STRUCTURES IN SUPERCRITICAL SCALE-FREE PERCOLATION

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Scale-free percolation is a percolation model on $\mathbb{Z}^d$ which can be used to model real-world networks. We prove bounds for the graph distance in the regime where vertices have infinite degrees. We fully characterize transience versus recurrence for dimension 1 and 2 and give sufficient conditions for transience in dimension 3 and higher. Finally, we show the existence of a hierarchical structure for parameters where vertices have degrees with infinite variance and obtain bounds on the cluster density.

1. Introduction. Random graphs are mathematical models commonly used to study real-world networks such as the world-wide web, social, financial, neural and biological networks. Many real-world networks exhibit the following two properties:

- *The small-world property*: distances within the network are very small in comparison to the number of nodes. With "small" we mean distances are at most of order of an iterated logarithm. Some real-world networks are even *ultra-small*, meaning that the distances are at most a double logarithm.
- *The scale-free property*: the number of connections per node behave statistically like a power-law. This implies that the variation is typically very high.

An example of a random graph model with these properties is the *Norros–Reittu random graph* [19] [see Figure 1(A)]. This model produces a random graph $G = (V, E)$ on a fixed set of vertices V, but with a random edge set $E \subset V \times V$ as follows: Every vertex $x \in V$ is assigned an i.i.d. random weight $W_x > 0$. Conditioned on the weights of its end-vertices, the edge $\{x, y\}$ is present in E with probability $p_{xy} = 1 - \exp(W_x W_y / \mathcal{N})$, independently of the status of other possible edges (here $\mathcal{N}$ is a normalizing constant). See [16] for more results on inhomogeneous random graphs.

These two properties are important, but the structure of many real-life networks, such as social networks, often have other features that influence the structure and formation of networks:

(A) Norros-Reittu random graph for $\tau = 1.95$ (B) Long-range percolation for $\alpha = 3.9, \lambda = 0.9$

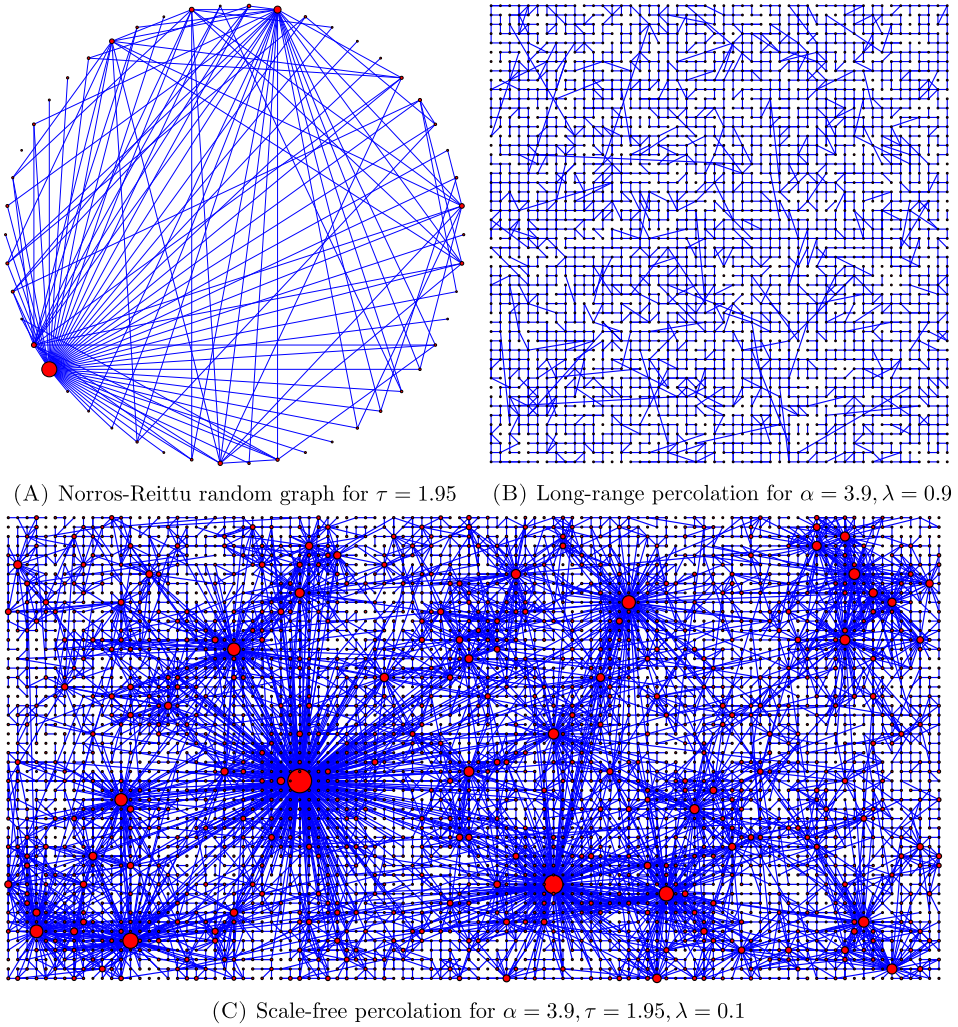(C) Scale-free percolation for $\alpha = 3.9, \tau = 1.95, \lambda = 0.1$

FIG. 1. *Simulations of the Norros–Reittu random graph* (A), *long-range percolation* (B) *and scale-free percolation. The size of the vertices is drawn proportionally to their weights.*

- *Geometric clustering*: in social networks, this manifests itself because people who are geographically close to each other are more likely to know each other, giving rise to formation of locally concentrated clusters within the network.
- *Hierarchies*: again in social networks, the more "important" people are, the more likely they know other important people, even if those people might be far away, giving rise to hierarchies within the network.

A well-known model that has geometric clustering and the connections over long distances required for the existence of hierarchies is *long-range percolation*

[LRP, see Figure 1(B)] [3–5, 15, 21]. LRP is a percolation model that produces random subgraphs of the graph $(\mathbb{Z}^d, \mathbb{Z}^d \times \mathbb{Z}^d)$ wherein an edge $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$ is (independently) *retained* with probability $p_{xy} \propto \lambda/|x - y|^\alpha$ for some positive constants λ and α, and removed otherwise. Thus, the connection probabilities are monotonically decreasing in α, and increasing in λ. For many choices of d and α, LRP has a *percolation phase transition* in λ, meaning that there exists $\lambda_c(d, \alpha) \in (0, \infty)$ such that when $\lambda > \lambda_c$ there exists an infinite cluster almost surely, whereas when $\lambda < \lambda_c$, all clusters are almost surely finite. When $\alpha \in (d, 2d)$, this model has the clustering property, as well as something akin to the small-world property [5]. It is, however, clearly not scale-free, since the decay of the degree distribution is faster than exponential.

Various models have been introduced in the recent years that combine three or four of the network properties described above. We mention, for instance, the models introduced by Aiello *et al.* [1], Flaxman, Frieze, and Vera [11] and Jacob and Mörters [17].

In this paper, we consider another model that has all four properties: *scale-free percolation* (SFP, also known as *heterogeneous long-range percolation*). SFP interpolates between long-range percolation and the Norros–Reittu random graph [see Figure 1(C)]. SFP was introduced by Deijfen, van der Hofstad and Hooghiemstra in [9]. We start with a formal definition of the model.

DEFINITION 1.1 (Scale-free percolation).   Consider the graph $(\mathbb{Z}^d, \mathbb{Z}^d \times \mathbb{Z}^d)$ for some fixed $d \geq 1$. Assign to each vertex $x \in \mathbb{Z}^d$ an i.i.d. weight W_x, where the weights follow a power-law distribution with parameter $\tau - 1$:

$$\mathbb{P}(W_x > w) = w^{-(\tau-1)}L(w), \qquad w > 0,$$

where L is a slowly-varying function [i.e., $L(wa)/L(w) \to 1$ for all $a > 0$ as $w \to \infty$, so the law of W_x is $(\tau - 1)$-regularly varying]. Conditionally on the weights, an edge $\{x, y\} \in \mathbb{Z}^d \times \mathbb{Z}^d$ is retained independently of all other edges with probability

$$p_{xy} = 1 - \exp\left(-\lambda \frac{W_x W_y}{|x - y|^\alpha}\right),$$

where $|x| = \|x\|_1$ and $\lambda, \alpha > 0$ are positive constants of the model.[2] The edge is removed otherwise. We call retained edges *open*, and removed edges *closed*. We denote the joint probability measure of edge occupation and weights by $\mathbb{P}_{(\lambda, W)}$ (where the subscript W refers to the *law* of the weights, not the actual values) and write just $\mathbb{P}$ if the parameters are clear from the context.

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[2]We choose to work with the ℓ_1-norm because it is a practical metric, but defining SFP with respect to any ℓ_p-norm with $p \in [1, \infty]$ gives qualitatively similar results.

Before we proceed with our results, let us briefly summarize some important features of SFP, as proved by Deijfen, van der Hofstad and Hooghiemstra [9], and by Deprez, Hazra and Wüthrich [10].

It turns out that the following parameter is frequently useful to describe the behaviour of SFP concisely:

$$(1.1) \qquad \gamma := \frac{\alpha(\tau - 1)}{d}.$$

Like long-range percolation, SFP on $\mathbb{Z}^d$ with parameter α and i.i.d. vertex weights whose law W is $(\tau - 1)$-regularly varying has a percolation phase-transition in λ at

$$(1.2) \qquad \lambda_c = \lambda_c(d, \alpha, W) := \inf\{\lambda > 0 | \text{there exists an infinite cluster } \mathcal{C}_\infty\}.$$

This phase transition is nontrivial, except when $d \geq 1$ and $\gamma < 2$, in which case $\lambda_c = 0$ and when $d = 1$, $\gamma > 2$, and $\alpha > 2d$, in which case $\lambda_c = \infty$ [9]. In the regime where SFP percolates, the infinite cluster $\mathcal{C}_\infty$ is almost surely unique [13]. Deprez *et al.* show that the percolation density of SFP is continuous when $\alpha \in (d, 2d)$: at $\lambda = \lambda_c$ there is no infinite cluster almost surely [10].

By the choice of the power-law distribution, this model is scale-free. Indeed, the degrees D follow a power-law of the form:

$$\mathbb{P}(D > s) = s^{-\gamma} \ell(s)$$

for some slowly varying function $\ell(s)$ [9]. This shows that the model behaves differently from long-range percolation. Many real-world networks are believed to have infinite variance degree distributions. SFP has infinite variance degrees when $\gamma < 2$. When $\gamma < 2$, SFP locally behaves like an ultra-small world [9].

Under the assumption that the weights are bounded away from 0, the probability that an edge is open in scale-free percolation with parameters α, τ and λ stochastically dominates the probability that an edge is open in long-range percolation with parameters α and some $\lambda' > 0$. Deprez *et al.* [10] use this domination to show that SFP locally has the small-world and clustering properties when $\alpha \in (d, 2d)$, analogous to long-range percolation [5].

2. Main results.

Distances within the infinite percolation cluster.   Given a graph $G = (V, E)$, the *graph distance on G* between any $x, y \in V$ is defined as

$$d_G(x, y) = \# \text{ edges in } E \text{ on a shortest path from } x \text{ to } y,$$

with the conventions that $d_G(x, x) = 0$ and $d_G(x, y) = \infty$ if x and y are not in the same connected component of the graph. We define the *diameter* of G as the maximal distance between two vertices in $G = (V, E)$, that is, $\text{diam} \, G = \max_{x, y \in V} d_G(x, y)$.

The infinite random subgraph $\mathcal{C}$ of $(\mathbb{Z}^d, \mathbb{Z}^d \times \mathbb{Z}^d)$ corresponding to the infinite component of supercritical SFP thus naturally produces a random metric on $\mathbb{Z}^d$. We write $d_{\mathcal{C}}$ for this metric. We write $x \wedge y$ for the minimum of x and y. Our first result is the proof of a conjecture by Deijfen *et al.* [9].

THEOREM 2.1 (Finite diameter in the infinite-degree cases). *Consider SFP on* $\mathbb{Z}^d$ *with* $d \geq 1$, $\lambda > 0$, *and with i.i.d. vertex weights whose law W satisfies for some* $\tau > 1$ *and some* $c > 0$,

$$(2.1) \qquad \mathbb{P}(W \geq w) \geq cw^{-(\tau-1)} \wedge 1, \qquad \text{for all } w > 0.$$

Then $\operatorname{diam} \mathcal{C} = 2$ *almost surely when* $\gamma \leq 1$, *and* $\operatorname{diam} \mathcal{C} \leq \lceil d/(d-\alpha) \rceil$ *almost surely when* $\alpha < d$.

Note that (2.1) implies $\mathbb{P}(W < c^{1/(\tau-1)}) = 0$, thus the weights are bounded away from 0. See Figure 2(A) for an overview of the graph distances in which we combine the results of the present paper and those of [9, 10]. Theorem 2.1 thus complements the characterization of distances. Our proof for the case $\alpha < d$ is based on the proof of a similar result for long-range percolation with $\alpha < d$ by Benjamini, Kesten, Peres and Schramm [3].

