Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2019, Vol. 55, No. 3, 1377–1401
https://doi.org/10.1214/18-AIHP922

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

A large deviation principle for empirical measures on Polish spaces: Application to singular Gibbs measures on manifolds

David García-Zelada

Université Paris-Dauphine, PSL Research University, CNRS, CEREMADE, 75016 PARIS, FRANCE. E-mail: garciazelada@ceremade.dauphine.fr

Received 16 January 2018; revised 11 May 2018; accepted 23 July 2018

Abstract. We prove a large deviation principle for a sequence of point processes defined by Gibbs probability measures on a Polish space. This is obtained as a consequence of a more general Laplace principle for the non-normalized Gibbs measures. We consider four main applications: Conditional Gibbs measures on compact spaces, Coulomb gases on compact Riemannian manifolds, the usual Gibbs measures in the Euclidean space and the zeros of Gaussian random polynomials. Finally, we study the generalization of Fekete points and prove a deterministic version of the Laplace principle known as Γ-convergence. The approach is partly inspired by the works of Dupuis and co-authors. It is remarkably natural and general compared to the usual strategies for singular Gibbs measures.

Résumé. On montre un principe de grandes déviations pour une suite de processus ponctuels définis par des mesures de probabilités de Gibbs dans un espace polonais. Il est obtenu comme conséquence d'un principe de Laplace pour des mesures de Gibbs non normalisées. On considère quatre applications: Des mesures de Gibbs conditionnées dans des espaces compacts, des gaz de Coulomb sur des variétés riemanniennes compactes, les mesures de Gibbs habituelles sur l'espace euclidien et les zéros des polynômes aléatoires gaussiens. Finalement, on étudie la généralisation des points Fekete et on prouve une version déterministe du principe de Laplace appelée Γ-convergence. Notre approche est partiellement inspirée par les travaux de Dupuis et ses coauteurs. C'est notablement naturelle et générale en comparaison avec les stratégies habituelles pour les mesures de Gibbs singulières.

MSC: 60F10; 60K35; 82C22; 30C15

Keywords: Gibbs measure; Coulomb gas; Empirical measure; Large deviation principle; Interacting particle system; Singular potential; Constant curvature; Relative entropy; Random polynomials; Fekete points

1. Introduction

The present article is inspired by part of the work of Dupuis, Laschos and Ramanan on large deviations for a sequence of point processes given by Gibbs measures associated to very general singular two-body interactions [15] but it differs from it in that we take a general sequence of interactions that includes, for instance, the interaction followed by the zeros of random polynomials as in [24]. We follow the philosophy of Dupuis and Ellis [14] about the use of variational formulas to make plausible and sometimes easier to find a Laplace principle. This philosophy has already been used by Georgii in [17] to treat a system of random fields on $\mathbb{Z}^d$ with interacting energies that converges uniformly to some limit functional.

We are interested in proving the Laplace principle and the large deviation principle for a very general sequence of energies in a not necessarily compact space. Part of our work has an overlap with the article of Berman [8] and it was developed independently. As in [8] the interest of this result is the generality of the sequence of energies: they do not need to be made of a two-body interaction potential but they may still be very singular. The key argument of the proof is a well-understood application of Jensen's inequality together with a general Laplace principle that has as its main ingredient a subadditivity property of the entropy. It is very simple compared to the ad hoc methods used in the usual proofs of the large deviation principles for Coulomb gases such as in [13,20,21] and [19]. In these methods, to prove a large deviation lower bound, the authors usually decompose the space in small regions and this decomposition may not be easy to achieve on a manifold and not so natural to look for. We give a more precise explanation of these methods in Remark 3.6.

Among the applications we can give we are particularly interested in explaining a simple case inspired by [6]. This is the case of a Coulomb gas on a two-dimensional Riemannian manifold. As a second application we study a large deviation principle for a conditional Gibbs measure, i.e. we fix the position of some of the particles and leave the rest of them random. The last applications we discuss are different proofs of already known results such as the special one-dimensional log-gas of [4] related to the Gaussian ensembles, the more general one-dimensional log-gas of [1, Section 2.6], the special two-dimensional log-gas [20] related to the Ginibre ensemble of random matrices and its generalization to an n-dimensional Coulomb gas in [13] and [15], the note in [19] about two-dimensional log-gases with a weakly confining potential and the Gaussian random polynomials of [24] and [11].

We now explain the contents of each section. The rest of Section 1 will be dedicated to the main definitions and assumptions we will need to state our results. Section 2 is about the usual mean-field case, the k-body interaction. We give sufficient conditions to be able to apply our result which will become important when we treat the Euclidean space case. In Section 3 we begin by giving an idea of the proofs which includes mainly a key variational formula. Then we give the proofs of the main theorem and of its corollary and we finish the section by giving some remarks about the usual proofs we may find in the literature. We discuss four particular examples in Section 4. More precisely, the conditional Gibbs measure, the Coulomb gas on a Riemannian manifold, a new way to obtain already known results in the Euclidean space about Coulomb gases and the assertion that the zeros of a Gaussian random polynomials may be treated by our main theorem. We conclude our article with Section 5 discussing a deterministic case which falls under the topic of Fekete points and which we consider as the natural deterministic analogue of the Laplace principle.

1.1. *Model*

Let M be a Polish space, i.e. a separable topological space metrizable by a complete metric. Endow it with the Borel σ-algebra associated to this topology, i.e. the least σ-algebra that contains the topology. Denote by $\mathcal{P}(M)$ the space of probability measures in M and endow it with the smallest topology such that $\mu \mapsto \int_M f\,d\mu$ is continuous for every bounded continuous function $f : M \to \mathbb{R}$. With this topology, $\mathcal{P}(M)$ is also a Polish space (see [10, Section 2.4]). This is called the weak topology. Suppose we have a sequence $\{W_n\}_{n\in\mathbb{N}}$ of symmetric measurable functions

$$W_n : M^n \to (-\infty, \infty]$$

and a sequence of non-negative numbers $\{\beta_n\}_{n\in\mathbb{N}}$ that converges to some $\beta \in (0, \infty]$. Fix a probability measure $\pi \in \mathcal{P}(M)$. We shall be interested in the asymptotic behavior of the *Gibbs measures* γ_n defined by

$$d\gamma_n = e^{-n\beta_n W_n}\, d\pi^{\otimes n}. \tag{1.1}$$

Define $\tilde{W}_n : \mathcal{P}(M) \to (-\infty, \infty]$ by

$$\tilde{W}_n(\mu) = \begin{cases} W_n(x_1, \dots, x_n) & \text{if } \mu \text{ is atomic with } \mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \\ \infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Stable sequence (S). *We shall say that the sequence $\{W_n\}_{n\in\mathbb{N}}$ is a* stable sequence *if it is uniformly bounded from below, i.e. if there exists $C \in \mathbb{R}$ such that*

$$W_n \geq C \quad \textit{for all } n \in \mathbb{N}.$$

Confining sequence (C). *We shall say that $\{W_n\}_{n\in\mathbb{N}}$ is a* confining sequence *if the following is true. Let $\{n_j\}_{j\in\mathbb{N}}$ be any increasing sequence of natural numbers and let $\{\mu_j\}_{j\in\mathbb{N}}$ be any sequence of probability measures on M. If there* *exists a real constant A such that*

$$\tilde{W}_{n_j}(\mu_j) \le A$$

for every $j \in \mathbb{N}$, *where* $\tilde{W}_n$ *is defined in* (1.2), *then* $\{\mu_j\}_{j\in\mathbb{N}}$ *is relatively compact in* $\mathcal{P}(M)$.

In order to study the behavior as $n \to \infty$ of γ_n we shall need a measurable function

$$W : \mathcal{P}(M) \to (-\infty, \infty].$$

Definition 1.1 (Macroscopic limit). Suppose that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S). We say that a measurable function $W : \mathcal{P}(M) \to (-\infty, \infty]$ is the *positive temperature macroscopic limit* of the sequence $\{W_n\}_{n\in\mathbb{N}}$ if the following two conditions are satisfied.

- **Lower limit assumption (A1).** For every sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on M that converges to some probability measure μ we have

$$\liminf_{n\to\infty} \tilde{W}_n(\mu_n) \ge W(\mu),$$

where $\tilde{W}_n$ is defined in (1.2).

- **Upper limit assumption (A2).** For each $\mu \in \mathcal{P}(M)$ we have that

$$\limsup_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \le W(\mu).$$

We say that W is the *zero temperature macroscopic limit* of the sequence $\{W_n\}_{n\in\mathbb{N}}$ if instead the *lower limit assumption* (A1) and the following condition are satisfied.

- **Regularity assumption (A2′).** Define the set of 'nice' probability measures

$$\mathcal{N} = \Big\{\mu \in \mathcal{P}(M) : D(\mu \parallel \pi) < \infty \text{ and } \limsup_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \le W(\mu)\Big\}. \tag{1.3}$$

For every $\mu \in \mathcal{P}(M)$ such that $W(\mu) < \infty$ we can find a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{N}$ such that $\mu_n \to \mu$ and $\limsup_{n\to\infty} W(\mu_n) \le W(\mu)$.

Now we are ready to state the Laplace principles and the large deviation principles.

1.2. *Main results*

Let $i_n : M^n \to \mathcal{P}(M)$ be the application defined by

$$i_n(x_1, \dots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}, \tag{1.4}$$

the usual continuous 'inclusion' of M^n in $\mathcal{P}(M)$. Define the *free energy* with parameter β as

$$F = W + \frac{1}{\beta} D(\cdot \parallel \pi) \tag{1.5}$$

(we suppose $0 \times \infty = 0$) where $D(\mu \parallel \nu)$ denotes the relative entropy of μ with respect to ν, also known as the Kullback–Leibler divergence i.e.

$$D(\mu \parallel \nu) = \int_M \frac{d\mu}{d\nu} \log\left(\frac{d\mu}{d\nu}\right) d\nu \tag{1.6}$$

if μ is absolutely continuous with respect to ν and $D(\mu \parallel \nu) = \infty$ otherwise.

Theorem 1.2 (Laplace principle). *Let $\{W_n\}_{n\in\mathbb{N}}$ be a stable sequence (S) and $W : \mathcal{P}(M) \to (-\infty, \infty]$ a measurable function. Take a sequence of positive numbers $\{\beta_n\}_{n\in\mathbb{N}}$ that converges to some $\beta \in (0, \infty]$.*

If $\beta < \infty$ suppose that W is the positive temperature macroscopic limit of $\{W_n\}_{n\in\mathbb{N}}$.

If $\beta = \infty$ suppose that W is the zero temperature macroscopic limit of $\{W_n\}_{n\in\mathbb{N}}$ and suppose that $\{W_n\}_{n\in\mathbb{N}}$ is a confining sequence (C).

Define the Gibbs measures γ_n by (1.1) and the free energy F by (1.5). Then, the following Laplace's principle is satisfied.

For every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f\circ i_n}\, d\gamma_n \xrightarrow[n\to\infty]{} - \inf_{\mu\in\mathcal{P}(M)} \big\{ f(\mu) + F(\mu) \big\}.$$

This Laplace principle implies the following large deviation principle.

