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Recurrence and transience properties of multi-dimensional diffusion processes in selfsimilar and semi-selfsimilar random environments*

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Abstract

We consider d-dimensional diffusion processes in multi-parameter random environments which are given by values at different d points of one-dimensional α-stable or (r, α)-semi-stable Lévy processes. From the model, we derive some conditions of random environments that imply the dichotomy of recurrence and transience for the d-dimensional diffusion processes. The limiting behavior is quite different from that of a d-dimensional standard Brownian motion. We also consider the direct product of a one-dimensional diffusion process in a reflected non-positive Brownian environment and a one-dimensional standard Brownian motion. For the two-dimensional diffusion process, we show the transience property for almost all reflected Brownian environments.

Keywords: Diffusion process in random environment; recurrence; transience; stable Lévy process; semi-stable Lévy process.
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1 Introduction and results

It is well-known that a multi-dimensional standard Brownian motion, consisting of d independent one-dimensional standard Brownian motions, is recurrent if $d = 1$ or 2, and transient otherwise. We consider such limiting behavior of multi-dimensional diffusion processes in stable and semi-stable Lévy environments.

Let $\mathcal{W}$ be the space of the $\mathbb{R}$-valued functions W that satisfy the following:

(I) $W(0) = 0$.
(II) W is right continuous and has left limits on $[0, \infty)$.
(III) W is left continuous and has right limits on $(-\infty, 0]$.
(IV) W is a non-zero process.

Following [19], we set a probability measure Q on $\mathcal{W}$ such that $\{W(x), x \geq 0, Q\}$ and $\{W(-x), x \geq 0, Q\}$ are independent, identical in law and strictly semi-stable Lévy processes with index $\alpha \in (0, 2]$, which have the following semi-selfsimilarity:

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$$\{W(x), x \in \mathbb{R}\} \stackrel{d}{=} \{a^{-1/\alpha} W(ax), x \in \mathbb{R}\} \quad \text{for some } a > 0, \tag{1.1}$$

where $\stackrel{d}{=}$ denotes the equality in all joint distributions. We call a and $a^{1/\alpha}$ in (1.1) an epoch and a span, respectively, and set

$$r = \inf\{a > 1 : a \text{ satisfies } (1.1)\}. \tag{1.2}$$

In this paper, we call (W, Q) an (r, α)-semi-stable Lévy environment. We remark that the trivial process, that is $W(x) = cx$ almost surely for $x \geq 0$ with $c \neq 0$, is also a non-zero, strictly semi-stable Lévy process (Remark 13.17 in [12]). If $r = 1$, (W, Q) is not only semi-selfsimilar, but selfsimilar. In this case, we call (W, Q) an α-stable Lévy environment. If $\alpha = 2$, then $r = 1$ and (W, Q) is a Brownian environment (Theorem 14.1 in [12]). Refer [12] for more properties of semi-stable Lévy processes.

For a fixed W, we consider a d-dimensional diffusion process starting at 0, $X_W = \{X_W^k(t), t \geq 0, k = 1, \ldots, d\}$ whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W(x_k)\} \frac{\partial}{\partial x_k} \right\}. \tag{1.3}$$

We regard values of W at different d points as constituting a multi-parameter environment. X_W is constructed by d independent standard Brownian motions with a scale transformation and a time change induced by W (c.f. [9]). Each component of X_W is symbolically described by

$$dX_W^k(t) = dB^k(t) - \frac{1}{2} W'(X_W^k(t))dt, \quad X_W^k(0) = 0, \quad \text{for } k = 1, \ldots, d,$$

where $B^k(t)$ is a one-dimensional standard Brownian motion independent of the environment (W, Q).

In the case where $d = 1$ and (W, Q) is a Brownian environment, Brox showed that the distribution of $(\log t)^{-2} X_W(t)$ converges weakly as $t \to \infty$ in [1]. This shows that X_W moves very slowly by the effect of the environment. This diffusion process is a continuous model of random walks in random environments studied by Solomon [14] and Sinai [13], and X_W is often called a Brox-type diffusion. Following Brox's result, Tanaka studied the cases of α-stable Lévy environments and showed the convergence theorem with the scaling $(\log t)^{-\alpha} X_W(t)$ under the assumption that $Q\{W(1) > 0\} > 0$ in [19]. Tanaka's results were extended to the cases of (r, α)-semi-stable Lévy environments in [16].

