_1k_2^3N^{8H+2}}{v_N^2(k_1N+k_2)^4} \sum_{i,j,k,l=0}^{N-1} \frac{\varphi_H(j-k)\varphi_{H+\frac{1}{2}}(i-l)\varphi_{H+\frac{1}{2}}(j-i)\varphi_{H+\frac{1}{2}}(k-l)}{N^{8H+3}} \\ &\le \frac{4k_1k_2^3}{v_N^2(k_1N+k_2)^4 N} \\ &\quad \sum_{i,k=0}^{N-1} \sum_{j,l\in\mathbb{Z}} \varphi_{H,N}(j-k)\varphi_{H+\frac{1}{2},N}(i-l)\varphi_{H+\frac{1}{2},N}(j-i)\varphi_{H+\frac{1}{2},N}(k-l) \\ &\le \frac{C}{N^4} \sum_{k\in\mathbb{Z}} (\varphi_{H,N} * \varphi_{H+\frac{1}{2},N})(k)(\varphi_{H+\frac{1}{2},N} * \varphi_{H+\frac{1}{2},N})(k) \\ &\le \frac{C}{N^4} \Big(\sum_{k\in\mathbb{Z}} (\varphi_{H,N} * \varphi_{H+\frac{1}{2},N})^2(k) \Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{Z}} (\varphi_{H+\frac{1}{2},N} * \varphi_{H+\frac{1}{2},N})^2(k) \Big)^{\frac{1}{2}} \\ &= \frac{C}{N^4} \|\varphi_{H,N} * \varphi_{H+\frac{1}{2},N}\|_{\ell^2(\mathbb{Z})} \|\varphi_{H+\frac{1}{2},N} * \varphi_{H+\frac{1}{2},N}\|_{\ell^2(\mathbb{Z})} \\ &\le \frac{C}{N^4} \|\varphi_{H,N}\|_{\ell^{\frac{4}{3}}(\mathbb{Z})} \|\varphi_{H+\frac{1}{2},N}\|^3_{\ell^{\frac{4}{3}}(\mathbb{Z})} \\ &= \frac{C}{N^4} \Big(\sum_{k=1-N}^{N-1} |\varphi_H(k)|^{\frac{4}{3}} \Big)^{\frac{3}{4}} \Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}} \Big)^{\frac{9}{4}}. \end{aligned}$$

Due to the bounds in (5.5), and the asymptotic behavior of $\varphi_{H+\frac{1}{2}}$ which shows that $\frac{1}{N^4}\Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}} \Big)^{\frac{9}{4}} \le CN^{6H-\frac{19}{4}}$, we get for $N \ge 3$:

$$P_2 \le C \begin{cases} N^{6H-\frac{19}{4}} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ (\log N)^{\frac{3}{4}} N^{6H-\frac{19}{4}} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

$$\Rightarrow \leq C \begin{cases} \frac{1}{N} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log N)^3}{N} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \tag{5.7}$$

Similarly,

$$\begin{aligned} P_3 &= \frac{4k_1^2 k_2^2 N^{8H+2}}{v_N^2 (k_1 N + k_2)^4} \sum_{i,j,k,l=0}^{N-1} \frac{\varphi_H(j-k)\varphi_H(i-l)\varphi_{H+\frac{1}{2}}(j-i)\varphi_{H+\frac{1}{2}}(k-l)}{N^{8H+2}} \\ &\leq \frac{C}{N^3} \Big(\sum_{k=1-N}^{N-1} |\varphi_H(k)|^{\frac{4}{3}} \Big)^{\frac{3}{2}} \Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}} \Big)^{\frac{3}{2}}, \end{aligned}$$

and since $\frac{1}{N^3} \Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}} \Big)^{\frac{3}{2}} \leq C N^{4H-\frac{7}{2}}$, we can write that for $N \geq 3$:

$$\begin{aligned} P_3 \leq C &\begin{cases} N^{4H-\frac{7}{2}} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ (\log N)^{\frac{3}{2}} N^{4H-\frac{7}{2}} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \\ \Rightarrow \leq C &\begin{cases} \frac{1}{N} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log N)^3}{N} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \end{aligned} \tag{5.8}$$

