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# Hypothesis testing of the drift parameter sign for fractional Ornstein–Uhlenbeck process

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**Abstract:** We consider the fractional Ornstein–Uhlenbeck process with an unknown drift parameter and known Hurst parameter $H$. We propose a new method to test the hypothesis of the sign of the parameter and prove the consistency of the test. Contrary to the previous works, our approach is applicable for all $H \in (0,1)$.

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## 1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and $B^H = \left\{B^H_t, t \in \mathbb{R}\right\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ on this probability space, that is a centered Gaussian process with covariance function

$$\mathbf{E} B^H_t B^H_s = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad t, s \in \mathbb{R}.$$

Since $\mathbf{E}\left(B^H_t - B^H_s\right)^2 = |t-s|^{2H}$ and the process $B^H$ is Gaussian, it has a continuous modification by Kolmogorov's theorem. In what follows we consider such modification.

The present paper deals with the inference problem associated with the Langevin equation

$$X_t = x_0 + \theta \int_0^t X_s\, ds + B_t^H, \quad t \geq 0, \tag{1}$$

where $x_0 \in \mathbb{R}$, and $\theta \in \mathbb{R}$ is an unknown drift parameter. This equation has a unique solution, see [5]. In order to avoid integration with respect to the fractional Brownian motion for $0 < H < 1/2$, we can write this solution in the following form

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H\, ds + B_t^H, \quad t \geq 0. \tag{2}$$

The process $X = \{X_t, t \geq 0\}$ is called a *fractional Ornstein–Uhlenbeck process* [5]. It is a Gaussian process, consequently its one-dimensional distributions are normal, with mean $x_0 e^{\theta t}$ and variance

$$v(\theta, t) = H \int_0^t s^{2H-1} \left( e^{\theta s} + e^{\theta(2t-s)} \right) ds,$$

see Lemma A.1 in Appendix.

The estimation problem for the drift parameter $\theta$ in the model (1) was studied in many works. We refer to the paper [15] for the extended survey of these results. The maximum likelihood estimators (MLE's) were studied in [13, 18, 19] for $H \geq 1/2$ and in the paper [20] for $H < 1/2$. Note that the MLE is hardly discretized because it contains the stochastic integrals with singular kernels. Therefore, several nonstandard estimators have been proposed recently. In particular, for $\theta < 0$ (the ergodic case) and $H \geq 1/2$ Hu and Nualart [11] constructed the analog of the least-squares estimator of the form

$$\hat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}, \tag{3}$$

where the integral $\int_0^T X_t dX_t$ is the divergence-type one. As an alternative, they considered the estimator

$$\hat{\theta}_T = -\left( \frac{1}{H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}. \tag{4}$$

In the non-ergodic case, when $\theta > 0$, Belfadli et al. [1] proposed the estimator

$$\hat{\theta}_T = \frac{X_T^2}{2\int_0^T X_t^2 dt} \tag{5}$$

and proved its strong consistency for $H > 1/2$. Later in [6] its strong consistency was obtained for all $H \in (0, 1)$. Note that the estimator (5) coincides with the estimator (3), where the integral $\int_0^T X_t dX_t$ is understood in the path-wise sense. In the papers [3, 4, 7, 8, 9, 12, 15, 21] the discretized estimators were considered. For the case $\theta < 0$ the drift parameter estimator based on polynomial variations was proposed in [10].

The methods of constructing the estimators and their asymptotic properties essentially depend on the sign of unknown drift parameter $\theta$. In particular, the estimator (4) is based on the ergodicity and does not work in the non-ergodic case. Similarly, the estimator (5) converges to zero if $\theta < 0$, see remark at the end of Sec. 3 in [11].

The above discussion motivates the hypothesis testing of the sign of drift parameter in the model (1). The interest to this problem is also connected with the stability properties of the solution to the equation (1), which also depend on the sign of $\theta$. For $H \geq 1/2$, this problem was studied by Moers [16]. He constructed a test using the estimator

$$\tilde{\theta}_{T,H} = \frac{X_T^2 - X_0^2}{2\int_0^T X_t^2 dt} - \left(\frac{1}{H\Gamma(2H)T}\int_0^T X_t^2 dt\right)^{-\frac{1}{2H}}. \tag{6}$$

The exact distribution of $\tilde{\theta}_{T,H}$ is not known, and the test is based on the asymptotic distribution of $T\tilde{\theta}_{T,H}$. The values of the corresponding test statistic should be compared with quantiles of the random variable

$$\frac{\left(B_1^H\right)^2}{2\int_0^1 \left(B_t^H\right)^2 dt} - \left(\frac{1}{H\Gamma(2H)}\int_0^1 \left(B_t^H\right)^2 dt\right)^{-\frac{1}{2H}}, \tag{7}$$

and the quantiles can be obtained by Monte Carlo simulation. The test can be used for testing three types of hypothesis: $H_0\colon \theta \geq 0$ against $H_1\colon \theta < 0$, $H_0\colon \theta \leq 0$ against $H_1\colon \theta > 0$, and $H_0\colon \theta = 0$ against $H_1\colon \theta \neq 0$. The consistency of the test is proved only for $H \in [1/2, 3/4)$ for a simple alternative $\theta_1 < 0$, and for $H \in [1/2, 1)$ for $\theta_1 > 0$. Tanaka [18, 19] considered the testing of the hypothesis $H_0\colon \theta = 0$ against the alternatives $H_1\colon \theta < 0$ and $H_1\colon \theta > 0$. He proposed tests based on the MLE (for both alternatives) and on the minimum contrast estimator (only for the ergodic case). Those tests were considered also for $H \geq 1/2$. To the best of our knowledge, there are no tests, suitable for the discrete-time observations of the process.