In view of the subdiffusive property of the Brox-type diffusion, we expect to see an exotic limiting behavior for multi-dimensional Brox-type diffusions. We now give a brief review of investigations related to multi-dimensional Brox-type diffusions. Fukushima *et al.* showed the recurrence of the diffusion process whose generator is

$$\frac{1}{2} e^{W(|\boldsymbol{x}|)} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ e^{-W(|\boldsymbol{x}|)} \frac{\partial}{\partial x_k} \right\},$$

where $|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_d^2}$ and W is a one-dimensional standard Brownian motion in [6]. In the case where the environment is Lévy's Brownian motion $W(\boldsymbol{x})$ with a multi-dimensional time, Tanaka showed the recurrence of the diffusion process for almost all environments in any dimension in [20]. These results were shown by Ichihara's recurrent test introduced in [8]. Mathieu studied asymptotic behavior of multi-dimensional diffusion processes in random environments by using the Dirichlet form and showed the convergence theorem for the case where the environment is a non-negative reflected Lévy's Brownian motion in [11]. Following this study, Kim obtained several limit theorems of the multi-dimensional diffusion processes in [10]. He also showed the convergence theorem in the case where the random environment consists of d independent one-dimensional reflected non-negative Brownian motions, which is a model studied in [17].

Recently, Gantert *et al.* showed the recurrence of d independent random walks in random environments satisfying a uniform ellipticity condition for the environments. They used estimates of quenched return probabilities to the origin of the one-dimensional random walks in random environments in [7]. In [18], the multi-dimensional diffusion process consisting of d independent Brox-type diffusions was studied and the recurrence of this process for almost all environments in any dimension was shown. The multi-dimensional diffusion process is a continuous time analogue of the random walks in [7]. We remark that the recurrence property of the multi-dimensional diffusion process is derived without the uniform ellipticity condition for the environments.

Following the previous studies, we consider limiting behavior of diffusion processes in (r,α)-semi-stable Lévy environments to be given by (1.1) and (1.2), which are extensions of the models studied in [7] and [18]. We call an increasing (r,α)-semi-stable or α-stable Lévy process a subordinator. We obtain some conditions of the random environments which imply the dichotomy of recurrence and transience of d-dimensional diffusion processes corresponding to the generator (1.3) as follows.

Theorem 1.1. (i) *If $\{-W(x), x \geq 0, Q\}$ is not a subordinator, then X_W is recurrent for almost all environments in any dimension.*

(ii) *If $\{-W(x), x \geq 0, Q\}$ is a subordinator, then X_W is transient for almost all environments in any dimension.*

Next, we consider d-dimensional diffusion processes consisting of d independent Brox-type diffusions. Let Q_k be the probability measure on $\mathcal{W}$ such that

(i) $\{W_k(-x_k), x_k \geq 0, Q_k\}$ is an α_k-stable Lévy process,
(ii) $\{W_k(x_k), x_k \geq 0, Q_k\}$ is a β_k-stable Lévy process,
(iii) the Lévy processes on positive and negative sides are independent.

We define an environment $(\mathbf{W},\mathbf{Q})$ by $\{(W_k,Q_k), k=1,\dots,d\}$ with each (W_k,Q_k) being independent. We remark that Suzuki studied the case where $d=1$ with independent an α-stable and a β-stable Lévy environments, and obtained some convergence theorems in [15].

For a fixed $\mathbf{W}$, we consider a d-dimensional diffusion process starting at 0, $X_{\mathbf{W}} = \{X^{(k)}_{W_k}(t), t \geq 0, k=1,\dots,d\}$ whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W_k(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W_k(x_k)\} \frac{\partial}{\partial x_k} \right\}. \tag{1.4}$$

On the d-dimensional diffusion processes, we obtain the following dichotomy theorem.