For the Norros–Reittu random graph, a similar result to Theorem 2.1 is known: van den Esker *et al.* [12] prove that when the weights are distributed as an infinite-mean power-law, then the diameter of the graph is almost surely 2 or 3 (more precise results are obtained under extra conditions).

Transience and recurrence. Graph distances are one way of characterizing the geometry of a graph. Another way of doing this is by studying the behaviour of random walk on the graph. The notions of *transience* and *recurrence* are particularly relevant.

DEFINITION 2.2 (Random walk, transience and recurrence). A simple random walk on a locally finite graph $G = (V, E)$ is a sequence $(X_n)_{n=0}^{\infty}$ with $X_0 \in V$ where X_{n+1} is chosen uniformly at random from the "neighbours" of X_n, that is,

$$X_{n+1} \in \{x \in V : \{x, X_n\} \in E\},$$

independently of $X_0, \ldots, X_{n-1}$. A graph is called *recurrent* if for every X_0 a random walk returns almost surely to its starting point X_0. A graph is called *transient* if it is not recurrent.

We prove the following two theorems, the results of which are summarized in the phase diagram in Figure 2(B).

THEOREM 2.3 (Transience in $d \geq 1$). *Consider SFP on* $\mathbb{Z}^d$ *with* $d \geq 1$, *i.i.d. vertex weights whose law W satisfies* (2.1), *either* $1 < \gamma < 2$ *or* $d < \alpha < 2d$, *or both, and* $\lambda > \lambda_c(d, \alpha, W)$. *Then the infinite cluster of SFP is transient almost surely.*
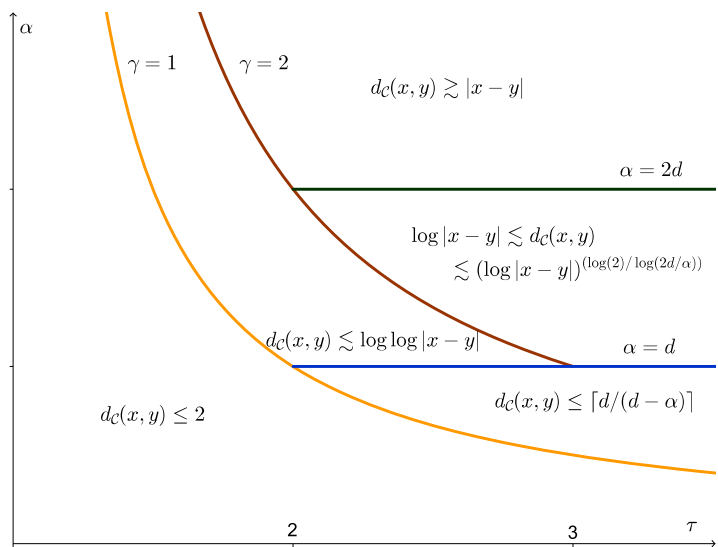
(A) Overview of graph distances, combined results of Theorem 2.1, [9] and [10]. By the notation $d_{\mathcal{C}}(x,y) \lesssim f(x,y)$ we mean that there exists a constant $c > 0$, such that $\lim_{|x-y|\to\infty} \mathbb{P}(d_{\mathcal{C}}(x,y) \leq cf(x,y)) = 1$. For $\gamma \in (1,2)$ and $\alpha > d$ stronger bounds have been proved [9, Theorem 5.1, 5.3].
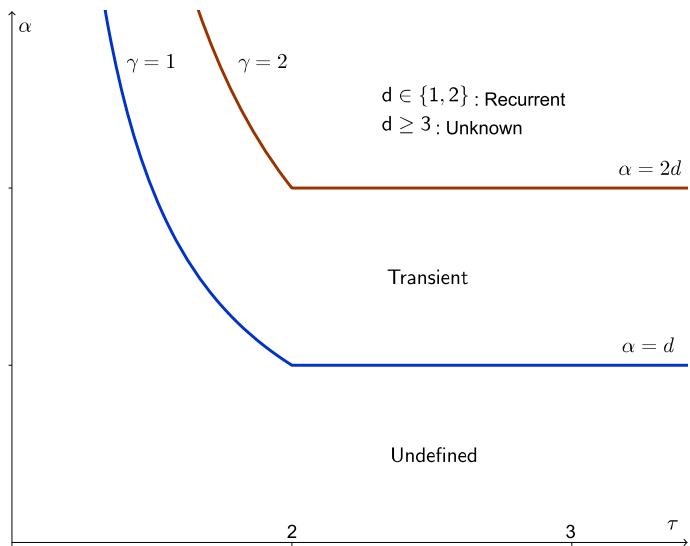
(B) Recurrent vs. transient. Results of Theorem 2.3 and 2.4

Fig. 2. *Phase diagrams. Transitions in γ and α.*

Recall Pólya's theorem, which states that the lattice of $\mathbb{Z}^d$ with nearest neighbour edges is recurrent if and only if $d \in \{1, 2\}$, and transient otherwise. Therefore, transience in these dimensions shows a dramatic difference to regular lattices. Berger [4] proved for LRP that the random walk is transient in one or two dimensions if and only if $\alpha \in (d, 2d)$. For SFP, the result is stronger: for any $\alpha > d$, there exists $\tau > 1$ such that the infinite cluster is transient.

THEOREM 2.4 (Recurrence in two dimensions). *Consider SFP on $\mathbb{Z}^d$ with $d = 2$, i.i.d. vertex weights whose law W satisfies*

$$(2.2) \qquad \mathbb{P}(W \geq w) \leq c w^{-(\tau-1)}, \qquad \text{for all } w \geq 0,$$

for some $\tau > 1$ and $c > 0$, $\alpha > 4$ and $\lambda > \lambda_c(2, \alpha, W)$, and such that either $\tau > 2$ or $\gamma > 2$, or both. Then the infinite percolation cluster is recurrent $\mathbb{P}_{(\lambda, W)}$-almost surely.

Note that, as mentioned before, in dimension 1 when $\gamma > 2$ and $\alpha > 2$ there is no infinite cluster almost surely [9], so in this case a random walk is trivially recurrent. We therefore give a full characterization of recurrence and transience of SFP in dimension one and two, while for $d \geq 3$ we only characterize it when $\alpha < 2d$ or $\gamma < 2$. For nearest-neighbour percolation, it is known that the infinite cluster is transient [14]. It would be interesting to verify whether this is true for other percolation models on $\mathbb{Z}^d$, in particular for scale-free percolation or long-range percolation.

Geometric clustering and hierarchies. We show that SFP has the geometric clustering property not only for $\alpha \in (d, 2d)$ as shown by Deprez *et al.* [10], Theorem 6, but also when $1 < \gamma < 2$. Moreover, these clusters can be organized in a hierarchical structure, a phenomenon that is also present in some real-life networks (see, e.g., [8], Chapter 13 or [2], Chapter 9). These hierarchical structures are not only present in finite boxes, they extend throughout $\mathbb{Z}^d$. Indeed, the infinite component of SFP contains an infinite subgraph exhibiting a prescribed hierarchy. We refer to Figure 3 for a simulation of SFP on $\mathbb{Z}$, where a hierarchical subgraph is clearly visible. We introduce the notion of a *hierarchically clustered tree*.

DEFINITION 2.5 (Hierarchically clustered trees). Fix $m \geq 1$ and $x \in \mathbb{Z}^d$. Define the box $\mathcal{Q}_m(x) := x + [0, m-1]^d \cap \mathbb{Z}^d$. Consider the set of trees $\mathcal{T}_{x,m}$ of all unrooted, connected, cycle-free subgraphs of $(\mathcal{Q}_m(x), \mathcal{Q}_m(x) \times \mathcal{Q}_m(x))$ [i.e., trees on $\mathcal{Q}_m(x)$], where each vertex v in such a tree is endowed with a weight $W_v \in \mathbb{R}$. Fix $\rho \in (0, 1]$ and $K > 0$. We call an element $T \in \mathcal{T}_{x,1}$ an $(x, 1, K, \rho)$-*hierarchically clustered tree* if $T = (\{x\}, \varnothing, \{W_x\})$ (i.e., T is the isolated vertex x with a weight). For $m \geq 2$, we call an element $T \in \mathcal{T}_{x,m}$ an (x, m, ρ, K)-*hierarchically clustered tree* if the following four properties hold:
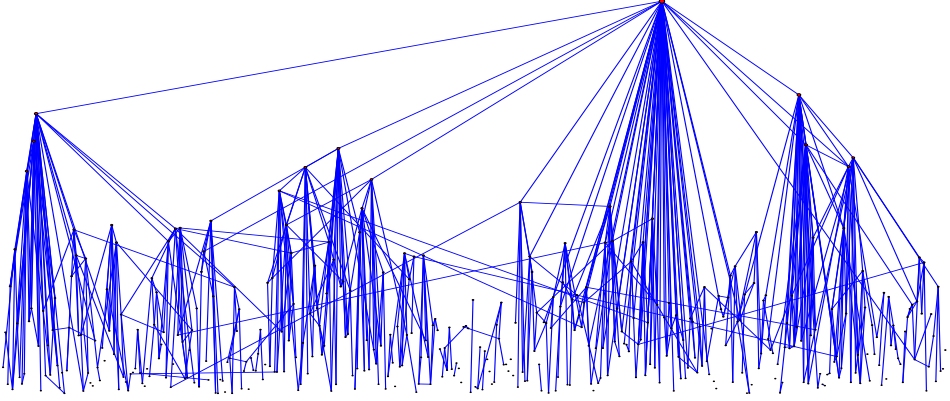
FIG. 3. *A simulation of scale-free percolation in $d = 1$. The vertex-height in the figure depends on the weight (logarithmically).* $\alpha = 2, \tau = 1.95, \lambda = 0.1$.

(1) (*Positive density*) T contains at least a fraction ρ of all the vertices in the box $\mathcal{Q}_m(x)$:

$$|V| > \rho m^d.$$

(2) (*Ultra-small world*) T is an ultra-small world in the sense that

$$\mathrm{diam}(T) \leq K \max\{1, \log \log m\}.$$

(3) (*Ordered weights*) If we root T at its maximum-weight vertex, then, for any vertex in the tree, the weights decrease step-by-step along the path from the root to that vertex.

(4) (*Spatial clustering*) If we remove any given edge from T, then there exists an $m' \leq m$ (depending on T and the removed edge) such that the two trees $T_1' = (V_1', E_1', W_1')$ and $T_2' = (V_2', E_2', W_2')$ that remain satisfy:

  (a) at least one (say T_1') is an (x', m', ρ, K)-hierarchically clustered tree for some $x' \in \mathcal{Q}_m(x)$, and
  (b) the other (say T_2') has its vertex set V_2' disjoint with the box on which T_1' is defined:

$$\mathcal{Q}_{m'}(x') \cap V_2' = \varnothing.$$

Note that condition (1) together with condition (2) implies that there exists $K' > 0$, such that for all $m \geq 1$

$$\mathrm{diam}(T) < K' \log \log |V_T|,$$

so hierarchically clustered trees combine a topological and a spatial version of the ultra-small world property.

THEOREM 2.6 (Hierarchically clustered trees). *Consider SFP on $\mathbb{Z}^d$ with $d \geq 1$, i.i.d. vertex weights whose law W satisfies* (2.1), *with $1 < \gamma < 2$, and any $\lambda > 0$. Let $\mathcal{S}_m$ denote the SFP configuration* inside *the cube $[0, m - 1]^d$. There exist $\xi > 0$, a density $0 < \rho \leq 1$, $K > 0$ and a constant $m_0 > 0$, such that*:

(1) *for all $m \geq m_0$,*

$$\mathbb{P}(\mathcal{S}_m \text{ contains a } (0, m, \rho, K)\text{-hierarchically clustered tree})$$

$$\geq 1 - \exp(-\rho m^{\xi}), \quad and$$

(2) *the infinite component $\mathcal{C}_\infty$ contains a.s. an infinite, connected, cycle-free subgraph $\mathcal{T}_\infty$ such that if we remove any given edge from $\mathcal{T}_\infty$, a finite and infinite connected component remain and there exist $x \in \mathbb{Z}^d$ and $m \geq 1$ such that the finite connected component is a (x, m, ρ, K)-hierarchically clustered tree.*

Related results and open questions.

Graph distance. This paper combined with [9, 10] gives bounds on the graph distance for every value in the parameter space, but the picture is not yet complete. We were not able to prove a nontrivial lower bound on the diameter in the regime where $\gamma < 2$ and $d > \alpha$, and it is not clear to us that the upper bound is sharp. And in the regime $\alpha \in (d, 2d)$ where $\gamma > 2$, there is a gap in the bounds on graph distances, since there the best known bounds are [9, 10]:

$$\lim_{|x| \to \infty} \mathbb{P}(c \log |x| \leq d_{\mathcal{C}}(0, x)$$

$$\leq c^{-1} (\log |x|)^{\log(2)/\log(2d/\alpha)}) = 1, \qquad \text{for some } c > 0.$$

What are the right asymptotics of $d_{\mathcal{C}}(0, x)$ in this regime?

