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THE FEYNMAN–KAC FORMULA AND HARNACK INEQUALITY FOR DEGENERATE DIFFUSIONS

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We study various probabilistic and analytical properties of a class of degenerate diffusion operators arising in population genetics, the so-called generalized Kimura diffusion operators Epstein and Mazzeo [*SIAM J. Math. Anal.* **42** (2010) 568–608; *Degenerate Diffusion Operators Arising in Population Biology* (2013) Princeton University Press; *Applied Mathematics Research Express* (2016)]. Our main results are a stochastic representation of weak solutions to a degenerate parabolic equation with singular lower-order coefficients and the proof of the scale-invariant Harnack inequality for nonnegative solutions to the Kimura parabolic equation. The stochastic representation of solutions that we establish is a considerable generalization of the classical results on Feynman–Kac formulas concerning the assumptions on the degeneracy of the diffusion matrix, the boundedness of the drift coefficients and the a priori regularity of the weak solutions.

1. Introduction. Generalized Kimura diffusion operators are a class of degenerate elliptic operators arising in Population Genetics as the infinitesimal generators of continuous limits of Markov chains Kimura (1957, 1964); Shimakura (1981); Ethier and Kurtz (1986); Karlin and Taylor (1981). A thorough study of the parabolic equations defined by generalized Kimura operators was initiated by C. Epstein and R. Mazzeo in Epstein and Mazzeo (2010, 2013), where the authors construct anisotropic Hölder spaces to prove existence, uniqueness and optimal regularity of solutions to the parabolic Kimura equation. In general, Kimura operators act on functions defined on manifolds with corners [Epstein and Mazzeo (2013), Section 2.1]. In adapted local coordinates, $z = (x, y) \in S_{n,m}$, the general-

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ized Kimura operator takes the form

$$\begin{aligned}\widehat{L}u = {} & \sum_{i=1}^{n}\big(x_i\widehat{a}_{ii}(x,y)u_{x_ix_i}+\widehat{b}_i(x,y)u_{x_i}\big)+\sum_{i,j=1}^{n}x_ix_j\bar{a}_{ij}(x,y)u_{x_ix_j}\\ & +\sum_{i=1}^{n}\sum_{l=1}^{m}x_i\widehat{c}_{il}(x,y)u_{x_iy_l}+\sum_{k,l=1}^{m}\widehat{d}_{kl}(x,y)u_{y_ky_l}+\sum_{l=1}^{m}\widehat{e}_l(x,y)u_{y_l},\end{aligned} \tag{1.1}$$

in a neighborhood of $(0,0)$, for $u \in C^2(S_{n,m})$. We let $S_{n,m} := \mathbb{R}_+^n \times \mathbb{R}^m$, $\mathbb{R}_+ := (0,\infty)$, with n and m nonnegative integers such that $n+m \geq 1$. The hypersurface boundary components of $S_{n,m}$ are defined by $\{x_i = 0\}$, for $i = 1,\dots,n$.

With $\widehat{x}_i = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, we denote the restriction

$$\beta_i(\widehat{x}_i, y) = \widehat{b}_i(x,y)|_{\{x_i=0\}}.$$

In normalized coordinates $a_{ii} = 1$, for all i. In these coordinates, the functions, $\{\beta_i\}$, are invariantly defined on $\partial S_{n,m}$, and are called the *weights* of the Kimura operator $\widehat{L}$. In applications to Genetics, a weight $\beta_i(\widehat{x}_i, y)$ is the aggregate mutation rate into the type i when $x_i = 0$ and the remaining types have frequencies $(\widehat{x}_i, y)$. If these weights are all constant, then the analysis is considerably simpler. Such operators have been examined in many important cases in Shimakura (1981). This is not a natural hypothesis, as it implies that the rates of mutation *into* a given type depend only on the target and not on the source.

In Epstein and Mazzeo (2016), an operator L closely related to $\widehat{L}$ is introduced, which is defined by a Dirichlet form. If the weights are all strictly positive, then the measure appearing in the Dirichlet form is a doubling measure, and one can establish that the Dirichlet form satisfies a scale-invariant L^2-Poincaré inequality. Using techniques of Moser as elaborated by Saloff-Coste, Grigor'yan and Sturm, one can then establish that nonnegative local solutions to $u_t - Lu = 0$ satisfy a Harnack inequality, as well as pointwise estimates for the heat kernel. The difference $L - \widehat{L}$ is a vector field tangent to the boundary, which is, in a precise sense, a lower order perturbation; see Section 5.2. If the weight $\beta_i(\widehat{x}_i, y)$ is nonconstant along the boundary component $\{x_i = 0\}$, then the coefficients of this vector field blow-up like $\ln x_i$. This explains the origin of the class of operators, defined in (1.10), whose study we continue in this paper. To illustrate the structure of the operator $\widehat{L}$, we begin with an example of the Wright–Fisher model for gene frequencies, which leads to a particular example of a Kimura diffusion.

EXAMPLE 1.1 (A Wright–Fisher model [Karlin and Taylor (1981), Chapter 15, Problem 33]). We consider a Wright–Fisher model for the frequencies of 3 types with mutation and no selection. The stochastic differential equation that describes the dynamics of the frequencies of 2 of the types, $\{X(t), Y(t)\}_{t\geq 0}$, is given by

$$dX(t) = \big(\alpha - (\alpha+\gamma)X(t)\big)\,dt + \sqrt{X(t)\big(1-X(t)\big)}\,dW_1(t),$$

$$dY(t) = \big(\beta - (\beta+\gamma)Y(t)\big)\,dt + \sqrt{Y(t)\big(1-Y(t)\big)}\,dW_2(t),$$

where α, β, γ are nonnegative constants determined by the mutation rates, and $\{W_i(t)\}_{t\geq 0}$, for $i = 1, 2$, are one-dimensional Brownian motions with correlation coefficient,

$$dW_1(t) \cdot dW_2(t) = \sqrt{X(t)Y(t)}/\sqrt{\big(1 - X(t)\big)\big(1 - Y(t)\big)}\,dt.$$

Because the drift coefficients of the processes $\{X(t)\}_{t\geq 0}$ and $\{Y(t)\}_{t\geq 0}$ are zero or point inside of the interval [0, 1] at the endpoints 0 and 1, where the variance of the processes vanishes, standard properties of one-dimensional diffusions imply that, almost surely,

$$0 \leq X(t), Y(t) \leq 1, \qquad \forall t \geq 0.$$

Moreover, an application of Itô's rule gives us that the process $Z(t) := X(t) + Y(t)$ satisfies

$$dZ(t) = \big(\alpha + \beta - (\alpha + \gamma)X(t) - (\beta + \gamma)Y(t)\big)\,dt + \sqrt{Z(t)\big(1 - Z(t)\big)}\,dW_3(t),$$

where $\{W_3(t)\}_{t\geq 0}$ is a one-dimensional Brownian motion defined by $\{W_1, W_2\}$. The same argument as above gives us that $0 \leq Z(t) \leq 1$, a.s. for all $t \geq 0$. It is now clear that we can model the frequency of the third type by the process $1 - X(t) - Y(t)$, and to understand this model, it is sufficient to characterize the dynamics of the process $\{X(t), Y(t)\}_{t\geq 0}$, whose state space is the two-dimensional simplex,

$$\Sigma_2 := \big\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y; x + y \leq 1\big\},$$

which is an example of a compact manifold with corners.

To understand the dynamics of the type frequency process, we study the infinitesimal generator of the process, which is given by

$$\begin{aligned}\widehat{L}u &= \frac{1}{2}x(1 - x)u_{xx} - xyu_{xy} + \frac{1}{2}y(1 - y)u_{yy} \\ &\quad + \big[\alpha - (\alpha + \gamma)x\big]u_x + \big[\beta - (\beta + \gamma)y\big]u_y, \qquad \forall (x, y) \in \Sigma_2.\end{aligned} \tag{1.2}$$

The study of the operator $\widehat{L}$ is nonstandard only in a neighborhood of the boundary of the domain. To facilitate this analysis, depending on the point that we fix on the boundary of Σ_2, we choose suitable adapted local coordinates to write the operator $\widehat{L}$ in the general form described in (1.1). For example, the form $\widehat{L}$ given in equation (1.2) is used in a neighborhood of the point (0, 0). In a neighborhood of the corner point (0, 1), the operator $\widehat{L}$ given by (1.2) does not obviously have the form (1.1), but a change of local coordinates, $(x', y') = (1 - (x + y), y)$, allows us to rewrite $\widehat{L}$ as

$$\begin{aligned}\widehat{L}'v &= \frac{1}{2}x'(1 - x')v_{x'x'} - x'y'v_{x'y'} \\ &\quad + \frac{1}{2}y'(1 - y')v_{y'y'} + \big[\beta - (\beta + \gamma)y'\big]v_{y'} \\ &\quad + \big[\beta + \gamma + (\alpha + \gamma)x' + (\alpha + \beta - 2\gamma)y'\big]v_{x'}, \qquad \forall (x', y') \in \Sigma_2,\end{aligned} \tag{1.3}$$

which is clearly of the more general form (1.1).

The weights of $\widehat{L}$, defined in (1.2), along $\{x=0\}$ and $\{y=0\}$ are the constants α and β, respectively. Along $x'=0$, the weight $\beta+\gamma+(\alpha+\beta-2\gamma)y'$ is nonconstant. It is the nonconstancy of weights that leads to the log-terms appearing below in the definition of generalized Kimura operators. If $\alpha, \beta, \beta+\gamma, \alpha+2\beta-\gamma$ are positive, then results in Epstein and Mazzeo (2013, 2016) imply that the process has a unique stationary measure, ν, which can be written

$$\nu = w(x,y)x^{\alpha-1}y^{\beta-1}(1-x-y)^{\beta+\gamma+(\alpha+\beta-2\gamma)y}\,dx\,dy, \tag{1.4}$$

for a bounded, nonnegative function $w(x,y)$.

In neighborhoods of the corner points $(0,0)$ and $(0,1)$, the operators $\widehat{L}$ given by (1.2), and $\widehat{L}'$ given by (1.3) can be extended to the space $S_{n,m}$, with $n=2$ and $m=0$. When choosing a boundary point of the form $(0,1/2)$, it is not difficult to see that the operator $\widehat{L}$ in (1.2) can be written in the form (1.1), and then extended to $S_{n,m}$, but now with $n=m=1$. Since the focus of this article is on local properties of solutions, it is natural to study Kimura operators (1.1) defined on the extended space $S_{n,m}$.

The main feature of the operator $\widehat{L}$ is that it is not strictly elliptic as we approach the boundary of the domain $S_{n,m}$, and not self-adjoint with respect to any obvious choice of measure. Hence, standard methods do not apply to understand the regularity properties of solutions to equations defined by $\widehat{L}$. The coefficient matrix corresponding to the second-order derivatives of the operator $\widehat{L}$ degenerates because the smallest eigenvalue of the second-order coefficient matrix tends to 0 at a rate proportional to the distance to the boundary of the domain. For this reason, the signs of the coefficient functions $\widehat{b}_i(z)$ along $\partial S_{n,m}$ play a crucial role in the regularity of solutions. In this article, we always assume that, for $i=1,\dots,n$, the drift coefficient $\widehat{b}_i(z)$ is a strictly positive function along $\{x_i=0\}\subset\partial S_{n,m}$. The precise technical conditions imposed on the coefficients of the operator $\widehat{L}$ are described in Assumption 5.1.

The initial motivation of our article was to prove that nonnegative solutions to the parabolic equation defined by generalized Kimura operators,

$$u_t - \widehat{L}u = 0 \qquad \text{on } (0,\infty)\times S_{n,m}, \tag{1.5}$$

satisfy a scale-invariant Harnack inequality.

THEOREM 1.2 (A scale-invariant Harnack inequality). *Suppose that Assumption 5.1 holds and let $c\in(\sqrt{2/3},1)$. Then there are positive constants, α, β, γ and $H=H(\bar{b},\delta,K,K_0,\Lambda,T)$, such that $\alpha>\beta$ and the following hold. Let $\Omega\subseteq S_{n,m}$ be an open set and $(t_1,t_2)\subset\mathbb{R}_+$. Let $Q:=(t_1,t_2)\times\Omega$. Assume that u is a nonnegative, continuous probabilistic solution to the equation $u_t-\widehat{L}u=0$ on Q, in the sense of Definition 5.5. Then for all $(s,z)\in Q$ and all $R>0$ such that*

$$\bigl(s-4R^2, s+R^2\bigr)\times B_{4R}(z)\subset Q, \tag{1.6}$$

we have that

$$\sup_{Q_\rho^-(s,z)} u \le H \inf_{Q_\rho^+(s,z)} u \qquad \forall \rho \in (0, cR), \tag{1.7}$$

where we let

$$Q_\rho^-(s,z) := \left(s-\alpha\rho^2, s-\beta\rho^2\right) \times B_\rho(z), \tag{1.8}$$

$$Q_\rho^+(s,z) := \left(s, s+\gamma\rho^2\right) \times B_\rho(z). \tag{1.9}$$

A different proof of Theorem 1.2, based on Moser's iteration method Moser (1961, 1964), is given in Epstein and Mazzeo (2016). The main difference between our method of proof and that of Epstein and Mazzeo (2016) is that we use a probabilistic approach due to K.-T. Sturm (1994), in which we view the operator $\widehat{L}$ as a lower-order perturbation of the *self-adjoint* generalized Kimura operator, L, defined by

$$\begin{aligned}
Lu = {} & \sum_{i=1}^n (x_i a_{ii} u_{x_i x_i} + b_i a_{ii} u_{x_i}) + \sum_{i,j=1}^n x_i x_j \tilde{a}_{ij} u_{x_i x_j} \\
& + \sum_{i=1}^n \sum_{l=1}^m 2 x_i c_{il} u_{x_i y_l} + \sum_{k,l=1}^m d_{lk} u_{y_l y_k} \\
& + \sum_{i=1}^n x_i \left(\partial_{x_i} a_{ii} + \sum_{j=1}^n (\tilde{a}_{ij} + \delta_{ij}\tilde{a}_{ii} + x_j \partial_{x_j} \tilde{a}_{ij} + \tilde{a}_{ij}(b_j - 1)) \right. \\
& \left. + \sum_{l=1}^m \partial_{y_l} c_{il} \right) u_{x_i} \\
& + \sum_{i=1}^n x_i \left[\sum_{j=1}^n \left(\partial_{x_i} b_j + \sum_{k=1}^n x_k \tilde{a}_{ik} \partial_{x_k} b_j + \sum_{l=1}^m c_{il} \partial_{y_l} b_j \right) \ln x_j \right] u_{x_i} \\
& + \sum_{l=1}^m \left[\sum_{i=1}^n (x_i \partial_{x_i} c_{il} + b_i c_{il}) \right. \\
& \left. + \sum_{k=1}^m \partial_{y_k} d_{lk} + \sum_{j=1}^n \left(\sum_{i=1}^n x_i c_{il} \partial_{x_i} b_j + \sum_{k=1}^m d_{lk} \partial_{y_k} b_j \right) \ln x_j \right] u_{y_l},
\end{aligned} \tag{1.10}$$

where $u \in C^2(S_{n,m})$ and δ_{ij} denotes the Kronecker delta symbol. The precise relationnship between the operators L and $\widehat{L}$ is described in §5.2. The difference $L - \widehat{L}$ is a vector field tangent to the boundary, with possibly singular coefficients. The operator L is defined by a symmetric Dirichlet form [see (2.1)]; if the weights are nonconstant, then log-terms arise from integration by parts.

While the results in Epstein and Mazzeo (2016) apply both to the operators L and $\widehat{L}$, in our work we appeal only to those results concerning the self-adjoint operator L, and *not* $\widehat{L}$. We believe that this observation is important because this suggests that it may be possible to generalize our approach to more general operators $\widehat{A}$, which can be written as a sum of a self-adjoint operator A and a lower-order, possibly singular, perturbation E, and exploit the fact that properties of self-adjoint operators are, generally speaking, easier to derive than those for nonself-adjoint operators.

We next comment on the main difficulty in adapting Sturm's approach to our framework in order to motivate two further results in our article, which are described in Theorems 1.3 and 1.6. The probabilistic proof of the Harnack inequality in Theorem 1.2 requires that we know that the solutions to the parabolic problems defined by the operators L and $\widehat{L}$ have a stochastic representation. This is particularly problematic for the generalized Kimura operator L, because as proven in Section 2.2, the weak solutions to the problem (2.10) only belong to a suitable weighted H^1 space. In particular, we do not know the regularity properties of the second-order derivatives of the solutions up to the boundary of the domain $S_{n,m}$. Their properties are not easy to derive as we can see that the operator L contains singular logarithmic terms. Thus, it is not possible to directly apply Itô's rule to establish the stochastic representation of solutions. Instead, we take a different approach, which circumvents the application of Itô's rule and uses a combination of probabilistic and analytic arguments described in Section 4.

To give the statement of the stochastic representation of solutions, we first need to define the generalized Kimura stochastic differential equation associated to the operator L:

$$\begin{aligned} dX_i(t) &= \left(g_i\big(Z(t)\big) + X_i(t)\sum_{j=1}^{n} f_{ij}\big(Z(t)\big)\ln X_j(t)\right)dt \\ &\quad + \sqrt{X_i(t)}\sum_{j=1}^{n+m}\sigma_{ij}\big(Z(t)\big)\,dW_j(t), \\ dY_l(t) &= \left(e_l\big(Z(t)\big) + \sum_{j=1}^{n} f_{l+n,j}\big(Z(t)\big)\ln X_j(t)\right)dt \\ &\quad + \sum_{j=1}^{n+m}\sigma_{l+n,j}\big(Z(t)\big)\,dW_j(t), \end{aligned} \tag{1.11}$$

for all $i = 1, \dots, n$ and $l = 1, \dots, m$, where $\{W(t)\}_{t\geq 0}$ is a $(n+m)$-dimensional Brownian motion, and we denote $Z = (X, Y)$. The relation between the generalized Kimura operator L, defined in (1.10), and the generalized Kimura stochastic differential equation (1.11) is described in Section 4.1. The existence and uniqueness of weak Markov solutions to the generalized Kimura equation (1.11) is established in Pop (2017a), Theorems 3.1 and 3.7.

Let $\Omega \subseteq S_{n,m}$ and denote $\underline{\Omega} := \Omega \cup (\partial\Omega \cap \partial S_{n,m})$. We also need to introduce the weighted Sobolev space $L^2(\underline{\Omega}; d\mu)$, which consists of measurable functions, $u : \underline{\Omega} \to \mathbb{R}$, that are L^2-integrable with respect to the measure $d\mu(z)$, defined by

$$d\mu(z) := \left(\prod_{i=1}^{n} x_i^{b_i(z)-1}\right) dz, \qquad \forall z = (x, y) \in S_{n,m}. \tag{1.12}$$

If the weights of L (or $\widehat{L}$) are positive, then the unique stationary measure for the associated process is locally of the form $w(z)\, d\mu(z)$, with w a bounded function; see (1.4). If $\widehat{L}$ is a standard Kimura diffusion, with weights $\{\widehat{b}_i(x, y)\}$ that are nonconstant along some boundary components, then the natural representation of the dual operator $\widehat{L}^t$ acting on measures of the form $f(z)\, d\mu(z)$ includes a first-order tangent vector field with logarithmically divergent coefficients. This provides a second reason why it is natural to consider operators with singular coefficients, such as those appearing in definition (1.10).

We establish the stochastic representation of weak solutions.

THEOREM 1.3 (Stochastic representation of weak solutions). *Suppose that the coefficients of the operator L satisfy Assumption 2.1. Let $\Omega \subseteq S_{n,m}$ be an open set. Given any function, $f \in L^2(\underline{\Omega}; d\mu)$, there is a set, $N \subset \underline{\Omega}$, of zero μ-measure, such that*

$$u(t, z) = \mathbb{E}_{\mathbb{P}^z}\left[f\big(Z(t)\big)\mathbf{1}_{\{t<\tau_\Omega\}}\right], \qquad \forall t \geq 0, \forall z \in \underline{\Omega} \backslash N, \tag{1.13}$$

where τ_Ω is defined by

$$\tau_\Omega := \inf\{t > 0 : Z(t) \notin \underline{\Omega}\}. \tag{1.14}$$

$\mathbb{P}^z$ is the probability distribution of the unique weak Markov solution, $\{Z(t)\}_{t\geq 0}$, to the Kimura equation (1.11), with initial condition $Z(0) = z$, and u is the unique weak solution to the initial-value problem,

$$\begin{aligned} u_t - Lu &= 0 \qquad \text{on } (0, \infty) \times \Omega, \\ u &= f \qquad \text{on } \{0\} \times \Omega. \end{aligned} \tag{1.15}$$

REMARK 1.4. In Section 2.2, we prove that the solutions to the initial-value problem (1.15) define a strongly continuous, contraction semigroup, $\{T_t^{\Omega}\}_{t\geq 0}$, on the weighted Sobolev space $L^2(\underline{\Omega}; d\mu)$. In other words, for all $f \in L^2(\underline{\Omega}; d\mu)$, the solution u to (1.15) can be represented in the form $u(t) = T_t^{\Omega} f$. Thus, the stochastic representation formula (1.13) is equivalent to

$$T_t^{\Omega} f(z) = \mathbb{E}_{\mathbb{P}^z}\left[f\big(Z(t)\big)\mathbf{1}_{\{t<\tau_\Omega\}}\right], \qquad \forall t \geq 0, \forall z \in \underline{\Omega} \backslash N. \tag{1.16}$$

REMARK 1.5. With more effort it can be shown that the set, N, of zero μ-measure appearing in the statement of Theorem 1.3 can be chosen to be the empty set; see Epstein and Pop (2014), Theorem 1.3, and Remark 4.5.