Theorem 1.2. (i) *If neither $\{-W_k(-x_k), x_k \geq 0, Q_k\}$ nor $\{-W_k(x_k), x_k \geq 0, Q_k\}$ is a subordinator for any k, then $X_{\mathbf{W}}$ is recurrent for almost all environments in any dimension.*

(ii) *If either $\{-W_k(-x_k), x_k \geq 0, Q_k\}$ or $\{-W_k(x_k), x_k \geq 0, Q_k\}$ is a subordinator for some k, then $X_{\mathbf{W}}$ is transient for almost all environments in any dimension.*

2 Proofs of Theorems

2.1 Proof of Theorem 1.1

We can show that the generator (1.3) is equal to

$$\frac{1}{2} \exp\left\{ \sum_{j=1}^{d} W(x_j) \right\} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\left\{ -\sum_{j=1}^{d} W(x_j) \right\} \frac{\partial}{\partial x_k} \right\} \tag{2.1}$$

through a simple calculation. Since semi-stable or stable Lévy processes are bounded on finite time intervals almost surely (cf. Chapter VIII in [3]), the Dirichlet form

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}^d} (\nabla f \cdot \nabla g) \exp\left\{ -\sum_{j=1}^{d} W(x_j) \right\} dx,\ \ f,g \in C_0^\infty(\mathbb{R}^d)$$

is closable on $L^2\left(\mathbb{R}^d, \exp\left\{-\sum_{j=1}^d W(x_j)\right\} dx\right)$ and its closure (denoted by $\mathcal{E}$ again) is a local regular Dirichlet form (see Section 3 in [5]). Furthermore, $\mathcal{E}$ is either transient or recurrent, and the property is the same as that of the diffusion process corresponding to the generator (2.1). Hence, we study limiting behavior of the process Y_W corresponding to the generator

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\left\{ -\sum_{j=1}^{d} W(x_j) \right\} \frac{\partial}{\partial x_k} \right\}. \tag{2.2}$$

The following proposition about a property of (r,α)-semi-stable Lévy environments is crucial for showing the recurrence of Y_W.

Proposition 2.1. *If $Q\{W(1) > 0\} > 0$, then for any positive a_0 and any d*

$$Q\left\{ \min_{1\le\theta\le r} \left\{ \inf_{\sigma\in S^{d-1}} \sum_{j=1}^{d} W(\theta\sigma_j) \right\} > a_0 \right\} > 0, \tag{2.3}$$

where S^{d-1} is the surface of the d-dimensional unit ball centered at the origin.

Proof. If $\{W(x)\}$ is a trivial strictly semi-stable or stable Lévy process, then the assumption that $Q\{W(1) > 0\} > 0$ implies that there exists a positive constant c such that $W(x) = cx$ almost surely (cf. Remark 13.17 in [12]). Hence, we obtain (2.3) for any d.

We next consider non-trivial cases.

In the case where $d = 1$, since $\{W(-x), x \ge 0, Q\}$ is the independent copy of $\{W(x), x \ge 0, Q\}$, it is sufficient to show that for any $a_0 > 0$

$$Q\left\{ \inf_{1\le x\le r} W(x) > a_0 \right\} > 0. \tag{2.4}$$

If $W(x)$ is the α-stable Lévy environment, then $r = 1$. This and the property of the support of the α-stable Lévy process (cf. Theorem 24.10 in [12]) imply the assertion. If $W(x)$ is the (r,α)-semi-stable environment, then we also have that $Q\{W(1) > a_0 + 1\} > 0$. Using the local boundedness of Lévy processes, for any $u > 0$ and any $M > 0$ we find that

$$Q\left\{ \sup_{0\le x\le u} |W(x)| < M \right\} > 0, \tag{2.5}$$

which and the Markov property of $\{W(x)\}$ at $x = 1$ imply that

$$Q\left\{ \inf_{1\le x\le r} \{W(x) - W(1)\} > -1 \,\middle|\, W(1) > a_0 + 1 \right\} > 0. \tag{2.6}$$

Hence, we obtain (2.4).

In the case where $d \ge 2$, we set $\phi(x) = (a_0 + 2d)x^2 - 2$. As $\sum_{j=1}^d \sigma_j^2 = 1$, we find that $\sum_{j=1}^d \phi(\theta\sigma_j) \ge a_0$ for any $\theta \in [1, r]$. Hence, it is sufficient to show that

$$Q\left\{ \min_{1\le\theta\le r} \left\{ \inf_{\sigma\in S^{d-1}} \sum_{j=1}^{d} \{W(\theta\sigma_j) - \phi(\theta\sigma_j)\} \right\} > 0 \right\} > 0. \tag{2.7}$$

In the same manner as the case where $d = 1$, we consider the positive side and shall show that

$$Q\left\{\inf_{0\le x\le r}\{W(x) - \phi(x)\} > 0\right\} > 0. \tag{2.8}$$

For $W(x)$, we define the first exit time from the interval $[-1, \phi(r) + 1]$ by τ.