Corollary 1.3 (Large deviation principle). *Suppose the same conditions as in Theorem 1.2. Define $Z_n = \gamma_n(M^n)$. Suppose $Z_n > 0$ for every n and notice that, as W_n is bounded from below, $Z_n < \infty$. Take the sequence of probability measures $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ defined by*

$$d\mathbb{P}_n = \frac{1}{Z_n}\, d\gamma_n. \tag{1.7}$$

For each $n \in \mathbb{N}$, let $i_n(\mathbb{P}_n)$ be the pushforward measure of $\mathbb{P}_n$ by i_n. Then the sequence $\{i_n(\mathbb{P}_n)\}_{n\in\mathbb{N}}$ satisfies a large deviation principle with speed $n\beta_n$ and with rate function

$$I = F - \inf F,$$

i.e. for every open set $A \subset \mathcal{P}(M)$ we have

$$\liminf_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{P}_n\big(i_n^{-1}(A)\big) \geq - \inf_{\mu\in A} I(\mu)$$

and for every closed set $C \subset \mathcal{P}(M)$ we have

$$\limsup_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{P}_n\big(i_n^{-1}(C)\big) \leq - \inf_{\mu\in C} I(\mu).$$

In the next section, Section 2, we shall study the usual case of k-body interaction. Section 4 will be about some more specific examples, such as the conditional Gibbs measure, the Coulomb gas on a compact Riemannian manifold, the usual Gibbs measures on a noncompact space such as the Euclidean space and the Gaussian random polynomials.

2. Example of a stable sequence: *k*-body interaction

We will give the most basic non-trivial example of a *stable sequence* (S). Take an integer $k > 0$ and a symmetric lower semicontinuous function bounded from below $G : M^k \to (-\infty, \infty]$. Define the symmetric measurable functions $W_n : M^n \to (-\infty, \infty]$ by

$$W_n(x_1, \dots, x_n) = \frac{1}{n^k} \sum_{\substack{\{i_1,\dots,i_k\}\subset\{1,\dots,n\} \\ \#\{i_1,\dots,i_k\}=k}} G(x_{i_1}, \dots, x_{i_k}),$$

and $W : \mathcal{P}(M) \to (-\infty, \infty]$ by

$$W(\mu) = \frac{1}{k!} \int_{M^k} G(x_1, \dots, x_k)\, d\mu(x_1) \cdots d\mu(x_k).$$

Proposition 2.1 (Stability, lower and upper limit assumption, (A1) and (A2)). *$\{W_n\}_{n\in\mathbb{N}}$ is a stable sequence (S), W is lower semicontinuous and the pair $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies the lower and upper limit assumption, (A1) and (A2).*

Proof. To see that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S) we notice that if $C \leq G$ then $\binom{n}{k}\frac{1}{n^k}C \leq W_n$. The lower semicontinuity of W is a consequence of the lower semicontinuity of G and the fact that it is bounded from below. Now, let us prove that $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies the *lower and upper limit assumption*, (A1) and (A2).

• **Lower limit assumption (A1).** Let $\mu \in \mathcal{P}(M)$. Take $N > 0$ and define $G_N = G \wedge N$. We will prove that

$$\tilde{W}_n(\mu) + \frac{N}{k!n^k}\left(n^k - \frac{n!}{(n-k)!}\right) \geq \frac{1}{k!}\int_{M^k} G_N(x_1,\dots,x_k)\,d\mu(x_1)\cdots d\mu(x_k), \tag{2.1}$$

where $\tilde{W}_n$ is the extension defined in (1.2). If $\tilde{W}_n(\mu) = \infty$ there is nothing to prove. If $\tilde{W}_n(\mu) < \infty$ then $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ for some $(x_1,\dots,x_n) \in M^n$. We have

$$\frac{1}{n^k}\sum_{\substack{\{i_1,\dots,i_k\}\subset\{1,\dots,n\}\\ \#\{i_1,\dots,i_k\}=k}} G_N(x_{i_1},\dots,x_{i_k}) + \frac{N}{k!n^k}\left(n^k - \frac{n!}{(n-k)!}\right) \geq \frac{1}{k!}\int_{M^k} G_N(x_1,\dots,x_k)\,d\mu(x_1)\cdots d\mu(x_k),$$

which due to the fact that $G \geq G_N$ implies the inequality (2.1).

Let $\mu_n \to \mu \in \mathcal{P}(M)$. Then, using the inequality (2.1) and taking the lower limit we get

$$\liminf_{n\to\infty} \tilde{W}_n(\mu_n) \geq \frac{1}{k!}\int_{M^k} G_N(x_1,\dots,x_k)\,d\mu(x_1)\cdots d\mu(x_k),$$

where we have used that G_N is lower semicontinuous and bounded from below. Finally, as G is bounded from below we can take N to infinity and use the monotone convergence theorem to get

$$\liminf_{n\to\infty} \tilde{W}_n(\mu_n) \geq \frac{1}{k!}\int_{M^k} G(x_1,\dots,x_k)\,d\mu(x_1)\cdots d\mu(x_k).$$

• **Upper limit assumption (A2).** For this it is enough to take $\mu \in \mathcal{P}(M)$ and notice that

$$\mathbb{E}_{\mu^{\otimes n}}[W_n] = \frac{1}{n^k}\binom{n}{k}\int_{M^k} G(x_1,\dots,x_k)\,d\mu(x_1)\cdots d\mu(x_k).$$

□

Now we give a sufficient condition for a k-body interaction to be a *confining sequence* (C).

Proposition 2.2 (k-body interaction and confining assumption). *Suppose $G(x_1,\dots,x_k)$ tends to infinity when $x_i \to \infty$ for all $i \in \{1,\dots,k\}$, i.e. suppose that for every $C \in \mathbb{R}$ there exists a compact set K such that $G|_{K^c\times\cdots\times K^c} \geq C$. Then $\{W_n\}_{n\in\mathbb{N}}$ is a confining sequence (C).*

Proof. Without loss of generality we can suppose G positive. Remember the definition of $\tilde{W}_n$ in (1.2). All we need is the following result.

Lemma 2.3 (Bound on the number of particles outside a compact set). *Suppose that G is positive. Take $n \in \mathbb{N}$, $A \in \mathbb{R}$ and $\mu \in \mathcal{P}(M)$ that satisfies*

$$\tilde{W}_n(\mu) \leq A.$$

If K is a compact set such that $G|_{K^c\times\cdots\times K^c} \geq C$ with $C > 0$, then

$$\mu(K^c) \leq \left(\frac{A}{C}k!\right)^{1/k} + \frac{k}{n}.$$

Proof. We first notice that $\mu = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ for some $(x_1, \ldots, x_n) \in M^n$. By the hypotheses we can see that

$$\frac{1}{n^k}[\text{number of } k\text{-combinations outside } K]C \leq \frac{1}{n^k}\sum_{\substack{\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}\\ \#\{i_1,\ldots,i_k\}=k}} G(x_{i_1}, \ldots, x_{i_k}) \leq A, \tag{2.2}$$

where [number of k-combinations outside K] denotes the cardinal of $\{S \subset \{1, \ldots, n\} : \#S = k \text{ and } \forall i \in S, x_i \notin K\}$. But, if m denotes the number of points among $x_1, \ldots, x_n$ outside K and if $k \leq m$, we have

$$\frac{(m-k)^k}{k!} \leq \frac{m!}{(m-k)!k!} = [\text{number of } k\text{-combinations outside } K]$$

which, along with the inequality (2.2), implies

$$\frac{m}{n} \leq \left(\frac{A}{C}k!\right)^{1/k} + \frac{k}{n}.$$

As $\mu = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ then $\mu(K^c) = \frac{m}{n}$ which concludes the proof. □

Then we can conclude using Prokhorov's theorem and the fact that every single probability measure is tight. □

Finally we notice that in the *regularity assumption* (A2′) we can replace finite entropy by absolute continuity with respect to π.

Proposition 2.4 (k-body interaction and regularity assumption). *Let*

$$\mathcal{N}_1 = \{\mu \in \mathcal{P}(M) : D(\mu \parallel \pi) < \infty\} \quad \textit{and}$$
$$\mathcal{N}_2 = \{\mu \in \mathcal{P}(M) : \mu \textit{ is absolutely continuous with respect to } \pi\}.$$

Suppose that for every μ with $W(\mu) < \infty$, there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{N}_2$ such that $\mu_n \to \mu$ and $W(\mu_n) \to W(\mu)$ then the same is true if we replace $\mathcal{N}_2$ by $\mathcal{N}_1$.

Proof. It is enough to prove that for every $\mu \in \mathcal{N}_2$ there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{N}_1$ such that $\mu_n \to \mu$ and $W(\mu_n) \to W(\mu)$. Let ρ be the density of μ with respect to π, i.e. $d\mu = \rho\, d\pi$. For each $n > 0$ define $\mu_n \in \mathcal{N}_1$ by $d\mu_n = \frac{\rho\wedge n\, d\pi}{\int_M \rho\wedge n\, d\pi}$. Then, by the monotone convergence theorem we can see that $\mu_n \to \mu$. And, again, by the monotone convergence theorem, by supposing $G \geq 0$, we can see that $W(\mu_n) \to W(\mu)$. □

3. Proof of the theorem

This section is dedicated to the proof of the main theorem, i.e. Theorem 1.2. We start giving a sketch of the proof.

3.1. *Idea of the proof*

We shall use the following very known result that tells us the Legendre transform of $D(\cdot \parallel \mu)$, defined in (1.6). See [14, Proposition 4.5.1] for a proof.

Lemma 3.1 (Legendre transform of the entropy). *Let E be a Polish probability space, μ a probability measure on E and $g : E \to (-\infty, \infty]$ a measurable function bounded from below. Then*

$$\log \mathbb{E}_\mu[e^{-g}] = -\inf_{\tau\in\mathcal{P}(E)}\{\mathbb{E}_\tau[g] + D(\tau \parallel \mu)\}.$$

Remember the definition of γ_n in (1.1) and F in (1.5). With the help of Lemma 3.1 we can write

$$\frac{1}{n\beta_n}\log\int_{M^n} e^{-n\beta_n f\circ i_n}\,d\gamma_n = \frac{1}{n\beta_n}\log\mathbb{E}_{\pi^{\otimes n}}\left[e^{-n\beta_n(f\circ i_n+W_n)}\right]$$
$$= -\inf_{\tau\in\mathcal{P}(M^n)}\left\{\mathbb{E}_{i_n(\tau)}[f]+\mathbb{E}_\tau[W_n]+\frac{1}{n\beta_n}D\left(\tau\parallel\pi^{\otimes n}\right)\right\},$$

where $i_n(\tau)$ denotes the pushforward measure of τ by i_n. So, we need to prove that

$$\inf_{\tau\in\mathcal{P}(M^n)}\left\{\mathbb{E}_{i_n(\tau)}[f]+\mathbb{E}_\tau[W_n]+\frac{1}{n\beta_n}D\left(\tau\parallel\pi^{\otimes n}\right)\right\}\xrightarrow[n\to\infty]{}\inf_{\mu\in\mathcal{P}(M)}\left\{f(\mu)+F(\mu)\right\}. \tag{3.1}$$

3.2. *Proof of Theorem 1.2: Case of finite β*

In this subsection we shall prove the Laplace principle Theorem 1.2 and the large deviation principle Corollary 1.3 for the case of finite β.

To prove this we need the following properties of the entropy. The first one is analogous to the *lower limit assumption (A1)*.

Lemma 3.2 (Lower limit property of the entropy). *Let $\{n_j\}_{j\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$. For each $j\in\mathbb{N}$ take $\tau_j\in\mathcal{P}(M^{n_j})$. If $i_{n_j}(\tau_j)\to\zeta\in\mathcal{P}(\mathcal{P}(M))$, then*

$$\mathbb{E}_\zeta\left[D(\cdot|\pi)\right]\le\liminf_{j\to\infty}\frac{1}{n_j}D\left(\tau_j\parallel\pi^{\otimes n_j}\right).$$

Proof. The idea of the proof is presented in [14]. We can also see [16]. It can be seen as equivalent to the large deviation upper bound of Sanov's theorem thanks to [22, Theorem 3.5]. □

And the second one is analogous to the notion of *confining sequence (C)*.