Our next result is a probabilistic reformulation of the Harnack inequality Epstein and Mazzeo (2016), Theorem 4.1, for nonnegative solutions to the parabolic equation defined by the operator L. The main difference between Epstein and Mazzeo (2016), Theorem 4.1, and our Theorem 1.6 is that, in the former result, the Harnack inequality (1.19) applies to functions belonging to suitable weighted H^1 Sobolev spaces (as defined in Section 2), while in the latter result, (1.19) applies to functions satisfying the stochastic representation (1.18). Thus, the a priori regularity properties of the functions $u(t, z)$ defined by (1.18) are not obvious. Along with Theorem 1.3, Theorem 1.6 is a key ingredient in our probabilistic proof of the Harnack inequality in Theorem 1.2. To state Theorem 1.6, we need the following notation. Let $Q := (t_1, t_2) \times \Omega$, and let

$$\eth Q := \big([t_1, t_2] \times \partial_1 \Omega\big) \cup \big(\{t_1\} \times \Omega\big), \tag{1.17}$$

where $\partial_1 \Omega := \partial\Omega \cap S_{n,m}$. We can now state the following.

THEOREM 1.6 (Harnack inequality for solutions defined by a stochastic representation). *Suppose that the coefficients of the operator L satisfy Assumption 2.1. There is a positive constant, K_0, such that the following hold. Let $g \in C(\overline{\eth Q})$ be a nonnegative function, and let u be the function defined by the stochastic representation*

$$u(t, z) := \mathbb{E}_{\mathbb{P}^z}\big[g\big(t - (t - t_1) \wedge \tau_\Omega\big), Z\big((t - t_1) \wedge \tau_\Omega\big)\big)\big] \qquad \forall (t, z) \in \bar{Q}, \tag{1.18}$$

where $\{Z(t)\}_{t\geq 0}$ is the unique weak Markov solution to the generalized Kimura stochastic differential equation (1.11), with initial condition $Z(0) = z$, and τ_Ω is as in (1.14). Then the function u satisfies the scale-invariant Harnack inequality, that is, for all $(t^0, z^0) \in \bar{Q}$ and $r > 0$, such that $Q_{2r}(t^0, z^0) \subset Q$, we have that

$$\operatorname*{ess\,sup}_{Q_r(t^0 - 2r^2, z^0)} u \leq K_0 \operatorname*{ess\,inf}_{Q_r(t^0, z^0)} u, \tag{1.19}$$

where the parabolic cylinder $Q_r(t^0, z^0)$ is defined in (4.16).

Our results, Theorems 1.3 and 1.6, are a considerable generalization of similar results concerning stochastic representation of solutions. In our article, the stochastic representation (1.13) holds for weak solutions u, with the property that u and u_{x_i} belong to a suitable weighted Sobolev space, and u_t is understood in a distributional sense (as described in Section 2.2). In particular, we do not have information about the regularity of the second-order derivatives $u_{x_i x_j}$, and so the standard method of applying Itô's rule to a sequence of regularized solutions to prove a stochastic representation formula is not applicable in our framework. The results known to us in the literature apply to functions having $C^{1,2}$ regularity [Karatzas and Shreve (1991), Theorem 5.7.6; Friedman (1975/1976), Theorem 6.5.1; Feehan and Pop (2015), Theorems 1.14, 1.15 and 1.17], or $W^{1,2}_p$ regularity [Sturm (1994), Theorem 4; Bensoussan and Lions (1982), Theorems 2.7.3 and 2.7.4; Portenko (1976), Section II.3], that is u, u_t, u_{x_i} and $u_{x_i x_j}$ are continuous functions, or they belong to a L^p space. Moreover, the operators considered in the aforementioned articles are strictly elliptic and the lower-order terms are bounded, measurable functions, while our operator L is degenerate and the lower-order terms have logarithmic singularities. The main advance in Theorem 1.3, over known results, is in the relaxation of the regularity assumptions of the solutions for which we establish the stochastic representation (1.13).

Stochastic representation of solutions defined by semigroups can also be derived in terms of Hunt processes [Fukushima, Oshima and Takeda (2011), Theorems 7.2.1 and 7.2.2], as opposed to solutions to stochastic differential equations. In the application of Theorem 1.3 to the proof of the Harnack inequality, we need to have the stochastic representation (1.13) in terms of the solutions to the stochastic differential equation. It is a nontrivial problem to prove that the Hunt process arising in the stochastic representation implied by Fukushima, Oshima and Takeda (2011), Theorem 7.2.1, is also a solution to a stochastic differential equation. In our proof of Theorem 1.3, given in Section 4.2 and Section 4.3, we establish that the Hunt process associated to the semigroup of solutions $u(t) = T_t^{\Omega} f$, where $u(0) = f$, given by Fukushima, Oshima and Takeda (2011), Theorem 7.2.1, has the same law as the unique weak Markov solution to the generalized Kimura equation (1.11), stopped upon leaving the region $\underline{\Omega}$. We are not aware of a reference to analogous results in the literature.

1.1. *Outline of the article.* The rest of our article is organized as follows. Our main results concerning the stochastic representation of solutions, Theorems 1.3 and 1.6, are proved in Section 4, while the Harnack inequality stated in Theorem 1.2 is proved in Section 5. Sections 2 and 3 contain results that lead to the proofs of our main results.

In Section 2, we introduce the parabolic problem defined by the generalized Kimura operator L on sub-domains Ω of $S_{n,m}$ and the notion of weak solution in Definition 2.2. We prove the existence and uniqueness of weak solutions in suitable weighted Sobolev spaces and we show that the weak solutions to the homogeneous initial-value problem (1.15) generate a strongly continuous, contraction semigroup, $\{T_t^{\Omega}\}_{t\geq 0}$. We also discuss the Dirichlet form associated to the generalized Kimura operator L and the semigroup $\{T_t^{\Omega}\}_{t\geq 0}$. Concepts from the theory of Dirichlet forms are used in Section 4.3 to give the proof of Theorem 1.3.

In Section 3, we study the properties of the fundamental solution for the semigroup $\{T_t\}_{t\geq 0}$, which play a key role in Section 4 to give the proofs of Theorems 1.3 and 1.6. Section 4 is organized into several parts. In Section 4.1, we describe the relation between the generalized Kimura operator L, defined in (1.10), and the generalized Kimura stochastic differential equation (1.11). In Section 4.2, we prove Theorem 1.3 in the particular case when $\Omega = S_{n,m}$. Because Theorem 1.3 cannot be obtained by a direct application of Itô's rule, due to the lack of regularity of the solutions $u(t) = T_t^{\Omega} f$, we adopt a different approach. We first establish the stochastic representation (1.13) by replacing $\{Z(t)\}_{t\geq 0}$ by a suitable Hunt process, $\{Z^0(t)\}_{t\geq 0}$, and then using the formalism of the martingale problem of Stroock and Varadhan, we show that the Hunt process $\{Z^0(t)\}_{t\geq 0}$ induces the same probability law on the canonical space as the unique weak Markov solution $\{Z(t)\}_{t\geq 0}$ to the Kimura stochastic equation (1.11). Theorem 4.1 is then obtained as a consequence of Proposition 4.2. We obtain the more general statement of Theorem 1.3 in Section 4.3, by combining Theorem 4.1 with a localization procedure from $S_{n,m}$ to sub-domains, $\Omega \subset S_{n,m}$. The localization procedure consists in studying the part of the Dirichlet form $(Q, H_0^1(\bar{S}_{n,m}; d\mu))$ on $\Omega \subset S_{n,m}$; see Fukushima, Oshima and Takeda (2011), Section 4.4. We conclude Section 4 with the proof of Theorem 1.6 in Section 4.4.

In Section 5, we adapt the method of the proof of Sturm (1994) to our framework to obtain the Harnack inequality for the Kimura operator $\widehat{L}$, defined in (1.1). In Section 1.2, we summarize the notation and conventions that are used throughout the article.

1.2. *Notation and conventions*. Let $\mathbb{Q}_+$ denote the set of positive rationals and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For all n, m and k positive integers, and $U \subset \mathbb{R}^n$ an open set, we denote by $C^k(U; \mathbb{R}^m)$ the set of functions $u : U \to \mathbb{R}^m$ that are k times continuously differentiable on U. We let $C^k(\bar{U}; \mathbb{R}^m)$ be the set of functions in $C^k(U; \mathbb{R}^m)$ with the property that all derivatives up to order k can be extended by continuity on $\bar{U}$ and the norm

$$\|u\|_{C^k(\bar{U};\mathbb{R}^m)} := \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq k}} \|D^\alpha u\|_{C(\bar{U};\mathbb{R}^m)} < \infty,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Let $a, b \in \mathbb{R}$. We denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

2. Initial-value problems, semigroups and Dirichlet forms. In this section, we state in Assumption 2.1 the conditions satisfied by the coefficients of the singular Kimura operator L given by (1.10). We then introduce in Definition 2.2 the notion of weak solution to the parabolic problem defined by the singular Kimura operator on a domain $\Omega \subseteq S_{n,m}$ and we recall the existence and uniqueness results for weak solutions to the initial-value problems defined by L. This allows us to introduce the semigroup, $\{T_t^{\Omega}\}_{t\geq 0}$, determined by the weak solutions and the associated Dirichlet form, $Q_{\Omega}(u, v)$.

2.1. *Assumptions on the coefficients of the singular Kimura operator L*. The particular choice of the operator L in Epstein and Mazzeo (2016) and in our work is motivated by the fact that it can be written in divergence form [L is a self-adjoint operator on $L^2(\bar{S}_{n,m}; d\mu)$]. That is, for all $u, v \in C_c^2(\bar{S}_{n,m})$, we have that

$$-(Lu, v)_{L^2(\bar{S}_{n,m};d\mu)} = Q(u, v),$$

where we recall that the weighted Sobolev space $L^2(\bar{S}_{n,m}; d\mu)$ consists of measurable functions, $u : \bar{S}_{n,m} \to \mathbb{R}$, that are L^2-integrable with respect to the measure $d\mu(z)$, defined by (1.12), and the symmetric bilinear form, $Q(u, v)$, is given by

$$Q(u, v) = \int_{S_{n,m}} q(u, v)(z)\, d\mu(z), \tag{2.1}$$

where we let

$$\begin{aligned} q(u, v)(z) := {} & \sum_{i=1}^{n} x_i a_{ii}(z) u_{x_i} v_{x_i} + \sum_{i,j=1}^{n} x_i x_j \tilde{a}_{ij}(z) u_{x_i} v_{x_j} \\ & + \sum_{i=1}^{n} \sum_{l=1}^{m} x_i c_{il}(z)(u_{x_i} v_{y_l} + u_{y_l} v_{x_i}) + \sum_{l,k=1}^{m} d_{lk}(z) u_{y_l} v_{y_k}. \end{aligned} \tag{2.2}$$

To state the technical assumptions satisfied by the bilinear form $Q(u, v)$, we introduce the following notation. For a set of indices, $I \subseteq \{1, \dots, n\}$, we let

$$\begin{aligned} M_I := \big\{ & z = (x, y) \in S_{n,m} : x_i \in (0, 1) \text{ for all } i \in I, \\ & \text{and } x_j \in [1, \infty) \text{ for all } j \in I^c \big\}, \end{aligned} \tag{2.3}$$

where we denote $I^c := \{1, \dots, n\} \backslash I$. We make the following assumptions about the coefficients of the bilinear form $Q(u, v)$ given by (2.1).

ASSUMPTION 2.1 (Coefficients of the operator L). There are positive constants, $\bar{b}$, δ and K, such that:

1. The coefficients $\operatorname{diag}(a_{ii}(z))$, $(\tilde{a}(z))$ and $(c(z))$ are chosen such that, for all $z \in \bar{M}_{\{1,\dots,n\}}$, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$, we have that

$$\begin{aligned} \delta\big(|\xi|^2 + |\eta|^2\big) \le {} & \sum_{i=1}^{n} a_{ii}(z) \xi_i^2 + \sum_{i,j=1}^{n} \tilde{a}_{ij}(z) \xi_i \xi_j \\ & + 2 \sum_{i=1}^{n} \sum_{l=1}^{m} c_{il}(z) \xi_i \eta_l + \sum_{l,k=1}^{m} d_{lk}(z) \eta_l \eta_k. \end{aligned} \tag{2.4}$$

Compare condition (2.4) with Pop (2017a), Condition (2.18) and Epstein and Mazzeo (2016), Condition (35).

2. Let $I \subsetneqq \{1, \dots, n\}$, $1 \le i, j \le n$, $1 \le l, k \le m$, $h \in I$, and $h' \in I^c$. For all $z \in \bar{M}_I$, we assume that

$$\begin{aligned} & \tilde{a}_{ij}(z) = 0, \qquad c_{il}(z) = 0, \qquad d_{lk}(z) = \delta_{lk}, \qquad b_i(z) = 1, \\ & \delta \le a_{hh}(z), \qquad x_{h'} a_{h'h'}(z) = 1. \end{aligned} \tag{2.5}$$

3. The coefficients $b_i(z)$ satisfy, for all $i = 1, \dots, n$,

$$b_i(z) \ge \bar{b} > 0, \qquad \forall z \in \partial S_{n,m} \cap \{x_i = 0\}. \tag{2.6}$$

4. The coefficients $a_{ii}(z), \tilde{a}_{ij}(z), b_i(z), c_{ik}(z), d_{kl}(z)$ together with their derivatives of any order are smooth and bounded functions on $\bar{S}_{n,m}$, for all $1 \leq i, j \leq n$ and for all $1 \leq k, l \leq m$, and we have that

$$\|a_{ii}\|_{C(\bar{S}_{n,m})} + \|\tilde{a}_{ij}\|_{C(\bar{S}_{n,m})} + \|c_{ik}\|_{C(\bar{S}_{n,m})} + \|d_{lk}\|_{C(\bar{S}_{n,m})} + \|b_i\|_{C(\bar{S}_{n,m})} \leq K. \tag{2.7}$$

The conditions imposed on the coefficients of the operator L in Assumption 2.1 ensure that they also satisfy the conditions assumed in Pop (2017a) and Epstein and Mazzeo (2016).

2.2. *The initial-value problem defined by the singular Kimura operator.* To formulate the notion of weak solution to the inhomogeneous initial-value problem,

$$\begin{aligned} u_t - Lu &= g \qquad \text{on } I \times \Omega, \\ u &= f \qquad \text{on } \{t_1\} \times \Omega, \end{aligned} \tag{2.8}$$

where $\Omega \subseteq S_{n,m}$ is an open set and $I := (t_1, t_2) \subset \mathbb{R}_+$, we need to introduce suitable function spaces. We denote $\partial_0\Omega$ the relative interior of $\partial\Omega \cap \partial S_{n,m}$, $\partial_1\Omega := \partial\Omega \cap S_{n,m}$, and $\underline{\Omega} := \Omega \cup \partial_0\Omega$, and we introduce a bilinear form on the open set Ω:

$$Q_\Omega(u, v) := \int_\Omega q(u, v)(z)\, d\mu(z), \qquad \forall u, v \in H^1_0(\underline{\Omega}; d\mu).$$

When $\Omega = S_{n,m}$, we simply denote Q_Ω by Q. We follow Sturm (1995), Section 1.3 A, letting $\mathcal{H} := L^2(\underline{\Omega}; d\mu)$, and $\mathcal{F} := H^1_0(\underline{\Omega}; d\mu)$ be the closure of C^1 functions with compact support in $\underline{\Omega}$, $C^1_c(\underline{\Omega})$, with respect to the norm,

$$\|u\|_{H^1(\Omega; d\mu)} := \left(Q_\Omega(u, u) + \|u\|^2_{\mathcal{H}}\right)^{1/2}.$$

Let $C(\bar{I}, \mathcal{H})$ be the space of continuous functions, $u : \bar{I} \to \mathcal{H}$, endowed with the norm

$$\|u\|_{C(\bar{I}, \mathcal{H})} := \sup_{t \in \bar{I}} \|u(t)\|_{\mathcal{H}} < \infty.$$

The space of functions $C(\bar{I}, \mathcal{F})$ is defined similarly to $C(\bar{I}, \mathcal{H})$ by replacing the space $\mathcal{H}$ with $\mathcal{F}$ in the preceding definition. We let $L^2(I, \mathcal{F})$ denote the space of measurable functions, u, on $I \times \underline{\Omega}$ endowed with the norm,

$$\|u\|_{L^2(I, \mathcal{F})} := \left(\int_I \|u(t)\|^2_{\mathcal{F}}\, dt\right)^{1/2} < \infty.$$

Let $\mathcal{F}^*$ denote the dual space of $\mathcal{F}$ with respect to the extension of the $L^2(\Omega; d\mu)$-pairing to $\mathcal{F} \times \mathcal{F}^*$. $H^1(I, \mathcal{F}^*)$ is the space of distributions, u, such that $u \in L^2(I, \mathcal{F}^*)$ and the distributional time derivative, $\frac{du}{dt}$, also belongs to $L^2(I, \mathcal{F}^*)$. We endow the space $H^1(I, \mathcal{F}^*)$ with the norm

$$\left(\int_I \left(\|u(t)\|^2_{\mathcal{F}^*} + \left\|\frac{du(t)}{dt}\right\|^2_{\mathcal{F}^*}\right) dt\right)^{1/2} < \infty.$$

Finally, we let $\mathcal{F}(I \times \underline{\Omega}) := L^2(I, \mathcal{F}) \cap H^1(I, \mathcal{F}^*)$. Following Sturm (1995), Section 1.4 A, Evans (1998), Section 7.1.1 b., Brezis (2011), Theorem 10.9, we define the notion of a weak solution as follows.

DEFINITION 2.2 (Weak solution). Let $f \in \mathcal{H}$ and $g \in L^2(I, \mathcal{F}^*)$. A function $u \in \mathcal{F}(I \times \underline{\Omega})$ is a solution to the inhomogeneous initial-value problem (2.8) if:

1. For all $v \in \mathcal{F}(I \times \underline{\Omega})$, we have that

$$\int_I Q_\Omega\big(u(t), v(t)\big)\,dt + \int_I \left(\frac{du(t)}{dt}, v(t)\right) dt = \int_I \big(g(t), v(t)\big)\,dt, \tag{2.9}$$

where $(\cdot, \cdot)$ denotes the dual pairing of $\mathcal{F}^*$ and $\mathcal{F}$.

2. The initial condition is satisfied in the $\mathcal{H}$-sense, that is, $\|u(t) - f\|_{\mathcal{H}} \to 0$ as $t \downarrow t_1$.

It is clear that the bilinear form $Q_\Omega(u, v)$ is coercive and continuous when $u, v \in \mathcal{F}$. Thus, we can apply Sturm (1995), Proposition 1.2, to obtain that for all $f \in \mathcal{H}$ there is a unique weak solution, $u \in \mathcal{F}(I \times \underline{\Omega})$, to the homogeneous initial-value problem:

$$\begin{aligned} u_t - Lu &= 0 \qquad \text{on } I \times \Omega, \\ u &= f \qquad \text{on } \{t_1\} \times \Omega. \end{aligned} \tag{2.10}$$

Moreover, there is a strongly continuous, symmetric, contraction semigroup on $\mathcal{H}$, $\{T_t^\Omega\}_{t \geq 0}$, such that the unique weak solution u can be represented in the form $u(t + t_1) = T_t^\Omega f$, for all $t \in (0, t_2 - t_1)$. For brevity, when $\Omega = S_{n,m}$ we denote T_t^Ω by T_t. Using (2.1) directly, and the definitions in Fukushima, Oshima and Takeda (2011), Properties $(\mathcal{E}.1)$-$(\mathcal{E}.7)$, it is easy to verify that the bilinear form $Q_\Omega(u, v)$ defines a Dirichlet form on the Hilbert space $H_0^1(\underline{\Omega}; d\mu) \subset L^2(\underline{\Omega}; d\mu)$, which is regular and strongly local.

From Sturm (1995), Proposition 1.2, and Duhamel's principle, we obtain the existence and uniqueness of a weak solution to the *inhomogeneous* initial-value problem (2.8). We state the result in the form that we need for later use.