Also,

$$\begin{aligned} P_4 &= \frac{4k_1^3 k_2 N^{8H+2}}{v_N^2 (k_1 N + k_2)^4} \sum_{i,j,k,l=0}^{N-1} \frac{\varphi_H(j-k)\varphi_H(i-l)\varphi_H(j-i)\varphi_{H+\frac{1}{2}}(k-l)}{N^{8H+1}} \\ &\leq \frac{C}{N^2} \Big(\sum_{k=1-N}^{N-1} |\varphi_H(k)|^{\frac{4}{3}} \Big)^{\frac{9}{4}} \Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}} \Big)^{\frac{3}{4}}. \end{aligned}$$

Note that $\frac{1}{N^2} \Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}} \Big)^{\frac{3}{4}} \leq N^{2H-\frac{9}{4}}$, so it implies that for $N \geq 3$:

$$\begin{aligned} P_4 \leq C &\begin{cases} N^{2H-\frac{9}{4}} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ (\log N)^{\frac{9}{4}} N^{2H-\frac{9}{4}} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \\ \Rightarrow \leq C &\begin{cases} \frac{1}{N} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log N)^3}{N} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \end{aligned} \tag{5.9}$$

And finally,

$$\begin{aligned} P_5 &= \frac{4k_2^4 N^{8H+2}}{v_N^2 (k_1 N + k_2)^4} \sum_{i,j,k,l=0}^{N-1} \\ &\quad \frac{\varphi_{H+\frac{1}{2}}(j-k)\varphi_{H+\frac{1}{2}}(i-l)\varphi_{H+\frac{1}{2}}(j-i)\varphi_{H+\frac{1}{2}}(k-l)}{N^{8H+4}} \\ &\leq \frac{C}{N^5}\Big(\sum_{k=1-N}^{N-1} |\varphi_{H+\frac{1}{2}}(k)|^{\frac{4}{3}}\Big)^3 \\ &\leq CN^{8H-6}. \end{aligned}$$

By the reason of $N^{8H-6} \leq \frac{1}{N}$ for all $H \in (\frac{1}{2}, \frac{5}{8}]$, it yields that for all $N \geq 3$:

$$P_5 \leq C \begin{cases} \frac{1}{N} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log N)^3}{N} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}), \end{cases} \tag{5.10}$$

Combining all the results (5.6)-(5.10) with (5.2), we infer that:

$$d\left(F_N; \mathcal{N}(0,1)\right) \leq C\sqrt{\mathbf{Var}\Big(\frac{1}{2}\|DF_N\|_{\mathcal{H}}^2\Big)} \leq C \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \sqrt{\frac{\log^3 N}{N}} & \text{if } H = \frac{5}{8} \\ N^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases}$$

which checked the desired bounds. □

Remark 7. • *The previous result has a discrete version which is mentioned in the Theorem 7.3.1 of [20], with the stationary Gaussian sequence* $\left(\frac{v_t(k)}{\sqrt{\mathbf{E}(v_t(k)^2)}}\right)_{k\in\mathbb{Z}}$*, this sequence indicates the normalized noise generated by the spatial wave solution such that* $v_t(k) := u(t, k+1) - u(t,k),\ k \in \mathbb{Z}$.