In the present paper we propose comparatively a simple test for testing the null hypothesis $H_0\colon \theta \leq 0$ against the alternative $H_1\colon \theta > 0$. The main advantage of our approach is that it can be used for any $H \in (0, 1)$. The distribution of the test statistic is computed explicitly, and the power of test can be found numerically for any given simple alternative. Also we consider the hypothesis testing $H_0\colon \theta \geq \theta_0$ against $H_1\colon \theta \leq 0$, where $\theta_0 \in (0, 1)$ is some fixed number. Unfortunately, our approach does not enable to test the hypothesis $H_0\colon \theta = 0$ against the two-sided alternative $H_1\colon \theta \neq 0$. The test is based on the observations of the process $X$ at two points: 0 and $T$. Therefore, it is applicable for both continuous and discrete cases.

**Remark 1.1.** The situation is similar to the model $Y_t = at + W_t$, where $W_t$ is a Wiener process. In this case the MLE of the drift parameter by observations at points $0 = t_0 \le t_1 \le \ldots \le t_N = T$ is given by

$$\hat{a} = \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} \left(Y_{t_{i+1}} - Y_{t_i}\right) = \frac{Y_T - Y_0}{T},$$

and depends on the observations at two points, see e.g. [2, p. 363]. Similarly, in Samuelson's model [17] of the form $\log S_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t$ with constant drift parameter $\mu$ and known volatility $\sigma$, the MLE of $\mu$ equals

$$\hat{\mu} = \frac{\log S_T - \log S_0}{T} + \frac{\sigma^2}{2}.$$

and also depends on the observations at two points.

The paper is organized as follows. In Section 2 the problem of hypothesis testing for the sign of the drift parameter $\theta$ is considered. Section 3 is devoted to numerics. In Appendix we get some auxiliary results. In particular, we calculate the first two moments of the fractional Ornstein–Uhlenbeck process.

## 2. Hypothesis testing of the drift parameter sign

### 2.1. Test statistic

For hypothesis testing of the sign of the parameter $\theta$ we construct a test based on the asymptotic behavior of the random variable

$$Z(t) = \frac{\log^+ \log |X_t|}{\log t}, \quad t > 1, \tag{8}$$

where $\log^+ x = \log x$ when $x > 1$ and $\log^+ x = 0$ otherwise. The following result explains the main idea. It is based on the different asymptotic behavior of the fractional Ornstein–Uhlenbeck process with positive drift parameter and negative one.

**Lemma 2.1.** *The value of $Z(t)$ converges a.s. to 1 for $\theta > 0$, and to 0 for $\theta \le 0$, as $t \to \infty$.*

*Proof.* For $\theta > 0$, by [6, Lemma 2.1] (see also [1] for $H > 1/2$),

$$e^{-\theta t} X_t \to \xi_\theta \quad \text{a.s., as } t \to \infty,$$

where $\xi_\theta = x_0 + \theta \int_0^\infty e^{-\theta s} B_s^H \, ds \simeq \mathcal{N}\left(x_0, \frac{H\Gamma(2H)}{\theta^{2H}}\right)$. Then

$$\log |X_t| - \theta t \to \log |\xi_\theta| \quad \text{a.s., as } t \to \infty,$$

where $\xi_\theta$ is a normal random variable, hence, $0 < |\xi_\theta| < \infty$ a.s. Therefore,

$$\frac{\log |X_t|}{t} \to \theta \quad \text{a.s., as } t \to \infty.$$

It means that there exists $\Omega' \subset \Omega$ such that $\mathbf{P}(\Omega') = 1$ and for any $\omega \in \Omega'$ there exists $t(\omega)$ such that for $t \geq t(\omega)$: $\log|X_t| > 0$. Hence, for $t \geq t(\omega)$ we have that

$$\left|\frac{\log^+ \log|X_t|}{\log t} - 1\right| = \left|\frac{\log\log|X_t|}{\log t} - 1\right|$$
$$= \left|\frac{\log\log|X_t| - \log t}{\log t}\right| = \left|\frac{\log\frac{\log|X_t|}{t}}{\log t}\right| \to 0$$

a. s., as $t \to \infty$. For $\theta \leq 0$, it follows from (19) that

$$|Z(t)| \leq \left|\frac{\log^+\left(\log\left(1 + t^H \log^2 t\right) + \log\zeta\right)}{\log t}\right|$$
$$= \left|\frac{\log\left(\log\left(1 + t^H \log^2 t\right) + \log\zeta\right)}{\log t}\right| \sim \left|\frac{\log\left(\log\left(t^H \log^2 t\right)\right)}{\log t}\right| \to 0,$$

as $t \to \infty$. □

The next result gives the cdf of $Z(t)$. Let $\Phi$ and $\varphi$ denote the cdf and pdf, respectively, of the standard normal variable.

