The Annals of Probability
2019, Vol. 47, No. 5, 2869–2893
https://doi.org/10.1214/18-AOP1325

GEOMETRIC STRUCTURES OF LATE POINTS OF A TWO-DIMENSIONAL SIMPLE RANDOM WALK

BY IZUMI OKADA

Tokyo University of Science

As Dembo (In *Lectures on Probability Theory and Statistics* (2005) 1–101 Springer, and *International Congress of Mathematicians, Vol. III* (2006) 535–558, Eur. Math. Soc.) suggested, we consider the problem of late points for a simple random walk in two dimensions. It has been shown that the exponents for the number of pairs of late points coincide with those of favorite points and high points in the Gaussian free field, whose exact values are known. We determine the exponents for the number of j-tuples of late points on average.

1. Introduction. This paper discusses the properties of special sites, called late points, in a two-dimensional random walk. The cover time is the time taken to randomly walk in $\mathbb{Z}_n^2 (= \mathbb{Z}^2/n\mathbb{Z}^2)$ and visit every point of $\mathbb{Z}_n^2$, and a late point of a random walk in $\mathbb{Z}_n^2$ is a point of $\mathbb{Z}_n^2$, where the first hitting time is nearly equal to the cover time in a certain specific sense. We denote the set of α-late points in $\mathbb{Z}_n^2$ as $\mathcal{L}_n(\alpha)$ for $0 < \alpha < 1$ as in [9] (see (2.1) in the next section) and obtain certain asymptotic forms of

$$\left|\left\{\vec{x} \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \le n^\beta \text{ for any } 1 \le i, l \le j\right\}\right| \tag{1.1}$$

for any $0 < \alpha, \beta < 1$ and $j \in \mathbb{N}$, where $\vec{x} := (x_1, \dots, x_j)$. We then solve the related problem posed in Open Problem 4 in [5] and Open Problem 4.3 in [6].

Approximately 60 years ago, Erdős and Taylor [13] proposed a problem concerning a simple random walk in $\mathbb{Z}^d$. Forty years later, Dembo, Peres, Rosen and Zeitouni [7, 8] solved it and other related problems by developing innovative proofs. These methods yielded results concerning late points in $\mathbb{Z}_n^2$, verified by Dembo et al. [9], which showed that the numbers of late points in clusters of different sizes have a variety of power growth exponents. These methods and tools have now been improved. Belius–Kistler [2] introduced a new multi-scale refinement of the 2nd moment method. In these estimates, it is more difficult to deal with the lower bound of some numbers than the upper one.

Conversely, in this study, it is more difficult to compute the upper bound. We explain why the results in [3, 9] cannot be easily extended to arbitrary j-tuples of points. In [3, 9], they estimated the probability that the pairs of points are late

Received March 2018; revised November 2018.
MSC2010 subject classifications. Primary 60G60; secondary 60J10.
Key words and phrases. Local time, late point, simple random walks.

points. The number of pairs of α-late points can be easily computed by this probability. However, because the probability is complex, the number of arbitrary j-tuples of late points cannot be easily computed by the probability that j-tuples of points are late points (see the explanation of the proofs for the main result in Section 2). Thus, we use a linear algebra approach by exploiting the relationship between the probability and ultrametric matrices. We find the relationship by estimating the probability with a Green's function.

Here, we explain the motivation for studying α-late points. We want to compare the asymptotic behavior of special points in a random walk and in the Gaussian free field (GFF) by understanding their similarity between the local time and the GFF. In fact, there are several known results concerning similarity. Eisenbaum et al. [12] showed a powerful equivalence law called the generalized second Ray–Knight theorem for a random walk and the GFF. Ding et al. [10, 11] showed a strong connection between the expected maximum of the GFF and the expected cover time. In addition, for $0 < \alpha < 1$, they used the set of α-high points in the GFF in $\mathbb{Z}_n^2$ (sites where the GFF takes high values) and α-favorite points in $\mathbb{Z}^2$ (sites where the local time is close to that of the most frequently visited site). Dembo et al. [9] and Brummelhuis and Hilhorst [3] estimated the number of pairs of α-late points, and Daviaud [4] estimated the α-high points. We show the corresponding results for the α-favorite points in our forthcoming paper. The similarity between α-late points, high points and favorite points are included in these estimations. In addition, we find that local times converge the GFF in long-time through the generalized second Ray–Knight theorem. We expect that the similarity helps us to understand the convergence.

2. Known results and main results. To state our main results, we introduce the following notation. Let d be the Euclidean distance and $\mathbb{N} := \{1, 2, \dots\}$. For $n \in \mathbb{N}$, let $D(x, r) := \{y \in \mathbb{Z}_n^2 : d(x, y) < r\}$ and for any $G \subset \mathbb{Z}_n^2$, $\partial G := \{y \in G^c : d(x, y) = 1 \text{ for some } x \in G\}$. For $x \in \mathbb{Z}_n^2$, we sometimes omit $\{\}$ when writing the one-point set $\{x\}$. Let $\{S_k\}_{k=1}^{\infty}$ be a simple random walk in $\mathbb{Z}_n^2$. Let P^x denote the probability of a simple random walk starting at x. For simplicity, we write P for P^0. Let $K(n, x)$ be the number of times visits for the simple random walk to x up to time n, that is, $K(n, x) = \sum_{i=0}^{n} 1_{\{S_i = x\}}$. For any $D \subset \mathbb{Z}_n^2$, let $T_D := \inf\{m \geq 1 : S_m \in D\}$. Let $\tau_n := \inf\{m \geq 0 : S_m \in \partial D(0, n)\}$. $\lceil a \rceil$ denotes the smallest integer n with $n \geq a$. We use the same notation for a simple random walk in $\mathbb{Z}^2$.

We introduce the known results for α-late points in $\mathbb{Z}_n^2$. Dembo et al. [8] estimated the asymptotic form of the cover time of a simple random walk in $\mathbb{Z}_n^2$ as follows:

$$\lim_{n \to \infty} \frac{\max_{x \in \mathbb{Z}_n^2} T_x}{(n \log n)^2} = \frac{4}{\pi} \qquad \text{in probability.}$$

For $0 < \alpha < 1$, we define the set of α-late points in $\mathbb{Z}_n^2$ such that

$$\mathcal{L}_n(\alpha) := \left\{ x \in \mathbb{Z}_n^2 : \frac{T_x}{(n \log n)^2} \geq \frac{4\alpha}{\pi} \right\}. \tag{2.1}$$

Brummelhuis and Hilhorst [3] estimated the average of (1.1) for $j = 2$, and Dembo et al. [9] estimated (1.1) in probability for $j = 2$. We extend this to a *full multifractal analysis*.

THEOREM 2.1. *For any* $0 < \alpha, \beta < 1$ *and* $j \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\log E[|\{\vec{x} \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \le n^\beta \textit{ for any } 1 \le i, l \le j\}|]}{\log n} = \hat{\rho}_j(\alpha, \beta),$$

where

$$\hat{\rho}_j(\alpha,\beta) := \begin{cases} 2 + 2(j-1)\beta - \dfrac{2j\alpha}{(1-\beta)(j-1)+1} & \left(\beta \le 1 + \dfrac{1-\sqrt{j\alpha}}{j-1}\right), \\ 2(j+1-2\sqrt{j\alpha}) & \left(\beta \ge 1 + \dfrac{1-\sqrt{j\alpha}}{j-1}\right). \end{cases}$$

REMARK 2.1. We are preparing a paper on the following result: for any $0 < \alpha, \beta < 1$ and $j \in \mathbb{N}$ in probability

$$\lim_{n\to\infty} \frac{\log |\{\vec{x} \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \le n^\beta \text{ for any } 1 \le i, l \le j\}|}{\log n} = \rho_j(\alpha, \beta),$$

where

$$\rho_j(\alpha,\beta) := \begin{cases} 2 + 2(j-1)\beta - \dfrac{2j\alpha}{(1-\beta)(j-1)+1} & \left(\beta \le \dfrac{j}{j-1}(1-\sqrt{\alpha})\right), \\ 4j(1-\sqrt{\alpha}) - 2j(1-\sqrt{\alpha})^2/\beta & \left(\beta \ge \dfrac{j}{j-1}(1-\sqrt{\alpha})\right). \end{cases}$$

An explanation of the difference in the exponents is given in [5, 6] for $j = 2$.

Now we provide an explanation of the proofs for the main result. In particular, we explain how this problem is connected to the linear algebra approach. Roughly speaking, certain asymptotic forms of (1.1) are determined using the hitting probabilities of j-points of a simple random walk. In addition, the hitting probabilities are determined by Green's functions of j-points, and the values of Green's functions of j-points behave with ultrametricity in long-time. Proposition 4.1 yields that we can reduce the configurations of j-points to those in an ultrametric position. That is why ultrametricity plays an important role in the main result.

Now we provide the details. For the proof of Theorem 2.1, we must find an appropriate estimate of

$$E\big[\big|\big\{\vec{x} \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \le n^\beta \text{ for any } 1 \le i, l \le j\big\}\big|\big] = \sum_{\substack{d(x_i,x_l)\le n^\beta,\\ x_i\in\mathbb{Z}_n^2, 1\le\forall i,l\le j}} P\big(\vec{x} \in \mathcal{L}_n(\alpha)^j\big). \tag{2.2}$$

Note that the position of a j-tuple point determines the value of $P(\vec{x} \in \mathcal{L}_n(\alpha)^j)$. This value can be expressed by a matrix constructed from $G_n(x,y) := \sum_{m=0}^{\infty} P^x(S_m = y, m < \tau_n)$ for $x, y \in D(0,n)$, which is the Green's function of the walk killed when it exits $D(0,n)$. We shall show that to achieve uniformity in $x_1, \dots, x_j \in D(0, n/3)$,

$$P(\vec{x} \in \mathcal{L}_n(\alpha)^j) \approx \exp\biggl(-2\alpha \log n \chi\biggl(\biggl(\frac{\pi G_n(x_i, x_l)}{2\log n}\biggr)_{1\le i,l\le j}\biggr)\biggr), \tag{2.3}$$

where $a_n \approx b_n$ means $\log a_n / \log b_n \to 1$ as $n \to \infty$ for any sequence and $\chi(A)$ is the summation over all the elements of A^{-1} for any regular matrix A. We explain the proof of (2.3) in step (I).

(I) The proof of (2.3):

In Section 4 (Proposition 4.2), we shall see the probability that $x_1, \dots, x_j$ in $D(0,n)$ will be uncovered by the walk under a certain condition determined by the crossing number between two large circles is used to estimate the left-hand side of (2.3). In Section 3, we obtain equations consisting of hitting probabilities and Green's functions (see (3.3)), which show that hitting probabilities can be expressed by certain cofactors of $(G_n(x_i, x_l))_{1\le i,l\le j}$ (see (3.1)). Finally, we find that the product is equal to the right-hand side of (2.3).