Let $0 < \alpha < 2$. Then, the distribution of $W(1)$ (denoted by μ) is strictly (r, α)-semi-stable or α-stable. Let $\nu(dx)$ be the Lévy measure of μ. We can write that $W = W_1 + W_2 + W_3$, where they are independent Lévy processes whose Lévy measures are restrictions of ν such that $\nu_1 := [\nu]_{\{x\ge\phi(r)+2\}}$, $\nu_2 := [\nu]_{\{x\le -1\}}$ and $\nu_3 := [\nu]_{\{-1<x<\phi(r)+2\}}$, respectively. For $k = 1$ and 2, we set the first jump time of W_k by τ_k and set the total mass of ν_k by J_k. We remark that the assumption implies that $J_1 > 0$. Then, we obtain that

$$Q\left\{\tau_1 < \frac{1}{\sqrt{a_0+2d}}\right\} = 1 - \exp\{-J_1/\sqrt{a_0+2d}\} > 0 \tag{2.9}$$

and

$$Q\left\{\tau_2 > \frac{1}{\sqrt{a_0+2d}}\right\} = \exp\{-J_2/\sqrt{a_0+2d}\} > 0. \tag{2.10}$$

Using the local boundedness of Lévy processes, we find that the probability that $W_3(x)$ does not exit from $[-1, \phi(r) + 1]$ with jumps induced by ν_3 until the time $1/\sqrt{a_0 + 2d}$ is positive. This positivity of the probability, (2.9) and (2.10) imply that

$$Q\left\{0 < \tau < \frac{1}{\sqrt{a_0+2d}} \text{ and } W(\tau) > \phi(r) + 1\right\} > 0. \tag{2.11}$$

As for $\alpha = 2$, $W(x)$ is a Brownian motion and (2.11) is obvious. We thus omit the proof.

Notice that $\phi(1/\sqrt{a_0 + 2d}) = -1$, which and (2.11) imply that

$$Q\left\{\inf_{0\le x\le\tau}\{W(x) - \phi(x)\} > 0 \text{ and } W(\tau) > \phi(r) + 1\right\} > 0. \tag{2.12}$$

The strong Markov property of $\{W(x)\}$, (2.12) and (2.5) imply that

$$Q\left\{\inf_{\tau\le x\le r}\{W(x) - W(\tau)\} > -1 \;\middle|\; \inf_{0\le x\le\tau}\{W(x) - \phi(x)\} > 0 \text{ and } W(\tau) > \phi(r) + 1\right\} > 0. \tag{2.13}$$

Hence, (2.12) and (2.13) imply (2.8). □

Proof of (i) of Theorem 1.1. In the case where $d = 1$, the scale function of the generator (2.2) is given by $\int_0^x \exp\{W(s)\}ds$. The assumption that $Q\{W(1) > 0\} > 0$ implies that $\limsup_{|s|\to\infty} W(s) = \infty$ almost surely. Hence, we obtain assertion (i) from a general theory of one-dimensional diffusion processes.

In the case where $d \ge 2$, we use Ichihara's recurrent test ([8] and [5]). It is sufficient to show that for almost all environments

$$\int_1^\infty s^{1-d}\left\{\int_{S^{d-1}} \exp\left\{-\sum_{j=1}^d W(s\sigma_j)\right\} d\sigma\right\}^{-1} ds = \infty, \tag{2.14}$$

where $d\sigma$ is the normalized uniform measure on S^{d-1}.

First, we consider the case of the (r,α)-semi-stable Lévy environment. For each $s \geq 1$, we can find $n \in \mathbb{Z}$ and $\theta \in [1,r)$ such that $s = r^n\theta$, which implies that the left-hand side of (2.14) is equal to

$$\sum_{n=0}^{\infty} r^{n(2-d)} \int_1^r \theta^{1-d} \left\{ \int_{S^{d-1}} \exp\left\{ -\sum_{j=1}^d W(r^n\theta\sigma_j) \right\} d\sigma \right\}^{-1} d\theta. \tag{2.15}$$