Lemma 3.3 (Confining property of the entropy). *Let $\{n_j\}_{j\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$. For each $j\in\mathbb{N}$ take $\tau_j\in\mathcal{P}(M^{n_j})$. If there exists a real constant C such that*

$$\frac{1}{n_j}D\left(\tau_j\parallel\pi^{\otimes n_j}\right)\le C$$

for every $j\in\mathbb{N}$, then the sequence $\{i_{n_j}(\tau_j)\}_{j\in\mathbb{N}}$ is tight.

Proof. The idea of the proof is presented in [14]. We can also see [16]. It can be seen as equivalent to the exponential tightness in Sanov's theorem thanks to [22, Theorem 3.3]. □

Without loss of generality, we can suppose $\beta_n=1$ for every n by redefinition of W_n and W. Then the *Gibbs measure* (1.1) and the *free energy* (1.5) are

$$d\gamma_n=e^{-nW_n}\,d\pi^{\otimes n}\quad\text{and}\quad F=W+D(\cdot\parallel\pi).$$

As explained in Section 3.1 we need to prove (3.1) which in this case is

$$\inf_{\tau\in\mathcal{P}(M^n)}\left\{\mathbb{E}_{i_n(\tau)}[f]+\mathbb{E}_\tau[W_n]+\frac{1}{n}D\left(\tau\parallel\pi^{\otimes n}\right)\right\}\xrightarrow[n\to\infty]{}\inf_{\mu\in\mathcal{P}(M)}\left\{f(\mu)+F(\mu)\right\}.$$

Proof of Theorem 1.2: Case of finite β. First, we will prove the *lower limit bound*

$$\liminf_{n\to\infty}\inf_{\tau\in\mathcal{P}(M^n)}\left\{\mathbb{E}_{i_n(\tau)}[f]+\mathbb{E}_\tau[W_n]+\frac{1}{n}D\left(\tau\parallel\pi^{\otimes n}\right)\right\}\ge\inf_{\mu\in\mathcal{P}(M)}\left\{f(\mu)+F(\mu)\right\}. \tag{3.2}$$

This is equivalent to say that for every increasing sequence of natural numbers $\{n_j\}_{j\in\mathbb{N}}$ if we choose, for each $j \in \mathbb{N}$, a probability measure $\tau_j \in \mathcal{P}(M^{n_j})$ we have

$$\lim_{j\to\infty}\left\{\mathbb{E}_{i_{n_j}(\tau_j)}[f]+\mathbb{E}_{\tau_j}[W_{n_j}]+\frac{1}{n_j}D\big(\tau_j \parallel \pi^{\otimes n_j}\big)\right\} \geq \inf_{\mu\in\mathcal{P}(M)}\big\{f(\mu)+F(\mu)\big\}, \tag{3.3}$$

where we can suppose that the limit exists and that it is finite and, in particular, the sequence is bounded from above.

Using that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S), we get that $\frac{1}{n_j}D(\tau_j \parallel \pi^{\otimes n_j})$ is uniformly bounded from above. By the *confining property of the entropy*, Lemma 3.3, we get that $i_{n_j}(\tau_j)$ is tight. By taking a subsequence using Prokhorov's theorem, we shall assume it converges to some $\zeta \in \mathcal{P}(\mathcal{P}(M))$. Then, by the *lower limit property of the entropy*, Lemma 3.2, we get

$$\mathbb{E}_\zeta\big[D(\cdot|\pi)\big] \leq \liminf_{j\to\infty}\frac{1}{n_j}D\big(\tau_j \parallel \pi^{\otimes n_j}\big).$$

As $\tilde{W}_n$ is measurable for every n (see [16, Proposition 7.6] for a proof) and the sequence $\{\tilde{W}_n\}_{n\in\mathbb{N}}$ is uniformly bounded from below we may use the *lower limit assumption* (A1) to get (see [22, Proposition 3.2])

$$\mathbb{E}_\zeta[W] \leq \liminf_{j\to\infty}\mathbb{E}_{\tau_j}[W_{n_j}].$$

Then, by taking the lower limit when j tends to infinity in (3.3), we obtain

$$\lim_{j\to\infty}\left\{\mathbb{E}_{i_{n_j}(\tau_j)}[f]+\mathbb{E}_{\tau_j}[W_{n_j}]+\frac{1}{n_j}D\big(\tau_j \parallel \pi^{\otimes n_j}\big)\right\} \geq \mathbb{E}_\zeta\big[f+W+D(\cdot \parallel \pi)\big] \geq \inf_{\mu\in\mathcal{P}(M)}\big\{f(\mu)+F(\mu)\big\}.$$

Now let us prove the *upper limit bound*

$$\limsup_{n\to\infty}\inf_{\tau\in\mathcal{P}(M^n)}\left\{\mathbb{E}_{i_n(\tau)}[f]+\mathbb{E}_\tau[W_n]+\frac{1}{n}D\big(\tau \parallel \pi^{\otimes n}\big)\right\} \leq \inf_{\mu\in\mathcal{P}(M)}\big\{f(\mu)+F(\mu)\big\}. \tag{3.4}$$

We need to prove that for every probability measure $\mu \in \mathcal{P}(M)$

$$\limsup_{n\to\infty}\inf_{\tau\in\mathcal{P}(M^n)}\left\{\mathbb{E}_{i_n(\tau)}[f]+\mathbb{E}_\tau[W_n]+\frac{1}{n}D\big(\tau \parallel \pi^{\otimes n}\big)\right\} \leq f(\mu)+F(\mu).$$

It is enough to find a sequence $\tau_n \in \mathcal{P}(M^n)$ such that

$$\limsup_{n\to\infty}\left\{\mathbb{E}_{i_n(\tau_n)}[f]+\mathbb{E}_{\tau_n}[W_n]+\frac{1}{n}D\big(\tau_n \parallel \pi^{\otimes n}\big)\right\} \leq f(\mu)+F(\mu).$$

We shall choose $\tau_n = \mu^{\otimes n}$. Then we know that, by the law of large numbers, we have the weak convergence $i_n(\tau_n) \to \delta_\mu$, so

$$\lim_{n\to\infty}\mathbb{E}_{i_n(\tau_n)}[f] = f(\mu).$$

In addition, by using that $D(\tau_n \parallel \pi^{\otimes n}) = nD(\mu \parallel \pi)$ and the *upper limit assumption* (A2) we get that

$$\limsup_{n\to\infty}\left\{\mathbb{E}_{i_n(\tau_n)}[f]+\mathbb{E}_{\tau_n}[W_n]+\frac{1}{n}D\big(\tau_n \parallel \pi^{\otimes n}\big)\right\} \leq f(\mu)+W(\mu)+D(\mu \parallel \pi)$$

completing the proof. $\square$

3.3. *Proof of Theorem* 1.2: *Case of infinite* β

In this subsection we provide a proof for Theorem 1.2 for the case of infinite β by modifying the proof used in the case of finite β. Recall that from the definition of *Gibbs measure* (1.1) and *free energy* (1.5) now we have

$$d\gamma_n = e^{-n\beta_n W_n}\, d\pi^{\otimes n} \quad \text{and} \quad F = W,$$

where $\beta_n \to \infty$.

We first notice that a *confining sequence* (C) satisfies an a priori stronger property.

Proposition 3.4 (Confining property of the expected value of the energy). *Assume that* $\{W_n\}_{n\in\mathbb{N}}$ *is a stable* (S) *and confining* (C) *sequence and take a sequence of probability measures* $\{\chi_j\}_{j\in\mathbb{N}}$ *on* $\mathcal{P}(M)$, *i.e.* $\chi_j \in \mathcal{P}(\mathcal{P}(M))$. *Suppose there exists an increasing sequence* $\{n_j\}_{j\in\mathbb{N}}$ *of natural numbers and a constant* $C < \infty$ *such that* $\mathbb{E}_{\chi_j}[\tilde{W}_{n_j}] \le C$ *for every* $j \in \mathbb{N}$. *Then* $\{\chi_j\}_{n\in\mathbb{N}}$ *is relatively compact in* $\mathcal{P}(\mathcal{P}(M))$.

Proof. The proof is left to the reader. See for instance [23, Lemma 2.1] for an idea or [16, Proposition 3.4] and [16, Proposition 7.4] for a full proof. □

Now we proceed with the proof of the theorem.

Proof of Theorem 1.2: Case of infinite β**.** Take $f : \mathcal{P}(M) \to \mathbb{R}$ bounded continuous. By Section 3.1 about the idea of the proof we need to obtain (3.1). We start proving the *lower limit bound*

$$\liminf_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)}[f] + \mathbb{E}_\tau[W_n] + \frac{1}{n\beta_n} D\big(\tau \parallel \pi^{\otimes n}\big) \right\} \ge \inf_{\mu\in\mathcal{P}(M)} \big\{ f(\mu) + W(\mu) \big\}.$$

As in the proof used in the case of finite β we want to see that for every increasing sequence of natural numbers $\{n_j\}_{j\in\mathbb{N}}$ and choosing for each $j \in \mathbb{N}$ a probability measure $\tau_j \in \mathcal{P}(M^{n_j})$ we have

$$\lim_{j\to\infty} \left\{ \mathbb{E}_{i_{n_j}(\tau_j)}[f] + \mathbb{E}_{\tau_j}[W_{n_j}] + \frac{1}{n_j\beta_{n_j}} D\big(\tau_j \parallel \pi^{\otimes n_j}\big) \right\} \ge \inf_{\mu\in\mathcal{P}(M)} \big\{ f(\mu) + W(\mu) \big\}, \tag{3.5}$$

where we can suppose that the limit exists and it is finite. As the entropy is non-negative we see that $\mathbb{E}_{\tau_j}[W_{n_j}] = \mathbb{E}_{i_{n_j}(\tau_j)}[\tilde{W}_{n_j}]$ is a bounded sequence and, since $\{W_n\}_{n\in\mathbb{N}}$ is a *confining sequence* (C), Proposition 3.4 tells us that $i_{n_j}(\tau_j)$ is relatively compact in $\mathcal{P}(\mathcal{P}(M))$. We continue as in the proof used in the case of finite β where now W is bounded from below by the *regularity assumption* (A2′) and because $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S).

The proof of the *upper limit bound* follows the same reasoning as in the case of finite β. Take $\mu \in \mathcal{P}(M)$. Following the arguments used in the case of finite β we can prove that

$$\limsup_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \mathbb{E}_{i_n(\tau)}[f] + \mathbb{E}_\tau[W_n] + \frac{1}{n\beta_n} D\big(\tau \parallel \pi^{\otimes n}\big) \right\} \le \inf_{\mu\in\mathcal{N}} \big\{ f(\mu) + W(\mu) \big\}, \tag{3.6}$$

where $\mathcal{N}$ was defined in (1.3). By the *regularity assumption* (A2′) we get

$$\inf_{\mu\in\mathcal{N}} \big\{ f(\mu) + W(\mu) \big\} = \inf_{\mu\in\mathcal{P}(M)} \big\{ f(\mu) + W(\mu) \big\}$$

completing the proof. □

3.4. *Proof of Corollary* 1.3

Proof of Corollary 1.3. We know that the large deviation principle is equivalent to the Laplace principle for the sequence $i_n(\mathbb{P}_n)$ if the rate function has compact level sets (see [14, Theorem 1.2.1] and [14, Theorem 1.2.3]).

If $\beta < \infty$ this is the case because the entropy has compact level sets (see [14, Lemma 1.4.3 (c)]) and W is a lower semicontinuos function bounded from below. The lower semicontinuity of W is a consequence of the *lower and upper limit assumption*, (A1) and (A2).

If β is infinite then there is no entropy term and we can use that $\{W_n\}_{n\in\mathbb{N}}$ is a *confining sequence* (C), and that $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies the *lower limit assumption* (A1) and the *regularity assumption* (A2′) to prove that W has compact level sets.