Next, we provide an explanation of the proof of Theorem 2.1 assuming (2.3). We explain the difficulty of the proof of the upper bound. In fact, by using (2.3), we find that the logarithm of (2.2) is asymptotically equal to that of the summation of

$$\exp\biggl(-2\alpha \log n \chi\biggl(\biggl(\frac{\pi G_n(x_i^{(n)}, x_l^{(n)})}{2\log n}\biggr)_{1\le i,l\le j}\biggr)\biggr) \tag{2.4}$$

over $(x_1^{(n)}, \dots, x_j^{(n)}) \in D(0, n/3)^j$, where $(x_1^{(n)}, \dots, x_j^{(n)})$ is an ultrametric space with the error term $n^{o(1)}$ as $n \to \infty$. Here, the ultrametric space with the error term $n^{o(1)}$ is the following set with an associated distance function: for any $1 \le i, l, p \le j$ with $i \neq l$, $l \neq p$ and $i \neq p$

$$d(x_i^{(n)}, x_l^{(n)}) \le \max\{d(x_l^{(n)}, x_p^{(n)}) n^{o(1)}, d(x_p^{(n)}, x_i^{(n)}) n^{o(1)}\}. \tag{2.5}$$

Then, the configuration of $x_1^{(n)}, \dots, x_j^{(n)}$ has a certain nesting structure. For example, if we estimate the upper bound of (2.2) for $j = 3$, we need to look at an equidistant configuration and the position that the one is far from the others. For $j = 3$, an equidistant configuration means a triple $(x_1^{(n)}, x_2^{(n)}, x_3^{(n)})$ such that as $n \to \infty$,

$$d(x_1^{(n)}, x_2^{(n)}) \approx d(x_2^{(n)}, x_3^{(n)}) \approx d(x_3^{(n)}, x_1^{(n)}).$$

For a general $j \in \mathbb{N}$, there are various positions of $x_1, \dots, x_j$. Therefore, when j increases, computing the upper bound in Theorem 2.1 becomes difficult. Subsequently, we developed the following unique step.

(II) The upper bound in Theorem 2.1 by assuming (2.3):

We need to find the leading term of (2.4) over $(x_1^{(n)}, \dots, x_j^{(n)})$ conditioned by (2.5). We will show that $(\pi G_n(x_i^{(n)}, x_l^{(n)})/(2\log n))_{1\le i,l\le j}$ is asymptotically close to the ultrametric matrix as $n \to \infty$ in a certain sense. Therefore, we define $\mathcal{M}_j$ (see Section 5), which is a certain set of $j \times j$-ultrametric matrices (see Section 3.3 in [1]), and estimate $\chi(A)$ for any A in $\mathcal{M}_j$ to further estimate $\chi((\pi G_n(x_i^{(n)}, x_l^{(n)})/(2\log n))_{1\le i,l\le j})$. Ultrametric matrices have come to the attention of some linear algebraists and have been used as models of systems that can be represented by a bifurcating hierarchical tree (see, e.g., [17]). In this study, we find new properties of $\mathcal{M}_j$. Proposition 5.3 yields the minimum of $\chi(A)$ for A in $\mathcal{M}_j$ under a certain condition. Finally, we obtain the result that the properties of $\mathcal{M}_j$ directly determine the asymptotic behavior of (2.4) and that the leading term comes from the equidistant configuration for any $j \in \mathbb{N}$.

3. Basic properties.

In this section, we use the preliminary results concerning a simple random walk that will be applied in later sections. In proofs given in the remainder of this paper, we use constants that may vary for different occurrences.

3.1. *Hitting probabilities*.

First we compute the probabilities that a simple random walk in $\mathbb{Z}_n^2$ does not hit a j-tuple point until a certain random time. Given the j distinct points $x_1, \dots, x_j$ of $\mathbb{Z}_n^2$ and a nonempty subset $\tilde{D}$ of $\mathbb{Z}_n^2$ that is disjoint from $X := \{x_1, \dots, x_j\}$, let $\tilde{\tau}$ denote a time when the walk enters $\tilde{D}$. For $1 \le i, l \le j$ and $y \notin X$, we define

$$q_{i,l} := \sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < \tilde{\tau} \wedge T_y),$$

$$Q := (q_{i,l})_{1\le i,l\le j}.$$

LEMMA 3.1. *For $1 \le u \le j$, it holds that*

$$P^{x_u}(T_y = T_X \wedge \tilde{\tau} \wedge T_y) = \sum_{i=1}^{j} \frac{(\textit{cofactor of } q_{u,i})}{\det(Q)} P^{x_i}(T_y = \tilde{\tau} \wedge T_y). \tag{3.1}$$

We have

$$\begin{aligned} \min_{1\le u\le j} P^{x_u}(T_y = \tilde{\tau} \wedge T_y)\chi(Q) &\le \sum_{u=1}^{j} P^{x_u}(T_y = T_X \wedge \tilde{\tau} \wedge T_y) \\ &\le \max_{1\le u\le j} P^{x_u}(T_y = \tilde{\tau} \wedge T_y)\chi(Q). \end{aligned} \tag{3.2}$$

Note that for any regular matrix A, $\chi(A)$ is the summation over all the elements of A^{-1}.

PROOF. Because the summation of both sides of (3.1) over $1 \le u \le j$ yields (3.2), it suffices to show (3.1). By decomposing the probability $P^{x_i}(T_y = \tilde{\tau} \wedge T_y)$ according to the last time the walk leaves the set X before $\tilde{\tau} \wedge T_y$, we obtain

$$P^{x_i}(T_y = \tilde{\tau} \wedge T_y) = \sum_{l=1}^{j} q_{i,l} P^{x_l}(T_y = T_X \wedge \tilde{\tau} \wedge T_y), \tag{3.3}$$

for $1 \le i \le j$. The matrix Q is regarded as the Green kernel for the Markov chain on X with the substochastic transition matrix $U := (u_{i,l})_{1 \le i,l \le j}$ given by

$$u_{i,l} := P^{x_i}(T_{x_l} = T_X < \tilde{\tau} \wedge T_y),$$

so that $UQ = Q - E$, where E denotes the unit matrix. Accordingly, Q is regular and

$$Q^{-1} := E - U. \tag{3.4}$$

Therefore, we have (3.1). □

Next, we introduce the estimates of the hitting probabilities for a simple random walk in $\mathbb{Z}^2$, as we only need estimates for "$\mathbb{Z}^2$" in this paper.

LEMMA 3.2. *To achieve uniformity in* $0 < r < |x| < R$,

$$\begin{aligned} P^x(T_0 < \tau_R) &= \frac{\log(R/|x|) + O(|x|^{-1})}{\log R}\bigl(1 + O\bigl((\log|x|)^{-1}\bigr)\bigr), \\ P^x(\tau_r < \tau_R) &= \frac{\log(R/|x|) + O(r^{-1})}{\log(R/r)}. \end{aligned} \tag{3.5}$$

PROOF. As per Exercise 1.6.8 in [14], or (4.1) and (4.3) in [18], we obtain the desired result. □

Next, we give the estimates of a Green's function. For $x, y \in D(0,n)$, $G_n(x,y)$ is a Green's function.

LEMMA 3.3. *For any* $x \in D(0,n)$

$$\begin{aligned} G_n(x,0) &= \sum_{m=0}^{\infty} P^x(S_m = 0, m < \tau_n) \\ &= \frac{2}{\pi}\log\left(\frac{n}{d(0,x)^+}\right) + O\bigl((d(0,x)^+)^{-1} + n^{-1} + 1\bigr), \end{aligned} \tag{3.6}$$

where τ_n *is the stopping time as we mentioned in Section 2 and* $a^+ = a \vee 1$. *In particular, for* $x, y \in D(0, n/3)$,

$$G_n(x,y) = \frac{2}{\pi}\log\left(\frac{n}{d(x,y)^+}\right) + O\bigl((d(x,y)^+)^{-1} + n^{-1} + 1\bigr). \tag{3.7}$$

PROOF. As per Proposition 1.6.7 in [14] or (2.1) in [18], we obtain (3.6). Therefore, for $x, y \in D(0, n/3)$,

$$
\begin{aligned}
G_n(x, y) &\le \sum_{m=0}^{\infty} P^{x-y}(S_m = 0, m < \tau_{4n/3}) \\
&= \frac{2}{\pi} \log\left(\frac{n}{d(x, y)^+}\right) + O\big((d(x, y)^+)^{-1} + n^{-1} + 1\big), \\
G_n(x, y) &\ge \sum_{m=0}^{\infty} P^{x-y}(S_m = 0, m < \tau_{2n/3}) \\
&= \frac{2}{\pi} \log\left(\frac{n}{d(x, y)^+}\right) + O\big((d(x, y)^+)^{-1} + n^{-1} + 1\big).
\end{aligned}
$$

Subsequently, we obtain (3.7). □

REMARK 3.1. In addition, with the aid of (3.6), the strong Markov property yields

$$
P(\tau_n < T_0) = \left(\sum_{m=0}^{\infty} P(S_m = 0, m < \tau_n)\right)^{-1} = \frac{\pi}{2 \log n}\big(1 + o(1)\big). \tag{3.8}
$$

4. Proof of Theorem 2.1. In this section, we provide the proof of Theorem 2.1. We give estimates for the proof of Theorem 2.1 in Section 4.1 and the proof of Theorem 2.1 in Section 4.2.

4.1. *Some estimates for the proof of Theorem* 2.1. To prepare estimates for the main result, we add the following definitions. We fix $j \in \mathbb{N}$. For $0 < \eta \le (1 - \beta) \wedge \beta$, let $\mathcal{M}_j = \mathcal{M}_j^{\beta,\eta}$ be the set of $j \times j$-matrices $(a_{i,l})_{1 \le i,l \le j}$ satisfying the following properties:

(a) symmetric,
(b) $a_{i,i} = 1$, $1 - \beta \le a_{i,l} \le 1 - \eta$ for any $1 \le i \ne l \le j$, and
(c) $a_{i,l} \ge \min\{a_{l,p}, a_{i,p}\}$ for any $1 \le i, l, p \le j$ with $i \ne l$, $l \ne p$, $p \ne i$.