LEMMA 2.3 (Existence and uniqueness of solutions to the inhomogeneous problem [Lions and Magenes (1972), Theorem 3.4.1 and Remark 3.4.3, Brezis (2011), Theorem 10.9]). *Let $f \in \mathcal{H}$ and $g \in L^2(I, \mathcal{H})$. Then there is a unique weak solution, $u \in \mathcal{F}(I \times \underline{\Omega})$, to the inhomogeneous initial-value problem* (2.8), *and we have that*

$$u(t + t_1) = T_t^\Omega f + \int_0^t T_{t-s}^\Omega g(s + t_1, \cdot)\,ds, \qquad \forall t \in [0, t_2 - t_1]. \tag{2.11}$$

3. Properties of the fundamental solution. In this section, we prove the existence of the fundamental solution associated to the semigroup $\{T_t\}_{t\geq 0}$, and some of its regularity properties that are extensively used in the sequel. In Lemmas 3.1, 3.2 and 3.4, we establish the main properties of the fundamental solution, which allow us to associate in Section 4.2 a suitable Hunt process to the semigroup $\{T_t\}_{t\geq 0}$. In Propositions 3.6 and 3.7, we prove a Hölder estimate and a L^q-distribution estimate satisfied by the fundamental solution, which enable us to establish that the Hunt process associated to the semigroup $\{T_t\}_{t\geq 0}$ is a solution to the generalized Kimura stochastic differential equation (1.11).

LEMMA 3.1 (Measurability of the fundamental solution). *There is a measurable function*, $p : (0,\infty) \times \bar{S}_{n,m} \times \bar{S}_{n,m} \to [0,\infty)$, *such that*

$$p(t,z,\cdot) \in L^2(\bar{S}_{n,m}; d\mu), \qquad \forall (t,z) \in (0,\infty) \times \bar{S}_{n,m}, \tag{3.1}$$

and we have that

$$\begin{aligned} T_t f(z) = \int_{S_{n,m}} p(t,z,w) f(w)\, d\mu(w), \\ \forall (t,z) \in (0,\infty) \times \bar{S}_{n,m}, \forall f \in L^2(\bar{S}_{n,m}; d\mu). \end{aligned} \tag{3.2}$$

LEMMA 3.2 (Regularity of the fundamental solution). *The fundamental solution*, p, *admits a modification*, $\bar{p}$, *that belongs to* $C((0,\infty) \times \bar{S}_{n,m} \times \bar{S}_{n,m})$, *and satisfies properties* (3.1) *and* (3.2), *with* p *replaced by* $\bar{p}$, *and the Chapman–Kolmogorov equations*,

$$\begin{aligned} \bar{p}(t+s,z,z^0) = \int_{S_{n,m}} \bar{p}(t,z,w)\bar{p}(s,w,z^0)\, d\mu(w), \\ \forall t,s > 0, \forall z, z^0 \in \bar{S}_{n,m}. \end{aligned} \tag{3.3}$$

REMARK 3.3. The following results in our article refer to the continuous modification of the fundamental solution, denoted by $\bar{p}$ in Lemma 3.2. In order to simplify notation, we denote the continuous modification $\bar{p}$ by p from now on.

LEMMA 3.4 (Transition probability densities). *For all* $(t,z) \in (0,\infty) \times \bar{S}_{n,m}$, *we have that* $p(t,z,\cdot) \in L^1(\bar{S}_{n,m}; d\mu)$. *Moreover*, p *is a nonnegative function*, *and satisfies*

$$\int_{S_{n,m}} p(t,z,w)\, d\mu(w) = 1, \qquad \forall (t,z) \in (0,\infty) \times \bar{S}_{n,m}. \tag{3.4}$$

The functions $p(t,z,w)\, d\mu(w)$ can therefore be viewed as transition probability densities.

REMARK 3.5. We note that partial statements of Lemmas 3.1 and 3.2 are established in Sturm (1995), Proposition 2.3(i) and (iii), where identities (3.2) and (3.3) are established for all $t > 0$ and for μ-*a.e.* $z, z^0 \in S_{n,m}$. In our proof of Lemmas 3.1 and 3.2, we prove the results *for all* $z, z^0 \in S_{n,m}$, which is important in the proof of Theorem 4.1, which is turn implies the main Theorem 1.3. The fact that identities hold for all $t > 0$ and for all $z, z^0 \in S_{n,m}$ allows us to view $p(t, z, \cdot)\, d\mu$ as transition probability densities and apply the Daniell–Kolmogorov theorem Baudoin (2014), Theorem 3.14, in the proof of Theorem 4.1.

To state the Hölder distribution estimates, we first need to introduce the *intrinsic distance*, ρ, induced by the bilinear form $Q(u, v)$ on $\bar{S}_{n,m}$; see also Pop (2017a), inequality (2.15), and Epstein and Mazzeo (2016), Identity (42). Given Assumption 2.1, there is a positive constant, c, such that for all sets of indices $I, J \subseteq \{1, \dots, n\}$ and all $z^0 \in \bar{M}_I$ and $z \in \bar{M}_J$, we have

$$\begin{aligned}
&c\Big(\max_{i\in I\cap J}\Big|\sqrt{x_i^0} - \sqrt{x_i}\Big| + \max_{k\in(I\cap J)^c}|x_k^0 - x_k| + \max_{l\in\{1,\dots,m\}}|y_l^0 - y_l|\Big)\\
&\quad\le \rho(z^0, z)\\
&\quad\le c^{-1}\Big(\max_{i\in I\cap J}\Big|\sqrt{x_i^0} - \sqrt{x_i}\Big| + \max_{k\in(I\cap J)^c}|x_k^0 - x_k|\\
&\qquad + \max_{l\in\{1,\dots,m\}}|y_l^0 - y_l|\Big).
\end{aligned} \tag{3.5}$$

For $z^0 \in \bar{S}_{n,m}$ and $r > 0$, we let

$$B_r(z^0) := \{z \in \bar{S}_{n,m} : \rho(z^0, z) < r\}, \tag{3.6}$$

denote the ball with center z^0 and radius r, with respect to the distance function ρ. When $z^0 = (0, 0)$, we write for brevity B_r instead of $B_r(0, 0)$. We can now state the following.

PROPOSITION 3.6 (Hölder distribution estimates). *There is a positive constant*, $\alpha_0 = \alpha_0(K, m, n)$, *so that for all* $\alpha \in (\alpha_0, \infty)$, *there is a positive constant*, $\gamma = \gamma(\alpha, K, m, n)$, *such that for all* $T > 0$, *we can find a positive constant*, $C = C(\alpha, \bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n, T)$, *with the property that*

$$\int_{S_{n,m}} \rho^\alpha(z^0, z)\, p(t, z^0, z)\, d\mu(z) \le C t^{n+m+\gamma}, \qquad \forall t \in (0, T], \forall z^0 \in \bar{S}_{n,m}. \tag{3.7}$$

We also have

PROPOSITION 3.7 (L^q-distribution estimates). *There is a positive constant, $q_0 = q_0(K, m, n) \in (1, 2)$, such that the following hold. For all $q \in [1, q_0)$, there is a positive constant, $\beta = \beta(q) < 1$, and for all $T > 0$, there is a positive constant, $C_0 = C_0(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n, q, T)$, such that*

$$\|p(t, z^0, \cdot)\|_{L^q(\bar{S}_{n,m};d\mu)} \leq C_0 t^{-\beta}, \qquad \forall z^0 \in \bar{S}_{n,m}, \forall t \in (0, T], \tag{3.8}$$

and

$$\int_0^T \|p(t, z^0, \cdot)\|_{L^q(\bar{S}_{n,m};d\mu)}\, dt \leq C_0, \qquad \forall z^0 \in \bar{S}_{n,m}. \tag{3.9}$$

REMARK 3.8. For comparison, we recall the analogous estimate for standard Brownian motion. For n-dimensional Brownian motion, direct calculations give us that, for all $\alpha, q > 0$, there is a positive constant, $C = C(\alpha, n)$, such that

$$\begin{aligned} &\int_{\mathbb{R}^n} |z - z^0|^\alpha p(t, z^0, z)\, dz = C t^{\alpha/2}, \\ &\int_{\mathbb{R}^n} |p(t, z^0, z)|^q\, dz = \frac{(2\pi)^{n(1-q)/2}}{q^{n/2}} t^{(1-q)n/2}, \\ &\qquad \forall z^0 \in \mathbb{R}^n, \forall t > 0. \end{aligned} \tag{3.10}$$

Thus, the conclusion of Propositions 3.6 and 3.7 hold in the case of Brownian motion, when $\alpha > 2n$ and $q \in [1, (n+2)/n)$.

The remainder of the section contains the technical proofs of Lemmas 3.1, 3.2 and 3.4, and of Propositions 3.6 and 3.7, which can be omitted on a first reading. We begin with the following.

PROOF OF LEMMA 3.1. By Lemma 2.3, for every function $f \in L^2(\bar{S}_{n,m}; d\mu)$, the homogeneous initial-value problem (2.8) has a unique solution, $u \in \mathcal{F}((0, \infty) \times \bar{S}_{n,m})$, which can be represented by $u(t) = T_t f$, for all $t \geq 0$. By Epstein and Mazzeo (2016), Corollary 4.1, the solution u is Hölder continuous on $(0, \infty) \times \bar{S}_{n,m}$, and so, the function $T_t f(z)$ is well defined at all points $(t, z) \in (0, \infty) \times \bar{S}_{n,m}$. By Sturm (1995), Theorem 2.1, for all $0 < t_1 < t_2$ and all compact sets, $K \subset \bar{S}_{n,m}$, there is a positive constant, $C = C(t_1, t_2, K)$, such that for all $t \in [t_1, t_2]$ and all $z \in K$, we have that

$$\begin{aligned} |T_t f(z)| &\leq C\left(\int_{t_1/2}^{t_2} \|T_s f\|^2_{L^2(\bar{S}_{n,m};d\mu)}\, ds\right)^{1/2} \\ &\leq C\left(t_2 - \frac{t_1}{2}\right)^{1/2} \|f\|_{L^2(\bar{S}_{n,m};d\mu)}, \end{aligned} \tag{3.11}$$

where in the second inequality we used the contraction property of $\{T_t\}_{t \geq 0}$. It follows that the map $L^2(\bar{S}_{n,m}; d\mu) \ni f \mapsto T_t f(z)$ is continuous, and so, there is a Borel measurable function, $p(t,z,\cdot)\in L^2(\bar{S}_{n,m};d\mu)$, such that identity (3.2) holds. Moreover, using the fact that

$$\|p(t,z,\cdot)\|_{L^2(\bar{S}_{n,m};d\mu)} = \sup_{\|f\|_{L^2(\bar{S}_{n,m};d\mu)}=1} |T_t f(z)|,$$

inequality (3.11) gives us that

$$\|p(t,z,\cdot)\|_{L^2(\bar{S}_{n,m};d\mu)} \leq C, \qquad \forall (t,z)\in[t_1,t_2]\times K. \tag{3.12}$$

Let $(t^0,z^0)\in(0,\infty)\times\bar{S}_{n,m}$ and $r>0$ be such that $t^0-4r^2>0$. We denote $Q_r(t^0,z^0):=(t^0-r^2,t^0)\times B_r(z^0)$. Let (t',z') and $(t''z'')$ be points in $Q_r(t^0,z^0)$. By Epstein and Mazzeo (2016), Corollary 4.1, there are positive constants, $\alpha\in(0,1)$ and $C=C(t^0,z^0,r)$, such that

$$|T_{t'}f(z')-T_{t''}f(z'')| \leq C\|T_t f\|_{L^\infty(Q_{2r}(t^0,z^0))}\bigl(\rho(z',z'')+\sqrt{|t'-t''|}\bigr)^\alpha.$$

From inequality (3.11), it follows that

$$|T_{t'}f(z')-T_{t''}f(z'')| \leq C\|f\|_{L^2(\bar{S}_{n,m};d\mu)}\bigl(\rho(z',z'')+\sqrt{|t'-t''|}\bigr)^\alpha. \tag{3.13}$$

Using the fact that

$$\|p(t',z',\cdot)-p(t'',z'',\cdot)\|_{L^2(\bar{S}_{n,m};d\mu)} = \sup_{\|f\|_{L^2(\bar{S}_{n,m};d\mu)}=1} |T_{t'}f(z')-T_{t''}f(z'')|,$$

inequality (3.13) gives us that, for all $(t',z'),(t'',z'')\in Q_r(t^0,z^0)$, we have

$$\|p(t',z',\cdot)-p(t'',z'',\cdot)\|_{L^2(\bar{S}_{n,m};d\mu)} \leq C\bigl(\rho(z',z'')+\sqrt{|t'-t''|}\bigr)^\alpha.$$

Because the point $(t^0,z^0)\in(0,\infty)\times\bar{S}_{n,m}$ was arbitrarily chosen, the preceding inequality implies that the function $p:(0,\infty)\times\bar{S}_{n,m}\times\bar{S}_{n,m}\to[0,\infty)$ is measurable, and satisfies property (3.1) and identity (3.2). □

PROOF OF LEMMA 3.2. For all $\varphi\in C_c^\infty(\bar{S}_{n,m})$, using the semigroup property and Epstein and Mazzeo (2016), Corollary 4.1, we have that $T_tT_s\varphi(z)=T_{t+s}\varphi(z)$, for all $t,s>0$ and all $z\in\bar{S}_{n,m}$. From (3.2), we can write the equality $T_tT_s\varphi(z)=T_{t+s}\varphi(z)$ in the form,

$$\begin{aligned}&\int_{S_{n,m}} p(t,z,w)\int_{S_{n,m}} p(s,w,z^0)\varphi(z^0)\,d\mu(z^0)\,d\mu(w)\\ &\quad=\int_{S_{n,m}} p(t+s,z,z^0)\varphi(z^0)\,d\mu(z^0),\end{aligned} \tag{3.14}$$

and so, identity (3.3) holds for all $t,s>0$, and all $z\in\bar{S}_{n,m}$, and for μ-almost every $z^0\in\bar{S}_{n,m}$. The symmetry property of the semigroup yields that

$$\begin{aligned}&p(t,z,z^0)=p(t,z^0,z),\\ &\forall t\in(0,\infty),\text{ for }(\mu\times\mu)\text{-almost all }(z^0,z)\in\bar{S}_{n,m}\times\bar{S}_{n,m}.\end{aligned} \tag{3.15}$$

To obtain the conclusion of Lemma 3.2, we first prove the following.

CLAIM 3.9. *There is a set $N \subset \bar{S}_{n,m}$ of zero μ-measure such that for all $t \in \mathbb{Q}_+$ and $z^0 \in \bar{S}_{n,m} \backslash N$, we have*

$$p(t, w, z^0) = p(t, z^0, w), \qquad \text{for } \mu\text{-a.e. } w \in \bar{S}_{n,m}, \tag{3.16}$$

and there is a countable and dense set, $A \subset \bar{S}_{n,m}$, such that $A \cap N = \varnothing$ and such that for all $t, s \in \mathbb{Q}_+$, $z \in A$ and $z^0 \in \bar{S}_{n,m} \backslash N$, we have

$$p(t+s, z, z^0) = \int_{S_{n,m}} p(t, z, w) p(s, w, z^0)\, d\mu(w). \tag{3.17}$$

PROOF. For $t \in \mathbb{Q}_+$, let $N_t \subset \bar{S}_{n,m} \times \bar{S}_{n,m}$ be a measurable set with zero $(\mu \times \mu)$-measure, such that identity (3.15) holds, for all $(z, z^0) \notin N_t$. Denote by $N^0 := \bigcup_{t \in \mathbb{Q}_+} N_t$. Then N^0 also has zero $(\mu \times \mu)$-measure, and identity (3.15) holds, for all $(z, z^0) \notin N^0$ and all $t \in \mathbb{Q}_+$. For all $k \in \mathbb{N}$, we let $K_k := [0, k]^n \times [-k, k]^m$, and we see that

$$(\mu \times \mu)(K_k^2 \backslash N^0) = (\mu \times \mu)(K_k^2) < \infty.$$

We denote $S_{z^0} := \{z \in K_k : (z, z^0) \in K_k^2 \backslash N^0\}$, and applying Folland (1999), Proposition 2.36, it follows that $\mu(S_{z^0}) = \mu(K_k)$, for μ-a.e. $z^0 \in K_k$. Thus, there is a set $N_k^1 \subset \bar{S}_{n,m}$ with zero μ-measure such that

$$p(t, w, z^0) = p(t, z^0, w), \qquad \forall t \in \mathbb{Q}_+, \text{for } \mu\text{-a.e. } w \in K_k, \forall z^0 \in K_k \backslash N_k^1.$$

Letting now $N^1 := \cup\{N_k^1 : k \in \mathbb{N}\}$, we have that N^1 has zero μ-measure, and

$$\begin{aligned} &p(t, w, z^0) = p(t, z^0, w), \\ &\qquad \forall t \in \mathbb{Q}_+, \text{for } \mu\text{-a.e. } w \in \bar{S}_{n,m}, \forall z^0 \in \bar{S}_{n,m} \backslash N^1. \end{aligned} \tag{3.18}$$

We now turn to the proof of the Chapman–Kolmogorov equations (3.17). Using the fact that identity (3.3) holds for all $t, s > 0$, and all $z \in \bar{S}_{n,m}$, and for μ-almost every $z^0 \in S_{n,m}$, we choose a countable and dense set, $A \subset \bar{S}_{n,m}$, such that $A \cap N^1 = \varnothing$, and for all $t, s \in \mathbb{Q}_+$ and $z \in A$, we choose a set $N_{t,s,z}^2$ with zero μ-measure such that $A \cap N_{t,s,z}^2 = \varnothing$ and identity (3.3) holds at all points $z^0 \in \bar{S}_{n,m} \backslash N_{t,s,z}^2$. We can now set $N^2 := \cup\{N_{t,s,z}^2 : t, s \in \mathbb{Q}_+, z \in A\}$, and $N := N^1 \cup N^2$, and we obtain that both identities (3.16) and (3.17) hold. We notice also that $A \cap N = \varnothing$. This completes the proof of the claim. $\square$

Let $t, s \in \mathbb{Q}_+$, $z \in A$, $z^0 \in \bar{S}_{n,m} \backslash N$, where the sets A and N are as in Claim 3.9. Because $p(s, z^0, \cdot) \in L^2(\bar{S}_{n,m}; d\mu)$, there is a unique weak solution, $u \in \mathcal{F}((0, T) \times \bar{S}_{n,m})$, where $T > 0$, to the homogeneous equation (2.8) with initial condition $f = p(s, z^0, \cdot)$. Using Epstein and Mazzeo (2016), Corollary 4.1, it follows that the solution u is continuous on $(0, T) \times \bar{S}_{n,m}$. Since $u(t) = T_t p(s, z^0, \cdot)$, the semigroup property (3.2) gives us that

$$u(t, z) = \int_{S_{n,m}} p(t, z, w) p(s, z^0, w)\, d\mu(w),$$

and properties (3.16) and (3.17) imply that $u(t,z) = p(t+s, z, z^0)$. Estimate (3.13) applied with $f := p(s, z^0, \cdot)$, and the property that

$$\|p(s, z^0, \cdot)\|_{L^2(\bar{S}_{n,m}; d\mu)} \leq 1,$$

for all $(s, z^0) \in (0, \infty) \times \bar{S}_{n,m}$, show that for all $t_1, t_2 > 0$, and all compact sets, $K \subset \bar{S}_{n,m}$, there are positive constants, $\alpha = \alpha(t_1, t_2, K) \in (0, 1)$ and $C = C(t_1, t_2, K)$, such that, for all $t', t'' \in [t_1, t_2] \cap \mathbb{Q}_+$, and for all $z', z'' \in K \cap A$ and all $z^0 \in \bar{S}_{n,m} \backslash N$, we have that

$$|p(t', z', z^0) - p(t'', z'', z^0)| \leq C\big(\rho(z', z'') + \sqrt{|t' - t''|}\big)^\alpha. \tag{3.19}$$

The preceding inequality together with the symmetry property (3.17) and the fact that $\mathbb{Q}_+ \times A \times (\bar{S}_{n,m} \backslash N)$ is dense in $(0, \infty) \times \bar{S}_{n,m} \times \bar{S}_{n,m}$ and $A \cap N = \varnothing$ give us that p admits a continuous modification on $(0, \infty) \times \bar{S}_{n,m} \times \bar{S}_{n,m}$, which is simply the unique continuous extension of p from $\mathbb{Q}_+ \times A \times (\bar{S}_{n,m} \backslash N)$ to $(0, \infty) \times \bar{S}_{n,m} \times \bar{S}_{n,m}$. We denote the continuous modification of p by $\bar{p}$. Inequality (3.19) yields that for all $t', t'' \in [t_1, t_2]$, and for all $z', z'' \in K$ and all $z^0 \in \bar{S}_{n,m}$,

$$|\bar{p}(t', z', z^0) - \bar{p}(t'', z'', z^0)| \leq C\big(\rho(z', z'') + \sqrt{|t' - t''|}\big)^\alpha, \tag{3.20}$$

where $C = C(t_1, t_2, K) > 0$ and $\alpha = \alpha(t_1, t_2, K) \in (0, 1)$. Using (3.18), we also have that

$$\bar{p}(t, z, w) = \bar{p}(t, w, z), \qquad \forall t > 0, \forall z, w \in \bar{S}_{n,m}. \tag{3.21}$$