**Lemma 2.2.** *For $t > 1$ the probability $g(\theta, x_0, t, c) = \mathbf{P}(Z(t) \leq c)$ is given by*

$$g(\theta, x_0, t, c) = \Phi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) + \Phi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) - 1, \tag{9}$$

*and $g$ is a decreasing function of $\theta \in \mathbb{R}$.*

*Proof.* Using Lemma A.1 and taking into account that $\log^+ x$ is a non-decreasing function, we obtain

$$\mathbf{P}(Z(t) \leq c) = \mathbf{P}\left(|X_t| \leq e^{t^c}\right) = \Phi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) - \Phi\left(\frac{-e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)$$
$$= \Phi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) + \Phi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) - 1.$$

Let us prove the monotonicity of the function $g$ with respect to $\theta$. Note that $g$ is an even function with respect to $x_0$. Therefore, it suffices to consider only the case $x_0 \ge 0$. The partial derivative equals

$$\begin{aligned}\frac{\partial g}{\partial \theta} &= \varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)\left(-x_0 t e^{\theta t} v^{-\frac{1}{2}}(\theta,t) - \frac{1}{2} v^{-\frac{3}{2}}(\theta,t) v'_\theta(\theta,t)\left(e^{t^c} - x_0 e^{\theta t}\right)\right)\\ &\quad + \varphi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)\left(x_0 t e^{\theta t} v^{-\frac{1}{2}}(\theta,t) - \frac{1}{2} v^{-\frac{3}{2}}(\theta,t) v'_\theta(\theta,t)\left(e^{t^c} + x_0 e^{\theta t}\right)\right)\\ &= -\frac{1}{2} v^{-\frac{3}{2}}(\theta,t) v'_\theta(\theta,t) e^{t^c}\left(\varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) + \varphi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)\right)\\ &\quad - x_0 e^{\theta t} v^{-\frac{3}{2}}(\theta,t)\left(t v(\theta,t) - \frac{1}{2} v'_\theta(\theta,t)\right)\\ &\quad \times\left(\varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) - \varphi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)\right).\end{aligned} \tag{10}$$

Since

$$v'_\theta(\theta,t) = H \int_0^t s^{2H-1}\left(s e^{\theta s} + (2t - s) e^{\theta(2t-s)}\right) ds > 0, \tag{11}$$

we see that the first term in the right-hand side of (10) is negative. Let us consider the second term. From (16) and (11) it follows that

$$t v(\theta,t) - \tfrac{1}{2} v'_\theta(\theta,t) = H \int_0^t s^{2H-1}\left(\left(t - \tfrac{1}{2}s\right) e^{\theta s} + \tfrac{1}{2} s e^{\theta(2t-s)}\right) ds > 0.$$

Since $\left|e^{t^c} - x_0 e^{\theta t}\right| \le e^{t^c} + x_0 e^{\theta t}$ for $x_0 \ge 0$, we have

$$\varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) - \varphi\left(\frac{e^{t^c} + x_0 e^{\theta_1 t}}{\sqrt{v(\theta,t)}}\right) \ge 0.$$

Thus, the second term in the right-hand side of (10) is non-positive. Hence, $\frac{\partial g}{\partial \theta} < 0$. □

### *2.2. Testing the hypothesis $H_0\colon \theta \le 0$ against $H_1\colon \theta > 0$*

To test $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$, we consider the following procedure: for a given significance level $\alpha$, and for sufficiently large value of $t$ we choose a threshold $c = c_t \in (0,1)$, see Lemma 2.3. Further, when $Z(t) \le c$ the hypothesis $H_0$ cannot be rejected, and when $Z(t) > c$ it is rejected. Below we will propose a technically simpler version of this test, without the computation of $c$, see Remark 2.4. The threshold $c$ can be chosen as follows.

Fix a number $\alpha \in (0,1)$, the significance level of the test. This level gives the maximal probability of a type I error, that is in our case the probability to reject the hypothesis $H_0 \colon \theta \le 0$ when it is true. By Lemma 2.2, for a threshold $c \in (0,1)$ and $t > 1$ this probability equals

$$\sup_{\theta \le 0} \mathbf{P}(Z(t) > c) = 1 - g(0, x_0, t, c).$$

Therefore, we determine $c_t$ as a solution to the equation

$$g(0, x_0, t, c_t) = 1 - \alpha. \tag{12}$$

The following result shows that for any $\alpha \in (0,1)$, it is possible to choose a sufficiently large $t$ such that $c_t \in (0,1)$.