• *A concrete interpretation is there exist a remarkable impact of the fractional Brownian motion on the spatial increments of the wave-solution, the result that we obtained shows that when,* $H \in (\frac{1}{2}, \frac{3}{4})$*, the quadratic variation of* B^H *and the spatial quadratic variation of the wave solution have the same rate of convergence to the normal distribution, although the fractional part of the noise is temporal and all our work is done in the spatial case. This reinforces the idea that the study of Gaussian processes does not necessarily rely on the analysis of filtrations, Markovian aspects etc.*

Remark 8. *Notice that the threshold* $\frac{5}{8}$ *is the same as in case of the fBm (see [20]). The same result as (5.1), modulo a change of the constant, holds for the different distances between the laws of random variables (e.g. Kolmogorov, Total Variations, Wasserstein ...).*

5.2. Optimal rate of convergence

In the case of the total variation distance (denoted d_{TV} in the sequel), we can obtain better estimates for the distance between F_N and the standard normal law than those in (5.1). This can be done via the main result in [21] (see also [19]) where is proved that the optimal rate of convergence to the standard normal law of a sequence $(F_N)_{N\geq 1}$ in the qth Wiener chaos, under the distance d_{TV}, is given by the quantity

$$M(N) := \max\left(\mathbf{E}\big(F_N^3\big), \sqrt{\mathbf{Var}\Big(\frac{1}{q}\|DF_N\|_{\mathcal{H}}^2\Big)}\right)$$

in the sense that

$$d_{TV}\left(F_N;\mathcal{N}(0,1)\right) \propto M(N) \tag{5.11}$$

where $u_n \propto v_n$ means that $0 < \overline{\lim\limits_{n\to\infty}}\, \dfrac{u_n}{v_n} < \infty$. Based on this result, we obtain the following sharp estimate for the rate of convergence of (4.10).

Lemma 4. *Fix $t \in (1,T]$ and assume that $H \in (\frac{1}{2}, \frac{3}{4})$. The sequence of random variables $(F_N)_{N\geq 1}$ (4.10) satisfies*

$$d_{TV}\left(F_N;\mathcal{N}(0,1)\right) \propto \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in (\frac{1}{2}, \frac{2}{3}) \\ \frac{(\log N)^2}{\sqrt{N}} & \text{if } H = \frac{2}{3} \\ N^{6H-\frac{9}{2}} & \text{if } H \in (\frac{2}{3}, \frac{3}{4}). \end{cases} \tag{5.12}$$

Proof: Recall that

$$F_N = I_2(f_N), \text{ with } f_N = \frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1N+k_2)} \sum_{j=0}^{N-1} \mathbb{1}_{[x_j,x_{j+1}]}^{\otimes 2} \in \mathcal{H}^{\odot 2}, \text{ and } \mathbf{E}(F_N^2) = 1.$$

We need to estimate the third moment of the sequence F_N. By using the product formula (4.5) and the isometry property (4.3), we have

$$\mathbf{E}\big(F_N^3\big) = 8\langle f_N \otimes_1 f_N, f_N\rangle_{\mathcal{H}}$$

Using the definition of the contraction (4.6),

$$f_N \otimes_1 f_N = \left(\frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1N+k_2)}\right)^2 \sum_{i,j=0}^{N-1} \mathbb{1}_{[x_i,x_{i+1}]} \otimes \mathbb{1}_{[x_j,x_{j+1}]} \langle \mathbb{1}_{[x_i,x_{i+1}]}, \mathbb{1}_{[x_j,x_{j+1}]}\rangle_{\mathcal{H}}$$

and so

$$\begin{aligned} &\langle f_N \otimes_1 f_N, f_N\rangle_{\mathcal{H}} \\ = &\left(\frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1N+k_2)}\right)^3 \end{aligned}$$

$$\sum_{i,j,k=0}^{N-1} \langle \mathbb{1}_{[x_i,x_{i+1}]}, \mathbb{1}_{[x_j,x_{j+1}]}\rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_i,x_{i+1}]} \otimes \mathbb{1}_{[x_j,x_{j+1}]}, \mathbb{1}^{\otimes 2}_{[x_k,x_{k+1}]}\rangle_{\mathcal{H}^{\otimes 2}}$$
$$= \left(\frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1N+k_2)}\right)^3$$
$$\sum_{i,j,k=0}^{N-1} \langle \mathbb{1}_{[x_i,x_{i+1}]}, \mathbb{1}_{[x_j,x_{j+1}]}\rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_i,x_{i+1}]}, \mathbb{1}_{[x_k,x_{k+1}]}\rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_k,x_{k+1}]}, \mathbb{1}_{[x_j,x_{j+1}]}\rangle_{\mathcal{H}}.$$