It remains to show that the continuous extension $\bar{p}$ satisfies properties (3.1), (3.2) and (3.3), with p replaced by $\bar{p}$. The previous argument together with the continuity estimate (3.20) and Epstein and Mazzeo (2016), Corollary 4.1, imply that $\bar{p}$ satisfies (3.2). Using the fact that $\{T_t\}_{t\geq 0}$ is a family of continuous operators on $L^2(\bar{S}_{n,m}; d\mu)$ and property (3.2), we obtain that (3.1) also holds for $\bar{p}$. Finally, property (3.17) gives us that $\bar{p}$ verifies (3.3). This completes the proof. □

We now give the Proof of Lemma 3.4

PROOF OF LEMMA 3.4. The fact that $p(t, z, \cdot)$ is a nonnegative function follows from Sturm (1995), Lemma 1.4. Using the contraction property in Sturm (1995), Proposition 1.6, of the semigroup $\{T_t\}_{t\geq 0}$ on $L^\infty(\bar{S}_{n,m}; d\mu)$, it follows that $p(t, z, \cdot) \in L^1(\bar{S}_{n,m}; d\mu)$. Property (3.4) is equivalent to

$$T_t 1(z) = 1, \qquad \forall (t, z) \in (0, \infty) \times \bar{S}_{n,m}. \tag{3.22}$$

Let $\{\varphi_k\}_{k\geq 1} \subset C_c^\infty(\bar{S}_{n,m})$ be a sequence of smooth functions such that $0 \leq \varphi_k \leq 1$, and $\varphi_k(z) = 1$ for $z \in B_k^e$, and $\varphi_k(z) = 0$ for $z \in \bar{S}_{n,m} \backslash B_{2k}^e$, where we denote $B_r^e := \{z \in \bar{S}_{n,m} : |z| < r\}$, for all $r > 0$. From (2.11), we have that the following identity holds in the $L^2(\bar{S}_{n,m}; d\mu)$-sense:

$$T_t \varphi_k = \varphi_k - \int_0^t T_s L \varphi_k \, ds.$$

Using the contraction property [Sturm (1995), Proposition 1.6] of the semigroup $\{T_t\}_{t\geq 0}$ on $L^p(\bar{S}_{n,m}; d\mu)$, for all $p \in [1, \infty]$, and the preceding identity, we obtain that

$$\|T_t\varphi_k - \varphi_k\|_{L^p(\bar{S}_{n,m};d\mu)} \leq t\|L\varphi_k\|_{L^p(\bar{S}_{n,m};d\mu)}, \qquad \forall k \geq 1.$$

From Assumption 2.1, it follows that there is a positive constant, $C = C(K, m, n, p)$, such that

$$\|L\varphi_k\|_{L^p(\bar{S}_{n,m};d\mu)} \leq Ck^{\frac{n+m}{p}-1}, \qquad \forall k \geq 1.$$

Choosing p large enough, we obtain that $\|L\varphi_k\|_{L^p(\bar{S}_{n,m};d\mu)} \to 0$, as $k \to \infty$, and so, there is a subsequence $\{T_t\varphi_k - \varphi_k\}_{k\geq 1}$ which converges to 0, μ-a.e. on $\bar{S}_{n,m}$.

Using the upper bound estimates of the fundamental solution [see (3.23) below], it follows that for all compact sets, $K \subset \bar{S}_{n,m}$, and all $t > 0$, we have

$$\sup_{z\in K}\left|T_t\varphi_k(z) - T_t 1(z)\right| \to 0, \qquad \text{as } k \to \infty.$$

By Epstein and Mazzeo (2016), Corollary 4.1, the functions $T_t\varphi_k$ belong to $C(\bar{S}_{n,m})$, for all $k \geq 1$, and so the function $T_t 1$ is continuous on $\bar{S}_{n,m}$. Since the sequence $(T_t\varphi_k - \varphi_k) \to 0$, μ-a.e. on $\bar{S}_{n,m}$, as $k \to \infty$, $\varphi_k \to 1$ pointwise on $S_{n,m}$, as $k \to \infty$, and $T_t 1$ is continuous on $\bar{S}_{n,m}$, it follows that identity (3.22) holds. This concludes the proof of identity (3.4). $\square$

Our next aim is to give the proofs of Propositions 3.6 and 3.7, with the aid of the supremum estimate established in Epstein and Mazzeo (2016), Corollary 4.3: there is a positive constant, C, such that

$$\begin{aligned} p(t, z^0, z) &\leq C\frac{e^{-\frac{1}{8t}\rho^2(z^0,z)}}{\sqrt{\mu(B_{\sqrt{t}}(z^0))\mu(B_{\sqrt{t}}(z))}} =: \tilde{p}(t, z^0, z), \\ &\qquad \forall t > 0, \forall z^0, z \in \bar{S}_{n,m}. \end{aligned} \tag{3.23}$$

To prove Propositions 3.6 and 3.7, we need a last elementary result concerning the μ-measure of balls with respect to the distance function ρ.

LEMMA 3.10 (Measure of the balls). *Assume that the coefficient $b \in C^1(\bar{S}_{n,m}; \mathbb{R}^n)$ satisfies condition (2.6). Then there are positive constants, C and r_0 depending on $\bar{b}$, $\|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}$, m, n, such that for all $r \in (0, r_0/2)$ and $z^0 \in \bar{S}_{n,m}$, we have that*

$$\begin{aligned} \frac{1}{C}r^{m+n}\prod_{i\in I(z^0)}\left|\sqrt{x_i^0} \vee r\right|^{2b_i(z^0)-1} &\leq \mu(B_r(z^0)) \\ &\leq Cr^{m+n}\prod_{i\in I(z^0)}\left|\sqrt{x_i^0} \vee r\right|^{2b_i(z^0)-1}, \end{aligned} \tag{3.24}$$

where we let

$$I(z^0) := \{i \in \{1, \dots, n\} : x_i^0 \in [0, r_0]\}. \tag{3.25}$$

PROOF. Because we assume that the coefficient b belongs to $C^1(\bar{S}_{n,m}; \mathbb{R}^n)$ and that condition (2.6) holds, there are positive constant, c and r_1 such that, for all $i = 1, \dots, n$, we have that $b_i(z) \ge \bar{b}/2$, for all $z = (x, y) \in \bar{S}_{n,m}$ such that $x_i \in (0, r_1)$, and $|b_i(z) - b_i(z^0)| \le cr$, for all $z \in B_r(z^0)$ and $r > 0$. Thus, there is a positive constant, $r_0 = r_0(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, small enough so that

$$\begin{aligned} &\frac{\bar{b}}{4} < \beta_i - cr \le b_i(z) \le \beta_i + cr, \\ &\qquad \forall z \in B_r(z^0), \forall r \in (0, r_0/2), \forall i \in I(z^0), \end{aligned} \tag{3.26}$$

where we denote $\beta_i := b_i(z^0)$, for all $i = 1, \dots, n$. We prove the first inequality in (3.24). From the preceding inequality, it follows that

$$\begin{aligned} &\mu(B_r(z^0)) \ge \int_{B_r(z^0)} \prod_{i \in I(z^0)}^{n} x_i^{\beta_i + cr - 1}\, dx_i \prod_{i \in I^c(z^0)}^{n} x_i^{b_i(z) - 1}\, dx_i \prod_{l=1}^{m} dy_l, \\ &\qquad \forall r \in (0, r_0/2), \end{aligned} \tag{3.27}$$

where we let $I^c(z^0) := \{1, \dots, n\} \backslash I(z^0)$. Using property (3.5) of the distance function ρ, it follows from the preceding inequality that there is a positive constant, $C_1 = C_1(r_0, \|b\|_{C(\bar{S}_{n,m})}, m, n)$, such that

$$\mu(B_r(z^0)) \ge C_1 r^{m + |I^c(z^0)|} \prod_{i \in I(z^0)} \int_{\{|\sqrt{x_i} - \sqrt{x_i^0}| < r\}} x_i^{\beta_i + cr - 1}\, dx_i,$$

from where the left-hand side of inequality (3.24) immediately follows. The right-hand side of inequality (3.24) is proved by a similar argument, and so, we omit the detailed proof. □

We now give the proof of Proposition 3.6.

PROOF OF PROPOSITION 3.6. Using (3.23), we see that it is sufficient to prove estimate (3.7) for $\tilde{p}(t, z^0, z)$ instead of $p(t, z^0, z)$. Notice that the positive constant T can be chosen as small as we like. Let r_0 be the positive constant appearing in the conclusion of Lemma 3.10. Without loss of generality, we may assume that T satisfies inequality (3.37). It follows from the left-hand side of inequality (3.24), using (3.37), that there is a positive constant, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, such that

$$\mu(B_{\sqrt{t}}(z)) \ge C t^{(m+n)/2} t^{nK}, \qquad \forall z \in \bar{S}_{n,m}, \forall t \in (0, T],$$

where we recall that $\|b_i\|_{C(\bar{S}_{n,m})} \leq K$ from condition (2.7). The supremum bound of the fundamental solution (3.23), together with the preceding inequality, gives us that

$$\begin{aligned}\int_{S_{n,m}} \rho^\alpha(z^0, z)\tilde{p}(t, z^0, z)\, d\mu(z) \\ \leq Ct^{-(m+n)/2-nK} \int_{S_{n,m}} \rho^\alpha(z^0, z)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z).\end{aligned} \tag{3.28}$$

We see that there is a positive constant, $C = C(\alpha, m, n)$, such that

$$\begin{aligned}\int_{S_{n,m}} \rho^\alpha(z^0, z)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z) \leq C\sum_{j=1}^n \int_{S_{n,m}} \rho^\alpha(x_j^0, x_j)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z) \\ + C\sum_{k=1}^m \int_{S_{n,m}} \rho^\alpha(y_k^0, y_k)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z).\end{aligned}$$

We estimate each term on the right-hand side of the preceding inequality. We show that there is a positive constant, $C = C(\alpha, \bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, such that for all $j = 1, \ldots, n$, we have

$$\int_{S_{n,m}} \rho^\alpha(x_j^0, x_j)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z) \leq Ct^{(m+\alpha)/2}, \tag{3.29}$$

and, for all indices $k = 1, \ldots, m$, we have that

$$\int_{S_{n,m}} \rho^\alpha(y_k^0, y_k)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z) \leq Ct^{(m+\alpha)/2}. \tag{3.30}$$

We outline the proof of estimate (3.29), but inequality (3.30) can be deduced by a similar argument, and so, we do not include the detailed proof.

Inequality (3.42) and definition (1.12) of the measure $d\mu(z)$ yield that there is a positive constant, $c = c(m, n)$, such that

$$\begin{aligned}\int_{S_{n,m}} \rho^\alpha(x_j^0, x_j)e^{-\frac{1}{8t}\rho^2(z^0,z)}\, d\mu(z) \\ \leq \int_0^\infty \rho^\alpha(x_j^0, x_j)e^{-\frac{c}{8t}\rho^2(x_j^0,x_j)}\varphi(x_j)\, dx_j \\ \times \prod_{\substack{i=1\\ i\neq j}}^n \int_0^\infty e^{-\frac{c}{8t}\rho^2(x_i^0,x_i)}\varphi(x_i)\, dx_i \prod_{k=1}^m \int_{\mathbb{R}} e^{-\frac{c|y_k^0-y_k|^2}{8t}}\, dy_k,\end{aligned} \tag{3.31}$$

where we recall the definition of the function φ in (3.43). From inequality (3.10) applied with $\alpha = 0$, we have that

$$\int_{\mathbb{R}} e^{-\frac{c|y_k^0-y_k|^2}{8t}}\, dy_k \leq Ct^{1/2}, \qquad \forall k = 1, \ldots, m, \tag{3.32}$$

while using again identity (3.10) and property (3.5) of the distance function ρ, there is a positive constant, $C = C(\bar{b}, K, m, n, T)$, such that for all $t \in (0, T]$, we have that

$$\int_0^\infty e^{-\frac{c}{8t}\rho^2(x_i^0, x_i)}\varphi(x_i)\,dx_i \leq C, \qquad \forall i = 1, \dots, n, i \neq j. \tag{3.33}$$

Thus, using the preceding two inequalities, estimate (3.31) becomes

$$\begin{aligned} &\int_{S_{n,m}} \rho^\alpha(x_j^0, x_j) e^{-\frac{c}{8t}\rho^2(z^0, z)}\,d\mu(z) \\ &\qquad \leq C t^{m/2} \int_0^\infty \rho^\alpha(x_j^0, x_j) e^{-\frac{c}{8t}\rho^2(x_j^0, x_j)} \varphi(x_j)\,dx_j. \end{aligned} \tag{3.34}$$

It remains to estimate the integral on the right-hand side of the preceding inequality. Using definition (3.43) of the function $\varphi(x_j)$, we write the integral on the right-hand side of inequality (3.34) as a sum of three integrals, $I_1 + I_2 + I_3$, where the integral I_1 is taken over the interval $(0, r_0/2)$, the integral I_2 is over $(r_0/2, 1)$, and the last integral is over $(1, \infty)$. We estimate integral I_1, which satisfies the inequality

$$I_1 \leq C t^{\alpha/2} \int_0^{r_0/2} \left(\frac{\rho(x_j^0, x_j)}{\sqrt{t}}\right)^\alpha e^{-\frac{8}{c}(\frac{\rho(x_j^0, x_j)}{\sqrt{t}})^2} x_j^{\bar{b}/4-1}\,dx_j,$$

where $C = C(\alpha, m, n)$ is a positive constant. Because the function $s \mapsto s^\alpha e^{-s^2}$ is bounded on $\bar{\mathbb{R}}_+$ and the function $s \mapsto s^{\bar{b}/4-1}$ is integrable on $(0, 1)$, we see that there is a positive constant, $C = C(\bar{b}, r_0, m, n)$, such that $I_1 \leq C t^{\alpha/2}$. A similar argument can be applied to estimate integrals I_2 and I_3 to prove that they satisfy the same estimate as I_1. Thus, using inequalities (3.34), (3.33) and (3.32), it follows that there is a positive constant, $C = C(\alpha, \bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, such that estimate (3.29) holds. A similar argument can be applied to prove that estimate (3.30) holds. Using (3.29) and (3.30) in inequality (3.28) gives us that

$$\int_{S_{n,m}} \rho^\alpha(z^0, z)\tilde{p}(t, z^0, z)\,d\mu(z) \leq C t^{(\alpha - n(2+K))/2}. \tag{3.35}$$

Choosing $\alpha_0 := 2(n+m) + n(2+K)$, the preceding inequality shows that, for all $\alpha \in (\alpha_0, \infty)$, there are positive constants, $C = C(\alpha, \bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ and $\gamma = \gamma(\alpha, K, m, n)$, such that estimate (3.7) holds. This completes the proof. □

We also have the following.

PROOF OF PROPOSITION 3.7. Notice that inequality (3.9) is a consequence of inequality (3.8), and so it is sufficient to prove that (3.8) holds. Using the supremum bound (3.23), it is sufficient to prove estimate (3.8) for $\tilde{p}(t, z^0, z)$. Thus, we will prove that there is a positive constant, $q_0 = q_0(K, m, n) \in (1, 2)$, such that for all $q \in [1, q_0)$, there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n, q, T)$ and $\beta = \beta(q) < 1$, such that we have

$$\|\tilde{p}(t, z^0, \cdot)\|_{L^q(S_{n,m};d\mu)} \leq C_0 t^{-\beta}, \qquad \forall z^0 \in \bar{S}_{n,m}, \forall t \in (0, T]. \tag{3.36}$$

Moreover, notice that the positive constant T can be chosen as small as we like. Let r_0 be the positive constant appearing in the conclusion of Lemma 3.10. Without loss of generality, we may assume that

$$\sqrt{T} \leq \frac{r_0}{2}. \tag{3.37}$$

It follows from the left-hand side of inequality (3.24), using (3.37), that there is a positive constant, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, such that

$$\begin{aligned} &\mu(B_{\sqrt{t}}(z)) \geq Ct^{(m+n)/2} \prod_{i \in I(z)} (x_i \vee t)^{b_i(z)-1/2}, \\ &\qquad\qquad\qquad\qquad \forall z \in \bar{S}_{n,m}, \forall t \in (0, T], \end{aligned} \tag{3.38}$$

where we recall the definition of the set of indices $I(z)$ in (3.25). To estimate $\|\tilde{p}(t, z^0, \cdot)\|_{L^q(S_{n,m};d\mu)}$, we consider the set

$$\begin{aligned} A_t(z^0) := \Big\{ &z = (x, y) \in S_{n,m} : \Big|\sqrt{x_i} - \sqrt{x_i^0}\Big| < t^\alpha \text{ for } i \in I(z^0), \\ &|x_i - x_i^0| < t^\alpha \text{ for } i \in I^c(z^0), \text{ and } |y_l - y_l^0| < t^\alpha \text{ for } l \in \{1 \dots, m\}\Big\}, \end{aligned}$$

where we choose $\alpha \in (0, 1/2)$, and we denote $A_t^c(z^0) := S_{n,m} \backslash A_t(z^0)$. We split the proof into two steps in which we estimate $\|\tilde{p}(t, z^0, \cdot)\|_{L^q(A_t^c(z^0);d\mu)}$ and $\|\tilde{p}(t, z^0, \cdot)\|_{L^q(A_t(z^0);d\mu)}$, respectively.

STEP 1 (Estimate of $\|\tilde{p}(t, z^0, \cdot)\|_{L^q(A_t^c(z^0);d\mu)}$). In this step, we prove that there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ and $c = c(m, n)$, such that

$$\|\tilde{p}(t, z^0, \cdot)\|_{L^q(A_t^c(z^0);d\mu)} \leq Ct^{-q(m+n)/2-qnK} e^{-\frac{qc}{t^{1-2\alpha}}}. \tag{3.39}$$

From inequalities (3.38) and (3.23), we have that

$$\begin{aligned} &\|\tilde{p}(t, z^0, \cdot)\|^q_{L^q(A_t^c(z^0);d\mu)} \\ &\quad \leq Ct^{-q(m+n)/2} \prod_{i \in I(z^0)} (x_i^0 \vee t)^{-q(b_i(z^0)-1/2)/2} \\ &\qquad \times \int_{A_t^c(z^0)} \prod_{j \in I(z)} (x_j \vee t)^{-q(b_j(z)-1/2)/2} e^{-\frac{q\rho^2(z,z^0)}{8t}} \, d\mu(z), \end{aligned} \tag{3.40}$$

which, with the aid of inequality (2.7), gives us

$$\|\tilde{p}(t,z^0,\cdot)\|^q_{L^q(A^c_t(z^0);d\mu)} \leq Ct^{-q(m+n)/2}t^{-qnK}\int_{A^c_t(z^0)} e^{-\frac{q\rho^2(z,z^0)}{8t}}\,d\mu(z). \tag{3.41}$$

Using the first inequality in (3.26), together with (2.5), (2.6) and (2.7), we obtain for all $i = 1,\dots,n$ that

$$\begin{aligned} x_i^{b_i(z)-1} \leq x_i^{\bar{b}/4-1}\mathbf{1}_{\{x_i\in(0,r_0/2)\}} + x_i^{-K-1}\mathbf{1}_{\{x_i\in[r_0/2,1)\}} \\ + \mathbf{1}_{\{x_i\in[1,\infty)\}}, \qquad \forall z \in S_{n,m}, \end{aligned} \tag{3.42}$$

and we denote the function on the right-hand side of the preceding inequality by

$$\begin{aligned} \varphi(x_i) = x_i^{\bar{b}/4-1}\mathbf{1}_{\{x_i\in(0,r_0/2)\}} + x_i^{-K-1}\mathbf{1}_{\{x_i\in[r_0/2,1)\}} \\ + \mathbf{1}_{\{x_i\in[1,\infty)\}}, \qquad \forall z \in S_{n,m}. \end{aligned} \tag{3.43}$$