**Lemma 2.3.** *Let $\alpha \in (0,1)$. Then there exists $t_0 \ge 1$ such that for all $t > t_0$ there exists a unique $c_t \in (0,1)$ such that $g(0, x_0, t, c_t) = 1 - \alpha$. Moreover $c_t \to 0$, as $t \to \infty$.*

*The constant $t_0$ can be chosen as the largest $t \ge 1$ that satisfies at least one of the following two equalities*

$$g(0, x_0, t, 0) = 1 - \alpha \quad \textit{or} \quad g(0, x_0, t, 1) = 1 - \alpha. \tag{13}$$

*Proof.* By Lemma A.3 *(iii)*, $v(0,t) = t^{2H}$. Then for $\theta = 0$ the formula (9) becomes

$$g(0, x_0, t, c) = \Phi\left(\frac{e^{t^c} - x_0}{t^H}\right) + \Phi\left(\frac{e^{t^c} + x_0}{t^H}\right) - 1. \tag{14}$$

For any $t > 1$, the function $g(0, x_0, t, c)$ is strictly increasing with respect to $c$. For $c = 0$ we have

$$g(0, x_0, t, 0) = \Phi\left(\frac{e - x_0}{t^H}\right) + \Phi\left(\frac{e + x_0}{t^H}\right) - 1 \to 2\Phi(0) - 1 = 0, \quad \text{as } t \to \infty.$$

Therefore, there exists $t_1 > 1$ such that $g(0, x_0, t, 0) < 1 - \alpha$ for all $t \ge t_1$.

Similarly, for $c = 1$

$$g(0, x_0, t, 1) = \Phi\left(\frac{e^t - x_0}{t^H}\right) + \Phi\left(\frac{e^t + x_0}{t^H}\right) - 1 \to 2\Phi(\infty) - 1 = 1, \quad \text{as } t \to \infty.$$

Therefore, there exists $t_2 > 1$ such that $g(0, x_0, t, 1) > 1 - \alpha$ for all $t \ge t_2$.

Thus, for any $t \ge t_0 = \max\{t_1, t_2\}$ there exists a unique $c_t \in (0,1)$ such that $g(0, x_0, t, c_t) = 1 - \alpha$.

To prove the convergence $c_t \to 0$, $t \to \infty$, consider an arbitrary $\varepsilon \in (0,1)$. Then

$$g(0, x_0, t, \varepsilon) = \Phi\left(\frac{e^{t^\varepsilon} - x_0}{t^H}\right) + \Phi\left(\frac{e^{t^\varepsilon} + x_0}{t^H}\right) - 1 \to 2\Phi(\infty) - 1 = 1, \text{ as } t \to \infty.$$

Arguing as above, we see that there exists $t_3 > 1$ such that for any $t > t_3$ the unique $c_t \in (0,1)$, for which $g(0, x_0, t, c_t) = 1 - \alpha$, belongs to the interval $(0, \varepsilon)$. This implies the convergence $c_t \to 0$, as $t \to \infty$.

It follows from (14) that $g(0, x_0, t, 0) = g(0, x_0, t, 1)$ for $t = 1$. As $t \to \infty$, we have $g(0, x_0, t, 0) \to 0$, $g(0, x_0, t, 1) \to 1$. Hence, at least one of the equalities (13) is satisfied for some $t \ge 1$ and the set of such $t$'s is bounded. □

**Remark 2.4.** Since the function $g(0, x_0, t, c)$ is strictly increasing with respect to $c$ for $t > 1$, we see that the inequality $Z(t) \le c_t$ is equivalent to the inequality $g(0, x_0, t, Z(t)) \le g(0, x_0, t, c_t) = 1 - \alpha$. Therefore, we do not need to compute the value of $c_t$. It is sufficient to compare $g(0, x_0, t, Z(t))$ with the level $1 - \alpha$.

**Algorithm 2.5.** *The hypothesis $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$ can be tested as follows.*

1. *Choose $0 < \alpha < 1$. Find $t_0$ defined in Lemma 2.3. The algorithm can be applied only in the case $t > t_0$.*
2. *Evaluate the statistic $Z(t)$ defined by (8).*
3. *Compute the value of $g(0, x_0, t, Z(t))$, see (14).*
4. *Do not reject the hypothesis $H_0$ if $g(0, x_0, t, Z(t)) \le 1 - \alpha$, and reject it otherwise.*

**Remark 2.6.** In fact, the condition $t > t_0$ is not too restrictive, since for reasonable values of $\alpha$, the values of $t_0$ are quite small, see Table 1.

Let us summarize the properties of the test in the following theorem.

**Theorem 2.7.** *The test described in Algorithm 2.5 is unbiased and consistent, as $t \to \infty$. For a simple alternative $\theta_1 > 0$ and moment $t > t_0$, the power of the test equals $1 - g(\theta_1, x_0, t, c_t)$, where $c_t$ can be found from (12).*

*Proof.* It follows from the monotonicity of $g$ with respect to $\theta$ (see Lemma 2.2) that for any $\theta_1 > 0$

$$\mathbf{P}(Z(t) > c_t) = 1 - g(\theta_1, x_0, t, c_t) > 1 - g(0, x_0, t, c_t) = \alpha.$$

This means that the test is unbiased. Evidently, for a simple alternative $\theta_1 > 0$ the power of the test equals $1 - g(\theta_1, x_0, t, c_t)$.