A strictly ultrametric matrix is a symmetric matrix with nonnegative entries that satisfies (c); in addition, $a_{i,i} > \max\{a_{i,k} : k \in \{1, \dots, i-1, i+1, \dots, j\}\}$ for any $1 \le i \le j$ (see [15]). Subsequently, any element in $\mathcal{M}_j$ is an ultrametric matrix. In Proposition 5.2, we will show any matrix in $\mathcal{M}_j$ is a regular matrix. Given the real-valued $j \times j$-matrices $M := (m_{i,l})_{1 \le i,l \le j}$ and $M' := (m'_{i,l})_{1 \le i,l \le j}$, let

$$
\begin{aligned}
\mathcal{E}[M, M'] &= \mathcal{E}[M, M'](j, n) \\
&:= \big\{\vec{x} \in (\mathbb{Z}_n^2)^j : m_{i,l} \le d(x_i, x_l) \le m'_{i,l} \text{ for any } 1 \le i \ne l \le j\big\}.
\end{aligned}
$$

Note that the set is independent of the diagonal elements of a matrix. When $m_{i,l} = a$ and $m'_{i,l} = a'$ for $1 \le i \ne l \le j$, we simply write $\mathcal{E}[(a),(a')]$. For $A \in \mathcal{M}_j$ and $\delta > 0$ let

$$\hat{\mathcal{E}}_\delta[A] = \hat{\mathcal{E}}_\delta[A](j,n) := \mathcal{E}\left[\left(\frac{1}{2^j} n^{1-a_{i,l}}\right)_{1\le i,l\le j}, \left(2^j n^{1-a_{i,l}+\delta}\right)_{1\le i,l\le j}\right].$$

By the following proposition, we find that it is possible to reduce the configuration of points to those in an ultrametric position.

PROPOSITION 4.1. *Fix $0 < \beta < 1$. For any $0 < \delta < 1 - \beta$, there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and $\vec{x} \in \mathcal{E}[(n^\eta),(n^\beta)]$, there exists $A \in \mathcal{M}_j^{\beta,\eta}$ such that $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$ holds.*

We will show Proposition 4.1 in Section 5.2. Next, we introduce our goal in this subsection.

PROPOSITION 4.2. *Fix $0 < \beta < 1$. For any $\varepsilon > 0$, there exist $C > 0$ and $0 < \delta < 1 - \beta$ such that for any $0 < \alpha < 1$, $x \in \mathbb{Z}_n^2$ and all sufficiently large $n \in \mathbb{N}$ that satisfy $A \in \mathcal{M}_j^{\beta,\eta}$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$, it holds that*

$$P\big(\vec{x} \in \mathcal{L}_n(\alpha)^j\big) \le C n^{-2\alpha\chi(A)+\varepsilon}$$

and for any $0 < \alpha < 1$, $x \in \mathbb{Z}_n^2$ with $x_1 \in D(0,n/10)^c$ and all sufficiently large $n \in \mathbb{N}$ that satisfy $A \in \mathcal{M}_j^{\beta,\eta}$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$, it holds that

$$C^{-1} n^{-2\alpha\chi(A)-\varepsilon} \le P\big(\vec{x} \in \mathcal{L}_n(\alpha)^j\big).$$

To show the proposition, we prepare notations and provide the lemma. For $k, n \in \mathbb{N}$ with $k \le n$, let $n_n = n_n(\alpha) := \lceil 2\alpha n^2 \log n \rceil$, $r_k := k!$, and $K_n := \lceil n^b r_n \rceil$ for $b \in [1,3]$. Let $\mathcal{R}_n^{x_1} = \mathcal{R}_n^{x_1}(\alpha)$ be the time until completion of the first $n_n(\alpha)$ excursions of the path from $\partial D(x_1, r_{n-1})$ to $\partial D(x_1, r_n)$ (see the definition of excursions in Lemma 2.3 in [9] et al.).

LEMMA 4.1. *Fix $0 < \beta < 1$. For any $\varepsilon > 0$, there exist $C > 0$ and $0 < \delta < 1 - \beta$ such that for any $0 < \alpha < 1$, $x \in \mathbb{Z}_{K_n}^2$, and all sufficiently large $n \in \mathbb{N}$ that satisfy $A \in \mathcal{M}_j^{\beta,\eta}$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[A](j, K_n)$, it holds that*

$$P\big(T_X > \mathcal{R}_n^{x_1}(\alpha)\big) \le C K_n^{-2\alpha\chi(A)+\varepsilon/2}.$$

PROOF. By the strong Markov property, it suffices to show that uniformity in $y_0 \in \partial D(x_1, r_{n-1})$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$,

$$P^{y_0}(T_X < T_{\partial D(x_1,r_n)}) = \frac{1+o(1)}{n}\chi(A). \tag{4.1}$$

Note that for $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$, $1 \le i, l \le j$, and $y_0 \in \partial D(x_1, r_{n-1})$

$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{\partial D(x_1,r_n)} \wedge T_{y_0}) \le G_{r_n}(x_i - x_1, x_l - x_1),$$
$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{\partial D(x_1,r_n)} \wedge T_{y_0}) \ge G_{r_{n-1}}(x_i - x_1, x_l - x_1).$$

Because $r_{n-1}/(2^j K_n^{\beta+\delta}) \ge 3$ for all sufficiently large $n \in \mathbb{N}$ and $x_i, x_l \in D(x_1, 2^j K_n^{\beta+\delta})$ hold, (3.7) yields

$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{\partial D(x_1,r_n)} \wedge T_{y_0}) = \frac{2}{\pi}\big(\log r_n - \log d(x_i, x_l)^+ + O(1)\big).$$

Subsequently, (3.5) yields that to achieve uniformity in $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$ and $y_0 \in \partial D(x_1, r_{n-1})$,

$$P^{x_i}(T_{y_0} < T_{\partial D(x_1,r_n)}) = \frac{1 + o(1)}{n}.$$

Therefore, to achieve uniformity in $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$,

$$\left|\frac{2}{\pi}\big(\log r_n - \log d(x_i, x_l)^+ + o(1)\big) - a_{i,l}\frac{2n\log n}{\pi}\right|$$
$$\le \max \frac{2n\log n}{\pi}|b_{i,l} - a_{i,l}| = o(1) n\log n,$$

where the above maximum is over $b_{i,l} = a_{i,l} + o(1)$ with $1 \le i, l \le j$. In addition, as per Remark 5.6, to achieve uniformity in $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$,

$$\left|\chi\left(\left(\frac{2}{\pi}\big(\log r_n - \log d(x_i, x_l)^+ + O(1)\big)\right)_{1\le i,l\le j}\right) - \frac{\pi}{2n\log n}\chi(A)\right|$$
$$= \frac{o(1)}{n\log n}. \tag{4.2}$$

Therefore, if we substitute $T_{\partial D(x_1,r_n)}$ and y for $\tilde{\tau}$ and y_0 in (3.2), (3.7) yields

$$\sum_{l=1}^{j} P^{x_l}(T_{y_0} < T_X \wedge T_{\partial D(x_1,r_n)}) = \frac{1 + o(1)}{n}\frac{\pi}{2n\log n}\chi(A). \tag{4.3}$$

Note that as per (3.8), we obtain

$$P^{y_0}(T_{y_0} < T_X \wedge T_{\partial D(x_1,r_n)}) = 1 - \frac{\pi + o(1)}{2n\log n}.$$

Subsequently,

$$\begin{aligned}
&P^{y_0}(T_X < T_{\partial D(x_1,r_n)})\\
&\quad = \sum_{i=0}^{\infty} P^{y_0}(T_{y_0} < T_X \wedge T_{\partial D(x_1,r_n)})^i P^{y_0}(T_X = T_{y_0} \wedge T_X \wedge T_{\partial D(x_1,r_n)})\\
&\quad = \frac{1}{1 - P^{y_0}(T_{y_0} < T_X \wedge T_{\partial D(x_1,r_n)})} P^{y_0}(T_X < T_{y_0} \wedge T_{\partial D(x_1,r_n)})\\
&\quad = \frac{2(1+o(1))n \log n}{\pi} P^{y_0}(T_X < T_{y_0} \wedge T_{\partial D(x_1,r_n)})\\
&\quad = \frac{2(1+o(1))n \log n}{\pi} \sum_{l=1}^{j} P^{x_l}(T_{y_0} < T_X \wedge T_{\partial D(x_1,r_n)}).
\end{aligned}$$

The last equality comes from the time-reversal of a simple random walk. Therefore, in view of (4.3), we have (4.1). □

PROOF OF PROPOSITION 4.2. Note that we only have to show the result for a sequence K_n, because $b \in [1,3]$ is arbitrary and thus K_n covers all sufficiently large integers. Fix $0 < \delta_1 < \alpha$. As per (3.19) in [9], there exist $c > 0$ and $\delta > 0$ such that for any $0 < \alpha < 1$, $n \in \mathbb{N}$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$,

$$P\left(\frac{4\alpha}{\pi}(K_n \log K_n)^2 < \mathcal{R}_n^{x_1}(\alpha - \delta_1)\right) \le c^{-1} \exp(-cn^2 \log n).$$

We find that for any $n \in \mathbb{N}$

$$\begin{aligned}
&P(\vec{x} \in \mathcal{L}_{K_n}(\alpha)^j)\\
&\quad \le P(T_X > \mathcal{R}_n^{x_1}(\alpha - \delta_1)) + P\left(\frac{4\alpha}{\pi}(K_n \log K_n)^2 < \mathcal{R}_n^{x_1}(\alpha - \delta_1)\right)\\
&\quad \le CK_n^{-2(\alpha-\delta_1)\chi(A)+\varepsilon/2} + c^{-1} \exp(-cn^2 \log n). \qquad (4.4)
\end{aligned}$$

Note that as per Lemma 4.1 in [9], for any $0 < \delta_2 < 1 - \alpha$, there exists $c > 0$ such that for any $n \in \mathbb{N}$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$,

$$P\left(\frac{4\alpha}{\pi}(K_n \log K_n)^2 > \mathcal{R}_n^{x_1}(\alpha + \delta_2)\right) \le c^{-1} \exp(-cn^2 \log n).$$

Then, we have that for $\vec{x} \in \hat{\mathcal{E}}_\delta[A]$ with $x_1 \in D(0, n/10)^c$,

$$\begin{aligned}
&P(\vec{x} \in \mathcal{L}_{K_n}(\alpha)^j)\\
&\quad \ge P(T_X > \mathcal{R}_n^{x_1}(\alpha + \delta_2)) - P\left(\frac{4\alpha}{\pi}(K_n \log K_n)^2 > \mathcal{R}_n^{x_1}(\alpha + \delta_2)\right)\\
&\quad \ge cK_n^{-2(\alpha+\delta_2)\chi(A)-\varepsilon/2} - c^{-1} \exp(-cn^2 \log n). \qquad (4.5)
\end{aligned}$$

Therefore, if we select sufficiently small $\delta_1, \delta_2 > 0$ for $\varepsilon > 0$, we obtain Proposition 4.2. □

4.2. *Proof of Theorem* 2.1.