Using property (3.5) of the distance function $\rho(z,z^0)$, together with definition (1.12) of the measure $d\mu(z)$, we obtain from (3.41) that there is a positive constant, $c = c(m,n)$, such that

$$\begin{aligned} &\|\tilde{p}(t,z^0,\cdot)\|^q_{L^q(A^c_t(z^0);d\mu)} \\ &\quad \leq Ct^{-q(m+n)/2-qnK} \\ &\qquad \times \Bigg(\sum_{i\in I(z^0)}\int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|>t^\alpha\}} e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}}\varphi(x_i)\,dx_i \\ &\qquad \times \prod_{\substack{j=1\\ j\neq i}}^n \int_0^\infty e^{-\frac{qc\rho^2(x_j,x_j^0)}{8t}}\varphi(x_j)\,dx_j \prod_{l=1}^m \int_{\mathbb{R}} e^{-\frac{qc\rho^2(y_l,y_l^0)}{8t}}\,dy_l \\ &\qquad + \sum_{i\in I^c(z^0)}\int_{\{|x_i-x_i^0|>t^\alpha\}} e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}}\varphi(x_i)\,dx_i \\ &\qquad \times \prod_{\substack{j=1\\ j\neq i}}^n \int_0^\infty e^{-\frac{qc\rho^2(x_j,x_j^0)}{8t}}\varphi(x_j)\,dx_j \prod_{l=1}^m \int_{\mathbb{R}} e^{-\frac{qc\rho^2(y_l,y_l^0)}{8t}}\,dy_l \\ &\qquad + \sum_{l=1}^m \prod_{i=1}^n \int_0^\infty e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}}\varphi(x_i)\,dx_i \\ &\qquad \times \int_{\{|y_l-y_l^0|>t^\alpha\}} e^{-\frac{qc\rho^2(y_l,y_l^0)}{8t}}\,dy_l \prod_{\substack{k=1\\ k\neq l}}^m \int_{\mathbb{R}} e^{-\frac{qc\rho^2(y_k,y_k^0)}{8t}}\,dy_k\Bigg). \end{aligned} \tag{3.44}$$

For brevity, we denoted by $\rho(x_i^0, x_i)$ the distance between the points z^0 and z that have the ith coordinates equal to x_i^0 and x_i, respectively, and all the other coordinates are equal, for all $i = 1, \dots, n$, and $x_i^0, x_i \in \bar{\mathbb{R}}_+$. We define similarly $\rho(y_l^0, y_l)$, for all $y_l^0, y_l \in \mathbb{R}$ and all $l = 1, \dots, m$. The parenthesis on the right-hand side of inequality (3.44) can be written as the sum of three terms, $I_1 + I_2 + I_3$. We show that there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ and $c = c(m, n)$, such that

$$I_1 + I_2 + I_3 \le Ce^{-\frac{cq}{t^{1-2\alpha}}}, \tag{3.45}$$

which implies estimate (3.39) by inequality (3.44). We will only give the detailed proof of the fact that

$$I_1 \le Ce^{-\frac{cq}{t^{1-2\alpha}}}, \tag{3.46}$$

because the estimates for the integrals I_2 and I_3 can be obtained by a similar argument. Let $i \in I(z^0)$. Using property (3.5) of the distance function ρ, there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ and $c = c(m, n)$, such that

$$\begin{aligned}\int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|>t^\alpha\}} e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}} x_i^{\bar{b}/4-1} \mathbf{1}_{\{x_i\in(0,r_0/2)\}}\, dx_i &\le e^{-\frac{qc}{8t^{1-2\alpha}}} \int_0^{r_0/2} x_i^{\bar{b}/4-1}\, dx_i \\ &\le Ce^{-\frac{qc}{t^{1-2\alpha}}}.\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}\int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|>t^\alpha\}} e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}} x_i^{-K-1} \mathbf{1}_{\{x_i\in[r_0/2,1)\}}\, dx_i &\le e^{-\frac{qc}{8t^{1-2\alpha}}} \int_{r_0/2}^1 x_i^{-K-1}\, dx_i \\ &\le Ce^{-\frac{qc}{t^{1-2\alpha}}},\end{aligned}$$

and we also have that

$$\begin{aligned}\int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|>t^\alpha\}} e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}} \mathbf{1}_{\{x_i\in[1,\infty)\}}\, dx_i &\le \int_1^\infty e^{-\frac{qc|x_i-x_i^0|^2}{8t}}\, dx_i \\ &\le Ce^{-\frac{qc}{t}},\end{aligned}$$

for some positive constant, $c = c(m, n)$. In the last inequality, we used the fact that $|x_i - x_i^0| \ge |1 - r_0| > 0$, since $x_i \in [1, \infty)$ and $x_0^i \in (0, r_0)$, where we recall that $i \in I(z^0)$ and the set of indices $I(z^0)$ is defined in (3.25). Using definition (3.43) of the function $\varphi(x_i)$, we see that there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ and $c = c(m, n)$, such that

$$\int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|>t^\alpha\}} e^{-\frac{qc\rho^2(x_i,x_i^0)}{8t}} \varphi(x_i)\, dx_i \le Ce^{-\frac{qc}{t^{1-2\alpha}}}.$$

We notice that, for all $j \in \{1, \dots, n\}$ and $l \in \{1, \dots, m\}$, we also have the very rough estimates

$$\int_0^\infty e^{-\frac{qc\rho^2(x_j, x_j^0)}{8t}} \varphi(x_j)\, dx_j + \int_{\mathbb{R}} e^{-\frac{qc\rho^2(y_l, y_l^0)}{8t}}\, dy_l \leq C,$$

where $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ is a positive constant. Thus, combining the preceding three inequalities it follows that estimate (3.46) holds. A similar argument implies estimate (3.45), and inequality (3.44) yields estimate (3.39). This completes the proof of Step 1.

STEP 2 (Estimate of $\|\tilde{p}(t, z^0, \cdot)\|_{L^q(A_t(z^0);d\mu)}$). In this step, we prove that there is a positive constant, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, such that

$$\begin{aligned} &\|\tilde{p}(t, z^0, \cdot)\|^q_{L^q(A_t(z^0);d\mu)} \\ &\qquad \leq C t^{(m+n)(\alpha - q/2)} \big(t^{nK(-q/2-\alpha q+2\alpha)} + t^{2nK(1-q)}\big), \end{aligned} \tag{3.47}$$

for all $t \in (0, T]$, where the positive constant T is chosen to satisfy conditions (3.49) and (3.52) below. From inequality (3.37), we may assume without loss of generality that the positive constant r_0 is small enough such that there is a positive constant, $C_1 = C_1(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$, with the property that $x_i \geq C_1$, for all $z = (x, y) \in A_t(z_0)$, $i \in I^c(z^0)$ and $t \in [0, T]$. Using the fact that the coefficient function $b(z)$ belongs to $C^1(\bar{S}_{n,m}; \mathbb{R}^n)$, and letting $c_1 = \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}$, we have that

$$\big|b_i(z) - b_i(z^0)\big| \leq c_1 t^\alpha, \qquad \forall z \in A_t(z^0), \forall i = 1, \dots, n. \tag{3.48}$$

From the first inequality in (3.26), we have that $b_i(z^0) \geq \bar{b}/4$, for all $i \in I(z^0)$. Choosing the positive constant T such that

$$T \leq \left(\frac{\bar{b}}{8c_2}\right)^{1/\alpha}, \tag{3.49}$$

we have that

$$b_i(z^0) - c_2 t^\alpha \geq \frac{\bar{b}}{8} > 0, \qquad \forall z = (x, y) \in A_t(z^0), \forall i \in I(z^0), \forall t \in [0, T],$$

where T satisfies the bounds (3.37) and (3.49). From (3.48), it follows that

$$\begin{aligned} &b_i(z^0) + c_2 t^\alpha \geq b_i(z) \geq b_i(z^0) - c_2 t^\alpha > 0, \\ &\qquad\qquad \forall z = (x, y) \in A_t(z^0), \forall i \in I(z^0). \end{aligned} \tag{3.50}$$

Choosing now $i \in I^c(z^0)$, we have that $x_i^0 \geq r_0$, and so, it follows that

$$x_i \geq x_i^0 - t^\alpha \geq r_0 - t^\alpha \geq \frac{r_0}{2}, \qquad \forall z = (x, y) \in A_t(z^0), \forall t \in [0, T], \tag{3.51}$$

where we choose T such that it satisfies the upper bound

$$T \leq \left(\frac{r_0}{2}\right)^{1/\alpha}. \tag{3.52}$$

Inequality (3.40) holds with $A_t^c(z^0)$ replaced by $A_t(z^0)$, and so, using property (3.5) of the distance function $\rho(z, z^0)$, definition (1.12) of the measure $d\mu(z)$, and inequalities (3.50) and (3.51), we obtain that there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n)$ and $c = c(m, n)$, such that

$$\begin{aligned}
&\|\tilde{p}(t, z^0, \cdot)\|^q_{L^q(A_t(z^0);d\mu)} \\
&\quad \leq C t^{-q(m+n)/2} \prod_{i\in I(z^0)} \left(x_i^0 \vee t\right)^{-q(b_i(z^0)-1/2)/2} \\
&\qquad \times \prod_{i\in I(z^0)} \int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|\leq t^\alpha\}} (x_i \vee t)^{-q(b_i(z^0)+c_2t^\alpha-1/2)/2} \\
&\qquad \times e^{-\frac{qc|\sqrt{x_i}-\sqrt{x_i^0}|^2}{8t}} x_i^{b_i(z^0)-c_2t^\alpha-1}\, dx_i \\
&\qquad \times \prod_{j\in I^c(z^0)} \int_{\{|x_j-x_j^0|\leq t^\alpha\}} e^{-\frac{qc|x_j-x_j^0|^2}{8t}}\, dx_j \prod_{l=1}^m \int_{\{|y_l-y_l^0|\leq t^\alpha\}} e^{-\frac{qc|y_l-y_l^0|^2}{8t}}\, dy_l.
\end{aligned}$$

The preceding inequality holds for all $t \in (0, T]$, where T satisfies both inequalities (3.49) and (3.52). The integrals in the last two product terms of the preceding inequality can all be bounded by t^α, and so, it follows that

$$\begin{aligned}
&\|\tilde{p}(t, z^0, \cdot)\|^q_{L^q(A_t(z^0);d\mu)} \\
&\quad \leq C t^{-q(m+n)/2} t^{\alpha(|I^c(z^0)|+m)} t^{q|I(z^0)|/2} \prod_{i\in I(z^0)} \left(x_i^0 \vee t\right)^{-qb_i(z^0)/2} \\
&\qquad \times \prod_{i\in I(z^0)} \int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|\leq t^\alpha\}} (x_i \vee t)^{-q(b_i(z^0)+c_2t^\alpha)/2} x_i^{b_i(z^0)-c_2t^\alpha-1}\, dx_i.
\end{aligned}$$

Direct calculations give us that there is a positive constant,

$$C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n),$$

such that for all $i \in I(z^0)$, we have that

$$\begin{aligned}
&\int_{\{|\sqrt{x_i}-\sqrt{x_i^0}|\leq t^\alpha\}} (x_i \vee t)^{-q(b_i(z^0)+c_2t^\alpha)/2} x_i^{b_i(z^0)-c_2t^\alpha-1}\, dx_i \\
&\qquad \leq C\left(\sqrt{x_i^0} + t^\alpha\right)^{-qb_i(z^0)+2b_i(z^0)}.
\end{aligned}$$

The preceding two inequalities yield

$$
\begin{aligned}
&\|\tilde{p}(t,z^0,\cdot)\|^q_{L^q(A_t(z^0);d\mu)} \\
&\quad \le Ct^{(m+|I^c(z^0)|)(\alpha-q/2)} \\
&\qquad \times \prod_{i\in I(z^0)} (x_i^0 \vee t)^{-qb_i(z^0)/2}\left(\sqrt{x_i^0}+t^\alpha\right)^{-qb_i(z^0)+2b_i(z^0)}.
\end{aligned}
\tag{3.53}
$$

Using condition (2.7) for the coefficients $b_i(z)$, we see that if $0 \le x_i^0 \le \sqrt{t}$, we have that

$$
(x_i^0 \vee t)^{-qb_i(z^0)/2}\left(\sqrt{x_i^0}+t^\alpha\right)^{-qb_i(z^0)+2b_i(z^0)} \le Ct^{K(-q/2-\alpha q+2\alpha)},
$$

while if $\sqrt{t} < x_i^0 \le r_0$, we obtain

$$
(x_i^0 \vee t)^{-qb_i(z^0)/2}\left(\sqrt{x_i^0}+t^\alpha\right)^{-qb_i(z^0)+2b_i(z^0)} \le Ct^{2\alpha K(1-q)}.
$$

The preceding two inequalities together with estimate (3.53) imply that

$$
\begin{aligned}
&\|\tilde{p}(t,z^0,\cdot)\|^q_{L^q(A_t(z^0);d\mu)} \\
&\quad \le Ct^{(m+n)(\alpha-q/2)}\left(t^{nK(-q/2-\alpha q+2\alpha)}+t^{2\alpha nK(1-q)}\right),
\end{aligned}
\tag{3.54}
$$

which immediately implies inequality (3.47) since we assume that $\alpha \in (0,1/2)$ and $q \in [1,\infty)$. This completes the proof of Step 2.

Combining inequalities (3.39) and (3.47), there are positive constants, $q_0 = q_0(K,m,n) \in (1,2)$ and $\alpha \in (0,1/2)$, such that for all $q \in [1,q_0)$, there are positive constants, $C = C(\bar{b}, \|b\|_{C^1(\bar{S}_{n,m};\mathbb{R}^n)}, m, n, q)$ and $\beta = \beta(q) < 1$, such that estimate (3.8) holds. This completes the proof. □

4. Stochastic representation of solutions. This section contains the proofs of Theorems 1.3 and 1.6. We describe in Section 4.1 the relationship between the generalized Kimura operator L, defined in (1.10), and the generalized Kimura stochastic differential equation (1.11). We recall that the stochastic representation in Theorem 1.3 cannot be obtained by a direct application of Itô's rule, and so our strategy of the proof is to first establish in Section 4.2 that the statement of Theorem 1.3 holds when $\Omega = S_{n,m}$, and then to extend this result in Section 4.3 to arbitrary sub-domains Ω of $S_{n,m}$, by using the part of a Dirichlet form; see Fukushima, Oshima and Takeda (2011), Section 4.4. Theorem 1.6 is proved in Section 4.4.

4.1. *The generalized Kimura stochastic differential equation.* We discuss the stochastic differential equation (1.11) and the relationship with the Kimura operator L defined in (1.10). The coefficients of the stochastic differential equation (1.11) are related to the coefficients of the differential operator L defined in (1.10), as follows. For all $i = 1, \dots, n$, $j = 1, \dots, n+m$, and $l = 1, \dots, m$, we let

$$
\begin{aligned}
g_i(z) &:= b_i a_{ii} + x_i \Bigg(\partial_{x_i} a_{ii} + \sum_{j=1}^{n} (\tilde{a}_{ij} + \delta_{ij}\tilde{a}_{ii} \\
&\qquad + x_j \partial_{x_j} \tilde{a}_{ij} + \tilde{a}_{ij}(b_j - 1)) + \sum_{l=1}^{m} \partial_{y_l} c_{il} \Bigg), \\
e_l(z) &:= \sum_{i=1}^{n} (x_i \partial_{x_i} c_{il} + b_i c_{il}) + \sum_{k=1}^{m} \partial_{y_k} d_{lk}, \\
f_{ij}(z) &:= \sum_{j=1}^{n} \Bigg(\partial_{x_i} b_j + \sum_{k=1}^{n} x_k \tilde{a}_{ik} \partial_{x_k} b_j + \sum_{l=1}^{m} c_{il} \partial_{y_l} b_j \Bigg), \\
f_{n+l,j}(z) &:= \sum_{i=1}^{n} x_i c_{il} \partial_{x_i} b_j + \sum_{k=1}^{m} d_{lk} \partial_{y_k} b_j.
\end{aligned}
\tag{4.1}
$$

To construct the dispersion coefficient matrix, $(\sigma(z))$, appearing in (1.11), we introduce the diffusion matrix, $(D(z))$, by letting, for all $i, j = 1, \dots, n$ and all $l, k = 1, \dots, m$,

$$
\begin{aligned}
D_{ii}(z) &:= 2\big(a_{ii}(z) + x_i \tilde{a}_{ii}(z)\big), \\
D_{ij}(z) &:= 2\sqrt{x_i x_j}\,\tilde{a}_{ij}(z), \qquad i \neq j, \\
D_{i,n+l}(z) = D_{n+l,i}(z) &:= 4\sqrt{x_i}\,c_{il}(z), \qquad D_{n+l,n+k}(z) = 2 d_{lk}(z).
\end{aligned}
\tag{4.2}
$$

We now argue that there is a matrix $(\sigma(z))$ such that

$$
\big(\sigma\sigma^*\big)(z) = D(z), \qquad \forall z \in \bar{S}_{n,m}. \tag{4.3}
$$

Notice that conditions (2.4) and (2.5) imply that the matrix $(D(z))$ is strictly elliptic. From Assumption 2.1, it follows that the coefficients a_{ii}, $\tilde{a}_{ij}$, c_{il} and d_{lk} are smooth functions of the variable $z = (x, y)$ on $\bar{S}_{n,m}$, and in particular they are smooth functions of the variable $(\sqrt{x}, y)$ on $\bar{S}_{n,m}$, where we denote

$$
\sqrt{x} = (\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_n}), \qquad \forall x \in \mathbb{R}^n_+. \tag{4.4}
$$

We obtain that the matrix D defined in (4.2) is smooth in the variables $(\sqrt{x}, y)$ on $\bar{S}_{n,m}$. We let $\widetilde{D}(\sqrt{x}, y) = D(x, y)$, for all $(x, y) \in \bar{S}_{n,m}$. Because the matrix $\widetilde{D}$ is strictly elliptic and smooth, we can build an extension from $\bar{S}_{n,m}$ to $\mathbb{R}^{n+m}$, which we denote the same as the matrix $\widetilde{D}$, such that the extended matrix remains strictly elliptic and smooth on $\mathbb{R}^{n+m}$. We can now apply Friedman

(1975/1976), Lemma 6.1.1, to the matrix $\widetilde{D}$, to obtain that there is a matrix $\widetilde{\sigma} \in C^{\infty}(\mathbb{R}^{n+m}; \mathbb{R}^{n+m} \times \mathbb{R}^{n+m})$, such that $\widetilde{\sigma}\widetilde{\sigma}^* = \widetilde{D}$. Letting now $\sigma(x^2, y) := \widetilde{\sigma}(x, y)$, we obtain identity (4.3) and that σ is a smooth function in the variables $(\sqrt{x}, y)$ on $\bar{S}_{n,m}$. The choice of the 'square root', $(\sigma(z))$, of the positive-definite matrix $(D(z))$ is irrelevant for the question of existence and uniqueness of weak solutions to the stochastic differential equations (1.11), as Karatzas and Shreve (1991), Problem 5.4.7, shows.

The preceding analysis, together with Assumption 2.1, implies that the hypotheses of Pop (2017a), Theorems 3.1 and 3.7, are verified. Thus, we obtain that, for all $z \in \bar{S}_{n,m}$, the generalized Kimura equation (1.11) has a unique weak solution, $\{Z(t)\}_{t\geq 0}$, with initial condition $Z(0) = z$, that satisfies the strong Markov property. We denote by $\mathbb{P}^z$ the probability law of the process $\{Z(t)\}_{t\geq 0}$, with initial condition $Z(0) = z$.

4.2. *Stochastic representation of solutions on $S_{n,m}$.* We prove Theorem 1.3 in the particular case when $\Omega = S_{n,m}$. We obtain a slightly stronger statement because we obtain that the set N of zero μ-measure set, appearing in the statement of Theorem 1.3, is empty when $\Omega = S_{n,m}$.

THEOREM 4.1 (Stochastic representation of weak solutions on $S_{n,m}$). *Suppose that the coefficients of the operator L satisfy Assumption 2.1. Given any function, $f \in L^2(\bar{S}_{n,m}; d\mu)$, we have that*

$$T_t f(z) = \mathbb{E}_{\mathbb{P}^z}\left[f\left(Z(t)\right)\right], \qquad \forall t \geq 0, \forall z \in \bar{S}_{n,m}. \tag{4.5}$$

Before we can give the proof of Theorem 4.1, we need to establish several intermediate results. We first prove that the stochastic representation (4.5) holds when the weak solution to the Kimura equation (1.11), $\{Z(t)\}_{t\geq 0}$, is replaced by a suitable Hunt process, $\{Z^0(t)\}_{t\geq 0}$.