Then we have to prove that, for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\frac{1}{n\beta_n} \log \mathbb{E}_{i_n(\mathbb{P}_n)}\big[e^{-n\beta_n f}\big] \xrightarrow[n\to\infty]{} -\inf_{\mu\in\mathcal{P}(M)} \big\{f(\mu) + F(\mu) - \inf F\big\}$$

or, using the measures γ_n,

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f\circ i_n} \frac{d\gamma_n}{Z_n} \xrightarrow[n\to\infty]{} -\inf_{\mu\in\mathcal{P}(M)} \big\{f(\mu) + F(\mu) - \inf F\big\}.$$

This we can achieve by using Theorem 1.2 twice, for f and for the zero function. □

Remark 3.5 (Other proof in the case of finite β). When treating the case $\beta < \infty$, the proof we are aware of is [9]. It uses a "quasi-continuity" of the energy and it seems somewhat specific to the logarithmic energy.

Remark 3.6 (Other proofs in the case of infinite β). The proofs that treat the case $\beta = \infty$ usually follow closely the approach we used for the large deviation upper bound. For the large deviation lower bound they proceed as follows. If A is an open set of $\mathcal{P}(M)$ and $\mu \in A$, they try to obtain

$$\liminf_{n\to\infty} \frac{1}{n\beta_n} \log \int_{i_n^{-1}(A)} e^{-n\beta_n W_n}\, d\pi^{\otimes n} \geq -W(\mu).$$

For this, they search pairwise disjoint sets $B_1, \dots, B_n$ such that $i_n(B_1 \times \cdots \times B_n) \subset A$ and such that $\max_{B_1\times\cdots\times B_n} W_n \simeq W(\mu)$. Then we may write

$$\int_{i_n^{-1}(A)} e^{-n\beta_n W_n}\, d\pi^{\otimes n} \geq \sum_{\sigma\in S_n} \int_{B_{\sigma(1)}\times\cdots\times B_{\sigma(n)}} e^{-n\beta_n W_n}\, d\pi^{\otimes n} \geq n!\pi(B_1)\cdots\pi(B_n)e^{-n\beta_n \max_{B_1\times\cdots\times B_n} W_n}.$$

If we are able to choose those sets such that $\pi(B_i) \geq \frac{C}{n}$ for some C independent of n we can obtain, using Stirling's formula,

$$\liminf_{n\to\infty} \frac{1}{n\beta_n} \log\big(n!\pi(B_1)\dots\pi(B_n)\big) \geq \liminf_{n\to\infty} \frac{1}{n\beta_n}(\log n! - n\log n) \geq 0$$

and conclude by using that $\lim_{n\to\infty} \max_{B_1\times\cdots\times B_n} W_n = W(\mu)$.

4. Applications

In this section we shall give the main applications we are thinking of: Conditional Gibbs measure, a Coulomb gas on a Riemannian manifold, the known results of Coulomb gases in Euclidean space and the zeros of Gaussian random polynomials.

4.1. *Conditional Gibbs measure*

In this subsection we treat the case of the Gibbs measure associated to a two-body interaction but with some of the points conditioned to be deterministic. We proceed by considering the deterministic points as a background charge and treat the interaction with this background as some potential energy that depends on n. More precisely, we use the following more general setup.

Let $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on a compact metric space M that converges to some probability measure $\nu \in \mathcal{P}(M)$. Suppose we have a lower semicontinuous function $G^E : M \times M \to (-\infty, \infty]$ that shall be thought of as the interaction energy between the particles and the environment and a symmetric lower semicontinuous function $G^I : M \times M \to (-\infty, \infty]$ that will be interpreted as the interaction energy between the particles. More precisely we define two kinds of energy.

External potential energy. The probability measure ν_n will interact with the n particles via the external potential $V_n : M \to \mathbb{R}$ defined by $V_n(x) = \int_M G^E(x, y)\, d\nu_n(y)$. This gives rise to the external energy $W_n^E : M^n \to (-\infty, \infty]$

$$W_n^E(x_1, \dots, x_n) = \frac{1}{n}\sum_{i=1}^{n} V_n(x_i)$$

with a macroscopic external energy $W^E : \mathcal{P}(M) \to (-\infty, \infty]$

$$W^E(\mu) = \int_{M\times M} G^E(x, y)\, d\mu(x)\, d\nu(y).$$

Internal potential energy. For each n we shall think of n particles interacting with the two-particle potential G^I. This would give rise to an internal energy $W_n^I : M^n \to (-\infty, \infty]$

$$W_n^I(x_1, \dots, x_n) = \frac{1}{n^2}\sum_{i<j}^{n} G^I(x_i, x_j)$$

and a macroscopic internal energy $W^I : \mathcal{P}(M) \to (-\infty, \infty]$

$$W^I(\mu) = \frac{1}{2}\int_{M\times M} G^I(x, y)\, d\mu(x)\, d\mu(y).$$

Total potential energy. For each n we define

$$W_n = W_n^E + W_n^I \quad \text{and} \quad W = W^E + W^I.$$

Then, it is not hard to see that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S) and W is a lower semicontinuous function. The example of a conditional Gibbs measure can be obtained essentially by choosing as ν_n the empirical measure of some points and $G^I = G^E$. So, a particular case of the next theorem is a Coulomb gas conditioned to all but an increasing number of points.

Theorem 4.1 (Varying environment). *Suppose that* $x \mapsto \int_M G^E(x, y)\, d\nu(y)$ *is continuous.*

Let

$$\tilde{\mathcal{N}} = \left\{\mu \in \mathcal{P}(M) : D(\mu \,\|\, \pi) < \infty \text{ and } y \mapsto \int_M G^E(x, y)\, d\mu(x) \text{ is continuous}\right\}$$

and suppose that for every $\mu \in \mathcal{P}(M)$ *such that* $W^I(\mu) < \infty$ *there exists a sequence* $\{\mu_n\}_{n\in\mathbb{N}}$ *of probability measures in* $\tilde{\mathcal{N}}$ *such that* $\mu_n \to \mu$ *and* $W^I(\mu_n) \to W^I(\mu)$.

Then W *is the zero temperature macroscopic limit of* $\{W_n\}_{n\in\mathbb{N}}$. *In particular, if we choose* $\beta_n \to \infty$, *Theorem* 1.2 *and Corollary* 1.3 *may be applied for* $(\{W_n\}_{n\in\mathbb{N}}, W)$.

Proof. Let us prove the *lower limit assumption* (A1).

Lower limit assumption (A1). By Proposition 2.1, we already know that $(\{W_n^I\}_{n\in\mathbb{N}}, W^I)$ satisfies the *lower limit assumption* (A1). We only need to check this for $(\{W_n^E\}_{n\in\mathbb{N}}, W^E)$.

If $\mu_n = i_n(x_1, \dots, x_n)$ then

$$\tilde{W}_n^E(\mu_n) = \int_M V_n \, d\mu_n = \int_{M \times M} G^E(x, y) \, d\mu_n(x) \, d\nu_n(y),$$

where $\tilde{W}_n^E$ is defined in (1.2). So, the *lower limit assumption* (A1) is a consequence of the lower semicontinuity of G^E.

Regularity assumption (A2′). To prove the *regularity assumption* (A2′) we take $\mu \in \mathcal{P}(M)$ such that $W(\mu) < \infty$. Then $W^I(\mu) < \infty$. By hypothesis, we know that there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures in $\tilde{\mathcal{N}}$ such that $\mu_n \to \mu$ and $W^I(\mu_n) \to W^I(\mu)$. As $x \mapsto \int_M G^E(x, y) \, d\nu(y)$ is continuous we also have that $W^E(\mu_n) \to W^E(\mu)$. So, $W(\mu_n) \to W(\mu)$.

We have to prove that the sequence we chose is in the set $\mathcal{N}$ defined in (1.3) by

$$\mathcal{N} = \Big\{\mu \in \mathcal{P}(M) : D(\mu \parallel \pi) < \infty \text{ and } \limsup_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \le W(\mu)\Big\},$$

i.e. we need to see that $\tilde{\mathcal{N}} \subset \mathcal{N}$.

Let $\mu \in \tilde{\mathcal{N}}$. Then

$$\begin{aligned}\mathbb{E}_{\mu^{\otimes n}}\big[W_n^E\big] &= \mathbb{E}_\mu[V_n] \\ &= \int_M \left(\int_M G^E(x, y) \, d\nu_n(y)\right) d\mu(x) \\ &= \int_M \left(\int_M G^E(x, y) \, d\mu(x)\right) d\nu_n(y).\end{aligned}$$

So, as $y \mapsto \int_M G^E(x, y) \, d\mu(x)$ is continuous, we get

$$\mathbb{E}_{\mu^{\otimes n}}\big[W_n^E\big] \xrightarrow[n\to\infty]{} \int_{M\times M} G^E(x, y) \, d\nu(y) \, d\mu(x) = W^E(\mu).$$

By Proposition 2.1 we already know that $\lim_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[W_n^I] = W^I(\mu)$ and then $\mu \in \mathcal{N}$. □

For the sake of completeness we treat the case of a Coulomb gas conditioned to all points but a finite fixed number of them. Again, by considering the deterministic points as a background charge we can use the following more general framework. Suppose we have two compact metric spaces M and N, a probability measure Π on N and two lower semicontinuous functions $G^E : N \times M \to (-\infty, \infty]$ and $G^I : N \to (-\infty, \infty]$. Let $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on M that converges to some probability measure $\nu \in \mathcal{P}(M)$. We will consider one particle in N interacting with the environment via G^E, i.e. via a potential energy $V_n : N \to (-\infty, \infty]$ defined by $V_n(x) = \int_M G^E(x, y) \, d\nu_n(y)$. This particle will also have a self-interaction given by $\lambda_n G^I$ where $\{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence that converges to zero. The case of a Coulomb gas conditioned to all but k particles may be obtained by essentially taking $N = M^k$, $\Pi = \pi^{\otimes k}$, $G^I(x_1, \dots, x_k) = \sum_{i<j} G(x_i, x_j)$, $G^E((x_1, \dots, x_k), y) = \sum_{i=1}^k G(x_i, y)$, $\lambda_n = \frac{1}{n}$ and ν_n as the empirical measure of the deterministic particles.

