The Annals of Applied Probability
2018, Vol. 28, No. 5, 3287–3289
https://doi.org/10.1214/18-AAP1393
Main article: https://doi.org/10.1214/13-AAP950
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ERRATUM: "PROPAGATION OF CHAOS IN NEURAL FIELDS" [*Ann. Appl. Probab.* **24** (2014) 1298–1328]

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This erratum indicates an erroneous proposition of expansion of the results in Appendix B of the paper [4] *Propagation of chaos in neural fields*, where possible extensions are announced of the result proved in the main text to less regular equations (namely, with nonglobally Lipschitz-continuous dynamics). We also provide here an equivalent representation of the law of the spatially extended McKean–Vlasov limit at each space location that simplifies measurability considerations.

Extension to nonglobally Lipschiz-continuous drifts. In [4], we demonstrate the convergence of a spatially extended neural network toward a mean-field equation under the assumption that the coefficients of each element's dynamics is globally Lipschitz-continuous (Assumption H1, page 1303). This setting is valid for a number of models in neuroscience, as discussed in Appendix A.

We proposed possible extensions of this results to models with nonglobally Lipschitz continuous drifts in Appendix B; however, the assumptions provided in the Appendix are too weak and extending the results under those assumptions is not valid. Indeed, existence and uniqueness of the limit not generally ensured for McKean–Vlasov diffusions with nonglobally Lipschitz continuous drifts, and counterexamples to the uniqueness were constructed by Scheutzow [3] (cited in this context in [1]). In this paper, the author indeed exhibits two McKean–Vlasov diffusions with very specific locally-Lipschitz drifts (constructed through infinite series of functions) that have at least two solutions. Therefore, assuming only local Lipschitz-continuity of the drift in a McKean–Vlasov diffusion is not enough to prove existence and uniqueness of solutions.

In cases arising in neuroscience, there is essentially one model that does not satisfy assumption (H1), the Fitzhugh–Nagumo (FhN) neuronal network model [equation (14) in Appendix A], for which each neuron has a dynamics characterized by a cubic drift. For this particular system, it was recently shown, in a setting similar to [4] but with nonspatial interactions and no delay, that there exists a unique solution to the associated McKean–Vlasov diffusion [1, 2]. The two contributions solve this problem using distinct methodologies: [1] is based on probability theory, and [2] uses functional analysis techniques. In the latter case, we

Received July 2017; revised February 2018.
MSC2010 subject classifications. 60F99, 60B10, 34C15.

demonstrated existence and uniqueness of solutions for all times considering the Fokker–Planck equation associated with the mean-field equation. The proof crucially relies on entropy estimates. In detail, assuming that the initial distribution has a finite entropy and sufficient integrability properties, it is shown that the entropy of the solution remains bounded for all times, which allows deriving an upper bound of the norm between two solutions at time t proportional to the norm of their initial distributions, in turn readily ensuring existence and uniqueness of solutions of the McKean–Vlasov equation. The spatially extended system with delays studied here is distinct from the model studied in [2] only in the mean-field interaction term, and we conjecture that under the hypotheses of boundedness and regularity of the interactions of the manuscript, the estimates of [2] could be extended to demonstrate the generalization proposed in Appendix B of [4] in the case of FhN systems.

Alternative representation of the limit process and measurability. We take advantage of this erratum to indicate a typographical error in the definition of the norm [equation (6)], which should read

$$\begin{aligned}\|Z\|^2_{\mathbb{L}^2_\lambda(\Gamma)} &= \mathcal{E}_{\hat{r}}\big[\mathbb{E}\big[\big|Z(\hat{r})\big|^2\big]\big]\\ &= \int_\Gamma \int_E |x|^2 p(r,dx)\,d\lambda(r) = \int_{\Gamma\times E} |x|^2(\lambda\otimes p)(dr,dx),\end{aligned}$$

where $\lambda \otimes p$ is the semi-direct product of λ with the kernel p (a similar notation can be used for $\|Z\|_{\mathcal{M}}$). Moreover, the norm of the chaotic Brownian motion [example (i), page 1307] has a norm bounded by $4T$ owing to Burkholder–Davis–Gundy's theorem (and not equal to T as initially indicated). As in the initial definition in equation (6), a similar inversion of the expectations $\mathbb{E}$ and $\mathcal{E}_{r'}$ appears in the first two inequalities for the control of the norm of the term A in the proof of Theorem 2. These expectations cannot be inverted because of the particular measurability property of the solution, whose law is measurable with respect to space, but whose trajectories are not.

These difficulties can be avoided and the proof of Theorem 2 can be simplified noting that there exists a measurable representation of that process. It is indeed easy to notice that the mean-field process $\bar{X}_t(r)$ has, for each fixed $r \in \Gamma$, the same law as the process $x_t(r)$ regular with respect to r:

$$\begin{cases} dx_t(r) = \Big(f\big(r,t,x_t(r)\big) + \displaystyle\int_\Gamma \int_E b\big(r,r',x_t(r),y\big) \\ \qquad\qquad \times\, p\big(t-\tau(r,r'),r'\big)(dy)\,d\lambda(r')\Big)\,dt \\ \qquad\qquad + \sigma(r)\,dW_t, \qquad t>0, \\ x_t(r) = x^0_t(r), \qquad t\in[-\tau,0], \end{cases} \tag{1}$$

where $p(t,r)$ is the Markov kernel from $(\Gamma, \mathcal{B}(\Gamma))$ to $(C, \mathcal{C})$ associated with the probability distribution of $x_t(r)$, W_t a standard m-dimensional Brownian motion (replacing the chaotic Brownian motion in the definition of $\bar{X}$), and $x_t^0(r)$ a random variable in $C([-\tau, 0], E)$ with law $\zeta_t^0(r)$ and which is measurable with respect to $r \in \Gamma$. For this process, we can show the following.

THEOREM. *For any* $(x_t^0(r), t \in [-\tau, 0], r \in \Gamma)$ *measurable with respect to* r *and with square-integrable law* $\zeta_t^0(r) \in \mathcal{M}^2([-\tau, 0], \mathbb{L}_\lambda^2(\Gamma))$, *the mean-field equation* (1) *has a unique strong solution on* $[-\tau, T]$ *for any* $T > 0$.

This theorem can be shown using exactly the same technique as the proof of Theorem 2, and avoids resorting on the notion of chaotic processes. We refer to [5], Theorem 1, where this technique was used to prove an analogous result in a comparable setting. This existence and uniqueness of solutions of (1) readily allows concluding on the existence and uniqueness of solutions for $X_t(r)$ for each fixed r, and on the measurability of the solution with respect to r of the paper, thus showing the existence and uniqueness of the Markov kernel p. Using the above representation saves all measurability bothers induced by considering spatially chaotic processes, and ensures that all estimates [in particular the norm given by equation (6) of the main text] are well defined.

Acknowledgement. We thank Wilhelm Stannat for his kind comments and suggestions.

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