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Errata to "Processes on unimodular random networks"*

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Abstract

We correct several statements and proofs in our paper, *Electron. J. Probab.* **12**, Paper 54 (2007), 1454–1508.

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Our paper had several minor misstatements and several somewhat incorrect proofs, which we rectify in this note.

(i) The statement in the introduction, "Unimodularity of a probability measure μ on rooted graphs is equivalent to the property that a reversible stationary distribution for this chain is given by the root-degree biasing of $\mu \ldots$," is incorrect. Unimodularity implies this reversibility, but is not implied by it. There is a similar loss of precision at the beginning of Section 4. The correct statement is the following: Let $\mu'(G, o) := \deg_G(o)\mu(G, o)$. Then unimodularity is equivalent to the distribution of the isomorphism class of (G, w_0, w_1) being the same as that of (G, w_1, w_0) when (G, w_0) has the distribution μ' and w_1 is a uniform random neighbor of w_0.

(ii) The proof of Theorem 3.1 was slightly incorrect. Instead of taking u, v in the same orbit, they should be any two vertices. After (3.5) is proved, one should deduce unimodularity from taking them in the same orbit. But they should still be general in order to deduce (3.4).

(iii) Preceding Corollary 4.3, A was called the infinitesimal generator, but it is the negative of the infinitesimal generator.

(iv) Formula (4.4) of Theorem 4.9 (which was changed inadvertently by the publisher from Proposition 4.9) is incorrect. The correct formula is

$$1 - \frac{2}{\overline{\deg}(\mu)} \,.$$

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This makes the last sentence of the published version of Theorem 4.9 obvious. The proof used stationarity of simple random walk, but mistakenly used μ as the stationary measure. Instead, of course, it is the degree-biased measure, σ. Formula (4.4) with σ in place of μ results, which then simplifies to the above.

(v) The proof of Proposition 4.10 has a slight error: when $1/\epsilon$ occurs, it should be ϵ.

(vi) The proof of Theorem 5.1 was not complete. In fact, the reduction to bounded weights is not needed since the inequality $\mathrm{Tr} f(S) \le \mathrm{Tr} f(T)$ holds for bounded increasing $f\colon \mathbb{R} \to \mathbb{R}$ and $0 \le S \le T$ that are Tr-measurable operators affiliated to Alg. This is proved like Lemma 4 of Brown and Kosaki (1990). Definitions are as follows. A closed densely defined operator is **affiliated** with Alg if it commutes with all unitary operators that commute with Alg. An affiliated operator T is called Tr**-measurable** if for all $\epsilon > 0$, there is an orthogonal projection $E \in \mathsf{Alg}$ whose image lies in the domain of T and $\mathrm{Tr}(E^\perp) < \epsilon$. The Laplacian A is Tr_μ-measurable because if E_n denotes the orthogonal projection to the space of functions that are nonzero only on those (G, o) where the sum of the edge weights at o is at most n, then $\lim_{n\to\infty} \mathrm{Tr}(E_n^\perp) = 0$ and $\|AE_n\| \le 2n$.

An alternative way to prove Theorem 5.1 is to modify the claim of the second paragraph that $P_t^{(i)}(o, o) = \lim_{n\to\infty} P_t^{(i,n)}(o, o)$ for $i = 1, 2$, which was, in fact, not correct (see below). We claim instead that there is a set of probability 1 for which $P_t^{(i)}(o, o) = \liminf_{n\to\infty} P_t^{(i,n)}(o, o)$ for $i = 1, 2$ and all $t > 0$ simultaneously, and also

$$\int P_t^{(i)}(o, o)\, d\mu_i(G, o) = \lim_{n\to\infty} \int P_t^{(i,n)}(o, o)\, d\mu_i(G, o) \tag{0.1}$$

for $i = 1, 2$. Of course, only (0.1) is needed to complete the proof of Theorem 5.1. Here, the superscript n refers to replacing each edge weight by 0 when the sum of the weights incident to its endpoints is larger than n (rather than what was described in the original paper). To verify these claims, let