Theorem 4.2 (A particle in a varying environment). *Suppose that $V : N \to \mathbb{R}$ defined by $V(x) = \int_M G^E(x, y) \, d\nu(y)$ is (bounded and) continuous. Let*

$$\tilde{\mathcal{N}} = \left\{\mu \in \mathcal{P}(N) : D(\mu \parallel \Pi) < \infty, \int_N G^I \, d\mu < \infty \textit{ and } y \mapsto \int_N G^E(x, y) \, d\mu(x) \textit{ is continuous}\right\}$$

and suppose that for every $z \in N$ there exists a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ in $\tilde{\mathcal{N}}$ such that $\mu_n \to \delta_z$. Take a sequence of non-negative numbers $\{\beta_n\}_{n\in\mathbb{N}}$ such that $\beta_n \to \infty$ and define the measures γ_n^c by

$$d\gamma_n^c = e^{-\beta_n(V_n + \lambda_n G^I)} \, d\Pi.$$

Then, we have the following Laplace principle. For every (bounded) continuous function $f : N \to \mathbb{R}$

$$\frac{1}{\beta_n} \log \int_N e^{-\beta_n f} \, d\gamma_n^c \xrightarrow[n \to \infty]{} - \inf_{x \in N} \{ f(x) + V(x) \}.$$

Proof. We use Lemma 3.1 to write

$$\frac{1}{\beta_n} \log \int_N e^{-\beta_n f} \, d\gamma_n^c = - \inf_{\mu \in \mathcal{P}(N)} \left\{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V_n] + \lambda_n \mathbb{E}_\mu\left[G^I\right] + \frac{1}{\beta_n} D(\mu \parallel \Pi) \right\}.$$

Following the same ideas used in the proofs of Theorem 1.2 and Theorem 4.1 we get

$$\liminf_{n \to \infty} \inf_{\mu \in \mathcal{P}(N)} \left\{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V_n] + \lambda_n \mathbb{E}_\mu\left[G^I\right] + \frac{1}{\beta_n} D(\mu \parallel \Pi) \right\} \geq \inf_{\mu \in \mathcal{P}(N)} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}$$

and

$$\limsup_{n \to \infty} \inf_{\mu \in \mathcal{P}(N)} \left\{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V_n] + \lambda_n \mathbb{E}_\mu\left[G^I\right] + \frac{1}{\beta_n} D(\mu \parallel \Pi) \right\} \leq \inf_{\mu \in \tilde{\mathcal{N}}} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}.$$

We shall think of N as included in $\mathcal{P}(N)$ by the application $z \mapsto \delta_z$. Then, by the continuity of V and f and as we are assuming that elements of N are approximated by elements of $\tilde{\mathcal{N}}$ we know that

$$\inf_{\mu \in \tilde{\mathcal{N}}} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \} = \inf_{\mu \in \tilde{\mathcal{N}} \cup N} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}.$$

As the infimum is achieved in N we get

$$\inf_{\mu \in \tilde{\mathcal{N}}} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \} = \inf_{x \in N} \{ f(x) + V(x) \} = \inf_{\mu \in \mathcal{P}(N)} \{ \mathbb{E}_\mu[f] + \mathbb{E}_\mu[V] \}$$

concluding the proof. □

4.2. *A Coulomb gas on a Riemannian manifold*

Let (M, g) be a compact oriented n-dimensional Riemannian manifold without boundary where g denotes the Riemannian metric. We shall define a continuous function $G : M \times M \to (-\infty, \infty]$ naturally associated to the Riemannian structure of M. This function along with the normalized volume form π of (M, g) will allow us to define the Gibbs measures γ_n of (1.1) and will put us in the context of Theorem 1.2.

For this we establish some notation. A signed measure Λ will be called a differentiable signed measure if it is given by an n-form or equivalently if it has a differentiable density with respect to π. From now on we shall identify $\Omega^n(M)$ with the space of differentiable signed measures. Denote by $\Delta : C^\infty(M) \to \Omega^n(M)$ the Laplacian operator, i.e. $\Delta = d * d$ where $*$ is the Hodge star operator or, equivalently, $\Delta f = \nabla^2 f \, d\pi$ where ∇^2 is the Laplace-Beltrami operator. The function G we will be interested in is given by the following result.

Proposition 4.3 (Green function). *Take any differentiable signed measure Λ. Then, there exists a symmetric continuous function $G : M \times M \to (-\infty, \infty]$ such that for every $x \in M$ the function $G_x : M \to (-\infty, \infty]$ defined by $G_x(y) = G(x, y)$ is integrable with respect to π and*

$$\Delta G_x = -\delta_x + \Lambda.$$

More explicitly, the previous equality can be written as follows. For every $f \in C^\infty(M)$ we have

$$\int_M G_x \Delta f = -f(x) + \int_M f \, d\Lambda.$$

Such a function will be called a Green function associated to Λ. Furthermore G is integrable with respect to $\pi \otimes \pi$. If μ is a differentiable signed measure then $\psi : M \to \mathbb{R}$ defined by $\psi(x) = \int_M G(x, y)\, d\mu(y)$ belongs to $C^\infty(M)$ and

$$\Delta\psi = -\mu + \mu(M)\Lambda.$$

In particular, we can get that G is bounded from below, $\int_M G_x\, d\Lambda$ does not depend on $x \in M$ and a Green function associated to Λ is unique up to an additive constant.

Proof. This result is well known if $\Lambda = \pi$. See for instance [2, Chapter 4]. Then if H is a Green function associated to π we define $\phi \in C^\infty(M)$ by $\phi(x) = \int_M H(x, y)\, d\Lambda(y)$ and the function $G : M \times M \to (-\infty, \infty]$ given by $G(x, y) = H(x, y) - \phi(x) - \phi(y)$ is a Green function associated to Λ. $\square$

We fix a differentiable signed measure Λ. For simplicity we choose the Green function G associated to Λ that satisfies $\int_M G_x\, d\Lambda = 0$ for every $x \in M$. Define $W_n : M^n \to \mathbb{R} \cup \{\infty\}$ by $W_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i<j}^n G(x_i, x_j)$ and $W : \mathcal{P}(M) \to (-\infty, \infty]$ by $W(\mu) = \frac{1}{2} \int_{M\times M} G(x, y)\, d\mu(x)\, d\mu(y)$. Because G is bounded from below and lower semicontinuous we may apply Proposition 2.1 about the k-body interaction. In particular, we obtain that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S), W is lower semicontinuous and $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

We can prove a strong form of the regularity assumption for W.

Proposition 4.4 (Regularity property of the Green energy). *Let $\mu \in \mathcal{P}(M)$. There exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of differentiable probability measures such that $\mu_n \to \mu$ and $W(\mu_n) \to W(\mu)$.*

Proof. We can assume $W(\mu) < \infty$, otherwise any sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of differentiable probability measures such that $\mu_n \to \mu$ will satisfy $W(\mu_n) \to W(\mu)$ due to the lower semicontinuity of W.

Using the proof of [3, Lemma 3.13] for the case of probability measures we know that the result is true for the Green function H associated to π. For general Λ, take $\phi \in C^\infty(M)$ defined by $\phi(x) = \int_M H(x, y)\, d\Lambda(y)$ as in the proof of Proposition 4.3. Then $G : M \times M \to (-\infty, \infty]$ given by $G(x, y) = H(x, y) - \phi(x) - \phi(y)$ is a Green function for Λ and for every $\mu \in \mathcal{P}(M)$ we have

$$\int_{M\times M} G(x, y)\, d\mu(x)\, d\mu(y) = \int_{M\times M} H(x, y)\, d\mu(x)\, d\mu(y) - 2 \int_M \phi\, d\mu.$$

From this relation and the result for H we get the result for G. $\square$

Then, $(\{W_n\}_{n\in\mathbb{N}}, W)$ is a nice model where Theorem 1.2 and the results of Section 4.1 can be used.

Corollary 4.5 (Macroscopic limit). *W is the zero temperature macroscopic limit and the positive temperature macroscopic limit of $\{W_n\}_{n\in\mathbb{N}}$, i.e. $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies all the conditions of Theorem 1.2. Additionally the results of Section 4.1 about the Conditional Gibbs measure may be applied.*

Now we shall enunciate a theorem that is our main motivation for choosing this model. Remember the definitions of i_n, (1.4), and $\mathbb{P}_n$, (1.7). Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables taking values in $\mathcal{P}(M)$ such that, for every $n \in \mathbb{N}$, X_n has law $i_n(\mathbb{P}_n)$. By studying the minimizers of the *free energy* F defined in (1.5) we can understand the possible limit points of $\{X_n\}_{n\in\mathbb{N}}$. In particular, if F attains its minimum at a unique probability measure μ_{eq}, we get

$$X_n \xrightarrow[n\to\infty]{\text{a.s.}} \mu_{\text{eq}}.$$

This is a consequence of Borel–Cantelli lemma and the large deviation principle in Corollary 1.3.

We specialize to the case of dimension two and finite β because the minimizer of F has a nice geometric meaning in this case.

Theorem 4.6 (Minimizer of the free energy). *Let ρ be a strictly positive differentiable function such that*

$$\Delta \log \rho = \beta \mu_{\mathrm{eq}} - \beta \Lambda, \tag{4.1}$$

where μ_{eq} denotes the probability measure defined by $d\mu_{\mathrm{eq}} = \rho\, d\pi$ (see [12] for the existence). Then $F(\mu_{\mathrm{eq}}) < F(\mu)$ for every $\mu \in \mathcal{P}(M)$ different from μ_{eq}. In particular, there exists only one strictly positive differentiable function that satisfies (4.1).

Remark 4.7 (Scalar curvature relation). The motivation for studying a 2-dimensional manifold is that μ_{eq} has a nice geometrical interpretation if we choose adequate Λ and β.

We shall suppose that $\chi(M)$, the Euler characteristic of M, is different from zero. If $\bar{g}$ is any metric, we denote by $R_{\bar{g}}$ the scalar curvature of $\bar{g}$. Choose

$$d\Lambda = \frac{R_g\, d\pi}{4\pi\chi(M)}.$$

It can be seen that if $\bar{g} = \rho g$, where $\int_M \rho\, d\pi = 1$, then

$$\Delta \log \rho = R_g\, d\pi - R_{\bar{g}}\rho\, d\pi.$$

With this identity we can prove that ρ is a solution to

$$R_{\bar{g}} = \bigl(4\pi\chi(M) + \beta\bigr) R_g \rho^{-1} - \beta,$$

where $\bar{g} = \rho g$ if and only if ρ is a solution to

$$\Delta \log \rho = \beta \mu_{\mathrm{eq}} - \beta \Lambda,$$

where $d\mu_{\mathrm{eq}} = \rho\, d\pi$. In particular, if $\chi(M) < 0$ and $\beta = -4\pi\chi(M)$ then $\bar{g}$ satisfies

$$R_{\bar{g}} = 4\pi\chi(M),$$

i.e. $\bar{g}$ is a metric with constant curvature. In other words, if $\beta = -4\pi\chi(M)$, the empirical measure converges almost surely to the volume form of the constant curvature metric conformally equivalent to the chosen metric.

The proof of Theorem 4.6 will be based on the fact that F is strictly convex and that we can calculate its derivative. We begin by proving its convexity.

Proposition 4.8 (Convexity of W). *W is convex.*

Proof. To prove the convexity it is enough to show that for every $\mu, \nu \in \mathcal{P}(M)$

$$\frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) \geq W\left(\frac{1}{2}\mu + \frac{1}{2}\nu\right) \tag{4.2}$$

due to the lower semicontinuity of W. If μ and ν are differentiable probability measures this is equivalent to

$$\int_M \|\nabla(f-g)\|^2\, d\pi \geq 0,$$

where $f(x) = \int_M G(x,y)\, d\mu(y)$ and $g(x) = \int_M G(x,y)\, d\nu(y)$. For general μ and ν we can conclude using Proposition 4.4, and taking lower limits in the inequality (4.2) for differentiable probability measures. $\square$

As $D(\cdot \,\|\, \pi)$ is strictly convex (see [14, Lemma 1.4.3]) we obtain that the *free energy* F of parameter $\beta < \infty$ is strictly convex.

Now we calculate the derivative of W and the entropy at μ_{eq}.

Lemma 4.9 (Derivative of W and the entropy). *Let μ be any probability measure different from μ_{eq} such that $F(\mu) < \infty$. Define*

$$\mu_t = t\mu + (1-t)\mu_{\text{eq}}, \quad t \in [0,1].$$

Then, $W(\mu_t)$ and $D(\mu_t \parallel \pi)$ are differentiable at $t = 0$, and

$$\frac{d}{dt} W(\mu_t)|_{t=0} = \int_{M\times M} G(x,y)\, d\mu_{\text{eq}}(x) \big(d\mu(y) - d\mu_{\text{eq}}(y)\big), \tag{4.3}$$

$$\frac{d}{dt} D(\mu_t \parallel \pi)|_{t=0} = \int_M \log\rho(y) \big(d\mu(y) - d\mu_{\text{eq}}(y)\big). \tag{4.4}$$

Proof. To get (4.3) we just notice that $W(\mu_t)$ is a polynomial of degree 2 and to obtain (4.4) we use the monotone convergence theorem as said for instance in [7, Proposition 2.11]. □

And now we are ready to finish the proof of Theorem 4.6.