Hierarchical structure. In Section 6 below, we determine that the bound on ξ in Theorem 2.6 is $\xi < \min\{d(2 - \gamma)/(\tau + 1), \frac{d}{2}(\tau + 2 - \sqrt{(\tau + 2)^2 - 4(2 - \gamma)})\}$. Biskup [5] shows a result rather similar to Theorem 2.6 on the clustering density for long-range percolation when $\alpha \in (d, 2d)$, where $\xi < d(2 - \alpha)$. The corresponding range for ξ for scale-free percolation would be $\xi < d(2 - \gamma)$. It might be possible to extend Theorem 2.6 to hold for this regime of ξ.

Scale-free percolation on the torus. Scale-free percolation is defined as a model on the infinite lattice $\mathbb{Z}^d$. A challenging question is the study of scale-free percolation, and in particular its critical behaviour, on the finite torus. Working on the torus keeps the translation invariance and provides the opportunity to compare the model to its nonspatial counterparts, such as the Norros–Reittu random graph [19].

Scale-free percolation on finite boxes is strongly related to geometric variants of the Norros–Reittu model or the Chung–Lu model. For example, Bringmann, Keusch and Lengler [7] introduce *geometric inhomogenous random graphs*, which

generalise a certain class of hyperbolic random graphs, and which could be described as "continuous SFP on the torus". Indeed, they do not use a grid, but place the points randomly. In a fairly general setup, where contrary to our model, the connection probability does not need to approach to 1 as $W_x W_y / |x - y|^\alpha$ goes to infinity; these authors prove that such graphs are ultra-small [6]. Moreover, they claim that their results also carry over to finite boxes. Because of this more general setup, it would be interesting (but possibly not straightforward) to see whether in their setting hierarchically clustered trees are also present.

Organization.   The proofs of the main results partly rely on a number of elementary properties of the vertex weights. We begin by proving these properties in Section 3. In Section 4, we prove the boundedness of the graph distance for $\alpha < d$ and $\gamma \leq 1$. In Section 5, we prove the random walk results, and in Section 6 we prove Theorem 2.6 on hierarchical clustering.

3. Preliminaries: Properties of the vertex weights.   We start by introducing some basic notation and definitions. Given two percolation configurations $\omega, \omega' \in \{0, 1\}^{\mathbb{Z}^d \times \mathbb{Z}^d}$, we write $\omega' \succcurlyeq \omega$ if $\omega'(e) = 1$ when $\omega(e) = 1$ for all $e \in \mathbb{Z}^d \times \mathbb{Z}^d$, that is, all edges that are open in ω are also open in ω'. We say that an event A is *increasing* if $\omega \in A$ implies $\omega' \in A$ for all $\omega' \succcurlyeq \omega$.

Given two random variables X and Y, we say that Y *stochastically dominates* X if for every $x \in \mathbb{R}$ the inequality $\mathbb{P}(X > x) \leq \mathbb{P}(Y > x)$ holds, and we write $X \preceq_d Y$.

LEMMA 3.1 (Stochastic domination for SFP).   *Let W and W' be random variables such that $W' \preceq_d W$. For any increasing event A,*

$$(3.1) \qquad \mathbb{P}_{(\lambda, W)}(A) \geq \mathbb{P}_{(\lambda, W')}(A).$$

This lemma can be proved with a straightforward coupling argument that we leave to the reader.

We commonly use Lemma 3.1 to simplify the law of W: If the law of W satisfies (2.1) and the law of W' satisfies

$$(3.2) \qquad \mathbb{P}(W' \geq w) = cw^{-(\tau-1)}, \qquad \text{for all } w \geq c^{1/(\tau-1)},$$

with the same constant c as in (2.1), then (3.1) holds.

The upcoming lemmas allow us to construct a coarse-graining argument in the proofs of Theorems 2.3 and 2.6.

LEMMA 3.2.   *Let W be a random variable with law given by (3.2). Let W'' be a random variable with law given by*

$$(3.3) \qquad \mathbb{P}(W'' \geq w) = w^{-(\tau-1)}, \qquad \text{for all } w \geq 1.$$

Then, for $y \geq c^{1/(\tau-1)}$, the conditional law of W given $\{W \geq y\}$ is the same as the law of yW'', that is, $\mathbb{P}(W \geq x \mid W \geq y) = \mathbb{P}(yW'' \geq x)$.

PROOF.   For $x \geq y$,

$$\mathbb{P}(W \geq x | W \geq y) = \left(\frac{y}{x}\right)^{\tau-1} = \mathbb{P}\left(W'' \geq \frac{x}{y}\right) = \mathbb{P}(yW'' \geq x). \qquad \square$$

LEMMA 3.3.   *Let* $\{W_i\}_{i=1}^{\infty}$ *be an i.i.d. sequence of random variables with law given by* (3.2). *Then, for all* $n \geq 1$ *and all* $K_2 \geq K_1 \geq c^{1/(\tau-1)}$,

$$\mathbb{P}\left(\max_{i=1,\dots,n} W_i \leq K_2 \Big| W_i \geq K_1 \text{ for } i = 1,\dots,n\right) \leq \exp\left(-n\left(\frac{K_1}{K_2}\right)^{\tau-1}\right).$$

PROOF.   Using that the weights are i.i.d., that $K_2 \geq K_1$, and that $1 - x \leq \exp(-x)$, we can bound the left-hand side by

$$\left(1 - \left(\frac{K_2}{K_1}\right)^{-(\tau-1)}\right)^n \leq \exp\left(-n\left(\frac{K_1}{K_2}\right)^{\tau-1}\right). \qquad \square$$

LEMMA 3.4.   *Fix an integer* $d \geq 1$ *and* $\alpha \in (0, \infty)$ *such that* $\gamma = \alpha(\tau - 1)/d < 2$. *Assign to each vertex in* $[0, N - 1]^d \subset \mathbb{Z}^d$ *an i.i.d. random variable with law satisfying* (2.1). *Let* $E_{N,\beta}$ *be the event that the box* $[0, N - 1]^d$ *contains at least* $\log N$ *vertices with weight larger than* $\beta N^{\alpha/2}$. *Then, for all* $\beta > 0$,

$$\mathbb{P}(E_{N,\beta}) \longrightarrow 1, \qquad \text{as } N \to \infty.$$

PROOF.   Let Y denote the number of vertices in $[0, N - 1]^d$ with weight exceeding $\beta N^{\alpha/2}$. By (2.1) and independence of the weights, we have $Y \succeq_d X$, where $X \sim \text{Bin}(N^d, c(\beta N^{\alpha/2})^{-(\tau-1)})$. Note that since $\gamma < 2$ we have $\mathbb{E}[X] = c\beta^{-(\tau-1)}N^{d(1-\gamma/2)} \gg \log N$ and $\text{Var}(X) \ll \mathbb{E}[X]^2$. It follows by the Paley–Zygmund inequality that (when N is sufficiently large),

$$\mathbb{P}(E_{N,\beta}) \geq \mathbb{P}(X \geq \log N) \geq \frac{(\mathbb{E}[X] - \log N)^2}{\text{Var}(X) + \mathbb{E}[X]^2} \longrightarrow 1. \qquad \square$$

We call any set that is a translate of $[0, N - 1]^d \subset \mathbb{Z}^d$ an N-box. We say that two N-boxes $\mathcal{Q}_1 = v_1 + [0, N - 1]^d$ and $\mathcal{Q}_2 = v_2 + [0, N - 1]^d$ are "k boxes away from each other" if $|v_1 - v_2| = kN$ (where we recall that $|\cdot|$ denotes the ℓ_1-norm).

LEMMA 3.5.   *Let* $d \geq 1$ *and* $k \geq 1$. *Consider two* N-*boxes* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *that are* k *boxes away from each other. For arbitrary* $u_1 \in \mathcal{Q}_1$ *and* $u_2 \in \mathcal{Q}_2$,

$$|u_1 - u_2| \leq 3dkN.$$

PROOF.   Let v_1 and v_2 be such that $\mathcal{Q}_1 = v_1 + [0, N - 1]^d$ and $\mathcal{Q}_2 = v_2 + [0, N - 1]^d$. Applying the triangle inequality twice, one obtains

$$|u_1 - u_2| \leq |v_1 - v_2| + |u_1 - v_1| + |v_2 - u_2| \leq kN + 2dN \leq 3dkN. \qquad \square$$

LEMMA 3.6.   *Fix $N \in \mathbb{N}$ and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two N-boxes that are k boxes away from each other such that $\mathcal{Q}_1 = Nv_1 + [0, N]^d$ and $\mathcal{Q}_2 = Nv_2 + [0, N]^d$ with $v_1, v_2 \in \mathbb{Z}^d$. Let $\beta > 0$ be given, the weights $\{W_x\}_{x \in \mathbb{Z}^d}$ be i.i.d. according to a law satisfying* (3.2), *and $\{W'_x\}_{x \in \mathbb{Z}^d}$ be i.i.d. with law* (3.3). *For $i = 1, 2$, write*

$$u_i = \arg\max_{u \in \mathcal{Q}_i} W_u.$$

Then

$$\mathbb{P}_{(\lambda, W)}(\{u_1, u_2\} \text{ is open}|W_{u_1}, W_{u_2} \geq \beta N^{\alpha/2}) \geq \mathbb{P}_{(\lambda\beta^2(3d)^{-\alpha}, W')}(\{v_1, v_2\} \text{ is open}).$$

PROOF.   Let $U \sim \text{Unif}[0, 1]$ denote a standard uniform random variable with c.d.f. $\mathbb{P}(U < x) = x$ for $x \in [0, 1]$. Then, by Definition 1.1,

$$\mathbb{P}_{(\lambda, W)}(\{u_1, u_2\} \text{ is open}|W_{u_1}, W_{u_2} \geq \beta N^{\alpha/2})$$
$$= \mathbb{P}^*\left(U < 1 - \exp\left(-\lambda \frac{W_1 W_2}{|u_1 - u_2|^\alpha}\right)\middle| W_1, W_2 \geq \beta N^{\alpha/2}\right),$$

where the probability measure $\mathbb{P}^*$ on the right-hand side is with respect to W_1 and W_2, which are i.i.d. with the same law as the elements of $\{W_x\}_{x \in \mathbb{Z}^d}$, and an independent random variable $U \sim \text{Unif}[0, 1]$. Using Lemmas 3.2 and 3.5, we bound the right-hand side from below by

$$\mathbb{P}^{**}\left(U < 1 - \exp\left(-\lambda\beta^2(3d)^{-\alpha}\frac{W'_1 W'_2}{k^\alpha}\right)\right),$$

where the probability measure $\mathbb{P}^{**}$ is with respect to W'_1 and W'_2 which are i.i.d. with the same law as the elements of $\{W'_x\}_{x \in \mathbb{Z}^d}$ and an independent random variable $U \sim \text{Unif}[0, 1]$.

On the other hand, since $|v_1 - v_2| = k$, by Definition 1.1 we also have

$$\mathbb{P}_{(\lambda\beta^2(3d)^{-\alpha}, W')}(\{v_1, v_2\} \text{ is open}) = \mathbb{P}^{**}\left(U < 1 - \exp\left(-\lambda\beta^2(3d)^{-\alpha}\frac{W'_1 W'_2}{k^\alpha}\right)\right).$$

$\square$

The claim thus follows.

4. Distances in the infinite degree case: Proof of Theorem 2.1.

PROOF OF THEOREM 2.1(1).   (*The case $\gamma \leq 1$.*) By translation invariance of the model, it suffices to show $\mathbb{P}(d_\mathcal{C}(0, x) \leq 2) = 1$. Since we assumed that the law of W satisfies (2.1), there exists a $c > 0$ such that $W_x \geq c^{1/(\tau-1)}$ for all $x \in \mathbb{Z}^d$ almost surely.

Fix $x \in \mathbb{Z}^d$. For $k \geq 1$, let $\mathcal{Q}_k$ denote the box centred at $x/2$ with sides of length $l_k := 2^k|x|$ and let $\mathcal{A}_k := \mathcal{Q}_k \setminus \mathcal{Q}_{k-1}$ with $\mathcal{A}_1 := \mathcal{Q}_1$. Note that there are $(2^d - 1)|x|^d 2^{d(k-1)}$ vertices in $\mathcal{A}_k$.
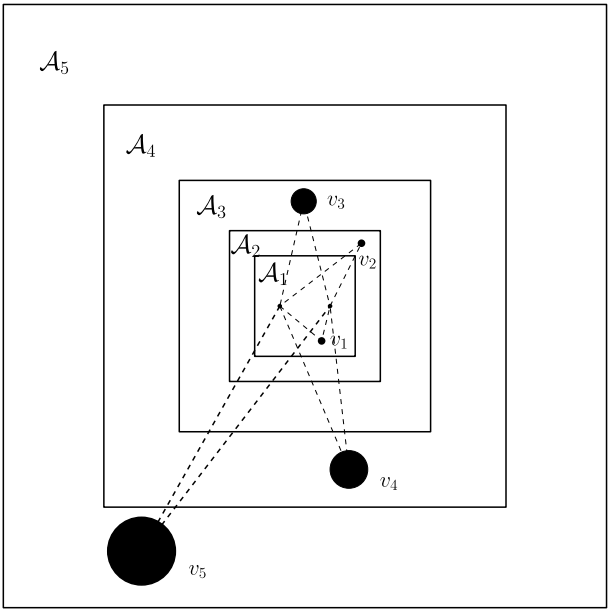
FIG. 4. *Construction for the proof of Theorem 2.1 for $d = 2$.*

We prove that the probability that the vertex with maximal weight for every $\mathcal{A}_k$ is connected to both 0 and x is strictly greater than some positive constant and let the result follow by Borel–Cantelli. Figure 4 shows a sketch of this construction.