$$B^{(i,n)}(x, y) := \begin{cases} A^{(i,n)}(x, y) & \text{if } x \ne y, \\ A^{(i)}(x, y) & \text{if } x = y, \end{cases}$$

and let $Q_t^{(i,n)}(x, y)$ be the transition kernel for the minimal process corresponding to the infinitesimal generator $-B^{(i,n)}$. Thus, when this process is at x, the rate of being killed is $A^{(i)}(x, x) - A^{(i,n)}(x, x)$, whereas the rate of moving is $A^{(i,n)}(x, x)$. An easy coupling argument shows that $P_t^{(i,n)}(x, y) \ge Q_t^{(i,n)}(x, y)$ and, because $P_t^{(i)}$ also are minimal processes, $P_t^{(i)}(x, y) = \lim_{n\to\infty} Q_t^{(i,n)}(x, y)$. Putting these together for $x = y = o$, we arrive at the inequality

$$P_t^{(i)}(o, o) \le \liminf_{n\to\infty} P_t^{(i,n)}(o, o) \tag{0.2}$$

for $i = 1, 2$ and all $t > 0$. By using Fatou's lemma, we see that (0.1) implies that equality holds in (0.2) a.s. for each $t > 0$ and hence for all $t \in \mathbb{Q}^+$ simultaneously. Since $P_t^{(i)}(o, o)$ and $P_t^{(i,n)}(o, o)$ are continuous and decreasing in t, we obtain that equality holds in (0.2) a.s. for all $t > 0$ simultaneously.

We now prove (0.1). Let E_n denote, as before, the orthogonal projection to the space of functions that are nonzero only on those (G, o) where the sum of the edge weights at o is at most n. Then $A^{(i,n)} E_n = A^{(i)} E_n$ for all n and $i = 1, 2$. Since $\lim_{n\to\infty} \mathrm{Tr}(E_n^\perp) = 0$, it follows that $\lim_{n\to\infty} A^{(i,n)} = A^{(i)}$ in the measure topology for each $i = 1, 2$; see Definition 1.5 of T. Fack and H. Kosaki (1986), Generalized s-numbers of τ-measurable operators, *Pacific J. Math.* **123**, 269–300. For $s \in [0, 1]$ and a Tr-measurable operator $T \ge 0$ with spectral resolution E_T, define

$$m_s(T) := \inf\bigl\{\lambda \ge 0\,;\ \mathrm{Tr}\bigl(E_T(\lambda, \infty)\bigr) \le 1 - s\bigr\}\,;$$

see Remark 2.3.1 of Fack and Kosaki (1968). A proof similar to that of Corollary 2.8 of Fack and Kosaki (1968) shows that for bounded monotone $f\colon \mathbb{R} \to \mathbb{R}$, we have

$$\mathrm{Tr}\big(f(T)\big) = \int_0^1 f\big(m_s(T)\big)\,ds\,. \tag{0.3}$$

Since $A^{(i,n)} \le A^{(i)}$, we have $m_s\big(A^{(i,n)}\big) \le m_s\big(A^{(i)}\big)$ by Lemma 2.5(iii) of Fack and Kosaki (1968). Therefore, $\lim_{n\to\infty} m_s\big(A^{(i,n)}\big) = m_s\big(A^{(i)}\big)$ by Lemma 3.4(ii) of Fack and Kosaki (1968). Now use $f(\lambda) := e^{-t\lambda}$ in (0.3) to obtain $\lim_{n\to\infty} \mathrm{Tr}\big(e^{-tA^{(i,n)}}\big) = \mathrm{Tr}\big(e^{-tA^{(i)}}\big)$, which is the same as (0.1).

For a counterexample to the claim in the proof of Theorem 5.1, but not in the context of a unimodular probability measure, consider the following birth and death chain: Let the weight on $(k, k+1)$ be a_k, where $a_0 := 1$ and $a_{k+1} := 2^{a_k}$. For the nth approximation, set $a_k^{(n)} := a_n \mathbb{1}_{\{k \le n\}}$. The original chain explodes, whereas the approximations are converging to the process "reflecting at infinity." One can describe the original chain as "absorbing at infinity."