It follows from the convergence $c_t \to 0$, as $t \to \infty$ (see Lemma 2.3), that $c_t < c$ for sufficiently large $t$ and some constant $c \in (0, 1)$. Taking into account the formula (9) and Lemma A.3 $(i)$, we get, as $t \to \infty$:

$$\begin{aligned}
1 &\ge 1 - g(\theta_1, x_0, t, c_t) \ge 1 - g(\theta_1, x_0, t, c) \\
&= 2 - \Phi\left(\frac{e^{t^c} - x_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) - \Phi\left(\frac{e^{t^c} + x_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) \\
&\to 2 - \Phi\left(-\frac{x_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) - \Phi\left(\frac{x_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) = 1.
\end{aligned}$$

Hence, the test is consistent. □

**Remark 2.8.** For the composite alternative $H_1\colon \theta > 0$, the power of the test of Algorithm 2.5 is small and equals

$$\inf_{\theta_1 > 0} (1 - g(\theta_1, x_0, t, c_t)) = 1 - g(0, x_0, t, c_t) = \alpha.$$

We can consider testing of $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta \ge \theta_0$, where $\theta_0 > 0$ is a fixed number. For this composite alternative the power of the test of Algorithm 2.5 equals

$$\inf_{\theta_1 \ge \theta_0} (1 - g(\theta_1, x_0, t, c_t)) = 1 - g(\theta_0, x_0, t, c_t)$$

and tends to 1, as $t \to \infty$.

### 2.3. Testing the hypothesis $H_0\colon \theta \ge \theta_0$ against $H_1\colon \theta \le 0$

Fix $\theta_0 \in (0, 1)$. Consider the problem of testing the hypothesis $H_0\colon \theta \ge \theta_0$ against alternative $H_1\colon \theta \le 0$.

**Algorithm 2.9.** *The hypothesis $H_0\colon \theta \ge \theta_0$ against the alternative $H_1\colon \theta \le 0$ can be tested as follows.*

1. *Choose $0 < \alpha < 1$. Find $\tilde{t}_0$ defined in Lemma 2.10. The algorithm can be applied only in the case $t > \tilde{t}_0$.*
2. *Evaluate the statistic $Z(t)$ defined by (8).*
3. *Compute the value of $g(\theta_0, x_0, t, Z(t))$, see (9).*
4. *Do not reject the hypothesis $H_0$ if $g(\theta_0, x_0, t, Z(t)) \ge \alpha$, and reject it otherwise.*

This algorithm is based on the following results. They can be proved similarly to the previous subsection.

**Lemma 2.10.** *Let $\alpha \in (0, 1)$. There exists $\tilde{t}_0 > 1$ such that for all $t > \tilde{t}_0$ there exists a unique $\tilde{c}_t \in (0, 1)$ such that*

$$g(\theta_0, x_0, t, \tilde{c}_t) = \alpha. \tag{15}$$

*In this case $\tilde{c}_t \to 1$, as $t \to \infty$.*

*The constant $\tilde{t}_0$ can be chosen as the largest $t > 1$ that satisfies at least one of the following two equalities*

$$g(\theta_0, x_0, t, 0) = \alpha \quad \text{or} \quad g(\theta_0, x_0, t, 1) = \alpha.$$

**Theorem 2.11.** *The test described in Algorithm 2.9 is unbiased and consistent, as $t \to \infty$. For a simple alternative $\theta_1 \le 0$ and moment $t > \tilde{t}_0$, the power of the test equals $g(\theta_1, x_0, t, \tilde{c}_t)$, where $\tilde{c}_t$ can be found from (15). For the composite alternative $H_1\colon \theta \le 0$, the power of the test equals $g(0, x_0, t, \tilde{c}_t)$ and tends to 1, as $t \to \infty$.*

**Remark 2.12.** The values of $\tilde{t}_0$ for various values of $\theta_0$ and $H$ are represented in Table 2. We see that if $\theta_0$ is too close to zero, then for small $H$, the condition $t > \tilde{t}_0$ does not hold for reasonable values of $t$.

**Remark 2.13.** If we have a confidence interval for $\theta$, then the value of $\theta_0 \in (0, 1)$ can be chosen less than or equal to a lower confidence bound (in the case when the latter is positive).