PROOF OF THE UPPER BOUND IN THEOREM 2.1. Fix $0 < \beta < 1$. Propositions 4.2 and 5.4 yield that for any $\varepsilon > 0$, there exists $C > 0$ such that for any $0 < \alpha < 1$, and $n \in \mathbb{N}$,

$$\sum_{\vec{x} \in \mathcal{E}[(n^\eta),(n^\beta)]} P(\vec{x} \in \mathcal{L}_n(\alpha)^j) \le Cn^{\hat{\rho}_j(\alpha,\beta)+\varepsilon}. \tag{4.6}$$

Now we extend the result for "$\mathcal{E}[(n^\eta),(n^\beta)]$" to "$\mathcal{E}[(0),(n^\beta)]$" by performing induction on $j \in \mathbb{N}$. We assume that for any $\varepsilon > 0$, there exists $C > 0$ such that for any $n \in \mathbb{N}$,

$$\sum_{(x_1,\dots,x_{j-1}) \in \mathcal{E}[(0),(n^\beta)]} P(x_1,\dots,x_{j-1} \in \mathcal{L}_n(\alpha)) \le Cn^{\hat{\rho}_j(\alpha,\beta)+(j-1)\varepsilon}.$$

For $j = 1$, according to Proposition 4.2, it is trivial that

$$\sum_{x \in \mathbb{Z}_n^2} P(x \in \mathcal{L}_n(\alpha)) \le Cn^{\hat{\rho}_1(\alpha,\beta)+\varepsilon}.$$

Let us assume that the claim holds for $j-1$ with $j \ge 2$. We show that the claim holds for j. For any $\varepsilon > 0$, we select $\eta > 0$ with $2\eta < \varepsilon$ and n_0 given in Proposition 4.1. Therefore, according to (4.6), Lemma A.1, and induction, we obtain that for any $n \ge n_0$,

$$\begin{aligned}
&E[|\{\vec{x} \in \mathcal{L}_n(\alpha)^j : d(x_i,x_l) \le n^\beta \text{ for any } 1 \le i,l \le j\}|] \\
&\quad = \sum_{\vec{x} \in \mathcal{E}[(0),(n^\beta)]} P(\vec{x} \in \mathcal{L}_n(\alpha)^j) \\
&\quad \le \sum_{\vec{x} \in \mathcal{E}[(n^\eta),(n^\beta)]} P(\vec{x} \in \mathcal{L}_n(\alpha)^j) \\
&\qquad + \sum_{(x_1,\dots,x_{j-1}) \in \mathcal{E}[(0),(n^\beta)](j-1)} P(x_1,\dots,x_{j-1} \in \mathcal{L}_n(\alpha))Cn^{2\eta} \\
&\quad \le Cn^{\hat{\rho}_j(\alpha,\beta)+\varepsilon} + Cn^{\hat{\rho}_{j-1}(\alpha,\beta)+(j-1)\varepsilon+2\eta} \le Cn^{\hat{\rho}_j(\alpha,\beta)+j\varepsilon}.
\end{aligned}$$

As it suffices to show it for all sufficiently large $n \in \mathbb{N}$, we obtain the desired result. $\square$

We write $A_r^{(j)}$ for $(a_{i,l})_{1 \le i,l \le j}$ if $a_{i,i} = 1$ and $a_{i,l} = r$ for $1 \le i \ne l \le j$ and $1-\beta \le r \le 1-\eta$. Note that $A_r^{(j)} \in \mathcal{M}_j$. In addition, $A_r^{(1)}$ is independent of r, and therefore, we sometimes write $A^{(1)}$.

PROOF OF THE LOWER BOUND IN THEOREM 2.1. It is trivial that $\chi(A_{1-l}^{(j)}) = j/(1+(j-1)(1-l))$. Fix $0 < \beta < 1$ and $\varepsilon > 0$ and pick $\delta > 0$ in Proposition 4.2.

If we consider $\vec{x} \in \mathcal{E}[(n^l), (5jn^l)]$ with $x_1 \in D(0, n/10)^c$ for $0 < \eta < l < 1$, then Proposition 4.2 yields that for any $0 < \alpha < 1$ and all sufficiently large $n \in \mathbb{N}$ with $5jn^l \le 2^j n^{l+\delta}$,

$$P(\vec{x} \in \mathcal{L}_n(\alpha)^j) \ge \exp\left(-\frac{2j\alpha \log n}{1 + (j-1)(1-l)} + o(\log n)\right).$$

Let

$$R := \left\{\vec{x} : x_1 \in \mathbb{Z}_n^2 \cap D\left(0, \frac{n}{10}\right)^c, \right.$$
$$\left. x_i \in x_1 + (0, 4(i-1)n^l) + D(0, n^l) \text{ for any } 2 \le i \le j\right\}.$$

Note that there exists $c > 0$ such that for any $n \in \mathbb{N}$,

$$|R| \ge cn^{2+2(j-1)l}.$$

In addition,

$$\mathcal{E}[(n^l), (5jn^l)] \supset R.$$

Therefore, Proposition 5.4 and the simple computation yield that for $\eta < s < 1$

$$\sum_{\vec{x} \in \mathcal{E}[(n^l),(5jn^l)]} P(\vec{x} \in \mathcal{L}_n(\alpha)^j)$$
$$\ge cn^{2+2(j-1)l} \times \exp\left(-\frac{2j\alpha \log n}{1 + (j-1)(1-l)} + o(\log n)\right).$$

As $2+2(j-1)l - 2\alpha j/(1+(j-1)(1-l))|_{l=(1+(1-\sqrt{j\alpha})/(j-1))\wedge\beta} = \hat{\rho}_j(\alpha, \beta)$ and η is arbitrary, we obtain the result. $\square$

5. Matrix argument. In this section, our goal is to arrive at Proposition 5.4, which is used in the proof of the upper bound in Theorem 2.1. We use only Propositions 5.4 and 5.5 in the proof of Theorem 2.1. To show Proposition 5.4, we prepare some propositions and lemmas in Section 5.1 and provide proofs in Section 5.2.

5.1. *Claims.* We first establish results that yield the properties of matrices in $\mathcal{M}_j$ and then those that link the properties of $\mathcal{M}_j$ with the main results. Note that (c) in the definition of $\mathcal{M}_j$ in Section 4.1 can be rewritten as

(d) for any $1 \le i, l, p \le j$ with $i \ne l, l \ne p, p \ne i$,

it holds that $a_{i,l} < a_{i,p} \Rightarrow a_{l,p} = a_{i,l}$

assuming (a) and (b). Hereafter, we simply write A for $(a_{i,l})_{1\le i,l\le j}$.

Now we introduce propositions that provide the properties of $\mathcal{M}_j$. For $j_k \in \mathbb{N}$, let $A_k := (a_{i,l}^{(k)})_{1\le i,l\le j_k} \in \mathcal{M}_{j_k}$ $(\forall k = 1, \dots, m)$ and $j = \sum_{k=1}^m j_k$. For the injective function $\sigma_k : \{1, \dots, j_k\} \to \{1, \dots, j\}$ $(\forall k = 1, \dots, m)$ with $\bigcup_{k=1}^m \operatorname{Im} \sigma_k =$

$\{1, \ldots, j\}$ and $s \leq \min\{a_{i,l}^{(k)} \mid k \in \{1, \ldots, m\}, i, l \in \{1, \ldots, j_k\}\}$, we let $A = A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$ if

$$a_{i,l} := \begin{cases} a^{(k)}_{\sigma_k^{-1}(i), \sigma_k^{-1}(l)} & (\forall i, l \in \operatorname{Im} \sigma_k, k = 1, \ldots, m), \\ s & \text{otherwise.} \end{cases}$$

Note that definitions yield $A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m} \in \mathcal{M}_j$ and $\min_{1 \leq i,l \leq j} a_{i,l} = s$. We define $(a_{i,l})_{1 \leq i,l \leq j} \cong ({a'}_{i,l})_{1 \leq i,l \leq j}$ if there exists a bijective function $\sigma : \{1, \ldots, j\} \to \{1, \ldots, j\}$ such that $a_{\sigma(i),\sigma(l)} = {a'}_{i,l}$ for any $1 \leq i, l \leq j$.

PROPOSITION 5.1. *It holds that for any $j \geq 2$ with $j \in \mathbb{N}$, $A \in \mathcal{M}_j$ satisfies the following: there exist $A_k \in \mathcal{M}_{j_k}$, σ_k for $k = 1, \ldots, m$ with $m \geq 2$ such that $A = A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$, where $s < \min\{a_{i,l}^{(k)} \mid k \in \{1, \ldots, m\}, i, l \in \{1, \ldots, j_k\}\}$.*

REMARK 5.1. We call $A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$ the maximal decomposition of A if $s < \min\{a_{i,l}^{(k)} \mid k \in \{1, \ldots, m\}, i, l \in \{1, \ldots, j_k\}\}$. We show that if ${A'}_1^{{\sigma'}_1} \boxplus_s \cdots \boxplus_s {A'}_{m'}^{{\sigma'}_{m'}}$ is the maximal decomposition of A, then $m = m'$ and there exists a bijective function $\tilde{\sigma} : \{1, \ldots, j\} \to \{1, \ldots, j\}$ such that $A_{\tilde{\sigma}(k)} \cong {A'}_k$ in Remark 5.4. Therefore, the maximal decomposition is uniquely determined in a certain sense. The maximal decomposition corresponds to clustering a j-tuple point by the maximal distance in the ultrametric space.

PROPOSITION 5.2. *Any element included in $\mathcal{M}_j$ is a regular matrix. In other words, for any $A \in \mathcal{M}_j$, there exists a unique solution $y_1, \ldots, y_j$ such that*

$$A \vec{y}^T = \vec{1}^T,$$

where $\vec{y} := (y_1, \ldots, y_j)$ and $\vec{1} := (1, \ldots, 1)$.

REMARK 5.2. References [15] and [16] demonstrated that a strictly symmetric ultrametric matrix is a regular matrix, and therefore, that the desired result had already been obtained. However, we provide another proof because the argument is used later.

We define Ξ inductively as follows: for $A \in \mathcal{M}_j$ whose maximal decomposition is $A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$,

$$\Xi(A) := \sum_{k=1}^{m} \Xi(A_k) + (m-1)(1-s),$$

where $\Xi(A) := 0$ for $A \in \mathcal{M}_1$.

REMARK 5.3. We simultaneously show the claim that Ξ is well defined and $\Xi(A) = \Xi(A')$ for $A \cong A'$ by performing induction on $j \in \mathbb{N}$. This is trivial for $j = 1$. We assume the claim for $1, \dots, j-1$ and show the claim for j. Subsequently, Remark 5.1 and the assumption yield that Ξ is well-defined for j. Note that if $A \cong A'$ holds and $A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$ is the maximal decomposition of A, there exists σ'_k for $1 \le k \le m$ such that $A' = A_1^{\sigma'_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma'_m}$. Therefore, we obtain $\Xi(A) = \Xi(A')$ for j and retain the claim.

Next, we observe the additional properties of the matrix included in $\mathcal{M}_j$.

PROPOSITION 5.3. *For any* $r \le j-1$

$$\min_{A \in \Xi^{-1}(\{r\})} \chi(A) = \chi\bigl(A^{(j)}_{1-r/(j-1)}\bigr) = \frac{j}{j-r}.$$

We provide the following lemmas concerning the configuration of points, which link the matrix argument with Proposition 5.4. To describe our goal in this section, we give the following lemma. Note that $(j-1)\eta \le \Xi(A) \le (j-1)\beta$ for $A \in \mathcal{M}_j^{\beta,\eta}$.