Proof of Theorem 4.6. As in Lemma 4.9, let μ be any probability measure different from μ_{eq} such that $F(\mu) < \infty$ and define

$$\mu_t = t\mu + (1-t)\mu_{\text{eq}}, \quad t \in [0,1].$$

Multiply (4.1) by $G(x,y)$ and integrate in one variable to get

$$-\log\rho(y) + \int_M \log\rho(x)\, d\Lambda(x) = \beta \int_M G(x,y)\rho(x)\, d\pi(x).$$

Then, we have that

$$\begin{aligned}
&\frac{d}{dt} F(\mu_t)|_{t=0} \\
&\quad = \int_{M\times M} G(x,y)\, d\mu_{\text{eq}}(x)\big(d\mu(y) - d\mu_{\text{eq}}(y)\big) + \frac{1}{\beta}\int_{M\times M} \log\rho(y)\big(d\mu(y) - d\mu_{\text{eq}}(y)\big) \\
&\quad = \int_M \left(\int_M G(x,y)\rho(x)\, d\pi(x) + \frac{1}{\beta}\log\rho(y)\right)\big(d\mu(y) - d\mu_{\text{eq}}(y)\big) \\
&\quad = \frac{1}{\beta}\int_M \left(\int_M \log\rho(x)\, d\Lambda(x)\right)\big(d\mu(y) - d\mu_{\text{eq}}(y)\big) \\
&\quad = \frac{1}{\beta}\left(\int_M \log\rho(x)\, d\Lambda(x)\right)\left(\int_M \big(d\mu(y) - d\mu_{\text{eq}}(y)\big)\right) = 0.
\end{aligned}$$

This implies, due to the strict convexity of $F(\mu_t)$ in t, that

$$F(\mu_{\text{eq}}) < F(\mu).$$ □

4.3. *Usual Coulomb gases*

In this subsection we provide different proofs to the large deviation principles associated to Coulomb gases studied in [19] and [15]. These models are usually motivated as describing the laws of eigenvalues of some random matrices and has as particular cases the models studied in [4,20,21] and [13]. We may see [1] for an introduction to random matrices. We would like to remark that the model studied in [5] may be treated by the same methods but does not fall directly in the regime of application of Theorem 1.2.

Suppose that l is a not necessarily finite measure on the Polish space M. Let $V : M \to (-\infty, \infty]$ and $G : M \times M \to (-\infty, \infty]$ be lower semicontinuous functions with G symmetric and such that the function defined by $(x, y) \mapsto G(x, y) + V(x) + V(y)$ is bounded from below. Define $H_n : M^n \to (-\infty, \infty]$ by

$$H_n(x_1, \dots, x_n) = \sum_{i<j}^{n} G(x_i, x_j) + n \sum_{i=1}^{n} V(x_i)$$

and $W : \mathcal{P}(M) \to (-\infty, \infty]$ by

$$W(\mu) = \frac{1}{2} \int_{M \times M} \bigl(G(x, y) + V(x) + V(y)\bigr)\, d\mu(x)\, d\mu(y).$$

Take a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ such that $\beta_n \to \infty$ and let γ_n be the Gibbs measure defined by

$$d\gamma_n = e^{-\frac{\beta_n}{n} H_n}\, dl^{\otimes n}.$$

We shall give some hypotheses that imply that γ_n satisfies a Laplace principle.

The first example is related to [19]. More precisely, if we choose $G(x, y) = -\beta \log \|x - y\|$, condition (1.7) of [19] implies the first three conditions of the following theorem (see the proof of Proposition 4.12 for an idea) and the last condition is a consequence the nature of the logarithmic interaction and the required continuity of V in [19]. We remark that there is a slight typo in [19]: we should require $\beta' > 2$ in dimension two.

Theorem 4.10 (Weakly confining case). *Take $\beta_n = n$. Suppose that*

- $\int_M e^{-V}\, dl < \infty$,
- *the function $(x, y) \mapsto G(x, y) + V(x) + V(y)$ is bounded from below*,
- *$G(x, y) + V(x) + V(y) \to \infty$ when $x, y \to \infty$ at the same time*, *and*
- *for every $\mu \in \mathcal{P}(M)$ such that $W(\mu) < \infty$, there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures absolutely continuous with respect to l such that $\mu_n \to \mu$ and $W(\mu_n) \to W(\mu)$.*

Then, for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n^2} \log \int_{M^n} e^{-n^2 f\circ i_n}\, d\gamma_n \xrightarrow[n\to\infty]{} - \inf_{\mu\in\mathcal{P}(M)} \bigl\{f(\mu) + W(\mu)\bigr\}.$$

Proof. Assume $\int_M e^{-V}\, dl = 1$ for simplicity. We notice that

$$d\gamma_n = e^{-(\sum_{i<j}^{n} G(x_i, x_j) + (n-1)\sum_{i=1}^{n} V(x_i))}\, d\bigl(e^{-V} l\bigr)^{\otimes n}.$$

If we define

$$\tilde{G}(x, y) = G(x, y) + V(x) + V(y)$$

and

$$W_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i<j}^{n} \tilde{G}(x_i, x_j)$$

we have

$$d\gamma_n = e^{-n^2 W_n}\, d\bigl(e^{-V} l\bigr)^{\otimes n}.$$

We now prove that $\{W_n\}_{n\in\mathbb{N}}$ satisfies the conditions necessary to apply Theorem 1.2.

Lower and upper limit assumption, (A1) and (A2). By hypotheses, $\tilde{G}$ is lower semicontinuous and bounded from below. We can apply Proposition 2.1 to get that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S) and that $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

Regularity assumption (A2′). Since $(\{W_n\}_{n\in\mathbb{N}}, W)$ satisfies the *upper limit assumption* (A2), the *regularity assumption* (A2′) does not depend on $\{W_n\}_{n\in\mathbb{N}}$ and we can use Proposition 2.4. Take $\mu \in \mathcal{P}(M)$ such that $W(\mu) < \infty$. Then, by hypothesis, there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures absolutely continuous with respect to l such that $\mu_n \to \mu$ and $W(\mu_n) \to W(\mu)$. As $W(\mu) < \infty$ we can assume $W(\mu_n) < \infty$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We want to prove that μ_n is absolutely continuous with respect to the measure defined by $e^{-V}\,dl$. For this it is enough to notice that $\mu_n(\{x \in M : V(x) = \infty\}) = 0$. We can see that the set $\{(x,y) \in M \times M : V(x) = \infty \text{ and } V(y) = \infty\}$ is included in the set $\{(x,y) \in M \times M : G(x,y) + V(x) + V(y) = \infty\}$. The latter has zero measure because $W(\mu) < \infty$ and we conclude by the definition of product measure.

Confining sequence (C). Using that $\tilde{G}(x,y) \to \infty$ when $x, y \to \infty$ at the same time and Proposition 2.2 we get that $\{W_n\}_{n\in\mathbb{N}}$ is a *confining sequence* (C).

We can finally apply Theorem 1.2. □

The second example is related to the article this work is inspired on, i.e. [15]. More precisely, Assumptions C1–C3 of [15, Theorem 1.6] imply the conditions of the following theorem. We remark that there is a slight typo in [15]: Assumption A should be changed by any weaker assumption that guarantees the finiteness of the Gibbs measures.

Theorem 4.11 (Strongly confining case). *Suppose that*

- *There exists $\xi > 0$ such that $\int_M e^{-\xi V}\,dl < \infty$,*
- *V is bounded from below,*
- *there exists $\varepsilon \in [0,1)$ such that $(x,y) \mapsto G(x,y) + \varepsilon V(x) + \varepsilon V(y)$ is bounded from below,*
- *the function $G(x,y) + V(x) + V(y)$ tends to infinity when $x, y \to \infty$ at the same time, and*
- *for every $\mu \in \mathcal{P}(M)$ such that $W(\mu) < \infty$, there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures absolutely continuous with respect to l such that $\mu_n \to \mu$ and $W(\mu_n) \to W(\mu)$.*

Then, for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f\circ i_n}\, d\gamma_n \xrightarrow[n\to\infty]{} -\inf_{\mu\in\mathcal{P}(M)} \bigl\{ f(\mu) + W(\mu) \bigr\}.$$

Proof. We can assume $\int_M e^{-\xi V}\,dl = 1$ for simplicity. Then we can write

$$d\gamma_n = e^{-\frac{\beta_n}{n}(\sum_{i<j}^n G(x_i,x_j) + (n - \frac{n}{\beta_n}\xi)\sum_{i=1}^n V(x_i))}\, d\bigl(e^{-\xi V} l\bigr)^{\otimes n},$$

which may only make sense for n large enough due to some positive and negative infinities. If we define

$$G^n(x,y) = G(x,y) + \frac{1}{n-1}\left(n - \frac{n}{\beta_n}\xi\right) V(x) + \frac{1}{n-1}\left(n - \frac{n}{\beta_n}\xi\right) V(y)$$

and

$$W_n(x_1,\dots,x_n) = \frac{1}{n^2} \sum_{i<j}^n G^n(x_i,x_j)$$

we have

$$d\gamma_n = e^{-n\beta_n W_n}\, d\bigl(e^{-\xi V} l\bigr)^{\otimes n}.$$

Now we can try to apply Theorem 1.2 to get the Laplace principle. Define

$$G_1(x,y)=G(x,y)+\varepsilon V(x)+\varepsilon V(y), \qquad W_n^1(x_1,\dots,x_n)=\frac{1}{n^2}\sum_{i<j}^{n}G_1(x_i,x_j),$$

$$G_2(x,y)=(1-\varepsilon)V(x)+(1-\varepsilon)V(y), \qquad W_n^2(x_1,\dots,x_n)=\frac{1}{n^2}\sum_{i<j}^{n}G_2(x_i,x_j)$$

and

$$a_n=\frac{1}{1-\varepsilon}\left(\frac{1}{n-1}\left(n-\frac{n}{\beta_n}\xi\right)-\varepsilon\right)\to 1.$$

This definitions allow us to write

$$W_n=W_n^1+a_nW_n^2.$$

We start by proving the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

Lower and upper limit assumption, (A1) and (A2). By the hypotheses, we can see that G_1 and G_2 are lower semicontinuous functions bounded from below. Then, we can apply Proposition 2.1 about the k-body interaction to get that $\{W_n^1\}_{n\in\mathbb{N}}$ and $\{W_n^2\}_{n\in\mathbb{N}}$ are *stable sequences* (S) and if we define the lower semicontinuous functions $W^1(\mu)=\frac{1}{2}\int_{M\times M}G_1(x,y)\,d\mu(x)\,d\mu(y)$ and $W^2(\mu)=\frac{1}{2}\int_{M\times M}G_2(x,y)\,d\mu(x)\,d\mu(y)$, then $(\{W_n^1\}_{n\in\mathbb{N}},W^1)$ and $(\{W_n^2\}_{n\in\mathbb{N}},W^2)$ satisfy the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

Then, as $a_n>0$ for n large enough, we get that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S) for n large enough. Noticing that

$$W^1(\mu)+W^2(\mu)=W(\mu)=\frac{1}{2}\int_{M\times M}\big(G(x,y)+V(x)+V(y)\big)\,d\mu(x)\,d\mu(y).$$

we obtain that $(\{W_n\}_{n\in\mathbb{N}},W)$ satisfies the *lower limit assumption* (A1) and the *upper limit assumption* (A2).

Confining sequence (C). By Proposition 2.2 about the confining assumption in the k-body interaction and by the fact that $G(x,y)+V(x)+V(y)\to\infty$ when $x,y\to\infty$ at the same time, we get that $\{W_n^1+W_n^2\}_{n\in\mathbb{N}}$ is a *confining sequence* (C). Along with the fact that $\{W_n^1\}_{n\in\mathbb{N}}$ and $\{W_n^2\}_{n\in\mathbb{N}}$ are *stable sequences* (S) and that $a_n\to 1$ this implies that $\{W_n\}_{n\in\mathbb{N}}$ is also a *confining sequence* (C).