PROPOSITION 4.2 (Stochastic representation using a Hunt process). *Suppose that the coefficients of the operator L satisfy Assumption 2.1. Then there is a filtered probability space, $(\mathcal{Z}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{\mathbb{P}^{0,z}\}_{z\in\bar{S}_{n,m}})$, and a progressively measurable process, $Z^0 : [0, \infty) \times \mathcal{Z} \to \bar{S}_{n,m}$, such that the following hold*:

(i) *For all $z \in \bar{S}_{n,m}$, almost surely with respect to the probability measure $\mathbb{P}^{0,z}$, we have that $Z^0(0) = z$ and $\{Z^0(t)\}_{t\geq 0}$ has continuous paths.*

(ii) *The process $\{Z^0(t)\}_{t\geq 0}$ has the strong Markov property.*

(iii) *For all functions, $f \in L^2(\bar{S}_{n,m}; d\mu)$, we have*

$$T_t f(z) = \mathbb{E}_{\mathbb{P}^{0,z}}\left[f\left(Z^0(t)\right)\right], \qquad \forall t \geq 0, \forall z \in \bar{S}_{n,m}. \tag{4.6}$$

PROOF. Lemmas 3.2 and 3.4 give us that the set $\{p(t, z, w)\, d\mu(w)\}$ forms a family of consistent transition probability densities on $\bar{S}_{n,m}$, and so the Daniell–Kolmogorov theorem implies that property (i) holds. Proposition 3.6 allows us to apply the Centsov–Kolmogorov theorem in Karatzas and Shreve (1991), Theorem 2.2.8, to conclude that we can assume without loss of generality that the process $\{Z^0(t)\}_{t\geq 0}$ has continuous paths. Moreover, the process $\{Z^0(t)\}_{t\geq 0}$ satisfies the Markov property and property (iii) holds when $f : S_{n,m} \to \mathbb{R}$ is a bounded, Borel measurable function. The continuity of the fundamental solution $\{p(t, z, w)\}$, established in Lemma 3.2, together with the supremum estimate (3.23), and the continuity of the paths of the process $\{Z^0(t)\}_{t\geq 0}$, allows us to apply Blumenthal and Getoor (1968), Theorem 1.8.11, to conclude that the process also has the strong Markov property. It remains to prove that property (iii) holds, not only when $f : S_{n,m} \to \mathbb{R}$ is a bounded, Borel measurable function, but also for all $f \in L^2(\bar{S}_{n,m}; d\mu)$. This follows by an approximation procedure by choosing a sequence of bounded, Borel measurable functions $\{f_k\}_{k\in\mathbb{N}}$ that converges in $L^2(\bar{S}_{n,m}; d\mu)$ to f. We omit the remaining details of the proof. □

We see that Theorem 4.1 follows from Proposition 4.2, if we prove that the law $\mathbb{P}^z$ of the unique weak solution, $\{Z(t)\}_{t\geq 0}$, to the Kimura equation (1.11), coincides with the law $\mathbb{P}^{0,z}$ of the Hunt process, $\{Z^0(t)\}_{t\geq 0}$, constructed in Proposition 4.2. We achieve this by using the formalism of the martingale problem associated to the operator L. Let $\mathcal{X} := C([0, \infty); \bar{S}_{n,m})$ be the space of continuous functions, $\omega : [0, \infty) \to \bar{S}_{n,m}$. For all $t \geq 0$, let $\mathcal{B}_t$ be the σ-algebra on $\mathcal{X}$ generated by the cylinder sets,

$$C := \{\omega \in \mathcal{X} : \omega(t_i) \in B_i,\, B_i \subseteq \bar{S}_{n,m} \text{ Borel measurable}, \forall i = 1, \dots, k\},$$

where $k \in \mathbb{N}$, and $0 \leq t_1 < \cdots < t_k \leq t$. We let $\mathcal{B} := \bigcup_{t\geq 0} \mathcal{B}_t$. We can now introduce the following.

DEFINITION 4.3 (A martingale problem associated to the operator L). Let $z \in \bar{S}_{n,m}$. A probability measure, $\mathbb{Q}^z$, on the filtered probability space $(\mathcal{X}, \{\mathcal{B}_t\}_{t\geq 0}, \mathcal{B})$ is a solution to the martingale problem associated to the operator L, if for all functions $\varphi \in C_c^\infty(\bar{S}_{n,m})$, the process defined by

$$M^\varphi(t, \omega) := \varphi(\omega(t)) - \varphi(\omega(0)) - \int_0^t L\varphi(\omega(r))\, dr, \qquad \forall t \geq 0, \forall \omega \in \mathcal{X}, \tag{4.7}$$

is a $\mathbb{Q}^z$-martingale with respect to the filtration $\{\mathcal{B}_t\}_{t\geq 0}$, and $\mathbb{Q}^z(\omega(0) = z) = 1$.

PROPOSITION 4.4 (Solutions to the martingale problem). *Let $z \in \bar{S}_{n,m}$. Then there is a solution, $\mathbb{Q}^z$, to the martingale problem in Definition 4.3. Moreover, we can choose $\mathbb{Q}^z$ so that it coincides with the probability law of the process $\{Z^0(t)\}_{t\geq 0}$, constructed in Proposition 4.2.*

PROOF. Let $\mathbb{Q}^z$ be the probability measure induced on the space $\mathcal{X}$ by the probability measure $\mathbb{P}^{0,z}$ constructed in Proposition 4.2. It is sufficient to show that the processes defined in (4.7) are $\mathbb{Q}^z$-martingales. We split the proof into two steps. For $\varphi \in C_c^\infty(\bar{S}_{n,m})$, we see that $L\varphi$ is unbounded on $S_{n,m}$ in general, and so we first prove that

$$\mathbb{E}_{\mathbb{Q}^z}\left[\int_0^t \left|L\varphi(\omega(r))\right| dr\right] < \infty, \qquad \forall t \geq 0, \tag{4.8}$$

using the L^q-distribution estimates established in Proposition 3.7. In the second step, we prove that the process $\{M^\varphi(t)\}_{t\geq 0}$ defined by (4.7) is indeed a $\mathbb{Q}^z$-martingale with respect to the filtration $\{\mathcal{B}_t\}_{t\geq 0}$.

From the expression (1.10) of the differential operator L and the fact that $\varphi \in C_c^\infty(\bar{S}_{n,m})$, we see that $L\varphi \in L^p(\bar{S}_{n,m}; d\mu)$, for all finite $p \geq 1$. Let $q_0 \in (1, 2)$ be the constant appearing in the conclusion of Proposition 3.7 and choose $q \in (1, q_0)$. Let $p \in (1, \infty)$ be the conjugate exponent of q. Using properties (4.6) and (3.2), we have that

$$\mathbb{E}_{\mathbb{Q}^z}\left[\int_0^t \left|L\varphi(\omega(r))\right| dr\right] = \int_0^t \int_{S_{n,m}} \left|L\varphi(w)\right| p(r, z, w)\, d\mu(w)\, dr,$$

and by Hölder's inequality, we obtain

$$\mathbb{E}_{\mathbb{Q}^z}\left[\int_0^t \left|L\varphi(\omega(r))\right| dr\right] \leq \int_0^t \|L\varphi\|_{L^p(\bar{S}_{n,m}; d\mu)} \|p(r, z, \cdot)\|_{L^q(\bar{S}_{n,m}; d\mu)}\, dr.$$

Inequality (3.9) shows that there is a positive constant, C_0, such that

$$\mathbb{E}_{\mathbb{Q}^z}\left[\int_0^t \left|L\varphi(\omega(r))\right| dr\right] \leq C_0 \|L\varphi\|_{L^p(\bar{S}_{n,m}; d\mu)},$$

which concludes the proof of inequality (4.8).

To prove that the process $\{M^\varphi(t)\}_{t\geq 0}$ defined by (4.7) is indeed a $\mathbb{Q}^z$-martingale it is sufficient to prove that, for all $0 \leq s \leq t$, we have

$$\mathbb{E}_{\mathbb{Q}^z}\left[\varphi(\omega(t)) | \mathcal{B}_s\right] = \varphi(\omega(s)) + \mathbb{E}_{\mathbb{Q}^z}\left[\int_s^t L\varphi(\omega(r))\, dr \Big| \mathcal{B}_s\right], \qquad \mathbb{Q}^z\text{-a.s.}, \tag{4.9}$$

which can be rewritten in the form

$$T_{t-s}\varphi(\omega(s)) - \varphi(\omega(s)) = \int_0^{t-s} T_r L\varphi(\omega(s))\, dr. \tag{4.10}$$

Applying Ethier and Kurtz (1986), Proposition 1.1.5, we know that the equality

$$T_{t-s}\varphi - \varphi = \int_0^{t-s} T_r L\varphi\, dr \tag{4.11}$$

holds in the $L^2(\bar{S}_{n,m}; d\mu)$-sense. Lemma 3.1 and our construction of the probability measure $\mathbb{Q}^z$ show that the marginal distributions of $\mathbb{Q}^z$ are given by $p(t, z, \cdot)\, d\mu$, and so they are absolutely continuous with respect to the weight $d\mu$. Thus, identity (4.11) implies (4.10). This concludes the proof. $\square$

We can now prove Theorem 4.1.

PROOF OF THEOREM 4.1. Applying the method of the proof of Karatzas and Shreve (1991), Proposition 5.4.6, adapted to our framework, we obtain that the solution, $\mathbb{Q}^z$, to the martingale problem constructed in Proposition 4.4 induces a solution, $\{Z(t)\}_{t\geq 0}$, to the Kimura equation (1.11), which has the same law as $\mathbb{Q}^z$. From part (ii) of Proposition 4.2, it follows that the process $\{Z(t)\}_{t\geq 0}$ has the strong Markov property, and so by Pop (2017a), Theorem 3.7, it is the unique weak solution to the Kimura equation (1.11) that has the strong Markov property and satisfies the initial condition $Z(0) = z$. Moreover, identity (4.6) implies (4.5). This completes the proof. $\square$

4.3. *Stochastic representation of solutions on sub-domains of $S_{n,m}$.* In this section, we use Theorem 4.1 together with the formalism of Dirichlet forms to give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. We prove the stochastic representation formula (1.13) with the aid of the part of the Dirichlet form $(Q, H^1_0(\bar{S}_{n,m}; d\mu))$ on $\underline{\Omega}$; see Fukushima, Oshima and Takeda (2011), Section 4.4. Let

$$\mathcal{F}_\Omega := \{u \in H^1_0(\bar{S}_{n,m}; d\mu) : u = 0 \text{ q.e. on } \bar{S}_{n,m} \backslash \underline{\Omega}\},$$

where $u = 0$ q.e. on $\bar{S}_{n,m} \backslash \underline{\Omega}$ means that the equality holds quasi-everywhere, that is, except for sets of capacity zero; see Fukushima, Oshima and Takeda (2011), Section 2.1, for the definition of the capacity of a set. By the observation following the proof of Fukushima, Oshima and Takeda (2011), Theorem 4.4.2, because the set $\bar{S}_{n,m} \backslash \underline{\Omega}$ is relatively open with respect to $\bar{S}_{n,m}$, it follows that the hypotheses of Fukushima, Oshima and Takeda (2011), Theorem 4.4.2, are satisfied, and so $(Q_\Omega, \mathcal{F}_\Omega)$ defines a Dirichlet form on $L^2(\underline{\Omega}; d\mu)$. Because sets of capacity zero have zero μ-measure, it follows that

$$\mathcal{F}_\Omega \subseteq \{u \in H^1_0(\bar{S}_{n,m}; d\mu) : u = 0 \ \mu\text{-a.e. on } \bar{S}_{n,m} \backslash \underline{\Omega}\},$$

and using the fact that the right-hand side from the preceding inclusion is equal to $H^1_0(\underline{\Omega}; d\mu)$, it follows that $\mathcal{F}_\Omega \subseteq H^1_0(\underline{\Omega}; d\mu)$. Since $(Q_\Omega, H^1_0(\underline{\Omega}; d\mu))$ is the smallest closed extension of Q_Ω on $L^2(\underline{\Omega}; d\mu)$ with core $C^\infty_c(\underline{\Omega})$, it follows that $\mathcal{F}_\Omega = H^1_0(\underline{\Omega}; d\mu)$. Thus, $(Q_\Omega, H^1_0(\underline{\Omega}; d\mu))$ is the part of the Dirichlet form $(Q, H^1_0(\bar{S}_{n,m}; d\mu))$ on the subset $\underline{\Omega}$. From Theorem 4.1 and Fukushima, Oshima and Takeda (2011), Theorem A.2.10, it follows that, for all $f \in L^2(\underline{\Omega}; d\mu)$ and all $t > 0$, we have

$$T^\Omega_t f(z) = \mathbb{E}_{\mathbb{P}^z}\left[f(Z(t))\mathbf{1}_{\{t<\tau_\Omega\}}\right] \qquad \text{for } \mu\text{-a.e. } z \in \underline{\Omega}. \tag{4.12}$$

We now fix $f \in L^2(\underline{\Omega}; d\mu)$. It remains to prove that there is a measurable set $N \subset \underline{\Omega}$ with zero μ-measure, such that identity (1.13) holds, where we recall that $u(t, z) = (T^\Omega_t f)(z)$. For all $t \in \mathbb{Q}_+$, using (4.12), there is a set $N_t \subset \underline{\Omega}$ with zero

μ-measure, such that (1.13) holds, for all $z \in \underline{\Omega}\backslash N_t$. Letting $N = \cup\{N_t : t \in \mathbb{Q}_+\}$, we have that N has zero μ-measure, and that

$$T_t^{\Omega} f(z) = \mathbb{E}_{\mathbb{P}^z}\big[f\big(Z(t)\big)\mathbf{1}_{\{t<\tau_\Omega\}}\big], \qquad \forall t \in \mathbb{Q}_+, \forall z \in \underline{\Omega}\backslash N. \tag{4.13}$$

We first show that the preceding identity holds at all $t > 0$ and $z \in \underline{\Omega}\backslash N$, when $f \in L^2(\underline{\Omega}; d\mu)$ is a *continuous* function. Let $(t, z) \in (0, \infty) \times (\underline{\Omega}\backslash N)$, and let $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_+$ be a decreasing sequence converging to t. By Epstein and Mazzeo (2016), Corollary 4.1, we have that $(T_{t_k}^{\Omega} f)(z)$ converges to $(T_t^{\Omega} f)(z)$, as t_k tends to t, for all $z \in \underline{\Omega}$, and so we only need to show that

$$\mathbb{E}_{\mathbb{P}^z}\big[f\big(Z(t_k)\big)\mathbf{1}_{\{t_k<\tau_\Omega\}}\big] \to \mathbb{E}_{\mathbb{P}^z}\big[f\big(Z(t)\big)\mathbf{1}_{\{t<\tau_\Omega\}}\big] \qquad \text{as } k \to \infty. \tag{4.14}$$

Because $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_+$ is a decreasing sequence converging to t, we have that $\mathbf{1}_{\{t_k<\tau_\Omega\}} \to \mathbf{1}_{\{t<\tau_\Omega\}}$, and because f is continuous and the paths of the process $\{Z(t)\}_{t\geq 0}$ are continuous, we have that $f(Z(t_k)) \to f(Z(t))$. Thus, we have the $\mathbb{P}^z$-a.s. convergence,

$$f\big(Z(t_k)\big)\mathbf{1}_{\{t_k<\tau_\Omega\}} \to f\big(Z(t)\big)\mathbf{1}_{\{t<\tau_\Omega\}}, \qquad \mathbb{P}^z\text{-a.s.} \tag{4.15}$$

We next show that the random variables $\{f(Z(t_k))\mathbf{1}_{\{t_k<\tau_\Omega\}}\}_{k\in\mathbb{N}}$ are uniformly integrable. We see that, by choosing $q \in (1, 2)$,

$$\mathbb{E}_{\mathbb{P}^z}\big[\big|f\big(Z(t_k)\big)\mathbf{1}_{\{t_k<\tau_\Omega\}}\big|^q\big] \leq \int_{S_{n,m}} p(t_k, z, w)\big|f(w)\big|^q \, d\mu(w),$$

where we extend f to $\bar{S}_{n,m}$ by letting $f = 0$ on $\bar{S}_{n,m}\backslash\underline{\Omega}$, and Hölder's inequality gives us that

$$\mathbb{E}_{\mathbb{P}^z}\big[\big|f\big(Z(t_k)\big)\mathbf{1}_{\{t_k<\tau_\Omega\}}\big|^q\big] \leq \|f\|_{L^2(\underline{\Omega};d\mu)}\big\|p(t_k, z, \cdot)\big\|_{L^{2/(2-q)}(\bar{S}_{n,m};d\mu)}.$$

The supremum estimate (3.23) gives us that there is a positive constant, $C = C(q, t, z)$, such that

$$\big\|p(t_k, z, \cdot)\big\|_{L^{2/(2-q)}(\bar{S}_{n,m};d\mu)} \leq C, \qquad \forall k \in \mathbb{N},$$

which yields

$$\mathbb{E}_{\mathbb{P}^z}\big[\big|f\big(Z(t_k)\big)\mathbf{1}_{\{t_k<\tau_\Omega\}}\big|^q\big] \leq C\|f\|_{L^2(\underline{\Omega};d\mu)}, \qquad \forall k \in \mathbb{N},$$

and so the family of random variables $\{f(Z(t_k))\mathbf{1}_{\{t_k<\tau_\Omega\}}\}_{k\in\mathbb{N}}$ is uniformly integrable, by the observation following Billingsley (1995), Theorem 16.13. Using (4.15) and the previous property, it follows from Billingsley (1995), Theorem 16.13, that (4.14) holds when $f \in L^2(\underline{\Omega}; d\mu)$ is *continuous*.

We now let f be an arbitrary function in $L^2(\underline{\Omega}; d\mu)$, and prove that there is a set $N \subset \underline{\Omega}$ with zero μ-measure, such that identity (1.13) holds, for all $t > 0$ and $z \in \underline{\Omega}\backslash N$. Let $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\underline{\Omega}; d\mu)$ be a sequence of continuous functions that converge in $L^2(\underline{\Omega}; d\mu)$ to f. Let $N_k \subset \underline{\Omega}$ be a set of zero μ-measure such that (1.13) holds with f replaced by f_k, for all $t > 0$ and $z \in \underline{\Omega}\backslash N_k$. Setting $N :=$

$\cup\{N_k : k \in \mathbb{N}\}$, we obtain a set of zero μ-measure. By Sturm (1995), Theorem 2.1, for all $0 < t_1 < t_2$ and all compact sets, $K \subset \bar{S}_{n,m}$, there is a positive constant, $C = C(t_1, t_2, K)$, such that for all $t \in [t_1, t_2]$ and all $z \in K$, we have that

$$\begin{aligned}\left|T_t^{\Omega}(f_k - f)(z)\right| &\leq C\left(\int_{t_1/2}^{t_2} \|T_s^{\Omega}(f_k - f)\|^2_{L^2(\underline{\Omega}; d\mu)}\, ds\right)^{1/2} \\ &\leq C\left(t_2 - \frac{t_1}{2}\right)^{1/2} \|f_k - f\|_{L^2(\underline{\Omega}; d\mu)},\end{aligned}$$

where in the second inequality we used the contraction property of the semigroup $\{T_t^{\Omega}\}_{t\geq 0}$. Thus, we clearly have that $T_t^{\Omega} f_k(z) \to T_t^{\Omega} f(z)$, for all $t > 0$ and all $z \in \underline{\Omega} \backslash N$. We also have

$$\begin{aligned}\left|\mathbb{E}_{\mathbb{P}^z}\left[f_k(Z(t))\mathbf{1}_{\{t<\tau_\Omega\}}\right] - \mathbb{E}_{\mathbb{P}^z}\left[f(Z(t))\mathbf{1}_{\{t<\tau_\Omega\}}\right]\right| &\leq \mathbb{E}_{\mathbb{P}^z}\left[\left|f_k(Z(t)) - f(Z(t))\right|\right] \\ &= T_t|f_k - f|(z) \qquad \text{(by (4.5))} \\ &\leq \|f_k - f\|_{L^2(\bar{S}_{n,m}; d\mu)},\end{aligned}$$

where in the last inequality we used the contraction property of $\{T_t\}_{t\geq 0}$, and we extend f_k and f by zero outside $\underline{\Omega}$. Thus, we also have that

$$\mathbb{E}_{\mathbb{P}^z}\left[f_k(Z(t))\mathbf{1}_{\{t<\tau_\Omega\}}\right] \to \mathbb{E}_{\mathbb{P}^z}\left[f(Z(t))\mathbf{1}_{\{t<\tau_\Omega\}}\right] \qquad \text{as } k \to \infty.$$

This concludes the proof that (1.13) holds, for all $t > 0$ and $z \in \underline{\Omega} \backslash N$. □

REMARK 4.5 (The set of measure zero appearing in Theorem 1.3). In our original proof of Theorem 1.3 in Epstein and Pop (2014), Theorem 1.3, we obtain the stochastic representation (1.13) with $N = \varnothing$. The appearance of the set N with zero μ-measure in Theorem 1.3 is due to our use of results from the theory of Dirichlet forms, while in Epstein and Pop (2014), Theorem 1.3, we apply more elementary methods, which allow us to obtain a stronger result. The stochastic representation (1.13), holding modulo a set of zero μ-measure, suffices for our purposes and in most of the probabilistic applications, but from an analytic point of view, it is often useful to know that the stochastic representation (1.13) holds at all points in the domain.