LEMMA 5.1. *For any* $\varepsilon > 0$ *and* $0 < \delta < e^{-j}\varepsilon$, *there exists* $C > 0$ *such that for any* $0 < t \le (j-1)\beta$, $A \in \Xi^{-1}(\{t\})$ *and* $n \in \mathbb{N}$,

$$\bigl|\hat{\mathcal{E}}_\delta[A]\bigr| \le C n^{2t+2+\varepsilon/2}.$$

To introduce Proposition 5.4, we prepare the following notation. As per Propositions 4.1 and 5.2, for $\delta > 0$, $\vec{x} \in \mathcal{E}[(n^\eta), (n^\beta)]$, and $n \ge n_0$, we can set

$$h = h_\delta(\vec{x}) := \inf\bigl\{\chi(B) : \vec{x} \in \hat{\mathcal{E}}_\delta[B], B \in \mathcal{M}_j^{\beta,\eta}\bigr\}.$$

PROPOSITION 5.4. *For any* $\varepsilon > 0$ *and* $0 < \delta < e^{-j}\varepsilon$, *there exists* $C > 0$ *such that for any* $n \in \mathbb{N}$,

$$\sum_{\vec{x} \in \mathcal{E}[(n^\eta),(n^\beta)]} n^{-2\alpha h_\delta(\vec{x})} \le C n^{\hat{\rho}_j(\alpha,\beta)+\varepsilon}.$$

PROOF. Note that Proposition 5.3 implies

$$\min_{\substack{\vec{x} \in \hat{\mathcal{E}}_\delta[A], \\ A \in \Xi^{-1}(\{t\})}} h_\delta(\vec{x}) = \min_{A \in \Xi^{-1}(\{t\})} \chi(A) = \frac{j}{j-t}.$$

For any $\delta > 0$ and $\varepsilon > 0$, there exist $C' := \lceil (\beta - \eta)/\delta \rceil$ and $C > 0$ such that for any $n \geq n_0$, the left-hand side of the desired formula is bounded by

$$\begin{aligned}(C')^j & \max_{0 \leq t \leq (j-1)\beta} \max_{A \in \Xi^{-1}(\{t\})} \sum_{\vec{x} \in \hat{\mathcal{E}}_\delta[A]} n^{-2\alpha h} \\ & \leq C \max_{0 \leq t \leq (j-1)\beta} n^{2t+2+\varepsilon/2} \max_{\substack{\vec{x} \in \hat{\mathcal{E}}_\delta[A], \\ A \in \Xi^{-1}(\{t\})}} n^{-2\alpha h} \\ & \leq C \max_{0 \leq t \leq (j-1)\beta} n^{2t+2+\varepsilon} n^{-2\alpha j/(j-t)}.\end{aligned}$$

The first inequality comes from Lemma 5.1 and the last one comes from Proposition 5.3. Therefore, we have

$$\max_{0 \leq t \leq (j-1)\beta} 2t + 2 - \frac{2\alpha j}{j-t} = \hat{\rho}_j(\alpha, \beta).$$

Because it is sufficient to show the claim for $n \geq n_0$, we obtain the desired result. □

5.2. *Proofs of various propositions and lemmas*. In this subsection, we provide proofs of the propositions and lemmas that are introduced in Sections 4.1 and 5.1.

First we provide the proof of Proposition 5.1.

PROOF OF PROPOSITION 5.1. Let $s := \min_{1 \leq i,l \leq j} (a_{i,l})_{1 \leq i,l \leq j}$. We define $k \sim k'$ if $a_{k,k'} > s$. First we show that $\sim$ constructs an equivalence class. Note that reflexive and symmetric relations are trivial owing to the definition of $\mathcal{M}_j$, and therefore, we show a transitive relation. Let us assume that $k_1 \sim k_2$ and $k_2 \sim k_3$. The definition of $\mathcal{M}_j$ yields $a_{k_1,k_3} \geq \min\{a_{k_1,k_2}, a_{k_2,k_3}\} > s$. Therefore, we obtain $k_1 \sim k_3$ and that $\{1, \dots, j\}/\sim$ is an equivalence class. Next, we show the claim. If $|\{1, \dots, j\}/\sim| = m$, we let $G_1, \dots, G_m$ be elements in $\{1, \dots, j\}/\sim$ and j_k be $|G_k|$ for $1 \leq k \leq m$. For any $1 \leq k \leq m$, we select some bijective function $\sigma_k : \{1, \dots, j_k\} \to G_k$. We set $A_k := (a_{i,l}^{(k)})_{1 \leq i,l \leq j_k}$ such that $a^{(k)}_{\sigma_k^{-1}(i), \sigma_k^{-1}(l)} = a_{i,l}$ for $1 \leq i, l \leq j_k$. Then, $A = A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$ and $s < \min\{a_{i,l}^{(k)} \mid k \in \{1, \dots, m\}, i, l \in \{1, \dots, j_k\}\}$ holds. Therefore, we obtain the desired result. □

REMARK 5.4. If $A_1^{\sigma_1} \boxplus_s \cdots \boxplus_s A_m^{\sigma_m}$ is the expression of the maximal decomposition of A, it is trivial that $\{\operatorname{Im} \sigma_k : 1 \leq k \leq m\} = \{1, \dots, j\}/\sim$ by the above proof. Then, it easily yields the uniqueness of the maximal decomposition and the claim in Remark 5.1.

Hereafter, we assume that $A = A_1^{\sigma_1} \boxplus_s A_2^{\sigma_2}$ unless otherwise stated. Note that that $A_1^{\sigma_1} \boxplus_s A_2^{\sigma_2}$ is not always the maximal decomposition of A. Let $g := |\operatorname{Im} \sigma_1|$ and $h := |\operatorname{Im} \sigma_2|$.

PROOF OF PROPOSITION 5.2. We show the claim and present the following conditions (A) and (B): when $j = 2$:

$$\text{(A)} \quad 1 - s\sum_{i=1}^{j} y_i > 0,$$

$$\text{(B)} \quad y_i > 0 \qquad \text{for any } 1 \le i \le j.$$

When $j = 1$, we change (A) to $1 - y_1 \ge 0$. As per symmetry, we need to show the result for only the case in which $\operatorname{Im}\sigma_1 = \{1, \dots, g\}$ and $\operatorname{Im}\sigma_2 = \{g+1, \dots, j\}$. We show the results by performing induction on $j \in \mathbb{N}$. It is trivial that the claim holds for $j = 1$ as $y_1 = 1$. Assuming that the claims (A) and (B) hold for $1, \dots, j-1$, we show that the claims (A) and (B) hold for j. As per symmetry, this assumption yields that $A_1 \in \mathcal{M}_g$ determines a unique solution $\vec{z} := (z_1, \dots, z_g)$ such that $A_1\vec{z}^T = \vec{1}^T$ and $A_2 \in \mathcal{M}_h$ determines a unique solution $\vec{z}' := (z'_1, \dots, z'_h)$ such that $A_2\vec{z}'^T = \vec{1}^T$. In addition, the assumption of (B) yields $z_i > 0$ for any $1 \le i \le g$ and $z'_i > 0$ for any $1 \le i \le h$. Therefore, it holds that $z_i + s\sum_{1\le l\le g, l\ne i} z_l \le 1$ for any $1 \le i \le g$ and $z_i + s\sum_{1\le l\le h, l\ne i} z'_l \le 1$ for any $1 \le i \le h$, and we are able to derive the following equation:

$$\sum_{i=1}^{g} z_i \le \frac{g}{1+(g-1)s}, \qquad \sum_{i=1}^{h} z'_i \le \frac{h}{1+(h-1)s}.$$

There exists $c > 0$ such that

$$1 - s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i \ge c. \tag{5.1}$$

This comes from $s \le 1 - \eta$. If we set

$$y_l = \frac{(1 - s\sum_{i=1}^{h} z'_i)z_l}{1 - s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} \qquad \text{for any } 1 \le l \le g,$$

$$y_l = \frac{(1 - s\sum_{i=1}^{g} z_i)z'_{l-g}}{1 - s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} \qquad \text{for any } g+1 \le l \le j,$$

we obtain a solution such that $A\vec{y}^T = \vec{1}^T$. Therefore, we have proved the existence of the solution. Hereafter, we observe the properties of $y_1, \dots, y_j$ by assuming their existence.

First we show (B). According to Proposition 5.1, it holds that

$$\sum_{i=1}^{g}(a_{l,i}y_i) + s\sum_{i=g+1}^{j} y_i = 1 \qquad \text{for any } 1 \le l \le g, \tag{5.2}$$

$$s\sum_{i=1}^{g} y_i + \sum_{i=g+1}^{j}(a_{l,i}y_i) = 1 \qquad \text{for any } g+1 \le l \le j. \tag{5.3}$$

Therefore, as per the definition of $z_1, \dots, z_g, z'_1, \dots, z'_h$, we obtain

$$\sum_{i=1}^{g} y_i = \left(1 - s \sum_{i=g+1}^{j} y_i\right) \sum_{i=1}^{g} z_i, \qquad \sum_{i=g+1}^{j} y_i = \left(1 - s \sum_{i=1}^{g} y_i\right) \sum_{i=1}^{h} z'_i.$$

A simple computation and (5.1) yield

$$\begin{aligned} \sum_{i=1}^{g} y_i &= \frac{\sum_{i=1}^{g} z_i - s \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}, \\ \sum_{i=g+1}^{j} y_i &= \frac{\sum_{i=1}^{h} z'_i - s \sum_{i=1}^{h} z'_i \sum_{i=1}^{g} z_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}, \end{aligned} \tag{5.4}$$

and therefore,

$$\sum_{i=1}^{j} y_i = \frac{\sum_{i=1}^{g} z_i + \sum_{i=1}^{h} z'_i - 2s \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}. \tag{5.5}$$

If we let $\tilde{s} = \min_{i,l \in \operatorname{Im} \sigma_1} a_{i,l}$, by assuming (A) and setting $g \geq 2$, we obtain

$$1 - \tilde{s} \sum_{i=1}^{g} z_i > 0. \tag{5.6}$$

Because $\tilde{s} \geq s$ for $g \geq 2$, (5.4) and (5.6) yield

$$1 - s \sum_{i=1}^{g} y_i = \frac{1 - s \sum_{i=1}^{g} z_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} \geq \frac{1 - \tilde{s} \sum_{i=1}^{g} z_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} > 0.$$

In addition, because $\tilde{s} > s$ for $g = 1$, (5.4) and (5.6) yield

$$1 - s y_1 = \frac{1 - s z_1}{1 - s^2 z_1 \sum_{i=1}^{h} z'_i} > \frac{1 - \tilde{s} z_1}{1 - s^2 z_1 \sum_{i=1}^{h} z'_i} \geq 0.$$

As per the definition of $y_{g+1}, \dots, y_j$, $z'_1, \dots, z'_h$ and (5.3), we have $(y_{g+1}, \dots, y_j) = (1 - s \sum_{i=1}^{g} y_i) \vec{z}'$. Subsequently, because we assume that the solution of $\vec{z}'$ satisfies $z'_i > 0$ for any $1 \leq i \leq h$, it holds that any solution $y_{g+1}, \dots, y_j$ satisfies $y_i > 0$ for any $g + 1 \leq i \leq j$. In addition, as per the same above-mentioned argument, the definitions of $y_1, \dots, y_g$, $z_1, \dots, z_g$ and (5.2) yield $y_i > 0$ for any $1 \leq i \leq g$, and therefore, (B) holds.