Regularity assumption (A2′). By an argument similar to the one given in the proof of Theorem 4.10 we can prove the *regularity assumption* (A2′) for W.

We have proved the conditions to apply Theorem 1.2. □

4.4. *Gaussian random polynomials*

In this subsection we will see that [24, Theorem 1] is a consequence of Corollary 1.3. Consider a probability measure $\nu\in\mathcal{P}(\mathbb{C})$ and a continuous function $\phi:\mathbb{C}\to\mathbb{R}$ such that

$$\liminf_{z\to\infty}\big\{\phi(z)-2\log\|z\|\big\}>-\infty. \tag{4.5}$$

Denote by $\mathbb{C}_n[z]$ the vector space of complex polynomials of degree less or equal than n and denote by $j_n:\mathbb{C}_n[z]\setminus\mathbb{C}_{n-1}[z]\to\mathcal{P}(\mathbb{C})$ the application that gives the empirical measure of the zeros of a polynomial, i.e. j_n is defined by

$$j_n(p)=\frac{1}{n}\sum_{i=1}^{n}\delta_{z_i}\quad\text{if } p(z)=a\prod_{i=1}^{n}(z-z_i)\text{ for some } a\neq 0.$$

We shall consider the complex Gaussian measure $\mathcal{G}_n$ with covariance $\langle \cdot, \cdot \rangle_n$ on $\mathbb{C}_n[z]$ given by

$$\langle p, q \rangle_n = \int_{\mathbb{C}} \bar{p}(z) q(z) e^{-n\phi(z)} \, d\nu(z),$$

where we have supposed that $\langle \cdot, \cdot \rangle_n$ is non-degenerate. We will see that the zeros of a random polynomial chosen according to $\mathcal{G}_n$ can be treated by Corollary 1.3. In other words, we are interested in the pushforward measure of the restriction of $\mathcal{G}_n$ to $\mathbb{C}_n[z] \setminus \mathbb{C}_{n-1}[z]$ by j_n, that we will denote by $j_n(\mathcal{G}_n)$ and that is still a probability measure because $\mathcal{G}_n(\mathbb{C}_{n-1}[z]) = 0$, and we want to write it in the form (1.1).

Proposition 4.12 (Gibbs measure form of the zeros of a random polynomial). *Define* $G : \mathbb{C} \times \mathbb{C} \to (-\infty, \infty]$ *by*

$$G(z, w) = -2 \log \|z - w\| + \phi(z) + \phi(w).$$

Then, by the condition (4.5), G *is a lower semicontinuous function bounded from below. Also, by* (4.5), $\int_{\mathbb{C}} e^{-2\phi(z)} \, d\mathcal{L}eb(z) < \infty$. *Define* $\pi \in \mathcal{P}(\mathbb{C})$ *by*

$$d\pi(z) = \frac{e^{-2\phi(z)}}{\int_{\mathbb{C}} e^{-2\phi(z)} \, d\mathcal{L}eb(z)} \, d\mathcal{L}eb(z),$$

the symmetric measurable function $w_n : \mathbb{C}^n \to (-\infty, \infty]$ *by*

$$w_n(z_1, \ldots, z_n) = \frac{1}{n^2} \sum_{i<j} G(z_i, z_j) + \frac{n+1}{n^2} \log\left(\int_{\mathbb{C}} e^{-\sum_{i=1}^n G(z, x_i)} \, d\nu(z) \right) \tag{4.6}$$

and the Gibbs measure γ_n *by*

$$d\gamma_n = e^{-n^2 w_n} \, d\pi^{\otimes n}.$$

Then the zeros of a random polynomial chosen according to $\mathcal{G}_n$ *follows the law* $\frac{\gamma_n}{\gamma_n(\mathbb{C}^n)}$. *More precisely,*

$$j_n(\mathcal{G}_n) = i_n\left(\frac{\gamma_n}{\gamma_n(\mathbb{C}^n)} \right),$$

where $i_n(\frac{\gamma_n}{\gamma_n(\mathbb{C}^n)})$ *denotes the pushforward measure of* $\frac{\gamma_n}{\gamma_n(\mathbb{C}^n)}$ *by* i_n.

Proof. The lower semicontinuity of G follows from the continuity of the logarithm and the continuity of ϕ. As $-2 \log \|z - w\| \geq -2 \log 2 - 2 \log \|z\| - 2 \log \|w\|$ if $\|z\|, \|w\| \geq 1$ and using (4.5) we know that G is bounded from below. By (4.5) there exists $C > 0$ such that $e^{-2\phi(z)} \leq C \|z\|^{-4}$ if $\|z\|$ is large enough and we obtain that $\int_{\mathbb{C}} e^{-2\phi(z)} \, d\mathcal{L}eb(z) < \infty$.

The statement about $j_n(\mathcal{G}_n)$ is a consequence of [11, Theorem 5.1] and the fact that

$$\begin{aligned} &\prod_{i<j} \|z_i - z_j\|^2 \left(\int_{\mathbb{C}} \prod_{i=1}^n \|z - z_i\|^2 e^{-n\phi(z)} \, d\nu(z) \right)^{-(n+1)} d\mathcal{L}eb^{\otimes n}(z_1, \ldots, z_n) \\ &\quad = e^{-\sum_{i<j} G(z_i, z_j)} \left(\int_{\mathbb{C}} e^{-\sum_{i=1}^n G(z_i, z)} \, d\nu(z) \right)^{-(n+1)} \prod_{i=1}^n d\pi^{\otimes n}(z_1, \ldots, z_n). \end{aligned}$$

□

The energy in (4.6) is a sum of an energy of the 2-body interaction type and a different kind of energy that we will try to understand. Under appropriate conditions in ϕ, the authors of [24] extend G to $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$ so we shall only consider compact spaces.

Consider $G : M \times M \to (-\infty, \infty]$ a lower semicontinuous function on a compact metric space M. Consider $\nu \in \mathcal{P}(M)$ a probability measure on M and denote its support by $K \subset M$. Define $W_n : M^n \to [-\infty, \infty)$ by

$$W_n(x_1, \dots, x_n) = \frac{1}{n} \log\left(\int_M e^{-\sum_{i=1}^n G(z,x_i)} \, d\nu(z) \right)$$

and $W : \mathcal{P}(M) \to [-\infty, \infty)$ by

$$W(\mu) = -\inf_{x \in K} \left\{ \int_M G(x, y) \, d\mu(y) \right\}.$$

Notice that $\{W_n\}_{n\in\mathbb{N}}$ is uniformly bounded from above and that it is not immediate to say that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S).

Lemma 4.13 (Upper limit properties). *W is upper semicontinuous and for each $\mu \in \mathcal{P}(M)$ we have that*

$$\limsup_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \le W(\mu). \tag{4.7}$$

Proof. *W is upper semicontinuous*. This can be seen as a consequence of the lower semicontinuity of the function $T : M \times \mathcal{P}(M) \to (-\infty, \infty]$ defined by $T(x, \mu) = \int_M G(x, y) \, d\mu(y)$ as follows. Suppose $\mu_n \to \mu$ in $\mathcal{P}(M)$ and take $x_n \in K$ such that $T(x_n, \mu_n) \le \inf_{x\in K} T(x, \mu_n) + \frac{1}{n}$. Then

$$\liminf_{n\to\infty} T(x_n, \mu_n) \le \liminf_{n\to\infty} \left[\inf_{x\in K} T(x, \mu_n) \right].$$

Take a subsequence such that $\lim_{j\to\infty} T(x_{n_j}, \mu_{n_j}) = \liminf_{n\to\infty} T(x_n, \mu_n)$ where, by taking a further subsequence if necessary, we may assume that x_n converge to some $x_\infty \in K$. The lower semicontinuity of T implies that $T(x_\infty, \mu) \le \lim_{j\to\infty} T(x_{n_j}, \mu_{n_j})$ and so

$$\inf_{x\in K} T(x, \mu) \le \liminf_{n\to\infty} \left[\inf_{x\in K} T(x, \mu_n) \right].$$

Proof of (4.7). Notice that

$$\tilde{W}_n(\hat{\mu}) = \frac{1}{n} \log\left(\int_M e^{-n \int_M G(z,x) \, d\hat{\mu}(x)} \, d\nu(z) \right)$$

if $\hat{\mu} \in i_n(M^n)$, where $\tilde{W}_n$ is defined by (1.2). Then, if $\hat{\mu} \in i_n(M^n)$, we have

$$\tilde{W}_n(\hat{\mu}) \le -\inf_{z\in K} \int_M G(z, x) \, d\hat{\mu}(x) = W(\hat{\mu}).$$

Let $\mu \in \mathcal{P}(M)$, then

$$\mathbb{E}_{\mu^{\otimes n}}[W_n] = \mathbb{E}_{i_n(\mu^{\otimes n})}[\tilde{W}_n] \le \mathbb{E}_{i_n(\mu^{\otimes n})}[W]$$

and so

$$\limsup_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[W_n] \le \limsup_{n\to\infty} \mathbb{E}_{i_n(\mu^{\otimes n})}[W] \le W(\mu)$$

by the upper semicontinuity and upper boundedness of W. □

We see that the *upper limit assumption*, (A2), with the sequence $\{W_n\}_{n\in\mathbb{N}}$ not necessarily a *stable sequence* (S), is satisfied in a very general context. This is not the case for the *lower limit assumption*, (A1) and we will state the two main conditions that allow us to obtain it.

Definition 4.14 (Bernstein–Markov condition). For any $\vec{x} = (x_1, \dots, x_n) \in M^n$ consider the function $s_{\vec{x}} : M \to \mathbb{R}$ defined by $s_{\vec{x}}(y) = e^{-\sum_{i=1}^n G(x_i, y)}$ and denote the support of ν by K. We say that (G, ν) satisfies the Bernstein–Markov condition if the following is true. For every $\varepsilon > 0$ there exists $C > 0$ such that

$$\sup_{y \in K} s_{\vec{x}}(y) \le C e^{\varepsilon n} \|s_{\vec{x}}\|_{L^1(M,\nu)}$$

for every $\vec{x} \in M^n$ and for every $n > 0$.

Definition 4.15 (Regular pair). We will say that the pair (G, K) is regular if the following is true. For every probability measure $\mu \in \mathcal{P}(M)$ and every $\varepsilon > 0$ there exists a probability measure $\nu \in \mathcal{P}(M)$ such that $\nu(\{x \in K : \int_M G(x, y)\, d\mu(y) \le \inf_{z \in K} \int_M G(z, y)\, d\mu(y) + \varepsilon\}) = 1$ and $x \mapsto \int_M G(x, y)\, d\nu(y)$ is finite and continuous.

Our Bernstein–Markov condition is an easy consequence of the Bernstein–Markov condition in the case of random polynomials (see [24, Lemma 9]) and our regular pair condition is a consequence of the non-thinness of K (see the proof of the second part of [24, Lemma 26]).

Proposition 4.16 (Lower semicontinuity and lower boundedness). *Suppose the pair (G, K) is regular. Then W is lower semicontinuous and bounded from below.*

Proof. *W is bounded from below*. The regular pair condition implies, in particular, that there exists a probability measure $\nu \in \mathcal{P}(M)$ supported on K such that $x \mapsto \int_M G(x, y)\, d\nu(y)$ is continuous. So,

$$\begin{aligned} W(\mu) &\ge -\int_K \left(\int_M G(x, y)\, d\mu(y) \right) d\nu(x) \\ &= -\int_M \left(\int_K G(x, y)\, d\nu(x) \right) d\mu(y) \\ &\ge -\sup_{y \in M} \int_K G(x, y)\, d\nu(x), \end{aligned}$$

where we have used Fubini's theorem. As $x \mapsto \int_K G(x, y)\, d\nu(y)$ is continuous, it is bounded from above and we have thus proved that W is bounded from below.