For each $k \in \mathbb{N}$, let v_k be the vertex in $\mathcal{A}_k$ with maximal weight and let E_k be the event that v_k is connected by an open edge to both 0 and x. Let $a_k := 2^{\frac{dk}{\tau-1}}$ and denote $F_k := \{W_{v_k} \geq a_k\}$, which is an increasing event.

Using Lemma 3.1 and Lemma 3.3 with $K_1 = c^{1/(\tau-1)}$ and $K_2 = 2^{\frac{dk}{\tau-1}}$, we can bound

$$
(4.1) \quad
\begin{aligned}
\mathbb{P}_{(\lambda,W)}(F_k) &\geq \mathbb{P}_{(\lambda,W')}(F_k) \geq 1 - \exp\bigl(-c(2^d - 1)|x|^d 2^{d(k-1)} 2^{-\frac{dk}{\tau-1}(\tau-1)}\bigr) \\
&= 1 - \exp\bigl(-c(1 - 2^{-d})|x|^d\bigr),
\end{aligned}
$$

where the measure $\mathbb{P}_{(\lambda,W')}$ refers to a model where all weights are distributed as in (3.2). The right-hand side of (4.1) is bounded below by some $\delta > 0$ uniformly in k.

Observe that $|v_k|, |v_k - x| \leq dl_k$ and recall that $\tau > 1$ and $\gamma \leq 1$. Write $\varepsilon = c^{1/(\tau-1)}$. We can bound the probabilities on the events E_k by conditioning on F_k as follows:

$$
\begin{aligned}
\mathbb{P}_{(\lambda,W)}(E_k|F_k) &\geq \mathbb{P}_{(\lambda,W')}(E_k|F_k) \geq \mathbb{P}_{(\lambda,a_k)}(E_k) \geq \left(1 - \exp\left(-\frac{\lambda \varepsilon a_k}{(dl_k)^\alpha}\right)\right)^2 \\
&= \left(1 - \exp\left(-\frac{\lambda \varepsilon}{(d|x|)^\alpha} 2^{dk/(\tau-1) - k\alpha}\right)\right)^2
\end{aligned}
$$

$$\geq \frac{1}{4}\left(\left(\frac{\lambda\varepsilon}{(d|x|)^\alpha}2^{dk(1/(\tau-1)-\alpha/d)}\right)^2 \wedge 1\right)$$

$$= \left(\left(\frac{\lambda\varepsilon}{2(d|x|)^\alpha}\right)^2(4^{d(1-\gamma)/(\tau-1)})^k\right) \wedge \frac{1}{4}$$

$$\geq \left(\left(\frac{\lambda\varepsilon}{2(d|x|)^\alpha}\right)^2\right) \wedge \frac{1}{4} =: \eta.$$

This bound is independent of k and of the weights $\{W_x\}_{x\in\mathbb{Z}^d}$. It follows that $\mathbb{P}_{(\lambda,W)}(E_k|F_k) \geq \eta$ and, therefore,

$$\mathbb{P}_{(\lambda,W)}(E_k) = \mathbb{P}_{(\lambda,W)}(E_k|F_k)\mathbb{P}_{(\lambda,W)}(F_k) \geq \eta\delta > 0.$$

Observe that the events E_k are independent of each other, hence we obtain the result for $\gamma \leq 1$ using the lemma of Borel–Cantelli. □

PROOF OF THEOREM 2.1(2). (*The case $\alpha < d$.*) By translation invariance it again suffices to show that

$$\mathbb{P}_{(\lambda,W)}\big(d_\mathcal{C}(0,x) \leq \lceil d/(d-\alpha)\rceil\big) = 1$$

for all $x \in \mathbb{Z}^d$. Recall that the assumption (2.1) on the law of W implies that $W \geq c^{1/(\tau-1)}$ almost surely. Note that $\{d_\mathcal{C}(0,x) \leq \lceil d/(d-\alpha)\rceil\}$ is an increasing event. Hence, by Lemma 3.1,

$$\mathbb{P}_{(\lambda,W)}\big(d_\mathcal{C}(0,x) \leq \lceil d/(d-\alpha)\rceil\big) \geq \mathbb{P}_{(\lambda,c^{1/(\tau-1)})}\big(d_\mathcal{C}(0,x) \leq \lceil d/(d-\alpha)\rceil\big).$$

Observe that SFP with constant vertex weights is equivalent to long-range percolation with the same d and α and some possibly different parameter λ'.

Benjamini *et al.* [3], Example 6.1, show that the diameter of the infinite cluster in long-range percolation with $\alpha < d$ for any $\lambda > 0$ is equal to $\lceil d/(d-\alpha)\rceil$ almost surely. Our claim about SFP therefore follows. □

5. Transience versus recurrence.

Transience proof. The proof of Theorem 2.3 is inspired by Berger's proof of transience for long-range percolation [4], Theorem 1.4(II). We use in particular a multiscale ansatz which roots back to the work of Newman and Schulman [18] for long-range percolation.

The case $1 < \gamma < 2$. In view of Lemma 3.1, we may assume (3.2) rather than (2.1) without loss of generality. We show that the infinite cluster of SFP almost surely contains a transient subgraph. The proof has two steps:

(1) We first assume that λ is large enough. With small probability, we remove some vertices from the graph independently of each other. Then we use a multi-scale ansatz: we group vertices into finite boxes, and call boxes "good" or "bad" according to the weights and edge structure *inside* the box. We iterate this process by considering larger boxes, which we call good or bad according to the number of good boxes in them and the edges between vertices in those boxes. This will imply transience for large values of λ.

(2) To couple the original model (for *any* $\lambda > \lambda_c$) to the model of the first step, we use a coarse-graining argument: We "zoom out" by considering large boxes of vertices and only considering the vertices with maximum weight in the boxes. We show that, with high probability, the weights of these vertices are so high, that the graph, only defined on these vertices, dominates a graph as described in the first step.

We use [4], Lemma 2.7, which describes a sufficient structure for a graph to be transient. To this end, we introduce the notion of a "renormalized graph":

We start with some notation. Given a graph $G = (V, E)$ and a sequence $\{C_n\}_{n=1}^{\infty}$ let $V_l(j_l, \ldots, j_1)$ with $l \in \mathbb{N}$ and $j_n \in \{1, \ldots, C_n\}$ be a subset of the vertex set V. Now let for $l \geq m$:

$$V_l(j_l, \ldots, j_m) = \bigcup_{j_{m-1}=1}^{C_{m-1}} \cdots \bigcup_{j_1=1}^{C_1} V_l(j_l, \ldots, j_1).$$

We call the sets $V_l(j_l, \ldots, j_m)$ *bags*, and the numbers C_n *bag sizes*.

DEFINITION 5.1.   We say that the graph $G = (V, E)$ is *renormalized for the sequence* $\{C_n\}_{n=1}^{\infty}$ if we can construct an infinite sequence of graphs such that the vertices of the *lth stage graph* are labelled by $V_l(j_l, \ldots, j_1)$ for all $j_n \in \{1, \ldots, C_n\}$, and such that for every $l \geq m > 2$, every $j_l, \ldots, j_{m+1}$, and all pairs of distinct $u_m, w_m \in \{1, \ldots, C_m\}$ and $u_{m-1}, w_{m-1} \in \{1, \ldots, C_{m-1}\}$ there is an edge in G between a vertex in $V_l(j_l, \ldots, j_{m+1}, u_m, u_{m-1})$ and a vertex in $V_l(j_l, \ldots, j_{m+1}, w_m, w_{m-1})$.

The underlying intuition is that every nth stage bag contains C_n $(n-1)$-stage bags, which contains again C_{n-1} $(n-2)$-stage bags. Every pair of $(n-2)$-stage bags in an n stage bag is connected by an edge between one of the vertices in the bags (see Figure 5).

LEMMA 5.2 (Berger [4], Lemma 2.7).   *A graph renormalized for the sequence C_n is transient if $\sum_{n=1}^{\infty} C_n^{-1} < \infty$.*

The lemma follows from the proof of [4], Lemma 2.7.
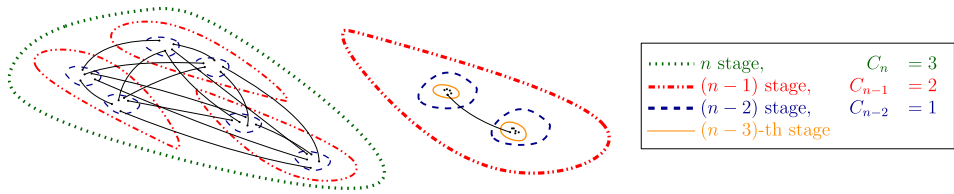
FIG. 5.　*n and (n − 1) stage bag of a renormalized graph.*

PROPOSITION 5.3. *Consider scale-free percolation with $\gamma < 2$ and weight distribution satisfying* (3.3). *Independently of this, perform an i.i.d. Bernoulli site percolation on the vertices of $\mathbb{Z}^d$, colouring a vertex "green" with probability $\mu \in (0, 1]$.*

Then the subgraph of the infinite scale-free percolation cluster that is induced by the green vertices has a (unique) infinite component $\mathcal{C}_{\lambda,\mu}$. There exists $\mu_0 < 1$ and $\lambda_0 > 0$, such that $\mathcal{C}_{\lambda,\mu}$ is transient for $\mu \geq \mu_0$ and $\lambda \geq \lambda_0$ almost surely.

The proof exploits a multiscale technique. Indeed, we proceed by showing that $\mathcal{C}_{\lambda,\mu}$ contains a renormalized subgraph that is transient. Therefore, $\mathcal{C}_{\lambda,\mu}$ is also transient.

PROOF OF PROPOSITION 5.3.　For all $n \in \mathbb{N}$, let

$$D_n := 2(n + 1)^2, \qquad C_n := (n + 1)^{2d},$$

and

$$u_n := d^{\alpha/2}(n + 2)^{d(2-\gamma)/2} 2^{(n+2)\alpha/2}((n + 3)!)^{\alpha}.$$

We partition the lattice $\mathbb{Z}^d$ into disjoint boxes of side length D_1, so that each such box contains D_1^d vertices, and call these the 1-*stage* boxes. (By convention, we call vertices of $\mathbb{Z}^d$ the 0-*stage* boxes.) We view these boxes as the vertices of a renormalized lattice. Now cover the lattice again, grouping together $(D_2)^d$ 1-stage boxes to form 2-*stage* boxes with sides of length D_2. Continue in this fashion, so that the *n-stage* boxes form a covering of $\mathbb{Z}^d$ by translates of $[0, \prod_{k=1}^n D_k - 1]^d$.

We call a 0-stage box "good" if the vertex associated with it is green.

For every stage $i \geq 1$, we define rules for a box to be "good" or "bad", depending only on the weights W_x and the edges of $\mathcal{C}$ inside the box. This implies that disjoint boxes are good or bad independently of each other.

A 1-stage box is good if it contains at least C_1 good 0-stage boxes and one of the vertices in these boxes has weight at least u_1. For each good 1-stage box, call the maximum-weight vertex, having weight at least u_1, and call it 1-*dominant*.

For $n \geq 2$, say that an n-stage box $\mathcal{Q}$ is good if the following three conditions are satisfied:

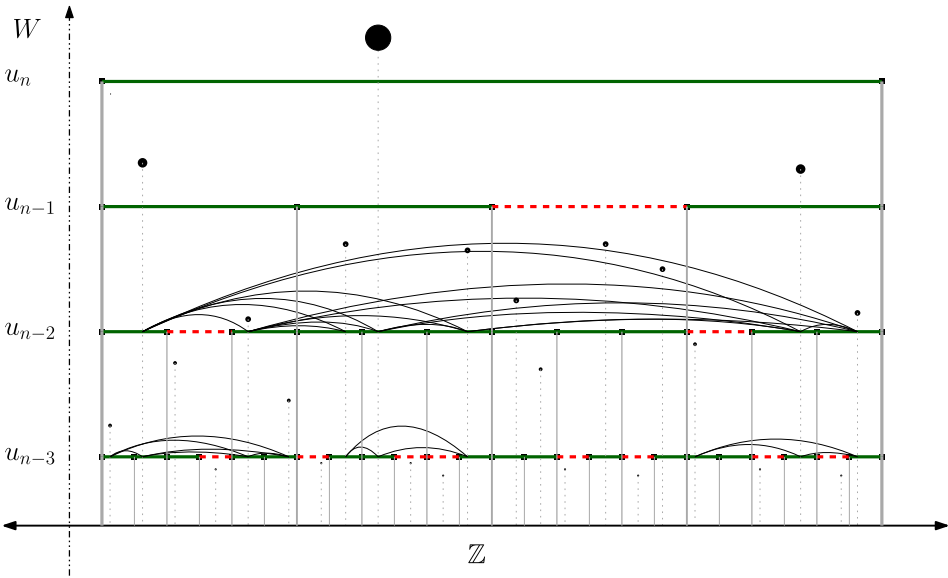(E) At least C_n of the $(n − 1)$-stage boxes in $\mathcal{Q}$ are good.