We set a measure preserving transformation for (W,Q) such that for $n = 0,1,2,\ldots$

$$T_nW(s) := r^{-n/\alpha}W(r^ns), \quad s \geq 1.$$

Then, the family of transformations $\{T_n\}$ is ergodic (for detail, see [4]). Setting

$$M(n) := \min_{1\leq\theta\leq r} \left\{ \inf_{\sigma\in S^{d-1}} \sum_{j=1}^d T_nW(\theta\sigma_j) \right\}, \tag{2.16}$$

we obtain that

$$\begin{aligned}
(2.15) &= \sum_{n=0}^{\infty} r^{n(2-d)} \int_1^r \theta^{1-d} \left\{ \int_{S^{d-1}} \exp\left\{ -r^{n/\alpha} \sum_{j=1}^d T_nW(\theta\sigma_j) \right\} d\sigma \right\}^{-1} d\theta \\
&\geq C_d^{-1} \sum_{n=0}^{\infty} r^{n(2-d)} \int_1^r \theta^{1-d} \exp\{r^{n/\alpha}M(n)\} d\theta \\
&\geq C_{d,r} \sum_{n=0}^{\infty} \exp\{n(2-d)\log r + r^{n/\alpha}M(n)\},
\end{aligned} \tag{2.17}$$

where C_d and $C_{d,r}$ denote the surface area of the unit sphere and a positive constant determined by d and r, respectively. We choose m_0 which satisfies that for any n

$$n(2-d)\log r + r^{n/\alpha}m_0 \geq 0.$$

Then,

$$(\text{the right-hand side of } (2.17)) \geq C_{d,r} \lim_{N\to\infty} N \frac{1}{N} \sum_{n=0}^{N} 1_{[m_0,\infty)}(M(n)). \tag{2.18}$$

Since $Q(W(1) > 0) > 0$, we can use Proposition 2.1. This and the ergodicity of $\{T_n\}$ imply that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} 1_{[m_0,\infty)}(M(n)) \\
&= E[1_{[m_0,\infty)}(M(0))] \\
&= Q\left\{ \min_{1\leq\theta\leq r} \left\{ \inf_{\sigma\in S^{d-1}} \sum_{j=1}^d W(\theta\sigma_j) \right\} > m_0 \right\} > 0 \qquad Q\text{-almost surely},
\end{aligned} \tag{2.19}$$

which and (2.18) show (2.14) for almost all environments.

Second, we consider the case of the α-stable Lévy environment, which is semi-selfsimilar for any $a > 0$ in (1.1). Fixing $\tilde{r} > 1$ as an epoch of the stable Lévy environment, we obtain that for any positive m_0 and any d

$$Q\left\{ \min_{1\leq\theta\leq\tilde{r}} \left\{ \inf_{\sigma\in S^{d-1}} \sum_{j=1}^d W(\theta\sigma_j) \right\} > m_0 \right\} > 0$$

in the same manner as that for showing Proposition 2.1. Instead of r in (2.15), we use $\tilde{r}$ and obtain (2.14) for almost all environments. $\square$

Proof of (ii) of Theorem 1.1. According to Ichihara's transient test introduced in [8], it is sufficient to show that

$$E\left[\int_1^\infty s^{1-d}\exp\left\{\sum_{j=1}^d W(s\sigma_j)\right\}ds\right]<\infty \tag{2.20}$$

on a subset of S^{d-1} with positive surface measure.

In the case where $d=1$, we show (2.20) for the subset $\{1\}\subset S^0$. As $\{-W(x),x\geq 0,Q\}$ is a subordinator, $\exp\{-W(s)\}\geq 1$ for any $s\geq 1$. By Theorem 24.11 in [12], we obtain that

$$\begin{aligned}&(\text{the left-hand side of } (2.20))\\&=\int_1^\infty E\left[\exp\{-(-W(s))\}\right]ds\\&=\int_1^\infty \exp\left\{s\left(\int_0^\infty(e^{-x}-1)\tilde{\nu}(dx)-\gamma_0\right)\right\}ds,\end{aligned} \tag{2.21}$$

where $\tilde{\nu}(dx)$ is the Lévy measure of the distribution of $-W(1)$ and γ_0 is the constant term of the drift of $\{-W(x)\}$.

If $\{-W(x)\}$ is a trivial strictly semi-stable or stable Lévy process, then $\gamma_0>0$ and $\tilde{\nu}\equiv 0$. We thus obtain (2.20).