W is lower semicontinuous. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures converging to some $\mu \in \mathcal{P}(M)$. We want to prove that $\liminf_{n \to \infty} W(\mu_n) \ge W(\mu)$. For $\varepsilon > 0$ the regular pair condition says that there exists $\nu \in \mathcal{P}(M)$ supported in K such that

$$\int_M G(x, y)\, d\mu(y) \le \inf_{z \in K} \int_M G(z, y)\, d\mu(y) + \varepsilon \tag{4.8}$$

for ν-almost every x and $x \mapsto \int_M G(x, y)\, d\nu(y)$ is bounded continuous. Then integrating 4.8 with respect to ν we get

$$\int_{M \times M} G(x, y)\, d\nu(x)\, d\mu(y) \le \inf_{z \in K} \int_M G(z, y)\, d\mu(y) + \varepsilon$$

but, as $x \mapsto \int G(x, y)\, d\nu(y)$ is bounded continuous we have that

$$\int_M \left(\int_M G(x, y)\, d\nu(x) \right) d\mu_n(y) \to \int_M \left(\int_M G(x, y)\, d\nu(x) \right) d\mu(y).$$

As ν is supported in K we know that

$$\inf_{z \in K} \int_M G(z, y)\, d\mu_n(y) \le \int_M \left(\int_M G(x, y)\, d\mu_n(y) \right) d\nu(x).$$

Taking the upper limit and using Fubini's theorem we get

$$\limsup_{n\to\infty}\left[\inf_{z\in K}\int_M G(z,y)\,d\mu_n(y)\right]\leq \inf_{z\in K}\int_M G(z,y)\,d\mu(y)+\varepsilon.$$

As this is true for every $\varepsilon>0$ we conclude the proof. □

Proposition 4.17 (Stability and lower limit assumption). *Suppose (G,K) is regular and that (G,ν) satisfies the Bernstein–Markov condition. Then $\{W_n\}_{n\in\mathbb{N}}$ is a stable sequence (S) and the pair $(\{W_n\}_{n\in\mathbb{N}},W)$ satisfies the lower limit assumption, (A1).*

Proof. If we take the logarithm on both sides of the Bernstein–Markov condition, we get

$$-\inf_{y\in K}\frac{1}{n}\sum_{i=1}^{n}G(x_i,y)\leq\frac{1}{n}\log(C)+\varepsilon+W_n(\vec{x}).$$

Equivalently, we have that

$$W(\mu)\leq\frac{1}{n}\log(C)+\varepsilon+\tilde{W}_n(\mu)$$

and, in particular, as W is bounded from below, we obtain that $\{W_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S). If $\mu_n\to\mu$ then

$$W(\mu)\leq\liminf_{n\to\infty}W(\mu_n)\leq\varepsilon+\liminf_{n\to\infty}\tilde{W}_n(\mu_n).$$

As this is true for every $\varepsilon>0$ we conclude the proof. □

The following corollary immediately implies [24, Theorem 1].

Corollary 4.18 (Zero temperature macroscopic limit). *Suppose that (G,K) is regular and that (G,ν) satisfies the Bernstein–Markov condition. Suppose also that for every probability measure $\mu\in\mathcal{P}(M)$ such that $\int_{M\times M}G(x,y)\,d\mu(x)\,d\mu(y)<\infty$ there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on M such that $D(\mu_n\,\|\,\pi)<\infty$ for every $n\in\mathbb{N}$ and such that*

$$\lim_{n\to\infty}\int_{M\times M}G(x,y)\,d\mu_n(x)\,d\mu_n(y)=\int_{M\times M}G(x,y)\,d\mu(x)\,d\mu(y). \tag{4.9}$$

Define $w_n:M^n\to(-\infty,\infty]$ and $w:\mathcal{P}(M)\to(-\infty,\infty]$ by

$$w_n(z_1,\dots,z_n)=\frac{1}{n^2}\sum_{i<j}G(z_i,z_j)+\frac{n+1}{n^2}\log\left(\int_{\mathbb{C}}e^{-\sum_{i=1}^{n}G(z,x_i)}\,d\nu(z)\right)$$

and

$$w(\mu)=\frac{1}{2}\int_{M\times M}G(x,y)\,d\mu(x)\,d\mu(y)-\inf_{x\in K}\left\{\int_K G(x,y)\,d\mu(y)\right\}.$$

Then w is the zero temperature macroscopic limit of $\{w_n\}_{n\in\mathbb{N}}$.

Proof. Using Propositions 4.17 and 2.1 we obtain that w_n is well defined, $\{w_n\}_{n\in\mathbb{N}}$ is a *stable sequence* (S) and that $(\{w_n\}_{n\in\mathbb{N}},w)$ satisfies the *lower limit assumption*, (A1). The *regularity assumption*, (A2′), is implied by Proposition 2.1, the continuity of w and (4.9). □

5. Fekete points and the zero temperature deterministic case

We begin by a fact which standard proof can be found in [16].

Proposition 5.1 (Convergence of the infima). *If W is the positive temperature macroscopic limit or the zero temperature macroscopic limit of a stable (S) and confining (C) sequence $\{W_n\}_{n\in\mathbb{N}}$ then*

$$\inf W_n \to \inf W.$$

In particular we get the following consequence.

Theorem 5.2 (Deterministic Laplace principle). *If W is the positive temperature macroscopic limit or the zero temperature macroscopic limit of a stable (S) and confining (C) sequence $\{W_n\}_{n\in\mathbb{N}}$ then for every bounded continuous function $f : M \to \mathbb{R}$*

$$\inf\{W_n + f \circ i_n\} \to \inf\{W + f\}.$$

Proof. It is enough to notice that if W is the positive temperature macroscopic limit (respectively, the zero temperature macroscopic limit) of the sequence $\{W_n\}_{n\in\mathbb{N}}$ then $W + f$ is the positive temperature macroscopic limit (respectively, the zero temperature macroscopic limit) of the sequence $\{W_n + f\}_{n\in\mathbb{N}}$ and use Proposition 5.1. □

This may be seen as a natural analogue of the Laplace principle. It is just (3.1) without the entropy term (as if β_n were infinity). This analogue is related to the notion of Γ-convergence (see [18] for an introduction to this topic) as is said in the following remark.

Remark 5.3 (Γ-convergence). Theorem 5.2 can be used to prove the Γ-convergence of the sequence $\tilde{W}_n$ defined in (1.2) (see [18, Theorem 9.4]). In fact, the confining property of $\{W_n\}_{n\in\mathbb{N}}$ is not needed as we can obtain the Γ-convergence from the following standard statement if we take A_n to be equal to the graph of $\tilde{W}_n$.

Let E be a measurable space. Take a sequence $\{A_n\}_{n\in\mathbb{N}}$ of measurable sets in E and choose $x \in E$. The following affirmations are equivalent.

(a) There exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables taking values in E such that

$$\forall n \in \mathbb{N}, \quad \mathbb{P}(X_n \in A_n) = 1 \quad \text{and} \quad X_n \xrightarrow{\mathbb{P}} x.$$

(b) There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in E such that

$$\forall n \in \mathbb{N}, \quad x_n \in A_n \quad \text{and} \quad x_n \to x.$$

Acknowledgements

I would like to thank Raphaël Butez, Djalil Chafaï, Adrien Hardy and the anonymous referees for their useful remarks.

References

[1] G. W. Anderson, A. Guionnet and O. Zeitouni. *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge University Press, Cambridge, 2010. MR2760897

[2] T. Aubin. *Some Nonlinear Problems in Riemannian Geometry*. *Springer Monographs in Mathematics*. Springer-Verlag, Berlin, 1998. MR1636569

[3] C. Beltrán, N. Corral and J. G. Criado del Rey. Discrete and continuous Green energy on compact manifolds. Preprint. Available at https://arxiv.org/abs/1702.00864.

[4] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields* **108** (4) (1997) 517–542. MR1465640

[5] G. Ben Arous and O. Zeitouni. Large deviations from the circular law. *ESAIM Probab. Stat.* **2** (1998) 123–134. MR1660943

[6] R. J. Berman. Kähler–Einstein metrics emerging from free fermions and statistical mechanics. *J. High Energy Phys.* **2011** (10) (2011) 106, 31 pp. MR2909692
[7] R. J. Berman. A thermodynamical formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics. *Adv. Math.* **248** (2013) 1254–1297. MR3107540
[8] R. J. Berman. On large deviations for Gibbs measures, mean energy and gamma-convergence. *Constr. Approx.* **48** (1) (2018) 3–30. MR3825945
[9] T. Bodineau and A. Guionnet. About the stationary states of vortex systems. *Ann. Inst. Henri Poincaré Probab. Stat.* **35** (2) (1999) 205–237. MR1678526
[10] V. S. Borkar. *Probability Theory. An Advanced Course*. *Universitext*. Springer-Verlag, New York, 1995. MR1367959
[11] R. Butez. Large deviations for the empirical measure of random polynomials: Revisit of the Zeitouni–Zelditch theorem. *Electron. J. Probab.* **21** (2016) 73, 37 pp. MR3592204
[12] J.-B. Castéras. A mean field type flow part I: Compactness of solutions to a perturbed mean field type equation. *Calc. Var. Partial Differ. Equ.* **53** (1–2) (2015) 221–246. MR3336318
[13] D. Chafaï, N. Gozlan and P.-A. Zitt. First-order global asymptotics for confined particles with singular pair repulsion. *Ann. Appl. Probab.* **24** (6) (2014) 2371–2413. MR3262506
[14] P. Dupuis and R. S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. *Wiley Series in Probability and Statistics: Probability and Statistics*. A Wiley-Interscience Publication, New York, 1997. MR1431744
[15] P. Dupuis, V. Laschos and K. Ramanan. Large deviations for empirical measures generated by Gibbs measures with singular energy functionals. Preprint. Available at https://arxiv.org/abs/1511.06928.
[16] D. García-Zelada. A large deviation principle for empirical measures on Polish spaces: Application to singular Gibbs measures on manifolds. Preprint. Available at https://arxiv.org/abs/1703.02680v1.
[17] H.-O. Georgii. Large deviations and maximum entropy principle for interacting random fields on $\mathbb{Z}^d$. *Ann. Probab.* **21** (4) (1993) 1845–1875. MR1245292
[18] D. Gianni and Maso. *An Introduction to* Γ*-Convergence*. *Progress in Nonlinear Differential Equations and Their Applications* **8**. Birkhäuser, Boston, 1993. MR1201152
[19] A. Hardy. A note on large deviations for 2D Coulomb gas with weakly confining potential. *Electron. Commun. Probab.* **17** (2012) 19, 12 pp. MR2926763
[20] F. Hiai and D. Petz. Logarithmic energy as an entropy functional. In *Advances in Differential Equations and Mathematical Physics* 205–221. *Contemporary Mathematics* **217**. American Mathematical Society, Providence, RI, 1998. MR1606719
[21] F. Hiai and D. Petz. A large deviation theorem for the empirical eigenvalue distribution of random unitary matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **36** (1) (2000) 71–85. MR1743092
[22] M. Mariani. A Γ-convergence approach to large deviations. Preprint. Available at https://arxiv.org/abs/1204.0640.
[23] E. Sandier and S. Serfaty. From the Ginzburg–Landau model to vortex lattice problems. *Comm. Math. Phys.* **313** (3) (2012) 635–743. MR2945619
[24] O. Zeitouni and S. Zelditch. Large deviations of empirical measures of zeros of random polynomials. *Int. Math. Res. Not.* **2010** (20) (2010) 3935–3992. MR2738347