FIG. 6.   *Sketch of the renormalization in Theorem* 2.3 *in* $d = 1$ *for* $D_n = 4$, $D_{n-1} = 3$, $D_{n-2} = 2$, $C_n = 3$, $C_{n-1} = 2$, $C_{n-2} = 1$. *"Good" boxes are marked with a solid line; "bad" boxes have a dashed line.*

(F) Any two $(n-2)$-dominant vertices that are in a good $(n-1)$-stage box within the same n-stage are connected by an edge.

(G) There is an $(n-1)$-dominant vertex in one of its good $(n-1)$-stage boxes, with weight at least u_n. Call this vertex n-dominant.

For each good n-stage box, choose the maximum weight vertex and call it the n-*dominant* vertex if its weight is at least u_n. (A vertex may be dominant for different values of n.) See Figure 6 for a sketch of this definition.

Note that by construction, the subgraph of $\mathcal{C}$ induced by the vertices that are in a good n-stage box for every $n \geq 0$ is a graph renormalized by a sequence of bag sizes $\{C_n\}$ that satisfies the transience condition of Lemma 5.2. Our aim is therefore to show that almost surely such a subgraph exists.

Define $E_n(v)$, $F_n(v)$ and $G_n(v)$ to be the events that conditions (E), (F) and (G) hold for the n-stage box containing the vertex v. To simplify notation, define $E_n := E_n(0)$, $F_n := F_n(0)$ and $G_n := G_n(0)$. We write $L_n(v)$ and L_n for the events that the nth stage boxes containing v and 0, respectively, are good. By translation invariance it is sufficient to show that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} L_n\right) > 0.$$

The events L_n are positively correlated, hence it is sufficient to show that

$$\prod_{n=1}^{\infty} \mathbb{P}(L_n) > 0.$$

We bound

(5.1) $$\mathbb{P}(L_n^c) \leq \mathbb{P}(E_n^c) + \mathbb{P}(F_n^c | E_n) + \mathbb{P}(G_n^c | E_n).$$

First, we give an upper bound for $\mathbb{P}(F_n^c | E_n)$. Recall that we use the ℓ_1-norm for distance in the definition of the edge-probabilities of SFP. The ℓ_1-distance between two vertices in the same n-stage box is at most

$$d \prod_{k=1}^{n} D_k = d2^n ((n+1)!)^2.$$

The probability that two good $(n-2)$-stage boxes are *not* connected by an open edge between its $(n-2)$-dominant vertices (which have weight at least u_{n-2}) is therefore at most

$$\exp\left(-\lambda d^{-\alpha} u_{n-2}^2 \prod_{k=1}^{n} D_k^{-\alpha}\right)$$

$$= \exp\left(-\lambda d^{-\alpha} \left(d^{\alpha/2} n^{d(2-\gamma)/2} 2^{n\alpha/2} [(n+1)!]^\alpha\right)^2 2^{-n\alpha} ((n+1)!)^{-2\alpha}\right)$$

$$= \exp\left(-\lambda n^{d(2-\gamma)}\right).$$

There are

$$\binom{D_n^d D_{n-1}^d}{2} < 4^d (n+1)^{4d}$$

pairs of $(n-2)$-stage boxes inside an n-stage box, so there can be at most $4^d(n+1)^{4d}$ edges between $(n-2)$-dominant vertices inside a good n-stage box. It follows by taking the union bound that

(5.2) $$\mathbb{P}(F_n^c | E_n) \leq \exp\left(d \log(4) + 4d \log(n+1) - \lambda n^{d(2-\gamma)}\right).$$

We proceed by establishing an upper bound on $\mathbb{P}(G_n^c | E_n)$. There exists a constant $c_1 > 0$ such that

(5.3)
$$\frac{u_{n-1}}{u_n} = 2^{-\alpha/2} \left(\frac{n+1}{n+2}\right)^{d(2-\gamma)/2} \frac{1}{(n+3)^\alpha}$$
$$\geq c_1 (n+1)^{-\alpha}.$$

Note that any good n-stage box contains at least C_n $(n-1)$-dominant vertices that all have weight larger than u_{n-1}. Using (5.3), Lemma 3.3, and $\gamma = \alpha(\tau -$

$1)/d$, gives for some $c_2 > 0$ that

$$\mathbb{P}(G_n^c | E_n) \leq \exp\left(-C_n\left(\frac{u_{n-1}}{u_n}\right)^{\tau-1}\right)$$

(5.4)
$$\leq \exp\left(c_1^{\tau-1}(n+1)^{2d-\alpha(\tau-1)}\right)$$

$$\leq \exp\left(-c_2 n^{d(2-\gamma)}\right).$$

The last term we bound is $\mathbb{P}(E_n^c)$. All $(n-1)$-stage boxes are good independent of each other with probability $\mathbb{P}(L_{n-1})$. Let $X \sim \text{Bin}(D_n^d, \mathbb{P}(L_{n-1}))$ be binomially distributed, so that $\mathbb{P}(E_n^c) = \mathbb{P}(X < C_n)$. We use Chernoff's bound that if $X \sim \text{Bin}(m, p)$, $\theta \in (0, 1)$, then $\mathbb{P}(X < (1-\theta)mp) \leq \exp(-\frac{1}{2}\theta^2 mp)$. For our model, this obtains

$$\mathbb{P}(E_n^c) \leq \exp\left(-\frac{1}{2}\left(1 - \frac{1}{2^d \mathbb{P}(L_{n-1})}\right)^2 \mathbb{P}(L_{n-1}) D_n^d\right)$$

(5.5)
$$\leq \exp\left(-2^{-d-1}\left(2^d \mathbb{P}(L_{n-1}) - 1\right)^2 (n+1)^{2d}\right).$$

Combining (5.1), (5.2), (5.4) and (5.5), gives that

$$\mathbb{P}(L_n^c) \leq \exp\left(d\log(4) + 4d\log(n+1) - \lambda n^{d(2-\gamma)}\right) + \exp\left(-c_2 n^{d(2-\gamma)}\right)$$
$$+ \exp\left(-2^{-d-1}\left(2^d \mathbb{P}(L_{n-1}) - 1\right)^2 (n+1)^{2d}\right).$$

If λ large enough (say larger than λ_0), there exists n_0 such that for $n \geq n_0$

(5.6) $\quad \mathbb{P}(L_n^c) \leq 2\exp\left(-c_2 n^{d(2-\gamma)}\right) + \exp\left(-2^{-d-1}\left(2^d \mathbb{P}(L_{n-1}) - 1\right)^2 (n+1)^{2d}\right).$

Define the sequence

$$\ell_n := 1 - (n+1)^{-3/2}$$

and observe that

(5.7)
$$\prod_{n=1}^{\infty} \ell_n > 0.$$

For any fixed $n_1 > n_0$, we can find $\lambda_0 > 0$ and $\mu_0 < 1$ such that $\mathbb{P}(L_{n_1}) \geq \ell_{n_1}$, because L_{n_1} depends only on the weights and edges inside a *finite* box. We further bound (5.6) for all $n > n_1$ by

$$\mathbb{P}(L_n^c) \leq \exp\left(-c_2 n^{d(2-\gamma)}\right) + \exp\left(c_3(n+1)^{2d}\right)$$

(5.8)
$$\leq (n+1)^{-3/2} = 1 - \ell_n,$$

where c_3 is a positive constant, and choose n_1 so large, that the last bound in (5.8) holds. Thus, using (5.7), (5.8) and $\mathbb{P}(L_n) > 0$ for all n, yields that

$$\prod_{n=1}^{\infty} \mathbb{P}(L_n) = \prod_{n=1}^{n_1} \mathbb{P}(L_n) \prod_{n=n_1+1}^{\infty} \mathbb{P}(L_n) \geq \prod_{n=1}^{n_1} \mathbb{P}(L_n) \prod_{n=n_1+1}^{\infty} \ell_n > 0.$$

With probability 1 the graph contains a cluster of good vertices that can be renormalized for the sequence C_n. By Lemma 5.2, this cluster is transient itself, since showing transience for a subgraph is enough for transience on the whole graph [20], Section 9. □

The case $d < \alpha < 2d$. We need two lemmas from the literature, which are complementary to the case $\alpha \in (d, 2d)$ of Proposition 5.3 and Lemma 3.4.

LEMMA 5.4 (Deprez, Hazra and Wüthrich [10], Lemma 9). *Assume $\gamma > 1$ and let $\alpha \in (d, 2d)$. Choose $\lambda > \lambda_c$ and let $\alpha' \in [\alpha, 2d)$. For every $\mu \in [0, 1)$ and $\beta > 0$, there exists $M_0 \geq 1$ such that for all $m \geq M_0$:*

$$\mathbb{P}(|\mathcal{C}_m| \geq \beta m^{\alpha'/2}) \geq \mu,$$

where $\mathcal{C}_m$ is the largest connected component in $[0, m-1]^d$.

Note that [10], Lemma 9, is proven for the exact power law distribution of the weights in (3.3). This is not a problem, since Lemma 3.1 implies that the result extends to a weight distribution satisfying (2.1) when $c \geq 1$. For $c < 1$, and percolation parameter $\lambda' > 0$, we observe that the model is equivalent to the case where $c = 1$ and $\lambda = \lambda' c'^{-2/(\tau-1)}$, since if the law of W satisfies (3.2) for some $c > 0$ and for W' it holds for $w \geq 1$ that $\mathbb{P}(W' > w) = w^{-(\tau-1)}$, then $W \overset{d}{=} c^{-1/(\tau-1)} W'$. Hence, we can scale the parameters such that $c = 1$ and apply Lemma 3.1.

LEMMA 5.5 (Berger [4], Lemma 2.7). *Let $d \geq 1$, $\alpha \in (d, 2d)$ and $\lambda > 0$. Consider the long-range percolation model on $\mathbb{Z}^d$ in which every two vertices, x and y, are connected by an open edge with probability:*

$$1 - \exp\left(-\frac{\lambda}{|x - y|^\alpha}\right),$$

independently of other edges, and every vertex is good with probability $\mu < 1$, independently of all other vertices.

There exist $\mu_1 < 1$ and λ_1, such that if $\lambda \geq \lambda_1$ and $\mu \geq \mu_1$, the infinite cluster on the good vertices is transient.

LEMMA 5.6. *Consider scale-free percolation with weight distribution satisfying (2.1). Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two N-boxes that are k boxes away from each other. Let $\beta > 0$ be given. Moreover, assume that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contain connected components $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, of size at least $\beta N^{\alpha/2}$. Then*

$$\mathbb{P}(\mathcal{C}_1 \text{ connected by an open edge to } \mathcal{C}_2 \mid |\mathcal{C}_1|, |\mathcal{C}_2| \geq \beta N^{\alpha/2})$$

$$\geq 1 - \exp\left(-\frac{\lambda(3d)^{-\alpha}\beta^2 c^{2/(\tau-1)}}{k^\alpha}\right).$$

PROOF. Since we assumed (2.1), all weights are at least $c^{1/(\tau-1)}$. By Lemma 3.5, we get for arbitrarily chosen vertices $u \in \mathcal{C}_1$, $v \in \mathcal{C}_2$ that

$$\mathbb{P}(\{u, v\} \text{ is closed}) \leq \exp\left(-\lambda(3d)^{-\alpha} c^{2/(\tau-1)} \frac{1}{kN^\alpha}\right).$$

Since both clusters contain at least $\beta N^{\alpha/2}$ vertices, there are at least $\beta^2 N^\alpha$ possible edges. We obtain

$$\mathbb{P}(\mathcal{C}_1 \text{ not connected by an open edge to } \mathcal{C}_2 \,||\mathcal{C}_1|, |\mathcal{C}_2| \geq \beta N^{\alpha/2})$$

$$\leq \exp\left(-\lambda(3d)^{-\alpha} c^{2/(\tau-1)} \frac{1}{kN^\alpha}\right)^{\beta^2 N^\alpha}$$

$$= \exp\left(-\lambda(3d)^{-\alpha} \beta^2 c^{2/(\tau-1)} \frac{1}{k^\alpha}\right). \qquad \square$$

Note that this is the $\alpha \in (d, 2d)$-counterpart of Lemma 3.6.

PROOF OF THEOREM 2.3. The previous lemmas readily imply the result for sufficiently large λ, and we are left to extend this to all $\lambda > \lambda_c$, which we achieve via coarse-graining.