Secondly we show (A). The fact that $y_i > 0$ for any $1 \leq i \leq j$ and $\sum_{i=1}^{j} a_{l,i} y_i = 1$ for any $1 \leq l \leq j$ yields $s \sum_{i=1}^{j} y_i < 1$ allows us to obtain the desired results.

Now we turn to prove the uniqueness of the solution using the result of (B) that we already obtained. In general, it is known that

$$V := \{\vec{y} : A \vec{y}^T = \vec{1}^T\} = x_0 + \operatorname{Ker} A,$$

where x_0 is a characteristic solution for $A\vec{y}^T = \vec{1}^T$. As per the result of (B), it holds that $\{v = (v_1, \dots, v_j) : v_i > 0\} \supset V$. Because V is a linear space, the equation $\operatorname{Ker} A \neq \{0\}$ is contradictory. Subsequently, $\operatorname{Ker} A = \{0\}$, and therefore, we have the desired claim. □

To show Proposition 5.3, we use the two lemmas. We argue with the values of Ξ and χ in Proposition 5.3 and the following lemmas. Because Ξ and χ are independent of σ_1 and σ_2, we omit σ_1 and σ_2. For example, we write $A_1 \boxplus_s A_2$ for $A_1^{\sigma_1} \boxplus_s A_2^{\sigma_2}$.

LEMMA 5.2. *Consider* $A, \overline{A} \in \mathcal{M}_j$ *such that* $A = A_1 \boxplus_s A_2$ *and* $\overline{A} = \overline{A}_1 \boxplus_s \overline{A}_2$. *Subsequently, if*

$$\chi(A_1) \geq \chi(\overline{A}_1) \quad \textit{and} \quad \chi(A_2) \geq \chi(\overline{A}_2),$$

it holds that

$$\chi(A) \geq \chi(\overline{A}).$$

PROOF. Note that it suffices to prove the case that $\chi(A_1) = \chi(\overline{A}_1)$ because we can prove the claim by repeating the same proof. Let

$$g(t, b, c) := \frac{b + c - 2tbc}{1 - t^2 bc},$$

where $1 - t^2 bc > 0$. It is found that g monotonically increases in c because a simple computation yields

$$\frac{\partial g}{\partial c} = \frac{(1 - tb)^2}{(1 - t^2 bc)^2} \geq 0.$$

Note that if we consider A and $\vec{y}$ such that $A\vec{y}^T = \vec{1}^T$, $\chi(A) = \sum_{i=1}^{j} y_i$ holds; then, (5.5) yields

$$\chi(A) = g\bigl(s, \chi(A_1), \chi(A_2)\bigr). \tag{5.7}$$

Because $g(t, b, c)$ monotonically increases in c, the assumption yields the desired result. □

LEMMA 5.3. *Consider* $r \leq j - 1$, $0 \leq \gamma \leq \gamma_1 \leq \gamma_2 \leq 1$ *with* $r = (g - 1)(1 - \gamma_2) + (h - 1)(1 - \gamma_1) + 1 - \gamma$ *and* $g + h = j$ *with* $g, h \geq 1$, *which satisfy* $A_{\gamma_2}^{(g)} \boxplus_\gamma A_{\gamma_1}^{(h)} \in \Xi^{-1}(\{r\})$. *Then, fixing the values* γ_2 *and* r, $\chi(A_{\gamma_2}^{(g)} \boxplus_\gamma A_{\gamma_1}^{(h)})$ *is minimized at* $\gamma = \gamma_1$.

PROOF. When we fix the values γ_2 and r, we find that $r = \Xi(A_{\gamma_2}^{(g)} \boxplus_\gamma A_{\gamma_1}^{(h)}) = (g - 1)(1 - \gamma_2) + (h - 1)(1 - \gamma_1) + 1 - \gamma$ is a constant, and therefore, we obtain

$(g-1)\gamma_2 + (h-1)\gamma_1 + \gamma$. Subsequently, if we set $p := (g-1)\gamma_2 + 1$ and $q := (h-1)\gamma_1 + \gamma + 1$, we find that p and q are constants. Note that $0 \le \gamma \le (q-1)/h$. In addition,

$$\chi(A_{\gamma_2}^{(g)}) = \frac{g}{(g-1)\gamma_2 + 1}, \qquad \chi(A_{\gamma_1}^{(h)}) = \frac{h}{(h-1)\gamma_1 + 1}.$$

According to (5.7), it holds that

$$\begin{aligned} f(\gamma) &:= \chi(A_{\gamma_2}^{(g)} \boxplus_\gamma A_{\gamma_1}^{(h)}) \\ &= \frac{g(q-\gamma) + hp - 2\gamma gh}{(q-\gamma)p - \gamma^2 gh}. \end{aligned}$$

It suffices to show the claim that f monotonically decreases in $0 \le \gamma \le (q-1)/h$. A simple computation yields

$$\begin{aligned} &\frac{\partial f(\gamma)}{\partial \gamma} \\ &\quad = \frac{((q-\gamma)p - \gamma^2 gh)(-g-2gh) - (g(q-\gamma) + hp - 2\gamma gh)(-p-2\gamma gh)}{((q-\gamma)p - \gamma^2 gh)^2}. \end{aligned}$$

Set

$$\tilde{f} := -g^2 h(1+2h)\left(\gamma - \frac{gq+hp}{g+2gh}\right)^2 + \frac{h(gq+hp)^2}{(1+2h)} - 2qpgh + hp^2. \tag{5.8}$$

Note that it holds that

$$\frac{q-1}{h} \le \text{the apex (summit) of } \tilde{f} = \frac{(gq+hp)}{g(1+2h)} \tag{5.9}$$

because $h(gq+hp) - (q-1)g(1+2h) = h(g+h)(1-\gamma_2) \ge 0$. The first inequality comes from $-(q-1) \ge -((h-1)\gamma_2 + \gamma_2) = -h\gamma_2$ and the second one comes from $\gamma_2 \le 1$. Therefore, we obtain (5.9). In addition, we claim

$$\tilde{f}\left(\frac{q-1}{h}\right) \le 0 \tag{5.10}$$

because $h\tilde{f}((q-1)/h) = -(ph - g(q-1))(2gh - g(q-1) - hp) \le 0$. The inequality comes from $ph - g(q-1) \ge gh\gamma_2 \ge 0$ and $2gh - g(q-1) - hp \ge 2gh - gh - hp \ge 0$. Therefore, as per (5.8), (5.9) and (5.10), we obtain for $0 \le \gamma \le (q-1)/h$, $\tilde{f}(\gamma) \le 0$ and the desired result. □

PROOF OF PROPOSITION 5.3. It is trivial that $\chi(A_{1-r/(j-1)}^{(j)}) = j/(j-r)$ holds; therefore, we show only $\min_{A \in \Xi^{-1}(\{r\})} \chi(A) = \chi(A_{1-r/(j-1)}^{(j)})$. We prove the result by performing induction on $j \in \mathbb{N}$. If $j = 1$ or 2, it is obvious that the claim holds. We assume that the claim holds for $1, \dots, j-1$ and show the claim for j. For $g \wedge h = 1$, Lemma 5.3 yields the desired result. It suffices to show the result for $j \geq 4$ with $g, h \geq 2$.

For any $r \leq j-1$ and $A = A_1 \boxplus_s A_2 \in \Xi^{-1}(\{r\})$, we select γ_1, γ_2 and γ, which satisfy $\gamma = s$, $\Xi(A_1) = \Xi(A^{(g)}_{\gamma_2}) = (g-1)(1-\gamma_2)$ and $\Xi(A_2) = \Xi(A^{(h)}_{\gamma_1}) = (h-1)(1-\gamma_1)$. Without loss of generality, we can assume that $\gamma_1 \leq \gamma_2$. Note that $A^{(g)}_{\gamma_2} \boxplus_\gamma A^{(h)}_{\gamma_1} \in \Xi^{-1}(\{r\})$. According to Lemma 5.2 and this assumption, we obtain

$$\chi\big(A^{(g)}_{\gamma_2} \boxplus_\gamma A^{(h)}_{\gamma_1}\big) \leq \chi(A). \tag{5.11}$$

In addition, we consider $\tilde{\gamma}_1$ satisfying $(h-1)\gamma_1 + \gamma = h\tilde{\gamma}_1$. Note that $A^{(g)}_{\gamma_2} \boxplus_{\tilde{\gamma}_1} A^{(h)}_{\tilde{\gamma}_1} \in \Xi^{-1}(\{r\})$ and $\gamma_1 \geq \tilde{\gamma}_1 \geq \gamma$. As per Lemma 5.3, we obtain

$$\chi\big(A^{(g)}_{\gamma_2} \boxplus_{\tilde{\gamma}_1} A^{(h)}_{\tilde{\gamma}_1}\big) \leq \chi\big(A^{(g)}_{\gamma_2} \boxplus_\gamma A^{(h)}_{\gamma_1}\big). \tag{5.12}$$

Note that for any σ_1 and σ_2, we can select σ_3, σ_4, σ_5 and σ_6 such that

$$\big(A^{(g)}_{\gamma_2}\big)^{\sigma_1} \boxplus_{\tilde{\gamma}_1} \big(A^{(h)}_{\tilde{\gamma}_1}\big)^{\sigma_2} = \big(\big(A^{(g)}_{\gamma_2}\big)^{\sigma_3} \boxplus_{\tilde{\gamma}_1} \big(A^{(h-1)}_{\tilde{\gamma}_1}\big)^{\sigma_4}\big)^{\sigma_5} \boxplus_{\tilde{\gamma}_1} \big(A^{(1)}\big)^{\sigma_6}.$$

In addition, we consider $\tilde{\gamma}_2$ satisfying $(g-1)\gamma_2 + h\tilde{\gamma}_1 = (j-2)\tilde{\gamma}_2 + \tilde{\gamma}_1$. Note that $A^{(j-1)}_{\tilde{\gamma}_2} \boxplus_{\tilde{\gamma}_1} A^{(1)} \in \Xi^{-1}(\{r\})$ and $\gamma_2 \geq \tilde{\gamma}_2 \geq \tilde{\gamma}_1$. According to Lemma 5.2 and the assumption, we obtain

$$\chi\big(A^{(j-1)}_{\tilde{\gamma}_2} \boxplus_{\tilde{\gamma}_1} A^{(1)}\big) \leq \chi\big(A^{(g)}_{\gamma_2} \boxplus_{\tilde{\gamma}_1} A^{(h)}_{\tilde{\gamma}_1}\big). \tag{5.13}$$