Table 1
*Value of $t_0$ for various $H$ and $\alpha$ ($x_0 = 1$)*

| $H$ | $\alpha = 0.01$ | $\alpha = 0.05$ |
|---|---|---|
| 0.1 | 1.2157 | 1.5310 |
| 0.2 | 1.2313 | 1.2373 |
| 0.3 | 1.2492 | 1.1526 |
| 0.4 | 1.2699 | 1.1124 |
| 0.5 | 1.2940 | 1.0889 |
| 0.6 | 1.3224 | 1.0736 |
| 0.7 | 1.3561 | 1.0627 |
| 0.8 | 1.3968 | 1.0547 |
| 0.9 | 1.4462 | 1.0485 |

Table 2
*Values of $\tilde{t}_0$ for various $H$ and $\theta_0$ ($x_0 = 1$, $\alpha = 0.05$)*

| $H$ | $\theta_0 = 0.1$ | $\theta_0 = 0.05$ | $\theta_0 = 0.01$ | $\theta_0 = 0.001$ | $\theta_0 = 0$ |
|---|---|---|---|---|---|
| 0.1 | 32.433 | 65.242 | 326.47 | 3193.6 | $2.3369 \cdot 10^{16}$ |
| 0.2 | 32.667 | 64.721 | 307.43 | 2719.1 | $1.5287 \cdot 10^{8}$ |
| 0.3 | 31.994 | 61.728 | 271.64 | 2073.5 | 285 900 |
| 0.4 | 30.592 | 57.078 | 227.99 | 1387.8 | 12 364 |
| 0.5 | 28.659 | 51.413 | 181.64 | 778.94 | 1878.1 |
| 0.6 | 26.375 | 45.233 | 137.06 | 382.06 | 534.70 |
| 0.7 | 23.903 | 38.967 | 98.759 | 189.71 | 217.96 |
| 0.8 | 21.386 | 32.995 | 69.618 | 104.11 | 111.19 |
| 0.9 | 18.946 | 27.621 | 49.408 | 63.576 | 65.878 |

## 3. Numerical illustrations

In this section we illustrate the performance of our algorithms by simulation experiments. We choose the initial value $x_0 = 1$ for all simulations.

In Tables 1–2 the values of $t_0$ and $\tilde{t}_0$ for various $H$ and $\theta_0$ are given.

We simulate fractional Brownian motion at the points $t = 0, h, 2h, 3h, \dots$ and compute the approximate values of the Ornstein–Uhlenbeck process as the solution to the equation (1), using Euler's approximations. For various values of $\theta$ we simulate $n = 1000$ sample paths with the step $h = 1/10000$. Then we apply our algorithms, choosing the significance level $\alpha = 0.05$. In Table 3 the empirical rejection probabilities of the test of Algorithm 2.5 for the hypothesis testing $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$ for $H = 0.3$ and $H = 0.7$ are reported.

Then we test the same hypothesis using the test of Moers [16] for $H = 0.7$. By Monte Carlo simulations for 20 000 sample paths of the process $\{B_t^H, t \in [0,1]\}$ we estimate the $(1-\alpha)$-quantile $\psi_{1-\alpha}$ of the distribution (7) for $\alpha = 0.05$. Then we compare the statistic $T\tilde{\theta}_{T,H}$ (see (6)) with the value of this quantile and reject the hypothesis $H_0\colon \theta \le 0$ if $T\tilde{\theta}_{T,H} > \psi_{1-\alpha}$. We obtained that $\psi_{0.95} \approx 0.827946$. The empirical rejection probabilities for this test are given in Table 4. We see

TABLE 3
*Empirical rejection probabilities of the test of Algorithm 2.5 for the hypothesis testing $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$ for $H = 0.3$ and $H = 0.7$*

| $\theta$ | $-0.1$ | $-0.05$ | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 |
|---|---|---|---|---|---|---|---|---|---|
| | | | | **$H = 0.3$** | | | | | |
| $T = 20$ | 0.000 | 0.003 | 0.043 | 0.341 | 0.701 | 0.880 | 0.973 | 0.982 | 0.996 |
| $T = 40$ | 0.000 | 0.000 | 0.043 | 0.675 | 0.952 | 0.995 | 0.999 | 1.000 | 1.000 |
| $T = 60$ | 0.000 | 0.000 | 0.039 | 0.860 | 0.994 | 1.000 | 1.000 | 1.000 | 1.000 |
| $T = 80$ | 0.000 | 0.000 | 0.048 | 0.940 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $T = 100$ | 0.000 | 0.000 | 0.049 | 0.986 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | **$H = 0.7$** | | | | | |
| $T = 20$ | 0.000 | 0.001 | 0.058 | 0.284 | 0.540 | 0.800 | 0.910 | 0.967 | 0.979 |
| $T = 40$ | 0.000 | 0.000 | 0.050 | 0.581 | 0.889 | 0.984 | 0.998 | 1.000 | 1.000 |
| $T = 60$ | 0.000 | 0.000 | 0.042 | 0.782 | 0.980 | 1.000 | 0.999 | 1.000 | 1.000 |
| $T = 80$ | 0.000 | 0.000 | 0.047 | 0.908 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 |
| $T = 100$ | 0.000 | 0.000 | 0.048 | 0.959 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

TABLE 4
*Empirical rejection probabilities of the test of Moers [16] for the hypothesis testing $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$ for $H = 0.7$*

| $\theta$ | $-0.1$ | $-0.05$ | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 |
|---|---|---|---|---|---|---|---|---|---|
| $T = 20$ | 0.001 | 0.013 | 0.085 | 0.370 | 0.706 | 0.873 | 0.947 | 0.976 | 0.992 |
| $T = 40$ | 0.000 | 0.004 | 0.095 | 0.682 | 0.948 | 0.993 | 0.999 | 1.000 | 1.000 |
| $T = 60$ | 0.000 | 0.002 | 0.092 | 0.881 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 |
| $T = 80$ | 0.000 | 0.000 | 0.105 | 0.948 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| $T = 100$ | 0.000 | 0.000 | 0.089 | 0.977 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

that comparing to our algorithm, the test of Moers has bigger power, i. e., it works a bit better when the alternative is true. But for $\theta = 0$, the necessary significance level $\alpha = 0.05$ is not achieved.