We next consider non-trivial cases. Since $\{-W(x)\}$ is a strictly semi-stable Lévy process, Theorem 14.7 in [12] implies that $\gamma_0\equiv 0$. For the environment, we obtain that

$$\begin{aligned}&(\text{the right-hand side of } (2.21))\\&\leq\int_1^\infty\exp\left\{-s\int_1^\infty(1-e^{-x})\tilde{\nu}(dx)\right\}ds\\&\leq\int_1^\infty\exp\{-s(1-e^{-1})\tilde{\nu}((1,\infty))\}ds\\&<\infty.\end{aligned}$$

For the α-stable Lévy environment, we use semi-selfsimilarity of the environment and can show the assertion in the same way.

In the case where $d\geq 2$, for a fixed W the components of X_W are independent since X_W is constructed by d independent standard Brownian motion. Hence we obtain the transience in any dimension. □

2.2 Proof of Theorem 1.2

In the same reason as mentioned in the proof of Theorem 1.1, we study limiting behavior of the process $Y_{\mathbf{W}}(t)$ corresponding to the generator

$$\sum_{k=1}^d\frac{\partial}{\partial x_k}\left\{\exp\left\{-\sum_{j=1}^d W_j(x_j)\right\}\frac{\partial}{\partial x_k}\right\}$$

for $d\geq 2$.

Proof of (i) of Theorem 1.2. We set

$$\begin{aligned}&\boldsymbol{\epsilon}:=(\epsilon_j)_{j=1,\dots,d}\in\{-1,+1\}^d,\\&R_j=R_j(\epsilon_j):=\begin{cases}(-\infty,0]\,, & \epsilon_j=-1,\\ [0,\infty)\,, & \epsilon_j=+1,\end{cases}\\&\gamma_j=\gamma_j(\epsilon_j):=\begin{cases}\alpha_j, & \epsilon_j=-1,\\ \beta_j, & \epsilon_j=+1.\end{cases}\end{aligned}$$

For each $\boldsymbol{\epsilon}$, we have that for any $a > 0$

$$\{W_j(a^{\gamma_j} x_j), x_j \in R_j\} \stackrel{d}{=} \{aW_j(x_j), x_j \in R_j\}.$$

We also set $\mathbb{R}_{\boldsymbol{\epsilon}} := R_1 \times \cdots \times R_d \setminus \{(0,\ldots,0)\}$ and $S^{d-1}_{\boldsymbol{\epsilon}} := S^{d-1} \cap \mathbb{R}_{\boldsymbol{\epsilon}}$. On $\mathbb{R}_{\boldsymbol{\epsilon}}$, a point $\boldsymbol{x}$ and a pair $s > 0$ and $\sigma \in S^{d-1}_{\boldsymbol{\epsilon}}$ have one-to-one correspondence to each other such that $x_j = s^{\gamma_j}\sigma_j$. We thus denote a function of $\boldsymbol{x}$ by $s = s(\boldsymbol{x})$ which satisfies that

$$\sum_{j=1}^{d} \left(\frac{x_j}{s^{\gamma_j}}\right)^2 = 1.$$

We set

$$E_{\boldsymbol{\epsilon}}(s) := \int_{S^{d-1}_{\boldsymbol{\epsilon}}} \exp\left\{-\sum_{j=1}^{d} W_j(s^{\gamma_j}\sigma_j)\right\} \frac{\sum_{j=1}^{d}\left(\sigma_j/s^{\gamma_j-1}\right)^2}{\left(\sum_{j=1}^{d}\gamma_j\sigma_j^2\right)^2} d\sigma,\ s \geq 1,$$

$$J_{\boldsymbol{\epsilon}} := \int_1^{\infty} s^{1-(\gamma_1+\cdots+\gamma_d)} E_{\boldsymbol{\epsilon}}^{-1}(s)ds.$$