Finally, Lemma 5.3 yields

$$\chi\big(A^{(j)}_{1-r/(j-1)}\big) \leq \chi\big(A^{(j-1)}_{\tilde{\gamma}_2} \boxplus_{\tilde{\gamma}_1} A^{(1)}\big). \tag{5.14}$$

Note that $A^{(j)}_{1-r/(j-1)} \in \Xi^{-1}(\{r\})$. Therefore, as per (5.11), (5.12), (5.13) and (5.14), we obtain the desired result. $\square$

PROOF OF LEMMA 5.1. We prove the claim by performing induction on $j \in \mathbb{N}$. Because $\hat{\mathcal{E}}_\delta[A] = \mathbb{Z}^2_n$ for $j = 1$ and $|\hat{\mathcal{E}}_\delta[A]| \leq |\mathbb{Z}^2_n| \times Cn^{2t+2\delta}$ for $j = 2$ and $A \in \Xi^{-1}(\{t\})$, it is obvious that the desired result holds for $j = 1, 2$. We assume that the result holds for $1, \dots, j-1$ with $j \geq 3$ and show the result for j. It suffices to prove that for any $\varepsilon > 0$, $0 < \delta < \varepsilon$, and $L < \infty$, there exists $C > 0$ such that for any $x \in \mathbb{Z}^2_n$, $t \leq (j-1)\beta$, $A \in \mathcal{M}^{\beta,\eta}_j \cap \Xi^{-1}(\{t\})$ and $n \in \mathbb{N}$

$$\big|\mathcal{E}\big[(0), \big(Ln^{1-a_{i,l}+\delta}\big)_{1\leq i,l\leq j}\big]\big| \leq Cn^{2t+2+\exp(j)\varepsilon/2}. \tag{5.15}$$

First we show the claim for the case that $g \wedge h = 1$. Without loss of generality, we only prove it for $h = 1$. Let $t_1 := \Xi(A_1)$ and $(a^1_{i,l})_{1\leq i,l\leq j-1} := A_1$. Note that $t = \Xi(A) = \Xi(A_1) + 1 - s = t_1 + 1 - s$. Then, for any $0 < \delta < \varepsilon$, there exists $C > 0$ such that for any $n \in \mathbb{N}$,

$$\big|\mathcal{E}\big[(0), \big(Ln^{1-a^1_{i,l}+\delta}\big)_{1\leq i,l\leq j-1}\big]\big| \leq Cn^{2t_1+2+\exp(j-1)\varepsilon/2}, \tag{5.16}$$

and therefore,

$$\begin{aligned}\big|\mathcal{E}\big[(0),\big(Ln^{1-a_{i,l}+\delta}\big)_{1\le i,l\le j}\big]\big| &\le \big|\mathcal{E}\big[(0),\big(Ln^{1-a^1_{i,l}+\delta}\big)_{1\le i,l\le j-1}\big]\big|\times Cn^{2-2s+2\delta}\\ &\le Cn^{2-2s+2t_1+2+\exp(j-1)\varepsilon/2+2\delta}\\ &\le Cn^{2t+2+\exp(j)\varepsilon/2}.\end{aligned}$$

Then, we have proven the claim.

Next, we show the claim for $j \ge 4$ and $g \wedge h \ne 1$. For $k \ge 2$, $x \in \mathbb{Z}_n^2$ and $L > 0$, let

$$\begin{aligned}\tilde{\mathcal{E}}_{\delta,x}[A] &= \tilde{\mathcal{E}}_{\delta,x}[A](k,L)\\ &:= \big\{(x_2,\dots,x_k) : (x,x_2,\dots,x_k) \in \mathcal{E}\big[(0),\big(Ln^{1-a_{i,l}+\delta}\big)_{1\le i,l\le k}\big]\big\}.\end{aligned}$$

Note that

$$\mathcal{E}\big[(0),\big(Ln^{1-a_{i,l}+\delta}\big)_{1\le i,l\le j}\big] \subset \big\{\vec{x} : x_1 \in \mathbb{Z}_n^2,\,(x_2,\dots,x_j) \in \tilde{\mathcal{E}}_{\delta,x_1}[A]\big\},$$

and therefore,

$$\big|\mathcal{E}\big[(0),\big(Ln^{1-a_{i,l}+\delta}\big)_{1\le i,l\le j}\big]\big| \le \sum_{x\in\mathbb{Z}_n^2} \big|\tilde{\mathcal{E}}_{\delta,x}[A]\big|.$$

Then, it suffices to prove that for any $\varepsilon > 0$, $0 < \delta < \varepsilon$ and $L < \infty$, there exists $C > 0$ such that for any $x \in \mathbb{Z}_n^2$, $t \le (j-1)\beta$, $A \in \mathcal{M}_j^{\beta,\eta} \cap \Xi^{-1}(\{t\})$ and $n \in \mathbb{N}$

$$\big|\tilde{\mathcal{E}}_{\delta,x}[A]\big| \le Cn^{2t+\exp(j)\varepsilon/2}.$$

For any A, let $t_1 := \Xi(A_1)$ and $t_2 := \Xi(A_2)$, and therefore, $t = \Xi(A) = t_1 + t_2 + 1 - s$ holds. Note that it holds that

$$A_1 \in \mathcal{M}_g^{\beta,\eta} \cap \Xi^{-1}\big(\{t_1\}\big), \qquad A_2 \in \mathcal{M}_h^{\beta,\eta} \cap \Xi^{-1}\big(\{t_2\}\big).$$

Subsequently, as per the assumption, we find that for any $\varepsilon > 0$ and $0 < \delta < \varepsilon$, there exists $C > 0$ such that for any $x_1, x_{g+1} \in \mathbb{Z}_n^2$, and $n \in \mathbb{N}$, it holds that

$$\big|\tilde{\mathcal{E}}_{\delta,x_1}[A_1]\big| \le Cn^{2t_1+\exp(g)\varepsilon/2}, \qquad \big|\tilde{\mathcal{E}}_{\delta,x_{g+1}}[A_2]\big| \le Cn^{2t_2+\exp(h)\varepsilon/2}. \tag{5.17}$$

Therefore, if we let $\tilde{D} := \{x \in \mathbb{Z}_n^2 : d(x_1,x) \le Ln^{1-s+\delta}\}$, we have that for any $0 < \delta < \varepsilon$

$$\begin{aligned}&\big|\tilde{\mathcal{E}}_{\delta,x_1}[A]\big|\\ &\quad\le \sum_{x_{g+1}\in\tilde{D}} \big|\big\{(x_2,\dots,x_j) : (x_2,\dots,x_g) \in \tilde{\mathcal{E}}_{\delta,x_1}[A_1],\\ &\qquad (x_{g+2},\dots,x_j) \in \tilde{\mathcal{E}}_{\delta,x_{g+1}}[A_2]\big\}\big|\\ &\quad\le Cn^{2-2s+2\delta}\big|\tilde{\mathcal{E}}_{\delta,x_1}[A_1]\big| \times \big|\tilde{\mathcal{E}}_{\delta,x_{g+1}}[A_2]\big|\\ &\quad\le Cn^{2t_1+2t_2+2-2s+\exp(j)\varepsilon/2} = Cn^{2t+\exp(j)\varepsilon/2}.\end{aligned}$$

The last inequality comes from $2\delta + (\exp(g) + \exp(h))\varepsilon/2 < (2 + \exp(g)/2 + \exp(h)/2)\varepsilon < \exp(j)\varepsilon/2$ for $j \geq 4$. Therefore, we obtain the desired result. $\square$

REMARK 5.5. The reason why we set "L" in (5.15) instead of "2^j" is to ensure that we obtain (5.16) and (5.17).

PROOF OF PROPOSITION 4.1. We show the result by performing induction on $j \in \mathbb{N}$. It is obvious that the claim holds for $j = 1$. Let us assume that the claim holds for $j-1$ and consider any $\vec{x} \in \mathcal{E}[(n^\eta), (n^\beta)]$ to show the claim for j. We set $1 \leq i_0, l_0 \leq j$ such that $d(x_{i_0}, x_{l_0}) = \min_{1 \leq i \neq l \leq j} d(x_i, x_l)$. Without loss of generality, we set $j = l_0$. Then, as per the assumption, it is easy to extend the following: for $(x_1, \dots, x_{j-1}) \in \mathcal{E}[(n^\eta), (n^\beta)](j-1, n)$ and $\delta > 0$, there exists

$$(a_{i,l})_{1 \leq i,l \leq j-1} \in \mathcal{M}^{\beta,\eta}_{j-1} \tag{5.18}$$

such that for all sufficiently large $n \in \mathbb{N}$,

$$(x_1, \dots, x_{j-1}) \in \hat{\mathcal{E}}_{\delta/2}\big[(a_{i,l})_{1 \leq i,l \leq j-1}\big](j-1, n).$$

Set $\tilde{A} := (\tilde{a}_{i,l})_{1 \leq i,l \leq j}$ as follows:

$$\begin{cases} \tilde{a}_{i,l} = \tilde{a}_{l,i} := a_{i,l} & \text{for any } 1 \leq i, l \leq j-1, \\ \tilde{a}_{j,l} = \tilde{a}_{l,j} := a_{i_0,l} & \text{for any } 1 \leq l \leq j-1 \text{ with } l \neq i_0, \\ \tilde{a}_{j,i_0} = \tilde{a}_{i_0,j} := \left(1 - \dfrac{\log d(x_{i_0}, x_j)}{\log n} + \delta\right) \wedge (1-\eta), \\ \tilde{a}_{j,j} := 1. \end{cases}$$

We prove that $\tilde{A} \in \mathcal{M}^{\beta,\eta}_j$ and $\vec{x} \in \hat{\mathcal{E}}_\delta[\tilde{A}]$. We first prove that $\tilde{A} \in \mathcal{M}^{\beta,\eta}_j$. It is obvious that the definition of $\tilde{A}$ yields that $\tilde{A}$ is symmetric and $1 - \beta \leq \tilde{a}_{i,l} \leq 1 - \eta$ for any $1 \leq i \neq l \leq j$. Note that $\tilde{a}_{i_0,j} = \max_{1 \leq i,l \leq j} \tilde{a}_{i,l}$ holds as $\hat{g}(s) := \max\{b \in [1-\beta, 1-\eta] : 2^{-j+1} n^{1-b} \leq s \leq 2^{j-1} n^{1-b+\delta/2}\}$ is monotonically decreasing; $\hat{g}(d(x_{i_0}, x_j)) \leq \tilde{a}_{i_0,j}$ for all sufficiently large $n \in \mathbb{N}$ with $2^{j-1} \leq n^{\delta/2}$ and $\hat{g}(d(x_i, x_l)) \geq \tilde{a}_{i,l}$ hold for any $1 \leq i, l \leq j$ with $i \neq i_0, j$. We only prove that for any $1 \leq i, l, p \leq j$ with $i \neq l$, $l \neq p$, $p \neq i$, (d) $\tilde{a}_{i,l} < \tilde{a}_{i,p} \Rightarrow \tilde{a}_{l,p} = \tilde{a}_{i,l}$ is as follows:

1. $i, l, p \neq j$: (5.18) yields $\tilde{A} \in \mathcal{M}^{\beta,\eta}_j$. Therefore, we obtain (d).
2. $i = j$ and $p, l \neq i_0$: If $\tilde{a}_{j,l} < \tilde{a}_{j,p}$, $a_{i_0,l} < a_{i_0,p}$ holds. Therefore, (5.18) yields $\tilde{a}_{l,p} = a_{l,p} = a_{i_0,l} = \tilde{a}_{j,l}$.
3. ($p = j$ and $i, l \neq i_0$) or ($l = j$ and $l, i \neq i_0$): The proof is almost the same as above.