TABLE 5
*Empirical rejection probabilities of the test of Algorithm 2.9 for the hypothesis testing $H_0\colon \theta \ge \theta_0$ against the alternative $H_1\colon \theta \le 0$ for $\theta_0 = 0.1$, $H = 0.3$ and $H = 0.7$*

| $\theta$ | $-0.25$ | $-0.2$ | $-0.15$ | $-0.1$ | $-0.05$ | 0 | 0.1 | 0.15 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | **$H = 0.3$** | | | | |
| $T = 40$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.938 | 0.056 | 0.008 | 0.000 |
| $T = 60$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.052 | 0.005 | 0.000 |
| $T = 80$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.054 | 0.001 | 0.000 |
| $T = 100$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.054 | 0.001 | 0.000 |
| | | | | | **$H = 0.7$** | | | | |
| $T = 40$ | 1.000 | 1.000 | 1.000 | 1.000 | 0.978 | 0.689 | 0.051 | 0.006 | 0.001 |
| $T = 60$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.052 | 0.004 | 0.001 |
| $T = 80$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.051 | 0.003 | 0.000 |
| $T = 100$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.051 | 0.001 | 0.000 |

Table 6
*Empirical rejection probabilities of the test of Algorithm 2.9 for the hypothesis testing $H_0: \theta \geq \theta_0$ against the alternative $H_1: \theta \leq 0$ for $\theta_0 = 0.05$, $H = 0.7$*

| $\theta$ | $-0.25$ | $-0.2$ | $-0.15$ | $-0.1$ | $-0.05$ | 0 | 0.05 | 0.1 | 0.15 |
|---|---|---|---|---|---|---|---|---|---|
| $T = 40$ | 0.842 | 0.773 | 0.661 | 0.566 | 0.368 | 0.149 | 0.051 | 0.007 | 0.000 |
| $T = 60$ | 0.999 | 1.000 | 0.990 | 0.955 | 0.799 | 0.346 | 0.047 | 0.002 | 0.000 |
| $T = 80$ | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 0.719 | 0.044 | 0.001 | 0.000 |
| $T = 100$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.980 | 0.050 | 0.000 | 0.000 |

Tables 5 and 6 represent empirical rejection probabilities of the test of Algorithm 2.9 for $\theta_0 = 0.1$ and $\theta_0 = 0.05$, respectively. We see that the test power increases if $\theta_0$ increases. Also, the test power tends to 1, as the time horizon $T$ increases. Hence these simulation studies confirm the theoretical results on the consistency.

## Appendix A

### A.1. *One-dimensional distribution of the fractional Ornstein–Uhlenbeck process*

Let $\{X_t, t \geq 1\}$ be the fractional Ornstein–Uhlenbeck process defined by (1).

**Lemma A.1.** *The random variable $X_t$ has normal distribution $\mathcal{N}(x_0 e^{\theta t}, v(\theta, t))$, with variance*

$$v(\theta, t) = H \int_0^t s^{2H-1} \left( e^{\theta s} + e^{\theta(2t-s)} \right) ds. \tag{16}$$

*Proof.* Since $B_t^H$ is a centered Gaussian process, it immediately follows from (2) that $X_t$ has normal distribution with mean $x_0 e^{\theta t}$. Let us calculate its variance. We have

$$\begin{aligned}
\operatorname{Var} X_t &= \mathbf{E} \left( B_t^H + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H \, ds \right)^2 \\
&= \mathbf{E} \left[ \left( B_t^H \right)^2 \right] + 2\theta e^{\theta t} \int_0^t e^{-\theta s} \mathbf{E} \left[ B_t^H B_s^H \right] ds \\
&\quad + \theta^2 e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} \mathbf{E} \left[ B_s^H B_u^H \right] ds \, du \\
&= t^{2H} + \theta e^{\theta t} \int_0^t e^{-\theta s} \left( t^{2H} + s^{2H} - (t-s)^{2H} \right) ds \\
&\quad + \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} \left( s^{2H} + u^{2H} - |s-u|^{2H} \right) ds \, du \\
&= t^{2H} + e^{\theta t} t^{2H} \left( 1 - e^{-\theta t} \right) + \theta e^{\theta t} \int_0^t e^{-\theta s} s^{2H} \, ds
\end{aligned}$$

$$
\begin{aligned}
&- \theta e^{\theta t} \int_0^t e^{-\theta(t-u)} u^{2H}\, du + \theta^2 e^{2\theta t} \int_0^t e^{-\theta s} s^{2H}\, ds \int_0^t e^{-\theta u}\, du \qquad (17)\\
&- \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s-u|^{2H}\, ds\, du \\
&= e^{\theta t} t^{2H} + \theta e^{\theta t} \int_0^t e^{-\theta s} s^{2H}\, ds - \theta \int_0^t e^{\theta s} s^{2H}\, ds \\
&+ \theta e^{2\theta t} \left(1 - e^{-\theta t}\right) \int_0^t e^{-\theta s} s^{2H}\, ds \\
&- \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s-u|^{2H}\, ds\, du \\
&= e^{\theta t} t^{2H} - \theta \int_0^t e^{\theta s} s^{2H}\, ds + \theta e^{2\theta t} \int_0^t e^{-\theta s} s^{2H}\, ds \\
&- \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s-u|^{2H}\, ds\, du.
\end{aligned}
$$