When $\gamma < 2$, let λ_0 and μ_0 be the values that we obtain from Proposition 5.3. To apply the proposition for all $\lambda > \lambda_c = 0$, we partition $\mathbb{Z}^d$ into (hyper)cubes of side length N (for some large N to be determined below), which we call N-boxes. In every N-box, we identify the vertex in it with maximum weight and call it the dominant vertex. We now choose β large enough so that $\lambda\beta^2(3d)^{-\alpha} > \lambda_0$. Second, we call those N-boxes good that contain a vertex with weight at least $\beta N^{\alpha/2}$. We choose N large enough so that the probability that an N-box is good is larger than μ_0 using Lemma 3.4. Lemma 3.6 implies that the probability that the dominant vertices in two good N-boxes, being *k boxes away from each other*, are connected, is bounded from below by $\mathbb{P}_{(\lambda_0, W'')}(\{v_1, v_2\} \text{ is open})$, where $v_1, v_2 \in \mathbb{Z}^d$ such that $|v_1 - v_2| = k$, and where W_1'', W_2'' are i.i.d. distributed according to (3.3). Thus, the status of the edges between dominant vertices in good N-boxes stochastically dominates an SFP model on $\mathbb{Z}^d$ with parameters α, λ_0 and weight-law W'', combined with a site percolation of intensity μ_0, exactly as described in Proposition 5.3. We now apply Proposition 5.3 to obtain the result for the case $\gamma < 2$.

The case $\alpha \in (d, 2d)$ is analogous: When $\alpha \in (d, 2d)$, let λ_1 and μ_1 be the values that we obtain from Lemma 5.5. To apply the lemma, we partition $\mathbb{Z}^d$ into N-boxes again. Choose β large enough, using Lemma 5.6, such that two N-boxes being k-boxes away from each other, having clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ with size at least $\beta N^{\alpha/2}$ are connected by an open edge between $\mathcal{C}_1$ and $\mathcal{C}_2$ with probability at least $1 - \exp(-\lambda_1/k^\alpha)$. Call the N-boxes that contain a cluster of size at least $\beta N^{\alpha/2}$ the good boxes. Choose N large enough so that the probability that an N-box is

good is larger than μ_1, using Lemma 5.4. We thus find that the status of the edges between the dominant vertices of good N-boxes stochastically dominates an LRP model on $\mathbb{Z}^d$ with independent edge probabilities $p_{x,y} = 1 - \exp(-\lambda_1/|x - y|^\alpha)$, combined with a site-percolation of intensity μ_1. An application of Lemma 5.5 thus obtains the result for the case $\alpha \in (d, 2d)$.

We conclude that in both cases we found a subgraph of the infinite cluster on which the random walk is transient, and hence the random walk is transient on the infinite cluster itself; cf. [20], Section 9. $\quad\square$

Recurrence proof. We verify that we can apply the following lemma.

LEMMA 5.7 (Berger [4], Theorem 3.10). *Let $d = 2, \alpha \geq 2d = 4$ and let $(P_{i,j})_{i,j\in\mathbb{N}}$ be a family of probabilities, such that*

$$\limsup_{i,j\to\infty} \frac{P_{i,j}}{(i + j)^{-4}} < \infty.$$

Consider a shift invariant percolation model on $\mathbb{Z}^2$ on which the bond between (x_1, y_1) and (x_2, y_2) is open with marginal probability $P_{|x_1-x_2|,|y_1-y_2|}$. If there exists an infinite cluster, then this cluster is recurrent.

To bound the marginal probabilities, we need a bound on the expectation of the product of the weights.

LEMMA 5.8. *Assume that the weight-distribution satisfies* (2.2). *Let W_1, W_2 be two independent copies of the random variable W.*
If $\tau > 2$, there exists a constant $C > 0$, such that for $u \geq 1$:

$$\mathbb{E}\big[(W_1 W_2/u) \wedge 1\big] \leq Cu^{-1}.$$

If $\tau \leq 2$, then there exists a constant $C > 0$, such that

$$\mathbb{E}\big[(W_1 W_2/u) \wedge 1\big] \leq C \log(u) u^{-(\tau-1)}.$$

PROOF. The proof for $\tau > 2$ is straightforward. Observe that $\mathbb{E}[W] < \infty$, hence

$$\mathbb{E}\big[(W_1 W_2/u) \wedge 1\big] \leq \mathbb{E}[W]^2/u.$$

For $\tau \leq 2$, we prove the claim for weights that satisfy (3.2) for some $c \geq 1$. Lemma 3.1 then implies that the claim holds for weights that satisfy (2.2).

Let $H(u)$ denote the distribution function of $W_1 W_2$. From [9], Lemma 4.3, we have for some $C' > 0$ that

$$1 - H(u) \leq C' \log(u) u^{-(\tau-1)}.$$

By

$$\int_1^u v \, dH(v) \le \int_1^u \left(1 - H(v)\right) dv,$$

we obtain the result

$$\mathbb{E}\big[(W_1 W_2/u) \wedge 1\big] = 1 - H(u) + \frac{1}{u} \int_1^u v \, dH(v)$$

$$\le 1 - H(u) + \frac{1}{u} \int_1^u \left(1 - H(v)\right) dv$$

$$\le C' \log(u) u^{-(\tau-1)} + \frac{C'}{u} \int_1^u \log(v) v^{-(\tau-1)} \, dv$$

$$\le C' \log(u) u^{-(\tau-1)} + \frac{C''}{u} \log(u) u^{-(\tau-2)}$$

$$\le C \log(u) u^{-(\tau-1)}. \qquad \square$$

PROOF OF THEOREM 2.4.   Observe that the scale-free percolation measure $\mathbb{P}_{(\lambda, W)}$ is indeed shift invariant. According to Lemma 5.7, we need to prove that

$$\limsup_{k \to \infty} k^4 \mathbb{P}\big(\{(0,0), (i,j)\} \text{ is open}\big) < \infty$$

whenever $|i| + |j| = k$. For convenience, we treat only $(i, j) = (k, 0)$, the other cases follow analogously. Lemma 5.8 and the bound $1 - \exp(-x) \le x$ give

$$\limsup_{k \to \infty} k^4 \mathbb{P}\big(\{(0,0), (k,0)\} \text{ is open}\big) = \limsup_{k \to \infty} k^4 \mathbb{E}\big[1 - e^{-\lambda \frac{W_{(k,0)} W_{(0,0)}}{k^\alpha}}\big]$$

$$\le \limsup_{k \to \infty} k^4 \mathbb{E}\big[1 - e^{-\lambda \frac{W_{(k,0)} W_{(0,0)}}{k^\alpha}}\big]$$

$$\le \limsup_{k \to \infty} k^4 \mathbb{E}\bigg[\bigg(\lambda \frac{W_{(k,0)} W_{(0,0)}}{k^\alpha}\bigg) \wedge 1\bigg].$$

For $\tau > 2$, recall that $\alpha \ge 2d = 4$ and, therefore,

$$\limsup_{k \to \infty} k^4 \mathbb{E}\bigg[\bigg(\lambda \frac{W_{(k,0)} W_0}{k^\alpha}\bigg) \wedge 1\bigg] \le \limsup_{k \to \infty} C \lambda k^{4-\alpha} \le C\lambda < \infty.$$

For $\tau \le 2$ and $\gamma > 2$,

$$\limsup_{k \to \infty} k^4 \mathbb{E}\bigg[\lambda \frac{W_{(k,0)} W_0}{k^\alpha} \wedge 1\bigg] \le \limsup_{k \to \infty} k^4 C \lambda^{\tau-1} \log\big(k^\alpha/\lambda\big) k^{-\alpha(\tau-1)}$$

$$\le \limsup_{k \to \infty} C' \log(k) k^{4-\alpha(\tau-1)}$$

$$= \limsup_{k \to \infty} C' \log(k) k^{4-2\gamma}$$

$$= 0. \qquad \square$$

6. Hierarchical clustering: Proof of Theorem 2.6. For the proof of Theorem 2.6, we largely use the same two steps as for the proof of Theorem 2.3. First, we prove the result for large values of λ on an SFP model combined with an i.i.d. Bernoulli site percolation, but only along a sequence $\{m_n\}_{n=1}^\infty$ diverging to infinity. Then we prove the result in this SFP model for all sufficiently large m. Lastly, by a coarse graining argument, we extend the result to hold for all $\lambda > \lambda_c = 0$.

We assume the weights satisfy (3.2). Lemma 3.1 then implies that the claim holds for weights that satisfy (2.1). We start with stating and proving a proposition in which we consider a site-percolated version of SFP.

PROPOSITION 6.1. *Consider SFP on $\mathbb{Z}^d$ with $d \geq 1$, $\gamma < 2$, and weight distribution that satisfies (3.3). Independently of this, perform an i.i.d. Bernoulli site percolation on the vertices of $\mathbb{Z}^d$, colouring a vertex "green" with probability $\mu \in (0, 1]$. Denote by $\mathcal{C}_{\lambda,\mu}$ the (unique) infinite subgraph of the infinite scale-free percolation cluster induced by the green vertices. We call this the site-percolated SFP.*

There exist constants $\mu_1 < 1, \lambda_1 > 0, K, \rho > 0$ and $n_2 \in \mathbb{N}$, and sequence $\{m_n\}_{n=1}^\infty$ with $m_n \in \mathbb{N}$, such that for all ξ satisfying

$$(6.1) \qquad 0 < \xi < \min\left(\frac{d(2-\gamma)}{\tau+1}, \frac{d}{2}(\tau + 2 - \sqrt{(\tau+2)^2 - 4(2-\gamma)})\right),$$

the following hold:

(1) The probability that the site-percolated SFP configuration with parameters $\mu \geq \mu_1$ and $\lambda \geq \lambda_1$ contains a $(0, m_n, \rho, K)$-hierarchically clustered tree inside the box $[0, m_n - 1]^d$, is bounded from below by

$$1 - 3\exp(-\rho m_n^\xi).$$

(2) Site-percolated SFP with $\lambda \geq \lambda_1$ and $\mu \geq \mu_1$ has an infinite component $\mathcal{C}_{\lambda,\mu}$ almost surely. $\mathcal{C}_{\lambda,\mu}$ contains a.s. an infinite, connected, cycle-free subgraph $\mathcal{T}_\infty$ such that removing an arbitrary edge yields a finite and infinite connected component and there exist $x \in \mathbb{Z}^d$ and $m \geq 1$ such that the finite connected component is an (x, m, ρ, K)-hierarchically clustered tree.

PROOF. Let D_1 be a large integer to be determined later, and let $\{a_n\}_{n=1}^\infty$, with $a_n \in (0, 1]$, be a sequence also to be determined later, such that

$$(6.2) \qquad \rho := \prod_{n=1}^\infty a_n > 0.$$

Let

$$\xi' \in \left(\xi, \min\left(\frac{d(2-\gamma)}{\tau+1}, \frac{d}{2}(\tau + 2 - \sqrt{(\tau+2)^2 - 4(2-\gamma)})\right)\right).$$

The bound $\xi' < d\frac{2-\gamma}{\tau+1}$ implies

$$(6.3) \qquad \zeta := \frac{d - \xi'}{(\alpha + \xi')(\tau - 1) - (d - \xi')} > 1.$$

For all $n \geq 2$, let $D_n := \lceil D_{n-1}^{\zeta} \rceil$, so that we have the telescoping product:

$$(6.4) \qquad D_n \geq D_{n-1}^{\zeta} \geq D_1 \prod_{k=1}^{n-1} D_k^{\zeta-1} \geq D_1^{\zeta^{n-1}},$$

and let

$$u_n := \prod_{k=1}^{n} D_k^{\frac{d-\xi'}{\tau-1}} \quad \text{and} \quad C_n := a_n D_n^d.$$

We give a sequence of graphs for $\mathbb{Z}^d$ by the same procedure as in Proposition 5.3. We call the vertices the 0-stage boxes. We partition the lattice $\mathbb{Z}^d$ into boxes of side-length D_1, so that each box contains D_1^d vertices, and call these the 1-stage boxes. Iteratively, we group D_{n-1}^d $(n-1)$-stage boxes into n-stage boxes, so that the n-stage boxes form a covering of $\mathbb{Z}^d$ by translates of $[0, \prod_{k=1}^{n} D_k - 1]^d$.

We call a 0-stage box "good" if the associated vertex is green, and we call this vertex "0-dominant". For every stage $n \geq 1$, we define rules for a box to be "good" or "bad" and for a vertex to be "n-dominant" depending only on the weights W_x and colours of the vertices and on the edges of $\mathcal{C}_{\lambda,\mu}$ inside the box. This implies that disjoint boxes are good or bad independently of each other.

For $n \geq 1$, we inductively define that an n-stage box is good if the following three conditions hold:

(E) At least C_n of the $(n-1)$-stage boxes it contains are good.

(F) The maximum weight $(n-1)$-dominant vertex in one of its good $(n-1)$-stage boxes has weight at least u_n. Call this vertex n-dominant. (A vertex can be dominant for multiple stages.)

(G) All $(n-1)$-dominant vertices in the good $(n-1)$-stage boxes are connected to the n-dominant vertex by an edge in $\mathcal{C}_{\lambda,\mu}$.

Define $E_n(v)$, $F_n(v)$ and $G_n(v)$ to be the events that respectively (E), (F) and (G) hold for the n-stage box containing the vertex v. Denote by $L_n(v)$ the event that the n-stage box containing v is good, that is, $L_n(v) = E_n(v) \cap F_n(v) \cap G_n(v)$. To simplify notation, we write $E_n := E_n(0)$, $F_n := F_n(0)$, $G_n := G_n(0)$ and $L_n := L_n(0)$.