As in the proof of Lemma 3, the dominant part of $\langle \mathbb{1}_{[x_i,x_{i+1}]}, \mathbb{1}_{[x_j,x_{j+1}]}\rangle_{\mathcal{H}}$ is

$$\frac{\varphi_H(|i-j|)}{N^{2H}}$$

with φ_H given by (3.25). So

$$\begin{aligned}
&\langle f_N \otimes_1 f_N, f_N\rangle_{\mathcal{H}} \\
&\sim \left(\frac{N^{2H+\frac{1}{2}}}{\sqrt{v_N}(k_1N+k_2)}\right)^3 N^{-6H} \sum_{i,j,k=0}^{N-1} \varphi_H(i-j)\varphi_H(j-k)\varphi_H(k-i) \\
&\sim \frac{1}{v_N^{\frac{3}{2}} N^{\frac{3}{2}}} \sum_{i,j,k=0}^{N-1} \varphi_H(i-j)\varphi_H(j-k)\varphi_H(k-i).
\end{aligned}$$

This above quantity in the right-hand side already appeared in the case of the quadratic variations of the fBm. From [3] especially relation (6.57) (or Proposition 4.2 in [21]), we deduce the following

$$\mathbf{E}(F_N^3) \propto C \begin{cases} \frac{1}{\sqrt{N}} & \text{if } H \in (\frac{1}{2}, \frac{2}{3}) \\ \frac{(\log N)^2}{\sqrt{N}} & \text{if } H = \frac{2}{3} \\ N^{6H-\frac{9}{2}} & \text{if } H \in (\frac{2}{3}, \frac{3}{4}). \end{cases} \tag{5.13}$$

On the other hand, from the proof of Lemma 3 and Proposition 4.2 in [21] we can easily show that

$$\mathbf{Var}\left(\frac{1}{q}\|DF_N\|^2_{\mathcal{H}}\right) \propto C \begin{cases} \frac{1}{N} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log N)^3}{N} & \text{if } H = \frac{5}{8} \\ N^{8H-6} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). \end{cases} \tag{5.14}$$

From (5.11), (5.13) and (5.14), we obtain the conclusion. □

5.3. Estimation of the Hurst parameter H

The variations of a stochastic process play a crucial role in its probabilistic and statistical analysis. Amomg others, its asymptotic behavior is a fundamental tool in estimation theory.

We will show in this subsection that the asymptotic behavior of the sequence $(V_N)_{N\geq 1}$ defined in (4.1), is related to the asymptotic properties of a class of estimators for the Hurst parameter H. The construction has been used in [7] for the case of the fBm and in [28] and the references therein, for other selfsimilar processes.

The idea for estimating the parameter H of the fractional noise driven our wave equation, is based on the discrete spatial observations of the solution. As standard method is to construct an estimator based on the spatial quadratic variations of the process $(u(t,x))_{x\in[0,1]}$. For fixed $t \in (1,T]$, we observe $u(t,\frac{i}{N})$, $i = 0,\dots,N$, and we define the next sequence:

$$S_N := \frac{1}{N}\sum_{i=0}^{N-1}\left(u(t,\frac{j+1}{N}) - u(t,\frac{j}{N})\right)^2. \tag{5.15}$$

This implies that:

$$A_N := \mathbf{E}\Big(S_N\Big) = \frac{k_1}{N^{2H}} + \frac{k_2}{N^{2H+1}} = \frac{k_1}{N^{2H}}\left(1 + \frac{k_2}{k_1 N}\right) \tag{5.16}$$

with k_1, k_2 from (4.9) and thus

$$\begin{aligned} \log A_N = \log \mathbf{E}\Big(S_N\Big) &= \log k_1 - 2H\log N + \log\left(1 + \frac{k_2}{k_1 N}\right) \\ &\sim \log k_1 - 2H\log N, \ \text{ as } N \to \infty. \end{aligned} \tag{5.17}$$