By arguments of Ichihara's recurrent test in [8] and [5], if $J_{\boldsymbol{\epsilon}} = \infty$ for any $\boldsymbol{\epsilon}$ and almost all environments, then we obtain the assertion. Setting $\hat{\gamma} := \min\{\alpha_j, \beta_j, j = 1,\ldots,d\}$, we have that

$$\begin{aligned} J_{\boldsymbol{\epsilon}} &\geq \hat{\gamma}^2 \int_1^{\infty} s^{1-(\gamma_1+\cdots+\gamma_d)} \frac{s^{2(\hat{\gamma}-1)}}{\int_{S^{d-1}_{\boldsymbol{\epsilon}}} \exp\left\{-\sum_{j=1}^{d} W_j(s^{\gamma_j}\sigma_j)\right\} d\sigma} ds \\ &\geq C_{d,\hat{\gamma}} \int_1^{\infty} s^{2\hat{\gamma}-(1+\gamma_1+\cdots+\gamma_d)} \exp\left\{\inf_{\sigma\in S^{d-1}_{\boldsymbol{\epsilon}}} \sum_{j=1}^{d} W_j(s^{\gamma_j}\sigma_j)\right\} ds, \end{aligned} \tag{2.22}$$

where $C_{d,\hat{\gamma}}$ denotes a positive constant determined by d and $\hat{\gamma}$. We set a measure preserving transformation for (W_j, Q_j) such that for all $t \in \mathbb{R}$

$$T_t^{(j)} W_j(\sigma_j) := e^{-t} W_j(e^{t\gamma_j}\sigma_j),\ \sigma \in S^{d-1}_{\boldsymbol{\epsilon}}.$$

A direct product of strong mixing transformations is strong mixing (cf. Theorem 2 in [2, p. 229]), which implies that $\{T_t^{(j)}\}$ is ergodic. We also set

$$M_{\boldsymbol{\epsilon}}(t) := \inf_{\sigma\in S^{d-1}_{\boldsymbol{\epsilon}}} \sum_{j=1}^{d} T_t^{(j)} W_j(\sigma_j).$$

For each component of **W**, the positive side and the negative side are independent. In addition, W_j's are also independent. These and (2.8) imply that for any $d \geq 2$

$$\mathbf{Q}\left\{\inf_{\sigma\in S^{d-1}} \sum_{j=1}^{d} \{W_j(\sigma_j) - \phi(\sigma_j)\} > 0\right\} > 0.$$

Hence, we obtain that for any $\boldsymbol{\epsilon}$, any $a_0 > 0$ and any $t \in \mathbb{R}$

$$\mathbf{Q}\{M_{\boldsymbol{\epsilon}}(t) > a_0\} > 0. \tag{2.23}$$

We choose $m_0 > 0$ which satisfies that for any $t \geq 0$

$$t\{2\hat{\gamma} - (\gamma_1 + \cdots + \gamma_d)\} + e^t m_0 \geq 0.$$

Then, we obtain that

$$\begin{aligned}
&(\text{the right-hand side of } (2.22)) \\
&= C_{d,\hat{\gamma}} \int_0^\infty \exp\left[t\{2\hat{\gamma} - (\gamma_1 + \cdots + \gamma_d)\} + e^t M_{\boldsymbol{\epsilon}}(t)\right] dt \\
&\geq C_{d,\hat{\gamma}} \lim_{N\to\infty} N \frac{1}{N} \int_0^N 1_{[m_0,\infty)}(M_{\boldsymbol{\epsilon}}(t))dt. \qquad (2.24)
\end{aligned}$$

The ergodicity of $\{T_t^{(j)}\}$ and (2.23) imply that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \int_0^N 1_{[m_0,\infty)}(M_{\boldsymbol{\epsilon}}(t))dt \\
&= E\left[1_{[m_0,\infty)}(M_{\boldsymbol{\epsilon}}(0))\right] \\
&= \mathbf{Q}\left\{M_{\boldsymbol{\epsilon}}(0) > m_0\right\} > 0 \qquad \mathbf{Q}\text{-almost surely.}
\end{aligned}$$

This positivity and (2.24) imply that $J_{\boldsymbol{\epsilon}} = \infty$ for any $\boldsymbol{\epsilon}$ and almost all environments. □

Proof of (ii) of Theorem 1.2. If $\{-W_{j_2}(x_{j_2}), x_{j_2} \geq 0, Q_{j_2}\}$ is a subordinator for some j_2 $(1 \leq j_2 \leq d)$, then the proof of (ii) of Theorem 1.1 implies that the j_2-th component of $X_{\mathbf{W}}$ is transient for almost all environments. If $\{-W_{j_2}(-x_{j_2}), x_{j_2} \geq 0, Q_{j_2}\}$ is a subordinator, then for the j_2-th component of $X_{\mathbf{W}}$ we can also show the transience property for almost all environments by showing (2.20) for the subset $\{-1\} \subset S^0$. Hence, at least one component of $X_{\mathbf{W}}$ is transient for almost all environments, which implies assertion (ii). □

Remark 2.2. Even in the case where $(\mathbf{W}, \mathbf{Q})$ contains semi-stable Lévy environments instead of some of the stable Lévy environments, the both assertions of Theorem 1.2 hold. The proofs are almost same as above. To prove (i) we need to take a common span for all the semi-stable Lévy environments.