The last summand can be rewritten as follows

$$
\begin{aligned}
&\frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s-u|^{2H}\, ds\, du \\
&= \frac{\theta^2}{2} e^{2\theta t} \left( \int_0^t \int_0^s e^{-\theta s - \theta u} (s-u)^{2H}\, du\, ds + \int_0^t \int_s^t e^{-\theta s - \theta u} (u-s)^{2H}\, du\, ds \right) \\
&= \theta^2 e^{2\theta t} \int_0^t \int_0^s e^{-\theta s - \theta u} (s-u)^{2H}\, du\, ds = \theta^2 e^{2\theta t} \int_0^t \int_0^s e^{-2\theta s + \theta v} v^{2H}\, dv\, ds \\
&= \frac{\theta}{2} e^{2\theta t} \int_0^t e^{-\theta v} v^{2H}\, dv - \frac{\theta}{2} \int_0^t e^{\theta v} v^{2H}\, dv.
\end{aligned}
$$

Substituting the latter value into the above formula (17) for $\operatorname{Var} X_t$, we get

$$
\begin{aligned}
\operatorname{Var} X_t &= e^{\theta t} t^{2H} - \frac{\theta}{2} \int_0^t e^{\theta s} s^{2H}\, ds + \frac{\theta}{2} e^{2\theta t} \int_0^t e^{-\theta s} s^{2H}\, ds \\
&= e^{\theta t} t^{2H} - \frac{1}{2} \int_0^t s^{2H}\, d\left(e^{\theta s} + e^{2\theta t - \theta s}\right) \\
&= H \int_0^t s^{2H-1} \left(e^{\theta s} + e^{2\theta t - \theta s}\right) ds. \qquad \square
\end{aligned}
$$

**Remark A.2.** The variance of the fractional Ornstein–Uhlenbeck process was also computed in [6]. Here we derive a quite simple formula for it, which is more suitable for our purposes.

As a direct corollary from Lemma A.1, we get the asymptotical behavior of the function $v(\theta, t)$, as $t \to \infty$.

**Lemma A.3.** *(i) If $\theta > 0$, then $v(\theta, t) \sim \frac{H\Gamma(2H)}{\theta^{2H}} e^{2\theta t}$, as $t \to \infty$.*
*(ii) If $\theta < 0$, then $v(\theta, t) \to \frac{H\Gamma(2H)}{(-\theta)^{2H}}$, as $t \to \infty$.*

*(iii)* $v(0,t) = t^{2H}$, $t \geq 0$.

*Proof.* $(i)$ If $\theta > 0$, then by formula (16),

$$\frac{v(\theta,t)}{e^{2\theta t}} = He^{-2\theta t}\int_0^t s^{2H-1}e^{\theta s}ds + H\int_0^t s^{2H-1}e^{-\theta s}ds \to \frac{H\Gamma(2H)}{\theta^{2H}}, \text{ as } t \to \infty.$$

$(ii)$ Note that $v(\theta,t) = e^{2\theta t}v(-\theta,t)$, by (16). Then the convergence follows from $(i)$.

$(iii)$ The statement follows directly from (16). □

**Remark A.4.** The statement $(i)$ was proved in [11] for the case $H \in [1/2, 1)$ and in [6] for all $H \in (0,1)$. The statement $(ii)$ for $H > 1/2$ was obtained in [1].

### *A.2. Almost sure bounds for the fractional Ornstein–Uhlenbeck process*

**Lemma A.5** ([14, 15])**.** *There exists a nonnegative random variable $\zeta$ such that for all $s > 0$, the following inequalities hold true:*

$$\sup_{0\le s\le t} \left|B_s^H\right| \le \left(1 + t^H \log^2 t\right)\zeta, \tag{18}$$

*and for $\theta \le 0$*

$$\sup_{0\le s\le t} |X_s| \le \left(1 + t^H \log^2 t\right)\zeta. \tag{19}$$

*Moreover, $\zeta$ has the following property: there exists $C > 0$ such that $\mathbf{E}\exp\{x\zeta^2\} < \infty$, for any $0 < x < C$.*

*Proof.* The bound (18) was established in [14], see also Eq. (14) in [15]. It also implies that the inequality (19) holds for $\theta = 0$, since in this case $X_t = x_0 + B_t^H$. The bound (19) for $\theta < 0$ was obtained in [15, Eq. (19)]. □

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