Estimating A_N by S_N, we can construct the following estimator:

$$\widehat{H}_N = \frac{-\log S_N + \log k_1}{2\log N}. \tag{5.18}$$

An important relation that we can easily establish between V_N and S_N, for all $N \geq 1$, is at this form:

$$S_N = A_N\left(1 + \frac{V_N}{\sqrt{N}}\right). \tag{5.19}$$

So using the fact that $\log(1+x) \sim x$, for x close to 0, it gives that (because $\frac{V_N}{\sqrt{N}}$ converges almost surely to 0 as $N \to \infty$, see [28], Section 5.5):

$$\log S_N \sim \log A_N + \frac{V_N}{\sqrt{N}}, \ \text{ as } N \to \infty \tag{5.20}$$

Looking now at the expression (5.18) combined with results (5.20) and (5.17), we obtain:

$$H - \widehat{H}_N \sim \frac{V_N}{2\sqrt{N}\log N}, \ \text{ as } N \to \infty. \tag{5.21}$$

An immediate observation is that the behavior of the sequence $(V_N)_{N\geq 1}$ gives directly the behavior of $H - \widehat{H}_N$, and thus we get the following result.

Theorem 2. *Suppose that $H \in (\frac{1}{2}, \frac{3}{4})$, the estimator $\widehat{H}_N$ of the Hurst parameter H defined by (5.18) is strongly consistent, i.e.*

$$\widehat{H}_N \to_{N\to\infty} H \text{ almost surely.}$$

Moreover, $\widehat{H}_N$ is an asymptotically normal estimator such that:

$$\frac{2\sqrt{N}\log N}{\sqrt{v_N}}(H - \widehat{H}_N) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1). \tag{5.22}$$

Proof: Viewing the relation (5.21) and recalling that due to the Borel-Cantelli lemma, the sequence $(V_N)_{N\geq}$ converges almost surely to 0, as N goes to ∞, we obtain the strong consistency of the estimator $\widehat{H}_N$.

The second part of the theorem is deduced from Lemma 3 in which we already verified the asymptotic normality of the sequence $(V_N)_{N\geq 1}$. □

5.4. Statistical interpretation

Consider a tightly stretched string without slope and let x a point on the string at $t = 0$, i.e. in the equilibrium position. When the string vibrates, we can assume that the horizontal displacement of the point x is negligible, since there is no slope. So $u(t,x)$ represents the position at time t of the point x on the vibrating string, under a random force W^H (the fractional-white Gaussian noise).

Consequently, observing $u(t,x_i)$, $i = 0,..,N$ means that at a certain moment $t > 1$, we are able to observe the position of the string at x_i, which seems reasonable from the practical point of view. It is also worth to note that only from the discrete observation at one arbitrary time, it is possible to get the estimation of the Hurst parameter H and implicitly the estimation of the self-similarity index of the process $(u(t,x), t \geq 0)$ which is $H + \frac{1}{2}$. Moreover, as showed before, the estimator is strongly consistent and asymptotically normal.

As a final comment, let us notice that the parameter H characterizes the main properties of the process u is time and in space, see e.g. Lemma 2.12 and Remark 2.

In the next section we present an improvement of the CLT which is the ASCLT.