3 Transience of a two-dimensional diffusion process

In [7], Gantert *et al.* considered a two-dimensional random walk $\{X_n, Y_n, n \in \mathbb{N}\}$ such that $\{X_n\}$ is a random walk in the random environment studied by Sinai [13], $\{Y_n\}$ is a centered random walk that converges weakly to a strictly α-stable Lévy process with $\alpha \in (1, 2]$ under a suitable scaling, and they are independent. They showed the recurrence of the random walk for almost all environments. In [19], Tanaka studied a one-dimensional diffusion process in a reflected non-positive Brownian environment $(-|W|, Q)$ and showed the convergence theorem with the same scaling as that of the diffusion process studied by Brox [1]. Denoting Tanaka's diffusion process by $\{X_{-|W|}(t)\}$, we consider a direct product of $\{X_{-|W|}(t)\}$ and a one-dimensional standard Brownian motion (denoting by $\{B^2(t)\}$). For the two-dimensional diffusion process, we obtain the transience property.

Proposition 3.1. *We assume that $\{X_{-|W|}(t)\}$ and $\{B^2(t)\}$ are independent. Then, the two-dimensional diffusion process $\{X_{-|W|}(t), B^2(t)\}$ is transient for almost all environments.*

Proof. By applying Ichihara's transient test in [8] to the generator

$$\sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left\{ \exp\{|W(x_1)|\} \frac{\partial}{\partial x_k} \right\},$$

it is sufficient to show that for almost all environments there exists $I \subset [0,1]$ such that $|I| > 0$ and

$$\int_1^\infty \frac{1}{s} \exp\{-|W(su)|\} ds < \infty, \quad u \in I. \tag{3.1}$$

We first show that for a fixed $\epsilon \in (0,1)$ there exists $I \subset [\epsilon, 1]$ such that $|I| > 0$ and for any $u \in I$

$$E\left[\int_1^\infty \frac{1}{s} \exp\left\{-|W(su)|\right\} ds\right] < \infty. \tag{3.2}$$

Using the inequality

$$\int_u^\infty e^{-z^2/2} dz \le \frac{1}{u} e^{-u^2/2} \quad \text{for any } u > 0,$$

we obtain that for any $u \in [\epsilon, 1]$

$$\begin{aligned}
&(\text{the left-hand side of } (3.2)) \\
&= \int_1^\infty \frac{1}{s} E[\exp\{-\sqrt{su}|W(1)|\}] ds \\
&= \int_1^\infty \frac{1}{s} \left\{ \sqrt{\frac{2}{\pi}} e^{su/2} \int_{\sqrt{su}}^\infty e^{-z^2/2} dz \right\} ds \\
&\le \sqrt{\frac{2}{\pi}} \int_1^\infty \frac{1}{s} e^{su/2} \frac{1}{\sqrt{su}} e^{-su/2} ds \\
&\le \sqrt{\frac{2}{\pi\epsilon}} \int_1^\infty s^{-3/2} ds \ = \frac{2\sqrt{2}}{\sqrt{\pi\epsilon}}.
\end{aligned} \tag{3.3}$$

We next show that the exceptional set of probability zero does not depend on u. Fubini's theorem and the inequality (3.3) imply that

$$\begin{aligned}
&E\left[\int_\epsilon^1 \left(\int_1^\infty \frac{1}{s} \exp\left\{-|W(su)|\right\} ds\right) du\right] \\
&= \int_\epsilon^1 E\left[\int_1^\infty \frac{1}{s} \exp\left\{-|W(su)|\right\} ds\right] du \ \le \frac{2\sqrt{2}}{\sqrt{\pi\epsilon}}(1-\epsilon).
\end{aligned}$$

Hence, we obtain that for almost all environments and a fixed $\epsilon \in (0,1)$

$$\int_\epsilon^1 \left(\int_1^\infty \frac{1}{s} \exp\left\{-|W(su)|\right\} ds\right) du < \infty,$$

which implies (3.1). $\square$

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