(vii) The sentence after Proposition 6.3, "Recall that in Section 10 we have added a second independent uniform $[0,1]$ coordinate to the edge marks to form $\mu^{\mathrm{B}} \in \mathcal{U}$, the standard coupling of Bernoulli percolation on μ," should be replaced by the following: "More generally, we'd like to couple together all these Bernoulli percolation measures. We do this by using the canonical networks. We wish the second coordinates to be uniformly distributed on $[0,1]$ and independent (but the same at each endpoint of a given edge). For $0 \le i < j$, let $U_{i,j}$ be i.i.d. uniform $[0,1]$ random variables. Then for each canonical network $(G,0) \in \mathcal{G}_*$ and for each $0 \le i < j$, change the mark at each endpoint of the edge between i and j, if there is an edge, by adjoining a second coordinate equal to $U_{i,j}$. Let μ^{B} be the law of the resulting network class when $[G,0]$ has law μ. It is clear that μ^{B} is unimodular when μ is. We refer to μ^{B} as the **standard coupling of Bernoulli percolation on** μ. In the future, we shall not be explicit about how randomness is added to random networks."

(viii) The proof of Proposition 7.1 was not quite correct. Here is a correct version:

Proof. We begin by proving part of the third sentence, namely,

$$\text{every weak limit point of } \langle \mathsf{WUSF}(\mu_n) \rangle \text{ stochastically dominates } \mathsf{WUSF}(\mu)\,. \tag{0.4}$$

Given a positive integer R, let $\mathsf{UST}_R(\mu)$ be the uniform spanning tree on the wired ball of radius R about the root. (Although $\mathsf{UST}_R(\mu) \notin \mathcal{U}$, this will not affect our argument.) Identify the edges of the wired ball of radius R with the edges of the ball itself. By definition, we have $\mathsf{UST}_R(\mu) \Rightarrow \mathsf{WUSF}(\mu)$ as $R \to \infty$. Clearly, $\mathsf{UST}_R(\mu_n) \Rightarrow \mathsf{UST}_R(\mu)$ as $n \to \infty$. Furthermore, the intersection of $\mathsf{WUSF}(\mu_n)$ with the ball of radius R stochastically dominates $\mathsf{UST}_R(\mu_n)$ by a theorem of Feder and Mihail (1992). Therefore, every weak limit point of $\langle \mathsf{WUSF}(\mu_n) \rangle$ stochastically dominates $\mathsf{UST}_R(\mu)$ and therefore also $\mathsf{WUSF}(\mu)$.

Suppose now that μ is concentrated on recurrent networks. If μ is concentrated on networks with bounded degree, then so is $\mathsf{WUSF}(\mu)$, and the latter is also concentrated on recurrent networks by Rayleigh's monotonicity principle. By Theorem 4.9, the claim of the first sentence follows. If μ has unbounded degree, then let μ_n be the law of the component of the root when all edges incident to vertices of degree larger than n are deleted. Clearly $\mu_n \in \mathcal{U}$ and $\mu_n \Rightarrow \mu$. We have shown that $\overline{\deg}(\mathsf{WUSF}(\mu_n)) = 2$, so that (0.4) and Fatou's lemma yield that $\overline{\deg}(\mathsf{WUSF}(\mu)) \le 2$, whence equality results from Theorem 6.1.

Suppose next that μ is concentrated on transient networks. Then the proof of Theorem 6.5 of BLPS (2001) gives the same result.

Finally, if μ is concentrated on neither recurrent nor transient networks, then we may write μ as a mixture of two unimodular measures that are concentrated on recurrent or on transient networks and apply the preceding.

This proves the first sentence. The second sentence is a special case of the third, so it remains to finish the proof of the third. By Fatou’s lemma and Theorem 6.1, after what we have shown, we know that all weak limits of $\langle \mathsf{WUSF}(\mu_n) \rangle$ have expected degree 2, as does $\mathsf{WUSF}(\mu)$. Since all such weak limits lie in $\mathcal{U}$, (0.4) shows that all weak limits are equal. $\square$