6. Almost Sure Central Limit Theorem

The ASCLT was stated firstly by Lévy [17], then it was extensively treated by many other authors, for instance in [14]. It constitutes an improvement of the CLT. In the case of multiple stochastic integrals of fixed order $q \geq 2$ we have the following result from [2] (recall that the contraction is defined in (4.6)):

Theorem 3. *Fix $q \geq 2$, and let $(F_N)_{N\geq 1}$ be a sequence of random variables defined by $F_N := (I_q(f_N))_{N\geq 1}$ with $f_N \in \mathcal{H}^{\odot q}$, such that for all $N \geq 1$, $\mathbf{E}(F_N^2) = q!\|f_N\|^2_{\mathcal{H}^{\otimes q}} = 1$ and $\|f_N \otimes_r f_N\|_{\mathcal{H}^{\otimes 2(q-r)}}$ goes to 0, as N goes to ∞, for every $r = 1, \ldots, q-1$. Then, $F_N \xrightarrow{\mathcal{L}} Z \sim \mathcal{N}(0,1)$, as $N \to \infty$. Moreover, if the following two conditions are fulfilled:*

$$(A_1) \qquad \sum_{N\geq 2} \frac{1}{N\log^2 N} \sum_{l=1}^{N} \frac{1}{l} \|f_l \otimes_r f_l\|_{\mathcal{H}^{\otimes 2(q-r)}} < \infty, \text{ for every } 1 \leq r \leq q-1,$$

$$(A_2) \qquad \sum_{N\geq 2} \frac{1}{N\log^3 N} \sum_{m,l=1}^{N} \frac{|\ \mathbf{E}(F_m F_l)\ |}{ml} < \infty,$$

then $(F_N)_{N\geq 1}$ satisfies an ASCLT. In other words, almost surely, for all bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log N} \sum_{l=1}^{N} \frac{\varphi(F_l)}{l} \longrightarrow \mathbf{E}(\varphi(Z)), \text{ as } N \to \infty.$$

The main result of this section is summarized in the following proposition.

Proposition 2. *Fix $t \in (1, T]$ and assume that $H \in (\frac{1}{2}, \frac{3}{4})$. Then the sequence $(F_N)_{N\geq 1}$ given by (4.12), satisfies the ASCLT, as $N \to \infty$.*

Proof: Using Theorem 3, we only need to check the hypothesis (A_1) and (A_2), so from one hand we have:

$$\begin{aligned} f_l \otimes_1 f_l &= \frac{l^{4H+1}}{v_N (k_1 l + k_2)^2} \sum_{j,k=0}^{l-1} \mathbb{1}^{\otimes 2}_{[x_j, x_{j+1}]} \otimes_1 \mathbb{1}^{\otimes 2}_{[x_k, x_{k+1}]} \\ &= \frac{l^{4H+1}}{v_N (k_1 l + k_2)^2} \sum_{j,k=0}^{l-1} \langle \mathbb{1}_{[x_j, x_{j+1}]}, \mathbb{1}_{[x_k, x_{k+1}]} \rangle_{\mathcal{H}} \mathbb{1}_{[x_j, x_{j+1}]} \otimes \mathbb{1}_{[x_k, x_{k+1}]}, \end{aligned}$$

which leads to:

$$\begin{aligned} &\|f_l \otimes_1 f_l\|^2_{\mathcal{H}^{\otimes 2}} \\ = \ &\frac{l^{8H+2}}{v_N^2 (k_1 l + k_2)^4} \sum_{j,k,m,p=0}^{l-1} \langle \mathbb{1}_{[x_j, x_{j+1}]}, \mathbb{1}_{[x_k, x_{k+1}]} \rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_m, x_{m+1}]}, \mathbb{1}_{[x_p, x_{p+1}]} \rangle_{\mathcal{H}} \\ &\times \langle \mathbb{1}_{[x_j, x_{j+1}]}, \mathbb{1}_{[x_m, x_{m+1}]} \rangle_{\mathcal{H}} \langle \mathbb{1}_{[x_k, x_{k+1}]}, \mathbb{1}_{[x_p, x_{p+1}]} \rangle_{\mathcal{H}} \\ = \ &\frac{l^{8H+2}}{v_N^2 (k_1 l + k_2)^4} \sum_{j,k,m,p=0}^{l-1} \left[k_1 \frac{\varphi_H(j-k)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-k)}{N^{2H+1}} \right] \\ &\times \left[k_1 \frac{\varphi_H(m-p)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(m-p)}{N^{2H+1}} \right] \left[k_1 \frac{\varphi_H(j-m)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-m)}{N^{2H+1}} \right] \end{aligned}$$