4.4. *Stochastic representation of weak solutions and the Harnack inequality.* We now give the proof of Theorem 1.6, which establishes that functions defined by the stochastic representation (1.18) satisfy a scale-invariant Harnack inequality (1.19). We let $Q := (t_1, t_2) \times \Omega$, and recall that the parabolic portion of the boundary $\eth Q$ is defined in (1.17). For a point $(t^0, z^0) \in \mathbb{R} \times \bar{S}_{n,m}$, and a positive constant, r, we let

$$Q_r(t^0, z^0) := (t^0 - r^2, t^0) \times B_r(z^0). \tag{4.16}$$

PROOF OF THEOREM 1.6. We choose a sequence of nonnegative functions, $\{g_k\}_{k\geq 0} \subset C^\infty(\bar{Q})$, such that

$$\|g_k - g\|_{C(\overline{\eth Q})} \to 0, \qquad \text{as } k \to \infty, \tag{4.17}$$

and analogously to the representation (1.18), we define

$$\begin{aligned} u_k(t,z) &:= \mathbb{E}_{\mathbb{P}^z}\big[g_k\big(t-(t-t_1)\wedge\tau_\Omega, Z\big((t-t_1)\wedge\tau_\Omega\big)\big)\big], \\ &\qquad \forall (t,z) \in \bar{Q}, \forall k \geq 0. \end{aligned} \tag{4.18}$$

Using property (4.17), it immediately follows from the preceding equality that $\|u_k - u\|_{L^\infty(\bar{Q})} \to 0$, as $k \to \infty$, and so to establish (1.18), it is sufficient to prove that the Harnack inequality, (1.19), holds for each nonnegative function u_k. We now prove that u_k is a local weak solution to the equation $u_t - Lu = 0$ on Q. While the previous statement is usually trivial in standard proofs of this result, because the solutions are assumed to be regular enough, in our case we prove this fact with the aid of the stochastic representation (1.16) as follows. Applying Epstein and Mazzeo (2016), Theorem 4.1, we obtain that the Harnack inequality holds for u_k. Letting

$$h_k(t,z) := \partial_t g_k(t,z) - Lg_k(t,z), \qquad \forall (t,z) \in \bar{Q}, \forall k \in \mathbb{N}, \tag{4.19}$$

we apply Itô's rule to the process $\{g_k(t-r, Z(r))\}_{0\leq r\leq t-t_1}$ and we obtain that

$$\begin{aligned} g_k(t,z) &= \mathbb{E}_{\mathbb{P}^z}\big[g_k\big(t-(t-t_1)\wedge\tau_\Omega, Z\big((t-t_1)\wedge\tau_\Omega\big)\big)\big] \\ &\quad + \mathbb{E}_{\mathbb{P}^z}\left[\int_0^{(t-t_1)\wedge\tau_\Omega} h_k\big(t-r, Z(r)\big)\,dr\right]. \end{aligned}$$

Using definition (4.18) of the function $u_k(t,z)$ and property (1.16), we have that

$$\begin{aligned} u_k(t,z) &= g_k(t,z) - \int_0^{t-t_1} T^\Omega_{t-(r+t_1)} h_k(r+t_1,\cdot)(z)\,dr, \\ &\qquad \forall t \in (t_1,t_2), \text{ for } \mu\text{-a.e. } z \in \underline{\Omega}. \end{aligned}$$

From (4.19), we know that $g_k \in C^\infty(\bar{Q})$ solves the equation $u_t - Lu = h_k$ on Q. Using the fact that $h_k \in L^2((t_1,t_2), L^2(\underline{\Omega}; d\mu))$, it follows by Lemma 2.3 that the integral term on the right-hand side of the preceding identity is a weak solution to the inhomogeneous problem $u_t - Lu = -h_k$ on Q. Thus, the function u_k is a weak solution to the homogeneous problem $u_t - Lu = 0$ on Q, and applying Epstein and Mazzeo (2016), Theorem 4.1, we obtain that u_k satisfies the Harnack inequality. Letting k tend to ∞ in (4.18) and using the fact that $\|u_k - u\|_{L^\infty(\bar{Q})} \to 0$, as $k \to \infty$, it follows that the function u defined by (1.18) also satisfies the Harnack inequality (1.19). This completes the proof. □

We have the following corollary of Theorem 1.6. This is a technical result needed in the proof of Lemma 5.8.

COROLLARY 4.6. *There is a positive constant, K_0, such that the following hold. Let $T > t_1$ and let $g \in C(\overline{\eth Q})$ be a nonnegative function, and let*

$$u(t,z) := \mathbb{E}_{\mathbb{P}^z}\big[g\big(t-(t-t_1)\wedge\tau_\Omega, Z\big((t-t_1)\wedge\tau_\Omega\big)\big)\mathbf{1}_{\{(t-t_1)\wedge\tau_\Omega<T-t_1\}}\big], \quad \forall (t,z)\in\bar{Q}. \tag{4.20}$$

Then the function u satisfies the scale-invariant Harnack inequality, that is, for all $(t^0, z^0) \in \bar{Q}$ and $r > 0$, such that $Q_{2r}(t^0, z^0) \subset Q$, we have that the scale-invariant Harnack inequality (1.19) holds.

PROOF. Similar to the proof of Theorem 1.6, we let $\{g_k\}_{k\geq 0} \subset C^\infty(\bar{Q})$ be a sequence of nonnegative, smooth functions, such that

$$\begin{aligned} \|g-g_k\|_{C(\bar{Q}\cap\{t<T\})} &\to 0 \qquad \text{as } k\to\infty, \\ \|g_k\|_{C(\bar{Q}\cap\{t>T\})} &\to 0 \qquad \text{as } k\to\infty. \end{aligned} \tag{4.21}$$

Theorem 1.6 yields that the sequence of functions $\{u_k\}_{k\geq 0}$ defined by (4.18) satisfies the Harnack inequality (1.19), and it is sufficient to prove that $\|u_k - u\|_{L^\infty(\bar{Q})} \to 0$, as $k \to \infty$, in order to conclude that u defined by (4.20) also satisfies the Harnack inequality (1.19). From definition (4.20) of the function $u(t,z)$ and property (4.21), there is a positive constant, C, such that for all $(t,z) \in \bar{Q}$, we have that

$$\begin{aligned} \big|u(t,z)-u_k(t,z)\big| \leq{}& \|g-g_k\|_{C(\bar{Q}\cap\{t<T\})} + \|g_k\|_{C(\bar{Q}\cap\{t>T\})} \\ &+ C\mathbb{P}^z\big((t-t_1)\wedge\tau_\Omega = T-t_1\big). \end{aligned} \tag{4.22}$$

We can assume without loss of generality that $t \geq T$, and so $\mathbb{P}^z((t-t_1)\wedge\tau_\Omega = T-t_1) = \mathbb{P}^z(\tau_\Omega = T-t_1)$. Let $v \in \mathcal{F}((0, t-T+1)\times\underline{\Omega})$ be the unique weak solution to the homogeneous initial-value problem (2.10) with initial condition $v(0,\cdot) \equiv 1$ given by Sturm (1995), Proposition 1.2. Then Theorem 1.3 gives us that $v(s,z) = \mathbb{P}^z(\tau_\Omega > s)$, for all $(s,z) \in (0, t-T+1)\times\underline{\Omega}$. Using the fact that

$$\mathbb{P}^z(\tau_\Omega = s) = \lim_{\varepsilon\downarrow 0}\mathbb{P}^z(\tau_\Omega > s-\varepsilon) - \mathbb{P}^z(\tau_\Omega > s) = \lim_{\varepsilon\downarrow 0} v(s-\varepsilon,z) - v(s,z),$$

and that the function v is continuous by Epstein and Mazzeo (2016), Corollary 4.1, it follows that $\mathbb{P}^z(\tau_\Omega = s) = 0$. Thus, the preceding inequality together with (4.21) and (4.22) yield that $\|u_k - u\|_{L^\infty(\bar{Q})} \to 0$, as $k \to \infty$. We can now conclude that the function u defined in (4.20) satisfies the Harnack inequality (1.19), since each element of the sequence $\{u_k\}_{k\in\mathbb{N}}$ also satisfies (1.19). This completes the proof. □

5. Harnack's inequality for standard Kimura operators. In this section, we give a probabilistic proof to Theorem 1.2 by adapting the argument used to establish Sturm (1994), Theorem 1. We organize this section into three parts. In Section 5.1, we introduce the assumptions imposed on the coefficients of the operator $\widehat{L}$ and we review some properties of the solutions to the parabolic equation (1.5). In Section 5.2, we establish the connection between the differential operators L and $\widehat{L}$, and we introduce the notion of a probabilistic solution in Definition 5.5, which we then use in Section 5.3 to give the proof of Harnack's inequality in Theorem 5.6, and of the scale-invariant Harnack inequality in Theorem 1.2.

5.1. *Properties of solutions to the parabolic equation* $u_t - \widehat{L}u = 0$. We first introduce Assumption 5.1, which describes the conditions that we impose on the coefficients of the standard Kimura differential operator $\widehat{L}$. We then review the existence, uniqueness and regularity of solutions in anisotropic Hölder spaces to the inhomogeneous initial-value problem defined by the operator $\widehat{L}$,

$$
\begin{cases} u_t - \widehat{L}u = g & \text{on } (0, \infty) \times S_{n,m}, \\ u(0, \cdot) = f & \text{on } S_{n,m}, \end{cases} \tag{5.1}
$$

obtained in Epstein and Mazzeo (2014) and Pop (2017b). In Lemma 5.2, we establish the stochastic representation of the solutions in anisotropic Hölder spaces to problem (5.1).

ASSUMPTION 5.1 (Coefficients of the operator $\widehat{L}$). The coefficients $(\widehat{a}(z))$, $(\bar{a}(z))$, $(\widehat{b}(z))$, $(\widehat{c}(z))$ and $(\widehat{d}(z))$ satisfy Assumption 2.1 imposed on $(a(z))$, $(\widetilde{a}(z))$, $(b(z))$, $(c(z))$ and $(d(z))$, respectively.

The stochastic differential equations associated to the standard Kimura diffusion operator $\widehat{L}$ can be written in the form

$$
\begin{aligned} d\widehat{X}_i(t) &= \widehat{b}_i\big(\widehat{Z}(t)\big)\,dt + \sqrt{\widehat{X}_i(t)} \sum_{j=1}^{n+m} \widehat{\sigma}_{ij}\big(\widehat{Z}(t)\big)\,d\widehat{W}_j(t), \qquad \forall t > 0, \\ d\widehat{Y}_l(t) &= \widehat{e}_l\big(\widehat{Z}(t)\big)\,dt + \sum_{j=1}^{n+m} \widehat{\sigma}_{l+n,j}\big(\widehat{Z}(t)\big)\,d\widehat{W}_j(t), \qquad \forall t > 0, \end{aligned} \tag{5.2}
$$

for all $i = 1, \ldots, n$ and $l = 1, \ldots, m$, where $\{\widehat{W}(t)\}_{t\geq 0}$ is a $(n+m)$-dimensional Brownian motion. We denote $\widehat{Z}(t) = (\widehat{X}(t), \widehat{Y}(t))$. Similarly to the construction of the matrix $(\sigma(z))$ in Section 4.1, there is $(\widehat{\sigma}(z))$ such that $(\widehat{\sigma}\widehat{\sigma}^*)(z) = \widehat{D}(z)$, where for all $z \in \bar{S}_{n,m}$, $i, j = 1, \ldots, n$, and $l, k = 1, \ldots, m$, we let

$$
\begin{aligned} &\widehat{D}_{ii}(z) := 2\big(\widehat{a}_{ii}(z) + x_i \bar{a}_{ii}(z)\big), \\ &\widehat{D}_{ij}(z) := 2\sqrt{x_i x_j}\,\widehat{a}_{ii}(z), \qquad i \neq j, \\ &\widehat{D}_{n+l,i}(z) = \widehat{D}_{i,n+l}(z) := 4\sqrt{x_i}\,\widehat{c}_{il}(z), \qquad \widehat{D}_{n+l,n+k}(z) := 2\widehat{d}_{lk}(z). \end{aligned} \tag{5.3}
$$

It follows that the coefficients of the stochastic differential equation (5.2) satisfy Pop (2017a), Assumptions 2.1 and 2.6. We may then apply Pop (2017a), Propositions 2.2 and 2.4, to conclude that the standard Kimura stochastic differential equation (5.2) has a unique weak solution, $(\widehat{Z}(t))_{t\geq 0}$, on a probability space, $(\widehat{\Xi}, \widehat{\mathbb{P}}^z)$, for any initial condition $\widehat{Z}(0) = z$, where $z \in \bar{S}_{n,m}$.

In order to prove the stochastic representation of solutions to the initial-value problem (5.1), we first recall the existence, uniqueness and regularity of solutions in anisotropic Hölder spaces to equation (5.1) obtained in Pop (2017b). We remark that such results are also established in Epstein and Mazzeo (2016), Theorem 1.1, Epstein and Mazzeo (2014), Proposition 2.1, and Epstein and Mazzeo (2013), Theorem 10.0.2, but the framework in Pop (2017b) is closer to this article.

To define the anisotropic Hölder spaces, let $\alpha \in (0,1)$, $k \in \mathbb{N}$, $T > 0$, and $U \subseteq S_{n,m}$. We let $C^{\alpha}_{\mathrm{WF}}([0,T] \times \bar{U})$ be the Hölder space consisting of continuous functions, $u : [0,T] \times \bar{U} \to \mathbb{R}$, such that the norm

$$\begin{aligned}\|u\|_{C^{\alpha}_{\mathrm{WF}}([0,T]\times\bar{U})} &:= \|u\|_{C([0,T]\times\bar{U})} \\ &\quad + \sup_{\substack{(t^0,z^0),(t,z)\in[0,T]\times\bar{U}\\ (t^0,z^0)\neq(t,z)}} \frac{|u(t^0,z^0) - u(t,z)|}{\rho^{\alpha}((t^0,z^0),(t,z))} < \infty.\end{aligned} \tag{5.4}$$

For $I \subseteq \{1,\dots,n\}$, let $U \subseteq M_I$ be an open set. We let $C^{2+\alpha}_{\mathrm{WF}}([0,T] \times \bar{U})$ denote the Hölder space of functions, $u \in C^2([0,T] \times U)$, such that

$$\begin{aligned} u, u_{x_i}, u_t &\in C^{\alpha}_{\mathrm{WF}}\big([0,T] \times \bar{U}\big), &&\forall i = 1,\dots,n,\\ \sqrt{x_i x_j} u_{x_i x_j}, \sqrt{x_i} u_{x_i y_l}, u_{y_l y_k} &\in C^{\alpha}_{\mathrm{WF}}\big([0,T] \times \bar{U}\big), &&\forall i, j \in I, \forall l, k = 1,\dots,m,\\ \sqrt{x_i} u_{x_i x_j}, u_{x_j x_k} &\in C^{\alpha}_{\mathrm{WF}}\big([0,T] \times \bar{U}\big), &&\forall i \in I, \forall j, k \in I^c.\end{aligned}$$

We now let U is an arbitrary open set in $S_{n,m}$. We let $C^{2+\alpha}_{\mathrm{WF}}([0,T] \times \bar{U})$ denote the Hölder space consisting of functions $u \in C^2([0,T] \times U)$, satisfying the property that

$$u \restriction_{\bar{U}\cap\bar{M}_I} \in C^{2+\alpha}_{\mathrm{WF}}\big([0,T] \times (\bar{U} \cap \bar{M}_I)\big), \qquad \forall I \subseteq \{1,\dots,n\}.$$

The elliptic Hölder spaces $C^{\alpha}(\bar{U})$ and $C^{2+\alpha}_{\mathrm{WF}}(\bar{U})$ are defined analogously to their parabolic counterparts, and so we omit their definitions for brevity.

From Assumption 5.1, it follows that the coefficients of the differential operator (1.1) satisfy the hypotheses of Pop (2017b), Theorem 1.4. Thus, given $f \in C^{2+\alpha}_{\mathrm{WF}}(\bar{S}_{n,m})$ and $g \in C^{\alpha}_{\mathrm{WF}}([0,\infty) \times \bar{S}_{n,m})$, the inhomogeneous initial-value problem (5.1) has a unique solution, u, that belongs to $C^{2+\alpha}_{\mathrm{WF}}([0,\infty) \times \bar{S}_{n,m})$. We can now prove the following.

LEMMA 5.2 (Stochastic representation of solutions to equation (5.1) with respect to $\widehat{\mathbb{P}}^z$). *Let* $\alpha \in (0,1)$, $f \in C^{2+\alpha}_{WF}(\bar{S}_{n,m})$, $g \in C^{\alpha}_{WF}([0,\infty) \times \bar{S}_{n,m})$, *and*

$u \in C^{2+\alpha}_{WF}([0,\infty) \times \bar{S}_{n,m})$ be the unique solution to the inhomogeneous initial-value problem (5.1). *Let $\Omega \subseteq S_{n,m}$ be an open set, $I = (t_1, t_2) \subset \mathbb{R}_+$ be a bounded interval, and $Q := I \times \Omega$. Then we have that*

$$u(t,z) = \mathbb{E}_{\widehat{\mathbb{P}}^z}\big[u\big(t-(t-t_1)\wedge\widehat{\tau}_\Omega, \widehat{Z}\big((t-t_1)\wedge\widehat{\tau}_\Omega\big)\big)\big] + \mathbb{E}_{\widehat{\mathbb{P}}^z}\bigg[\int_0^{(t-t_1)\wedge\widehat{\tau}_\Omega} g\big(t-s, \widehat{Z}(s)\big)\,ds\bigg],$$

for all $(t,z) \in \bar{Q}$, where the stopping time $\widehat{\tau}_\Omega$ is defined by

$$\widehat{\tau}_\Omega := \inf\{t > 0 : \widehat{Z}(t) \notin \underline{\Omega}\}, \tag{5.5}$$

and the process $\{\widehat{Z}(t)\}_{t\geq 0}$ is the unique weak solution to the standard Kimura stochastic differential equation (5.2), *with initial condition $\widehat{Z}(0) = z$.*

PROOF. The proof is a direct consequence of Itô's rule applicable to the framework of Kimura stochastic differential equations and established in Pop (2017a), Proposition 2.10. We omit the detailed proof for brevity. □

5.2. *Connection between the differential operators L and $\widehat{L}$.* Lemma 5.2 shows that the homogeneous initial-value problem (5.1) admits solutions that can be expressed using the probability distribution, $\widehat{\mathbb{P}}^z$, of the unique weak solution, $\{\widehat{Z}(t)\}$, to the standard Kimura stochastic differential equation equation (5.2). In this section, using Girsanov's Theorem, we prove in Lemma 5.4 that the solutions to the homogeneous initial-value problem (5.1) have a stochastic representation that uses the probability distribution, $\mathbb{P}^z$, of the unique Markovian solution to a suitable Kimura stochastic differential equation with singular drift of the form (1.11), as opposed to that of weak solutions to the stochastic differential equation (5.2). This shows that our Definition 5.5 of probabilistic solutions is not vacuous. In Section 5.3, we use Definition 5.5 to prove in Theorem 5.6 that the Harnack inequality holds for nonnegative probabilistic solutions to equation (1.5).

We make a specific choice of the differential operator L of the form given by (1.10). We define the coefficients of the operator L in terms of the coefficients of the operator $\widehat{L}$ such that for all $i, j = 1, \ldots, n$ and all $l, k = 1, \ldots, m$, we have

$$a_{ii}(z) := \widehat{a}_{ii}(z), \qquad \tilde{a}_{ij}(z) := \bar{a}_{ij}(z),$$
$$c_{il}(z) := \frac{1}{2}\widehat{c}_{il}(z), \qquad d_{lk}(z) := \widehat{d}_{lk}(z),$$

and we choose the coefficients $b_i(z)$ in (1.10), such that $g_i(z) = \widehat{b}_i(z)$, where the coefficients $g_i(z)$ are defined in (4.1). With this choice of the coefficients $b_i(z)$ and $a_{ij}(z)$, we define $f_{ij}(z)$ as in (4.1), and so the stochastic differential equation

(1.11) becomes, for all $i=1,\dots,n$ and all $l=1,\dots,m$,

$$
\begin{aligned}
dX_i(t) &= \left(\widehat{b}_i\big(Z(t)\big) + X_i(t)\sum_{j=1}^n f_{ij}\big(Z(t)\big)\ln X_j(t)\right)dt \\
&\quad + \sqrt{X_i(t)}\sum_{j=1}^{n+m}\widehat{\sigma}_{ij}\big(Z(t)\big)\,dW_j(t), \\
dY_l(t) &= \left(e_l\big(Z(t)\big) + \sum_{j=1}^n f_{l+n,j}\big(Z(t)\big)\ln X_j(t)\right)dt \\
&\quad + \sum_{j=1}^{n+m}\widehat{\sigma}_{l+n,j}\big(Z(t)\big)\,dW_j(t),
\end{aligned}
\tag{5.6}
$$

where $\{W(t)\}_{t\geq 0}$ is a $(n+m)$-dimensional Brownian motion. Because the coefficients of the differential operator (1.1) satisfy Assumption 5.1, it follows that the preceding choice of the coefficients of the stochastic differential equation (5.2) satisfy Pop (2017a), Assumption 3.2. We may then apply Pop (2017a), Theorems 3.1 and 3.7, to conclude that the Kimura stochastic differential equation with logarithmic drift (5.6) has a unique Markov solution, $(Z(t))_{t\geq 0}$, on a probability space, $(\Xi, \mathbb{P}^z)$, for any initial condition $Z(0)=z$, where $z\in\bar{S}_{n,m}$.