Because $\tilde{A}$ is symmetric for j and i_0, (d) remains to be proven for the following cases:

1. $i = j, l = i_0$: The assumption is contradictory.
2. $i = j, p = i_0$: The result is trivial.
3. $l = j, p = i_0$: The assumption is contradictory.

Therefore, we obtain $\tilde{A} \in \mathcal{M}_j^{\beta,\eta}$. Finally, we prove $\vec{x} \in \hat{\mathcal{E}}_\delta[\tilde{A}]$. Note that the triangle inequality yields that for any $1 \le l \le j-1$ with $l \neq i_0$,

$$\begin{aligned} d(x_j, x_l) &\le d(x_j, x_{i_0}) + d(x_{i_0}, x_l) \\ &\le 2^{j-1} n^{1-\tilde{a}_{i_0,j}+\delta} + 2^{j-1} n^{1-a_{i_0,l}+\delta} \le 2^j n^{1-\tilde{a}_{j,l}+\delta}, \end{aligned}$$

and as $d(x_j, x_l) + d(x_j, x_{i_0}) \ge d(x_{i_0}, x_l)$ and $d(x_j, x_l) \ge d(x_j, x_{i_0})$,

$$d(x_j, x_l) \ge \frac{1}{2} d(x_{i_0}, x_l) \ge \frac{1}{2} \frac{1}{2^{j-1}} n^{1-\tilde{a}_{i_0,l}} = \frac{1}{2^j} n^{1-\tilde{a}_{j,l}}.$$

Therefore, as it is trivial that the corresponding result holds for $d(x_j, x_{i_0})$, we have $\vec{x} \in \hat{\mathcal{E}}_\delta[\tilde{A}]$ and we obtain the desired result. □

Therefore, we obtain Proposition 5.4. Finally, we provide the following proposition, which is used in Section 4.

PROPOSITION 5.5. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $A \in \mathcal{M}_j$ and $|\delta_{i,l}| \le \delta$ it holds that $(a_{i,l} + \delta_{i,l})_{i,l=1}^j$ is a regular matrix and*

$$\left|\chi(A) - \chi\left((a_{i,l} + \delta_{i,l})_{i,l=1}^j\right)\right| \le \varepsilon.$$

REMARK 5.6. It is trivial that Proposition 5.5 yields the following. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $n \in \mathbb{N}$, $A \in \mathcal{M}_j$ and $|\delta_{i,l}| \le \delta$, it holds that $(a_{i,l} + \delta_{i,l})_{i,l=1}^j$ is a regular matrix and

$$\left|\frac{\chi(A)}{n} - \chi\left((a_{i,l} + \delta_{i,l})_{i,l=1}^j\right)\right| \le \frac{\varepsilon}{n}.$$

PROOF OF PROPOSITION 5.5. First we prove that $\mathcal{M}_j$ is the closed set for each $j \in \mathbb{N}$. We consider $\max_{1 \le i,l \le j} |a_{i,l} - a'_{i,l}|$ for $(a_{i,l})_{1 \le i,l \le j}$, $(a'_{i,l})_{1 \le i,l \le j} \in \mathcal{M}_j$ as the metric on $\mathcal{M}_j$. Consider $(a^m_{i,l})_{1 \le i,l \le j} \in \mathcal{M}_j$ and $(a^\infty_{i,l})_{1 \le i,l \le j}$ such that $(a^m_{i,l})_{1 \le i,l \le j} \to (a^\infty_{i,l})_{1 \le i,l \le j}$ as $m \to \infty$. Note that it is trivial that $(a^\infty_{i,l})_{1 \le i,l \le j}$ satisfies (a) and (b). Therefore, it suffices to show that $(a^\infty_{i,l})_{1 \le i,l \le j}$ satisfies (d). First we assume that $a^\infty_{i,l} < a^\infty_{i,p}$ holds for some $1 \le i, l, p \le j$ with $i \neq l$, $l \neq p$, $p \neq i$. For $k \in \mathbb{N}$, set $m_k := \inf\{m > m_{k-1} : a^m_{i,l} > a^m_{i,p}\}$ with $m_1 = 1$ and $\inf \varnothing = 0$. Then, $\sup m_k < \infty$ and, as per the definition of $\mathcal{M}_j$, there exists $m_0 \in \mathbb{N}$ such that for any $m \ge m_0$, $a^m_{l,p} = a^m_{i,l}$. Therefore, $a^\infty_{l,p} = a^\infty_{i,l}$ holds. $\mathcal{M}_j$ is the closed set and compact.

Now we prove the desired result. Note that

$$\max_{1 \le i,l \le j} |a_{i,l}| \le 1. \tag{5.19}$$

In addition, Proposition 5.2 yields that $\det A \neq 0$ holds for $A \in \mathcal{M}_j$. By the compactness of $\mathcal{M}_j$, we have $\inf_{A \in \mathcal{M}_j} |\det A| \neq 0$. Therefore, (5.19) yields that there exists $\delta > 0$ such that for any $\delta_{i,l}$ with $|\delta_{i,l}| \leq \delta$,

$$\inf_{A \in \mathcal{M}_j} \left|\det(a_{i,l} + \delta_{i,l})_{1 \leq i,l \leq j}\right| \neq 0. \tag{5.20}$$

Thus, the first claim holds. Finally, it is trivial that (5.19) and (5.20) again yield the second claim, and therefore, we obtain the desired result. □

APPENDIX

A.1. Computation of exponents. In this section, we provide the estimation for the monotonicity of the exponents.

LEMMA A.1. *For any $j \geq 2$ and $0 < \alpha, \beta < 1$,*

$$\hat{\rho}_j(\alpha, \beta) - \hat{\rho}_{j-1}(\alpha, \beta) \geq 0.$$

REMARK A.1. As discussed previously, this result yields Theorem 2.1. In addition, the above lemma is equivalent to Theorem 2.1. For any $0 < \alpha, \beta < 1$,

$$\left|\{\vec{x} \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \leq n^\beta \text{ for any } 1 \leq i, l \leq j\}\right|$$

is monotonically increasing in $j \in \mathbb{N}$. Therefore, Theorem 2.1 naturally yields the above lemma.

REMARK A.2. We omit the proof because we only need long and elementary computations.

Acknowledgments. We are grateful to Professor Kazumasa Kuwada and Kohei Uchiyama for their interesting discussions. In addition, we would like to thank Yoshihiro Abe for his helpful comments.

REFERENCES

[1] ALESKEROV, F., GOLDENGORIN, B. and PARDALOS, P. M. (2014). *Clusters, Orders, and Trees: Methods and Applications. Springer Optimization and Its Applications* **92**. Springer, New York.

[2] BELIUS, D. and KISTLER, N. (2017). The subleading order of two dimensional cover times. *Probab. Theory Related Fields* **167** 461–552. MR3602852

[3] BRUMMELHUIS, M. J. A. M. and HILHORST, H. J. (1991). Covering of a finite lattice by a random walk. *Phys. A* **176** 387–408. MR1130067

[4] DAVIAUD, O. (2006). Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.* **34** 962–986. MR2243875

[5] DEMBO, A. (2005). Favorite points, cover times and fractals. In *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1869** 1–101. Springer, Berlin. MR2228383

[6] DEMBO, A. (2006). Simple random covering disconnection, late and favorite points. In *International Congress of Mathematicians*, *Vol*. *III* 535–558. Eur. Math. Soc., Zürich. MR2275695

[7] DEMBO, A., PERES, Y. and ROSEN, J. (2007). How large a disc is covered by a random walk in n steps? *Ann*. *Probab*. **35** 577–601. MR2308589

[8] DEMBO, A., PERES, Y., ROSEN, J. and ZEITOUNI, O. (2004). Cover times for Brownian motion and random walks in two dimensions. *Ann*. *of Math*. (2) **160** 433–464. MR2123929

[9] DEMBO, A., PERES, Y., ROSEN, J. and ZEITOUNI, O. (2006). Late points for random walks in two dimensions. *Ann*. *Probab*. **34** 219–263. MR2206347

[10] DING, J. (2014). Asymptotics of cover times via Gaussian free fields: Bounded-degree graphs and general trees. *Ann*. *Probab*. **42** 464–496. MR3178464

[11] DING, J., LEE, J. R. and PERES, Y. (2012). Cover times, blanket times, and majorizing measures. *Ann*. *of Math*. (2) **175** 1409–1471. MR2912708

[12] EISENBAUM, N., KASPI, H., MARCUS, M. B., ROSEN, J. and SHI, Z. (2000). A Ray–Knight theorem for symmetric Markov processes. *Ann*. *Probab*. **28** 1781–1796. MR1813843

[13] ERDŐS, P. and TAYLOR, S. J. (1960). Some problems concerning the structure of random walk paths. *Acta Math*. *Acad*. *Sci*. *Hung*. **11** 137–162. MR0121870

[14] LAWLER, G. F. (1991). *Intersections of Random Walks*. *Probability and Its Applications*. Birkhäuser, Boston, MA. MR1117680

[15] MARTÍNEZ, S., MICHON, G. and SAN MARTÍN, J. (1994). Inverse of strictly ultrametric matrices are of Stieltjes type. *SIAM J*. *Matrix Anal*. *Appl*. **15** 98–106. MR1257619

[16] NABBEN, R. and VARGA, R. S. (1994). A linear algebra proof that the inverse of a strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix. *SIAM J*. *Matrix Anal*. *Appl*. **15** 107–113. MR1257620

[17] ROBERTS, M. D. (2015). Ultrametric distance in syntax. *Prague Bull*. *Math*. *Linguist*. **103** 111–130.

[18] ROSEN, J. (2005). A random walk proof of the Erdős–Taylor conjecture. *Period*. *Math*. *Hungar*. **50** 223–245. MR2162811

DEPARTMENT OF INFORMATION SCIENCES
TOKYO UNIVERSITY OF SCIENCE
2641 YAMAZAKI
NODA, CHIBA, 278-0022
JAPAN
E-MAIL: izumiokada1205@gmail.com