$$
\begin{aligned}
& \times \left[k_1 \frac{\varphi_H(k-p)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(k-p)}{N^{2H+1}} \right] \\
\leq \quad & \frac{l^{8H+2}}{v_N^2 (k_1 l + k_2)^4} \sum_{j,k,m,p=0}^{l-1} \left[k_1 \frac{\varphi_H(j-k)}{l^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-k)}{l^{2H+1}} \right] \\
& \times \left[k_1 \frac{\varphi_H(m-p)}{l^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(m-p)}{l^{2H+1}} \right] \left[k_1 \frac{\varphi_H(j-m)}{l^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-m)}{l^{2H+1}} \right] \\
& \times \left[k_1 \frac{\varphi_H(k-p)}{l^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(k-p)}{l^{2H+1}} \right].
\end{aligned}
$$

By the same reasoning as in the proof of Lemma 3, we deduce that:

$$
\|f_l \otimes_1 f_l\|_{\mathcal{H}^{\otimes 2}} \leq C \begin{cases} \frac{1}{\sqrt{l}} & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) \\ \frac{(\log l)^{\frac{3}{2}}}{\sqrt{l}} & \text{if } H = \frac{5}{8} \\ l^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}), \end{cases} \tag{6.1}
$$

which implies that for every $H \in (\frac{1}{2}, \frac{3}{4})$:

$$
\sum_{l=0}^{\infty} \frac{\|f_l \otimes_1 f_l\|_{\mathcal{H}^{\otimes 2}}}{l} < \infty,
$$

and obviously the first condition (A_1) is satisfied.

From another hand, we note that:

$$
\begin{aligned}
\mathbf{E}(F_m F_l) &= 2\langle f_m, f_l \rangle_{\mathcal{H}^{\otimes 2}} \\
&= \frac{2 l^{2H+\frac{1}{2}} m^{2H+\frac{1}{2}}}{v_N (k_1 l + k_2)(k_1 m + k_2)} \sum_{j=0}^{m-1} \sum_{p=0}^{l-1} \langle \mathbb{1}_{[x_j, x_{j+1}]}, \mathbb{1}_{[x_p, x_{p+1}]} \rangle_{\mathcal{H}}^2 \\
&= \frac{2 l^{2H+\frac{1}{2}} m^{2H+\frac{1}{2}}}{v_N (k_1 l + k_2)(k_1 m + k_2)} \\
& \quad \sum_{j=0}^{m-1} \sum_{p=0}^{l-1} \left[k_1 \frac{\varphi_H(j-p)}{N^{2H}} + k_2 \frac{\varphi_{H+\frac{1}{2}}(j-p)}{N^{2H+1}} \right]^2.
\end{aligned}
$$

Using the same argument as the proof of the Lemma 2, it yields that:

$$
\begin{aligned}
\mathbf{E}(F_m F_l) &\leq C \frac{l^{2H+\frac{1}{2}} m^{2H+\frac{1}{2}}}{lm} \sum_{j=0}^{m-1} \sum_{p=0}^{l-1} \frac{\left(\varphi_H(j-p)\right)^2}{N^{4H}} \\
&\leq \frac{C}{\sqrt{ml}} \sum_{j=0}^{m-1} \sum_{p=0}^{l-1} \left(\varphi_H(j-p)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= C\sqrt{\frac{l}{m}}\sum_{k\in\mathbb{Z}}\Big(\varphi_H(k)\Big)^2 \\
&\leq C\sqrt{\frac{l}{m}},
\end{aligned}
$$

this holds since the above series $\sum_{k\in\mathbb{Z}}\Big(\varphi_H(k)\Big)^2 < \infty$, for all $H < \frac{3}{4}$.

Viewing Remark 3.3 in [2], this last result leads easily to the condition $(A2)$, and the proof is achieved. □

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