Applying Pop (2017a), Lemma 3.3, we obtain that the matrix $\widehat{\sigma}(z)$ is invertible. We denote by $\theta(z) := (\theta_1(z),\dots,\theta_{n+m}(z))$ the unique solution to the system of linear equations:

$$
\begin{cases}
\displaystyle\sum_{k=1}^{n+m}\widehat{\sigma}_{ik}(z)\theta_k(z) = \sqrt{x_i}\sum_{j=1}^n f_{ij}(z)\ln x_j, & \forall i=1,\dots,n \\
\displaystyle\sum_{k=1}^{n+m}\widehat{\sigma}_{n+l,k}(z)\theta_k(z) = \sum_{j=1}^n f_{n+l,j}(z)\ln x_j + \widehat{e}_l(z) - e_l(z), & \forall l=1,\dots,m,
\end{cases}
$$

for all $z\in\bar{S}_{n,m}$. It follows from Pop (2017a), Lemma 3.5, that:

LEMMA 5.3. *Suppose that the coefficients of the differential operator* (1.1) *satisfy Assumption* 5.1. *Then for all* $T>0$, *there is a positive constant*, $\Lambda = \Lambda(\bar{b},\delta,K,m,n,T)$, *such that*

$$
\mathbb{E}_{\widehat{\mathbb{P}}^z}\left[e^{9\int_0^T |\theta(\widehat{Z}(t))|^2\,dt}\right] \leq \Lambda, \qquad \forall z\in\bar{S}_{n,m}, \tag{5.7}
$$

where $\{\widehat{Z}(t)\}_{t\geq 0}$ *is the unique weak solution to the standard Kimura equation* (5.2), *with initial condition* $\widehat{Z}(0)=z$.

We obtain from Lemma 5.3 and Karatzas and Shreve (1991), Proposition 3.5.12 and Corollary 3.5.13, that the process

$$
\widehat{M}(t) := e^{\int_0^t \theta(\widehat{Z}(s))\cdot d\widehat{W}(s) - \frac{1}{2}\int_0^t |\theta(\widehat{Z}(s))|^2\,ds}, \qquad \forall t\in[0,T],
$$

is a $\widehat{\mathbb{P}}^z$-martingale. From Girsanov's theorem [Karatzas and Shreve (1991), Theorem 3.5.1], by letting

$$W(t) := \widehat{W}(t) - \int_0^t \theta(\widehat{Z}(s))\,ds, \qquad \forall t \in [0, T], \tag{5.8}$$

and defining a new probability measure $\mathbb{P}^z$ by

$$\frac{d\mathbb{P}^z}{d\widehat{\mathbb{P}}^z} = M(T), \tag{5.9}$$

we obtain that $\{W(t)\}_{t\geq 0}$ is a $\mathbb{P}^z$-Brownian motion. We also have that $(Z(t) := \widehat{Z}(t), W(t))$ is a weak solution to the generalized Kimura equation (5.6), with initial condition $Z(0) = z$, for all $0 \leq t \leq T$. Because $Z(t) := \widehat{Z}(t)$, for all $0 \leq t \leq T$, we also have that $\tau_\Omega = \widehat{\tau}_\Omega$, for all open sets $\Omega \subseteq S_{n,m}$, where the preceding two stopping times are defined by (1.14) and (5.5), respectively. From Pop (2017a), Lemma 3.8, it follows that the process

$$M(t) := e^{-\int_0^t \theta(Z(s))\cdot dW(s) - \frac{1}{2}\int_0^t |\theta(Z(s))|^2\,ds}, \qquad \forall t \in [0, T], \tag{5.10}$$

is a $\mathbb{P}^z$-martingale.

We can now state the stochastic representation of solution to the homogeneous initial-value problem with respect to the probability distribution $\mathbb{P}^z$. Identity (5.9) and Lemma 5.2 imply the following.

LEMMA 5.4 (Stochastic representation of solutions to equation (5.1) with respect to $\mathbb{P}^z$). *Suppose that the hypotheses of Lemma 5.2 hold. Then we have that*

$$\begin{aligned} u(t,z) &= \mathbb{E}_{\mathbb{P}^z}\big[M\big((t-t_1)\wedge\tau_\Omega\big)u\big(t-(t-t_1)\wedge\tau_\Omega, Z\big((t-t_1)\wedge\tau_\Omega\big)\big)\big], \\ &\quad + \mathbb{E}_{\mathbb{P}^z}\left[M\big((t-t_1)\wedge\tau_\Omega\big)\int_0^{(t-t_1)\wedge\tau_\Omega} g\big(t-s, Z(s)\big)\,ds\right], \\ &\qquad \forall (t,z) \in \bar{Q}, \end{aligned} \tag{5.11}$$

where the process $\{Z(t)\}_{t\geq 0}$ is the unique weak Markov solution to the generalized Kimura equation (5.6), *with initial condition $Z(0) = z$.*

The stochastic representation (5.11) shows that the parabolic problem $u_t - \widehat{L}u = 0$ admits probabilistic solutions in the sense of Sturm (1994), Definition (2.1). Thus, the following definition of probabilistic solutions to the Kimura equation (1.5) is not vacuous.

DEFINITION 5.5 (Probabilistic solution). Let $\Omega \subseteq S_{n,m}$ be an open set and $(t_1, t_2) \subset \mathbb{R}_+$. Let $Q := (t_1, t_2) \times \Omega$. We say that a *continuous* function, $u : Q \to \mathbb{R}$, is a probabilistic solution to the parabolic equation $u_t - \widehat{L}u = 0$ on Q, if for all open sets $Q' := (t_1', t_2') \times \Omega' \subseteq Q$, we have, $\forall (t,z) \in \bar{Q}'$, that

$$u(t,z) = \mathbb{E}_{\mathbb{P}^z}\big[M\big((t-t_1')\wedge\tau_{\Omega'}\big)u\big(t-(t-t_1')\wedge\tau_{\Omega'}, Z\big((t-t_1')\wedge\tau_{\Omega'}\big)\big)\big], \tag{5.12}$$

where $\mathbb{P}^z$ is the probability distribution of the unique weak Markov solution, $\{Z(t)\}_{t\geq 0}$, to the Kimura equation (1.11), with initial condition $Z(0)=z$, and the martingale $\{M(t)\}_{t\geq 0}$ is defined in (5.10).

5.3. *The proof of Harnack's inequality.* We use Definition 5.5 to prove that the Harnack inequality holds for nonnegative probabilistic solutions to equation (1.5). We have the following analogue of Sturm (1994), Theorem 1.

THEOREM 5.6 (Harnack inequality). *Suppose that Assumption 5.1 holds. Let $c \in (\sqrt{2/3}, 1)$ and $T > 0$. Then there is a positive constant, $H = H(\bar{b}, \delta, K, K_0, \Lambda, T)$, such that for all $(s,z) \in (0,T) \times \bar{S}_{n,m}$ and for all $R \in (0, \sqrt{s})$, if u is a nonnegative, continuous probabilistic solution to the parabolic equation (1.5) on $Q_R(s,z)$, we have that*

$$u(t,w) \leq Hu(s,z), \qquad \forall (t,w) \in \bigl(s - c^2R^2, s - 2R^2/3\bigr) \times B_{cR}(s,z). \tag{5.13}$$

Theorem 1.2 is essentially a direct consequence of Theorem 5.6.

PROOF OF THEOREM 1.2. Let $(s,z) \in Q$ and $R > 0$ be such that inclusion (1.6) holds. Let $c \in (\sqrt{2/3}, 1)$. Inequality (5.13) gives us, for all $r \in (0,R)$, that

$$u(t,w) \leq Hu(s,z), \qquad \forall (t,w) \in \bigl(s - c^2r^2, s - 2r^2/3\bigr) \times B_{cr}(s,z),$$

and by denoting $\rho := cr$, we obtain that for all $\rho \in (0, cR)$, we have that

$$u(t,w) \leq Hu(s,z), \qquad \forall (t,w) \in \left(s - \rho^2, s - \frac{2}{3c^2}\rho^2\right) \times B_\rho(s,z). \tag{5.14}$$

Let d be a positive constant chosen such that

$$d^2 < \max\left\{1, 4 - \frac{8}{3c^2}\right\}, \tag{5.15}$$

and let $(s', z') \in (s, s+d^2) \times B_\rho(z)$, where we assume that $\rho \in (0, cR)$. Inclusion (1.6) and our choice of the point (s', z') allows us to apply inequality (5.14) with (s', z') replacing (s,z) and 2ρ replacing ρ to obtain that

$$\begin{aligned} &u(t', w') \leq Hu(s', z'), \\ &\qquad \forall (t', w') \in \left(s' - 4\rho^2, s' - \tfrac{8}{3c^2}\rho^2\right) \times B_{2\rho}(z'). \end{aligned} \tag{5.16}$$

Notice that

$$\begin{aligned} &\left(s - \frac{8}{3c^2}\rho^2, s - (4-d^2)\rho^2\right) \times B_\rho(z) \\ &\quad \subseteq \cap \left(s' - 4\rho^2, s' - \frac{8}{3c^2}\rho^2\right) \times B_{2\rho}(z'), \end{aligned} \tag{5.17}$$

where the preceding intersection is taken over all points (s', z') in the set $(s, s + d^2\rho^2) \times B_\rho(z)$. We now set

$$\alpha := \frac{8}{3c^2}, \qquad \beta := 4 - d^2, \qquad \text{and} \qquad \gamma := d^2,$$

and notice that our choice of the positive constant d in (5.15) implies that $\alpha > \beta$. Then inequality (5.16) and property (5.17) give us that the scale-invariant Harnack inequality (1.7) holds. This completes the proof of Theorem 1.2. $\square$

We prove Theorem 5.6 with the aid of a series of lemmas. Our proof closely follows the arguments in Sturm (1994), Section 2, used to prove Sturm (1994), Theorem 1, and so we only include the details that are different from the ones in Sturm (1994). Complete proofs of these auxiliary results can be found in Epstein and Pop (2014), Section 7.3.

Let T be a positive constant, $(s, z) \in (0, T) \times \bar{S}_{n,m}$, and let $r \in (0, \sqrt{s})$. We let τ_r be the stopping time defined by

$$\tau_r := \inf\{t \geq 0 : Z(t) \notin \bar{B}_r(z)\}. \tag{5.18}$$

From identity (5.12), we obtain that the probabilistic solutions to the parabolic equation (1.5) on $Q_R(s, z)$, can be written in the form

$$\begin{aligned} u(t, w) = \mathbb{E}_{\mathbb{P}^w}\big[&M\big((t - s + r^2) \wedge \tau_r\big)u\big(t - (t - s + r^2) \wedge \tau_r, \\ &Z\big((t - s + r^2) \wedge \tau_r\big)\big)\big], \end{aligned} \tag{5.19}$$

for all $r \in (0, R)$ and all $(t, w) \in Q_r(s, z)$.

We begin with the analogue of Sturm (1994), Lemma 1.

LEMMA 5.7 (An estimate from below). *Assume that the hypotheses of Theorem 5.6 hold. We then have that*

$$u^{1/3}(s, z) \geq \Lambda^{-1/6}\mathbb{E}_{\mathbb{P}^z}\left[\frac{1}{R}\int_0^R u^{1/3}\big(s - r^2 \wedge \tau_r, Z(r^2 \wedge \tau_r)\big)\, dr\right]. \tag{5.20}$$

PROOF. To prove estimate (5.20), we can follow the same argument as in the proof of Sturm (1994), Lemma 1, replacing the use of Sturm (1994), inequality (2.2), with our Lemma 5.3. $\square$

Let K_0 be the positive constant appearing in the statement of Theorem 1.6. For simplicity, given $\rho \in (0, r)$ and $r \in (0, R)$, we denote

$$Q^r_\rho(s, z) := Q_\rho(s, z) \cap \big(0, s - 2r^2/3\big) \times S_{n,m}. \tag{5.21}$$

We have the following analogue of Sturm (1994), Lemma 3.

LEMMA 5.8 (Iterated Harnack inequality). *There are positive constants*, $C = C(K_0)$ *and* $m = m(K_0)$, *such that the following hold. Assume that the hypotheses of Theorem* 5.6 *hold. Let* $r \in (\sqrt{2}R/\sqrt{3}, R)$ *and for all* $(t, w) \in Q_r(s, z)$ *let*

$$\begin{aligned} v(t, w) := \mathbb{E}_{\mathbb{P}^w}\big[&u^6\big(t - \big(t - s + r^2\big) \wedge \tau_r, Z\big(\big(t - s + r^2\big) \wedge \tau_r\big)\big) \\ &\times \mathbf{1}_{\{t-(t-s+r^2)\wedge\tau_r < s-2R^2/3\}}\big]. \end{aligned} \tag{5.22}$$

Then, for all $\rho \in (\sqrt{2}R/\sqrt{3}, r)$, *we have that*

$$v(t, w) \le C\left(\frac{r}{r-\rho}\right)^m v(s, z), \qquad \forall (t, w) \in Q_\rho^R(s, z). \tag{5.23}$$

PROOF. Because the function u is assumed to be continuous by the hypotheses of Theorem 5.6, it follows from Corollary 4.6 that the function v defined in (5.22) satisfies the Harnack inequality (1.19). Using now an induction argument similar to that used to prove Sturm (1994), Inequality (2.8), we obtain the estimate in (5.23). We omit the detailed proof. $\square$

We now have the analogue of Sturm (1994), Lemma 2.

LEMMA 5.9 (An intermediate estimate from above). *There are positive constants*, $C = C(K_0)$ *and* $m = m(K_0)$, *such that the following hold. Assume that the hypotheses of Theorem* 5.6 *hold. Then, for all* $\rho \in (\sqrt{2}R/\sqrt{3}, R)$ *and all* $\eta \in (\rho, R)$, *we have that*

$$\begin{aligned} u^6(t, w) \le C\Lambda^3 &\frac{\eta^{m+1}}{(\eta-\rho)^{m+1}} \\ &\times \mathbb{E}_{\mathbb{P}^z}\left[\frac{1}{\eta}\int_0^\eta u^6\big(s - r^2 \wedge \tau_r, Z\big(r^2 \wedge \tau_r\big)\big)\right. \\ &\left.\times \mathbf{1}_{\{s-r^2\wedge\tau_r \le s-2R^2/3\}}\,dr\right], \end{aligned} \tag{5.24}$$

for all $(t, w) \in Q_\rho^R(s, z)$.

PROOF. To prove estimate (5.24), we proceed exactly as in the proof of Sturm (1994), Lemma 2, replacing the use of Sturm (1994), inequality (2.4), with our Lemma 5.3, and that of Sturm (1994), Lemma 3, with that of Lemma 5.8. $\square$

We conclude with the analogue of Sturm (1994), Lemma 4.

LEMMA 5.10 (An estimate from above). *Let* $c \in (\sqrt{2}/\sqrt{3}, 1)$. *Then there is a positive constant*, $C = C(c, K_0, \Lambda)$, *such that the following hold. Assume that the* *hypotheses of Theorem 5.6 hold. Then we have, for all* $(t, w) \in Q^R_{cR}(s, z)$,

$$u^{1/3}(t, w) \le C\mathbb{E}_{\mathbb{P}^z}\left[\frac{1}{R}\int_0^R u^{1/3}(s - r^2 \wedge \tau_r, Z(r^2 \wedge \tau_r)) \times \mathbf{1}_{\{s-r^2\wedge\tau_r \le s-2R^2/3\}}\,dr\right]. \tag{5.25}$$

PROOF. The method of the proof is based on that of Sturm (1994), Lemma 4, which in turn uses ideas of the Fabes and Stroock (1984), Proof of Lemma 3.2. We first remark that the factor $1/R$ was apparently omitted from Sturm (1994), Estimate (2.10). As written it cannot hold because, as $R \downarrow 0$, the right-hand side in Sturm (1994), Estimate (2.10), converges to 0, while the left-hand side remains fixed, and possibly positive. The right-hand side of Sturm (1994), Estimate (2.10), should be replaced by an averaging over $r \in (0, R)$ as follows: for all $\rho \in (\sqrt{2}R/\sqrt{3}, R)$, we let

$$I_R\left(\frac{\rho}{R}\right) := \mathbb{E}_{\mathbb{P}^z}\left[\frac{1}{\rho}\int_0^\rho u^6(s - r^2 \wedge \tau_r, Z(r^2 \wedge \tau_r))\mathbf{1}_{\{s-r^2\wedge\tau_r \le s-2R^2/3\}}\,dr\right]^{1/6}.$$

A close inspection of the arguments used to prove Sturm (1994), Lemma 4, allows us to see that they immediately adapt to this definition of I_R as above, and we conclude that estimate (5.25) holds. We omit the details of the proof. □

We can now give the following.

PROOF OF THEOREM 5.6. Inequality (5.13) follows from estimates (5.20) and (5.25). □

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REFERENCES

BAUDOIN, F. (2014). *Diffusion Processes and Stochastic Calculus*. European Mathematical Society (EMS), Zürich. MR3236242

BENSOUSSAN, A. and LIONS, J.-L. (1982). *Applications of Variational Inequalities in Stochastic Control*. *Studies in Mathematics and Its Applications* **12**. North-Holland, Amsterdam-New York. MR0653144

BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. Wiley, New York. MR1324786

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. *Pure and Applied Mathematics* **29**. Academic Press, New York. MR0264757

BREZIS, H. (2011). *Functional Analysis*, *Sobolev Spaces and Partial Differential Equations*. Springer, New York. MR2759829

EPSTEIN, C. L. and MAZZEO, R. (2014). C^0-estimates for diagonal degenerate diffusion operators arising in population biology. Preprint.

EPSTEIN, C. L. and MAZZEO, R. (2010). Wright–Fisher diffusion in one dimension. *SIAM J. Math*. *Anal*. **42** 568–608. MR2607922

EPSTEIN, C. L. and MAZZEO, R. (2013). *Degenerate Diffusion Operators Arising in Population Biology*. *Annals of Mathematics Studies* **185**. Princeton Univ. Press, Princeton, NJ. MR3202406

EPSTEIN, C. L. and MAZZEO, R. (2016). Harnack inequalities and heat-kernel estimates for degenerate diffusion operators arising in population biology. *Appl. Math. Res. Express. AMRX*. Available at arXiv:1406.1426 [math.AP].

EPSTEIN, C. L. and POP, C. A. (2014). Harnack inequalities for degenerate diffusions. Available at arXiv:1406.4759.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

EVANS, L. C. (1998). *Partial Differential Equations*. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR1625845

FABES, E. B. and STROOCK, D. W. (1984). The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations. *Duke Math*. *J*. **51** 997–1016. MR0771392

FEEHAN, P. M. N. and POP, C. A. (2015). Stochastic representation of solutions to degenerate elliptic and parabolic boundary value and obstacle problems with Dirichlet boundary conditions. *Trans*. *Amer*. *Math*. *Soc*. **367** 981–1031. MR3280035

FOLLAND, G. B. (1999). *Real Analysis*: *Modern Techniques and Their Applications*, 2nd ed. Wiley, New York. MR1681462

FRIEDMAN, A. (1975/1976). *Stochastic Differential Equations and Applications*, *Vols*. *I*, *II*. Academic, New York.

FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (2011). *Dirichlet Forms and Symmetric Markov Processes*, extended ed. *de Gruyter Studies in Mathematics* **19**. de Gruyter, Berlin. MR2778606

KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

KARLIN, S. and TAYLOR, H. E. (1981). *A Second Course on Stochastic Processes*. Academic, New York.

KIMURA, M. (1957). Some problems of stochastic processes in genetics. *Ann*. *Math*. *Stat*. **28** 882–901. MR0094267

KIMURA, M. (1964). Diffusion models in population genetics. *J. Appl*. *Probab.* **1** 177–232. MR0172727

LIONS, J.-L. and MAGENES, E. (1972). *Non-homogeneous Boundary Value Problems and Applications*. *Vol*. *I*. Springer, New York. MR0350177

MOSER, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm*. *Pure Appl*. *Math*. **14** 577–591. MR0159138

MOSER, J. (1964). A Harnack inequality for parabolic differential equations. *Comm*. *Pure Appl*. *Math*. **17** 101–134. MR0159139

POP, C. A. (2017a). Existence, uniqueness and the strong Markov property of solutions to Kimura diffusions with singular drift. *Trans. Amer. Math. Soc.* **369** 5543–5579. MR3646770

POP, C. A. (2017b). C^0-estimates and smoothness of solutions to the parabolic equation defined by Kimura operators. *J. Funct*. *Anal*. **272** 47–82. MR3567501

PORTENKO, N. I. (1976). Diffusion processes with unbounded drift coefficient. *Theory Probab. Appl*. **20** 27–37.

SHIMAKURA, N. (1981). Formulas for diffusion approximations of some gene frequency models. *J. Math*. *Kyoto Univ.* **21** 19–45. MR0606311

STURM, K.-T. (1994). Harnack's inequality for parabolic operators with singular low order terms. *Math. Z.* **216** 593–611. MR1288047

STURM, K.-T. (1995). Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math.* **32** 275–312. MR1355744

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