On the event that L_n occurs, we can construct a graph by the following procedure:

(1) Start with the set of all 0-dominant vertices inside the n-stage box containing v.
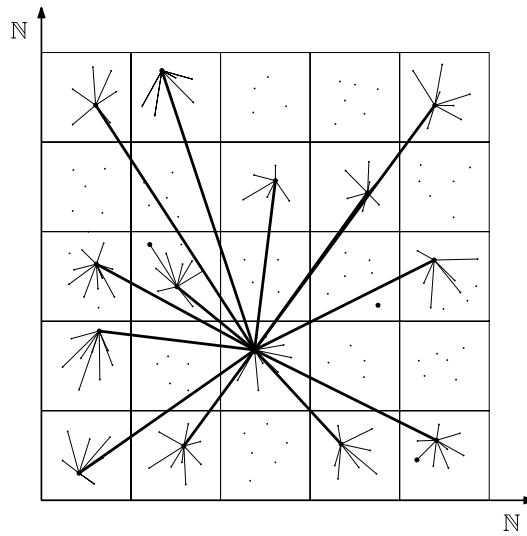
FIG. 7.   *Sketch of the renormalization in Proposition* 6.1 *for* $d = 2$.

(2) For every $i \in \{0, \dots, n-1\}$ connect each i-dominant vertex v to the $(i+1)$-dominant vertex inside the $(i+1)$-stage box that contains v (unless this creates a self-loop).

Figure 7 displays the first two stages of the construction. We sequence m_k and obtain a bound:

$$(6.5) \qquad m_n := \prod_{k=1}^{n} D_k \geq D_1^{\frac{\xi^n - 1}{\xi - 1}}.$$

We claim that the constructed connected component of the n-dominant vertex in the n-stage box of v is a tree that satisfies the conditions given in Definition 2.5. Since (6.2) holds, the event L_k implies that the intersection of the site-percolated SFP configuration and the cube $[0, m_k - 1]^d$ contains a tree with at least

$$\prod_{k=1}^{n} C_k = \prod_{k=1}^{n} a_k D_k^d \geq \rho m_n^d$$

vertices, which verifies Condition (1) of Definition 2.5. We obtain Condition (2) for some $K > 0$ because the box-size m_n as well as the number of vertices in $[0, m_n - 1]^d$ both grow double exponentially fast in n. Conditions (3) and (4) follow straightforwardly from the construction. We therefore conclude that Proposition 6.1(1) follows if we show that

$$(6.6) \qquad \mathbb{P}(L_n^c) \leq 3 \exp\left(-\rho \prod_{k=1}^{n} D_k^{\xi}\right),$$

and that Proposition 6.1(2) follows if we show that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} L_n\right) > 0.$$

The events L_n are positively correlated, hence it is sufficient for assertion (2) of the theorem that $\prod_{n=1}^{\infty} \mathbb{P}(L_n) > 0$, which follows from (6.6). It thus remains to prove (6.6).

We bound

$$(6.7) \qquad \mathbb{P}(L_n^c) \leq \mathbb{P}(E_n^c) + \mathbb{P}(F_n^c | E_n) + \mathbb{P}(G_n^c | E_n \cap F_n),$$

and analyze the terms separately.

We start with proving two bounds on $\mathbb{P}(F_n^c | E_n)$. The conditioning on E_n gives that the sum of cluster sizes in the good $(n-1)$-stage boxes is at least $\prod_{k=1}^{n} C_k \geq \prod_{k=1}^{n} a_k D_k^d$. Using Lemma 3.3 with $K_1 = 1$, $K_2 = u_n$, the definition of u_n yields that

$$(6.8) \qquad \mathbb{P}(F_n^c | E_n) \leq \exp\left(-u_n^{-(\tau-1)} \prod_{k=1}^{n} a_k D_k^d\right) = \exp\left(-\prod_{k=1}^{n} a_k D_k^{\xi'}\right).$$

Similarly, the conditioning on E_n gives that there are at least C_n vertices with weight at least u_{n-1}. Using Lemma 3.3 with $K_1 = u_{n-1}$, $K_2 = u_n$, the definition of u_n yields that

$$(6.9) \qquad \mathbb{P}(F_n^c | E_n) \leq \exp\left(-a_n D_n^d \left(D_n^{-\frac{d-\xi'}{\tau-1}}\right)^{\tau-1}\right) = \exp\left(-a_n D_n^{\xi'}\right).$$

Note that it is not a priori clear which of these two bounds is better, since this may depend on the choice of D_1, n, τ, ξ' and the sequence (a_k). Therefore, we use both bounds.

We move on to $\mathbb{P}(G_n^c | E_n \cap F_n)$. For $n \geq 2$, define

$$(6.10) \qquad \beta_n := u_n u_{n-1} \prod_{k=1}^{n} D_k^{-\alpha}.$$

Since $D_n = \lceil D_{n-1}^{\zeta} \rceil$ by definition, it follows that

$$D_{n-1} \geq \left(\frac{1}{2} D_n\right)^{1/\zeta}.$$

Substituting the values of u_n, u_{n-1}, ζ and (6.4) gives

$$\frac{\beta_n}{\beta_{n-1}} = \frac{u_n}{u_{n-2}} D_n^{-\alpha} = D_n^{-\alpha+\frac{d-\xi'}{\tau-1}} D_{n-1}^{\frac{d-\xi'}{\tau-1}} \geq \left(\frac{1}{2}\right)^{1/\zeta} D_n^{-\alpha+\frac{d-\xi'}{\tau-1}} \left(D_n^{\frac{(\alpha+\xi')(\tau-1)-(d-\xi)}{d-\xi'}}\right)^{\frac{d-\xi'}{\tau-1}}$$

$$= \left(\frac{1}{2}\right)^{1/\zeta} D_n^{-\alpha+\frac{d-\xi'}{\tau-1}} D_n^{\alpha+\xi'-\frac{d-\xi'}{\tau-1}} = \left(\frac{1}{2}\right)^{1/\zeta} D_n^{\xi'}.$$

It follows that for some $c > 0$,

$$\beta_n \geq c\left(\frac{1}{2}\right)^{(n-1)/\zeta} \prod_{k=1}^{n} D_k^{\xi'}.$$

The distance between two $(n-1)$-dominant vertices in the same n-stage box is at most $d \prod_{k=1}^{n} D_k$. There are at most D_n^d good $(n-1)$-dominant vertices (having weight at least u_{n-1}). Recalling (6.10), by the union bound and the conditioning on E_n and F_n, we thus obtain that

$$\mathbb{P}(G_n^c | E_n \cap F_n) \leq D_n^d \exp\left(-\lambda d^{-\alpha} u_n u_{n-1} \prod_{k=1}^{n} D_k^{-\alpha}\right)$$

$$(6.11) \qquad\qquad = D_n^d \exp(-\lambda d^{-\alpha} \beta_n)$$

$$\leq \exp\left(d \log(D_n) - c\lambda d^{-\alpha}\left(\frac{1}{2}\right)^{(n-1)/\zeta} \prod_{k=1}^{n} D_k^{\xi'}\right).$$

Since D_n grows double exponentially fast and $\xi' > \xi > 0$, we obtain for n sufficiently large (larger than n_0, say), that

$$(6.12) \qquad\qquad \mathbb{P}(G_n^c | E_n \cap F_n) \leq \exp\left(-\prod_{k=1}^{n} D_k^{\xi}\right).$$

We move on to $\mathbb{P}(E_n^c)$. All $(n-1)$-stage boxes are good independently of each other with probability $\mathbb{P}(L_{n-1})$. Let $X \sim \mathrm{Bin}(D_n^d, \mathbb{P}(L_{n-1}))$. Then

$$\mathbb{P}(E_n^c) = \mathbb{P}(X < C_n) = \mathbb{P}(X < a_n D_n^d).$$

As in (5.5), we apply Chernoff's bound and obtain

$$\mathbb{P}(E_n^c) \leq \exp\left(-\frac{(\mathbb{P}(L_{n-1}) - a_n)^2}{2\mathbb{P}(L_{n-1})} D_n^d\right),$$

whenever $0 < a_n < \mathbb{P}(L_{n-1})$. We now choose

$$(6.13) \qquad\qquad a_n := \mathbb{P}(L_{n-1})\left(1 - \sqrt{2} D_n^{-d/2} \prod_{k=1}^{n} D_k^{\xi'/2}\right).$$

We will show below that $a_n > 0$ for all $n \geq 1$ and that $\prod_{n=1}^{\infty} a_n > 0$, as required by (6.2). Assuming these inequalities, we have

$$(6.14) \qquad\qquad \mathbb{P}(E_n^c) \leq \exp\left(-\mathbb{P}(L_{n-1}) \prod_{k=1}^{n} D_k^{\xi'}\right),$$

so that using $\mathbb{P}(L_{n-1}) > a_n > \rho$, $\xi' > \xi$, (6.8), and (6.12) yields

$$\mathbb{P}(L_n^c) \leq \exp\left(-\mathbb{P}(L_{n-1}) \prod_{k=1}^{n} D_k^{\xi'}\right)$$

(6.15)
$$+ \exp\left(-\prod_{k=1}^{n} a_k \prod_{k=1}^{n} D_k^{\xi'}\right) + \exp\left(-\prod_{k=1}^{n} D_k^{\xi}\right)$$

$$\leq 3 \exp\left(-\prod_{k=1}^{n} a_k \prod_{k=1}^{n} D_k^{\xi}\right),$$

which gives the desired bound (6.6). For later reference, we note that if instead we apply (6.7), (6.9) and (6.12), then we obtain for n sufficiently large:

(6.16) $$\mathbb{P}(L_n^c) \leq \exp\left(-\mathbb{P}(L_{n-1}) \prod_{k=1}^{n} D_k^{\xi'}\right) + \exp(-a_n D_n^{\xi'}) + \exp\left(-\prod_{k=1}^{n} D_k^{\xi}\right).$$

All that remains is to show that $a_n > 0$ for all $n \geq 1$ and that $\prod_{n=1}^{\infty} a_n > 0$. For positivity of a_n, it is by (6.13) sufficient to show that for some $b > 0$ and D_1 sufficiently large:

(6.17)
$$D_n^{-d/2} \prod_{k=1}^{n} D_k^{\xi'/2} < \frac{1}{2\sqrt{2}} D_1^{-b\zeta^n}.$$

Since $\xi' > 0$, $\zeta > 1$, $\prod_{k=1}^{n} D_k \leq (D_{n+1}/D_1)^{1/(\zeta-1)}$ and $D_{n+1} = \lceil D_n^{\zeta} \rceil \leq 2D_n^{\zeta}$ by (6.4), we obtain

$$D_n^{-d} \prod_{k=1}^{n} D_k^{\xi'} \leq D_n^{-d} \left(\frac{D_{n+1}}{D_1}\right)^{\xi'/(\zeta-1)} = D_n^{-d} \left(\frac{\lceil D_n^{\zeta} \rceil}{D_1}\right)^{\xi'/(\zeta-1)}$$

$$\leq \left(\frac{2}{D_1}\right)^{\xi'/(\zeta-1)} D_n^{-d + \frac{\xi'\zeta}{(\zeta-1)}}.$$

We show that

$$-d + \xi' \frac{\zeta}{\zeta - 1} < 0.$$

By definition, $\xi' < \frac{d}{2}(\tau + 2 - \sqrt{(\tau + 2)^2 - 4(2 - \gamma)})$. So we derive, after rearranging terms and dividing by $\sqrt{d}$, that

$$-\frac{1}{\sqrt{d}}\xi' + \frac{\sqrt{d}}{2}(\tau + 2) > \frac{\sqrt{d}}{2}\sqrt{(\tau + 2)^2 - 4(2 - \gamma)}.$$

Squaring and substituting $\gamma = \alpha(\tau - 1)/d$ yield

$$\frac{1}{d}\xi'^2 - \xi'(\tau + 2) + 2d - \alpha(\tau - 1) > 0,$$

so that after rearranging

$$\frac{1}{d}(d - \xi')^2 > (\alpha + \xi')(\tau - 1) - (d - \xi').$$

Hence, we obtain the result after dividing by $(d - \xi')$ and, using (6.3), substituting ζ:

$$1 - \frac{\xi'}{d} > \frac{1}{\zeta},$$

which can be inverted to give

$$-d + \xi' \frac{\zeta}{\zeta - 1} < 0.$$

Hence, there exists a constant $b > 0$ such that, when we choose D_1 sufficiently large,

$$D_n^{-d/2} \prod_{k=1}^{n} D_k^{\xi'/2} < \frac{1}{2\sqrt{2}} D_n^{-b\zeta}.$$

By (6.4), we have $D_n \geq D_1^{\zeta^{n-1}}$, so (6.17) follows, and we may conclude that $a_n > 0$.

Observe that $\prod_{k=1}^{\infty} a_k > 0$ if and only if $\prod_{k=1}^{\infty} \mathbb{P}(L_k) > 0$, since a_n approaches $\mathbb{P}(L_n)$ double exponentially fast. Moreover, combining (6.13) and (6.17) gives

$$a_n \geq \frac{1}{2} \mathbb{P}(L_{n-1}),$$

so that, using (6.16), we can bound

(6.18)
$$\mathbb{P}(L_n^c) \leq \exp\left(-\mathbb{P}(L_{n-1}) \prod_{k=1}^{n} D_k^{\xi'}\right)$$
$$+ \exp\left(-\frac{1}{2}\mathbb{P}(L_{n-1}) D_n^{\xi'}\right) + \exp\left(-\prod_{k=1}^{n} D_k^{\xi}\right).$$

Define the sequence

$$\ell_n := 1 - (n+1)^{-3/2},$$

and observe that

(6.19)
$$\prod_{n=1}^{\infty} \ell_n > 0.$$

For any fixed $n_1 > n_0$, we can find $\lambda_0 > 0$ and $\mu_0 < 1$ such that $\mathbb{P}(L_{n_1}) \geq \ell_{n_1}$, because L_{n_1} depends only on the weights and edges inside a *finite* box. Since D_k

grows double exponentially fast, we further bound (6.18) for all $n > n_1$ by

$$\mathbb{P}(L_n^c) \leq \exp\left(-\left(1 - \frac{1}{\sqrt{2}}\right)\prod_{k=1}^{n} D_k^{\xi'}\right)$$

(6.20)
$$+ \exp\left(-\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)D_n^{\xi'}\right) + \exp\left(-\prod_{k=1}^{n} D_k^{\xi}\right)$$

$$\leq (n+1)^{-3/2} = 1 - \ell_n.$$

We thus choose n_1 so large that the last bound in (6.20) holds. Using (6.19), (6.20) and $\mathbb{P}(L_n) > 0$ for all n, yields that

$$\prod_{n=1}^{\infty} \mathbb{P}(L_n) = \prod_{n=1}^{n_1} \mathbb{P}(L_n) \prod_{n=n_1+1}^{\infty} \mathbb{P}(L_n) \geq \prod_{n=1}^{n_1} \mathbb{P}(L_n) \prod_{n=n_1+1}^{\infty} \ell_n > 0.$$

Recalling that, by (6.13) and (6.17), this is equivalent to (6.2), the bound (6.15) completes the proof. $\quad\square$

REMARK 6.2. Note that K in the above proof depends on the choice of ξ, that is, $K = K(\xi)$. A short calculation shows that $\inf_\xi K(\xi) = 2/|\log(\gamma - 1)|$ (this is achieved at $\xi, \xi' = 0^+$). This means that, for x, y in the hierarchically clustered tree,

$$d_{\mathcal{C}}(x, y) \leq \frac{2(1 + \varepsilon)}{|\log(\gamma - 1)|} \log\log\|x - y\|$$

for all $\varepsilon > 0$. We believe that this is also true w.h.p. for the other vertices in the infinite cluster. Deijfen *et al.* [9], Corollary 5.4, show that asymptotically, as $\|x - y\| \to \infty$, $2/|\log(\gamma - 1)|$ is indeed the correct constant.

To prove Theorem 2.6, we need extend the above claims from the specific sequence $\{m_n\}_{n=1}^{\infty}$ to *all* (sufficiently large) $m \in \mathbb{N}$. This extension is the content of the next lemma. After this lemma, we extend the claim to hold for all $\lambda > 0$ and weights following a power-law given by (3.2).

LEMMA 6.3. *Consider SFP on $\mathbb{Z}^d$ with $d \geq 1, \gamma < 2$, and weight distribution that satisfies (3.3). Independently of this, perform i.i.d. Bernoulli site percolation on the vertices of $\mathbb{Z}^d$, colouring a vertex "green" with probability $\mu \in (0, 1]$. Denote by $\mathcal{S}_{m,\lambda,\mu}$ the SFP-realization induced by the green vertices in the box $[0, m - 1]^d$. We call this the site-percolated SFP.*

Then there exist a density $\rho > 0$ and constants $\mu_1 < 1, \lambda_2 > 0$, and $K', m_0 \in \mathbb{N}$, such that for $m \geq m_0$, and parameters $\lambda \geq \lambda_2, \mu \geq \mu_1$, the probability that $\mathcal{S}_{m,\lambda,\mu}$ with parameters contains a $(0, m, \rho, K')$-hierarchically clustered tree inside the box $[0, m - 1]^d$ is bounded from below by

$$1 - 3\exp(-\rho m^\xi),$$

whenever $\xi < \min(\frac{d(2-\gamma)}{\tau+1}, \frac{d}{2}(\tau + 2 - \sqrt{(\tau + 2)^2 - 4(2 - \gamma)}))$.

PROOF.   Let the constants $\mu_1, \lambda_1, \xi', K$, the sequences $\{D_k\}, \{u_k\}, \{C_k\}$ and $\{m_k\}$ be as in Proposition 6.1, and ζ as in (6.3). Assume $\mu \geq \mu_1, \lambda \geq \lambda_1$, and let m be large enough (how large precisely will be determined in several steps). We define

$$(6.21) \qquad n = \sup\{i : m_i \leq m\}, \quad \text{and} \quad k = \left\lfloor \frac{m}{m_n} \right\rfloor,$$

both depending on m, and note that $n \to \infty$ as $m \to \infty$. Partition the box $[0, km_n]^d$ into k^d boxes of side-length m_n. We call these the n-boxes. Let v^* be the vertex in $[0, km_n]^d$ with maximum weight.

We use the same definition of good boxes and dominant vertices as in the proof of Proposition 6.1. So in particular, an n-box is good if its n-dominant vertex has weight at least $u_n = m_n^{(d-\xi')/(\tau-1)}$. We define E to be the event that at least $\frac{1}{2}k^d$ n-boxes are good, F the event that $W_{v^*} \geq (km_n)^{(d-\xi')/(\tau-1)}$, and G the event that every good n-box's n-dominant vertex is connected by an open edge to v^*. Let $L = E \cap F \cap G$.

Observe that the event L implies indeed that there exists a $(0, m, \rho, K)$-hierarchically clustered tree. Properties (1), (3) and (4) of Definition 2.5 readily follow from Proposition 6.1. For Property (2), we observe that, for any $n \in \mathbb{N}$, the diameter of the constructed tree, after connecting the separate trees via v^*, only increases by 2 in comparison to Proposition 6.1. Hence, there exists $K' > 0$, such that Property (2) is satisfied. We bound

$$(6.22) \qquad \mathbb{P}(L^c) \leq \mathbb{P}(E^c) + \mathbb{P}(F^c|E) + \mathbb{P}(G^c|E \cap F),$$

and analyze the three summands term by term.

By our assumption (3.3), all $(km_n)^d$ vertices have weight at least 1. By Lemma 3.3,

$$(6.23) \qquad \mathbb{P}(F^c|E) \leq \exp(-(km_n)^\xi).$$

The ℓ^1-distance between two vertices in $[0, km_n]^d$ is bounded above by dkm_n, and the number of n-dominant vertices is at most k^d. Therefore,

$$(6.24) \qquad \mathbb{P}(G^c|E \cap F) \leq k^d \exp\left(-\lambda d^{-\alpha} m_n^{2\frac{d-\xi'}{\tau-1}-\alpha} k^{\frac{d-\xi'}{\tau-1}-\alpha}\right).$$

We claim that

$$(6.25) \qquad m_n^{2\frac{d-\xi'}{\tau-1}-\alpha} k^{\frac{d-\xi'}{\tau-1}-\alpha} \geq c(km_n)^{\xi'}, \qquad \text{for some constant } c > 0.$$

Indeed, by our choice of n and k in (6.21), $D_{n+1} = \lceil D_n^\zeta \rceil < D_n^\zeta + 1$, and $m_n = \prod_{k=1}^n D_k$ by their definitions in (6.4) and (6.5), we have

$$\frac{k}{m_n^{\zeta-1}} \leq \frac{D_{n+1}}{m_n^{\zeta-1}} \leq \frac{D_n^\zeta + 1}{D_n^{\zeta-1} \prod_{k=1}^{n-1} D_k^{\zeta-1}} = \frac{D_n}{m_{n-1}^{\zeta-1}} + \prod_{k=1}^n D_k^{1-\zeta}.$$

Iterating this gives

$$\frac{k}{m_n^{\zeta-1}} \le \frac{D_{n+1}}{m_n^{\zeta-1}} \le \frac{D_n}{m_{n-1}^{\zeta-1}} + \prod_{k=1}^{n} D_k^{1-\zeta}$$

$$\le \frac{D_{n-1}}{m_{n-2}^{\zeta-1}} + \prod_{k=1}^{n-1} D_k^{1-\zeta} + \prod_{k=1}^{n} D_k^{1-\zeta} \le \cdots \le D_1 + \sum_{k=1}^{n}\left(\prod_{l=1}^{k} D_l\right)^{1-\zeta}.$$

Recall that by (6.3) $\zeta > 1$, and by (6.4) $\prod_{l=1}^{k} D_l \ge (1/D_1)D_1^{\zeta^k/(\zeta-1)}$, so

$$\frac{k}{m_n^{\zeta-1}} \le D_1 + \sum_{k=1}^{n}\left(\prod_{l=1}^{k} D_l\right)^{1-\zeta} \le D_1 + \sum_{k=1}^{\infty}\left(\prod_{l=1}^{k} D_l\right)^{1-\zeta}$$

(6.26)

$$\le D_1 + \sum_{k=1}^{\infty} D_1^{-\zeta^k} < \infty.$$

Hence, we can further estimate for some $c' > 0$

$$k \le c'm_n^{-1+\zeta} = c'm_n^{-1+\frac{d-\xi'}{(\alpha+\xi')(\tau-1)-(d-\xi')}} = c'm_n^{-1+\frac{d-\xi'}{\tau-1}(\alpha+\xi'-\frac{d-\xi'}{\tau-1})^{-1}},$$

so that

$$k^{\alpha+\xi'-\frac{d-\xi'}{\tau-1}} \le c'^{\alpha+\xi'-\frac{d-\xi'}{\tau-1}} m_n^{-(\alpha+\xi'-\frac{d-\xi'}{\tau-1})+\frac{d-\xi'}{\tau-1}},$$

and finally

$$c'^{\frac{d-\xi'}{\tau-1}-\xi'-\alpha}(km_n)^{\xi'} \le m_n^{2\frac{d-\xi'}{\tau-1}-\alpha} k^{\frac{d-\xi'}{\tau-1}-\alpha},$$

from which (6.25) follows. Consequently, by (6.24),

(6.27) $\mathbb{P}(G^c|E \cap F) \le k^d \exp(-\lambda d^{-\alpha} c'^{\frac{d-\xi'}{\tau-1}-\xi'-\alpha}(km_n)^{\xi'}) \le \exp(-(km_n)^{\xi}),$

where we choose λ sufficiently large for the second bound (this determines the value of λ_2).

It remains to bound $\mathbb{P}(E^c)$. By Proposition 6.1, there exists $\rho > 0$, such that for n large, n-boxes are good independently of each other with probability at least $1 - \exp(-\rho m_n^\xi)$. Let $X \sim \text{Bin}(k^d, 1 - \exp(-\rho m_n^\xi))$. Writing out the binomial distribution, using $\binom{n}{k} \le n^k$ and $1 - \exp(-x) \le 1$, further bounding the sum by its maximum and the number of terms, we obtain

$$\mathbb{P}(E^c) = \mathbb{P}\left(X < \frac{1}{2}k^d\right) = \sum_{l=0}^{\lfloor\frac{1}{2}k^d\rfloor} \binom{k^d}{l} \exp(-\rho m_n^\xi)^{k^d-l}(1 - \exp(-\rho m_n^\xi))^l$$

$$\le \left(\frac{1}{2}k^d + 1\right)k^{\frac{1}{2}dk^d} \exp\left(-\frac{1}{2}\rho k^d m_n^\xi\right)$$

$$= \exp\left(\log\left(\frac{1}{2}k^d + 1\right) + \frac{1}{2}dk^d \log(k) - \frac{\rho}{2}k^d m_n^\xi\right).$$

For any ε with $0 < \varepsilon < d - \xi$, we can take m large enough so that (using $k^{d-\xi} \geq 1$)

$$(6.28) \qquad \mathbb{P}(E^c) \leq \exp\left(-\frac{\rho}{2^{1+\varepsilon}}(km_n)^\xi\right).$$

Combining (6.22), (6.23), (6.27) and (6.28) gives that for m sufficiently large,

$$\mathbb{P}\left(\mathcal{S}_{km_n} \text{ contains a } \left(0, km_n, \frac{\rho}{2}, K\right)\text{-hierarchically clustered tree}\right)$$

$$\geq 1 - 3\exp\left(-\frac{\rho}{2^{1+\varepsilon}}(km_n)^\xi\right).$$

From the construction, it follows that

$$\frac{1}{2} \leq \frac{km_n}{m} \leq 1,$$

so that for m large

$$\mathbb{P}\left(\mathcal{S}_m \text{ contains a } \left(0, m, \frac{\rho}{2^{d+1}}, K'\right)\text{-hierarchically clustered tree}\right)$$

$$\geq \mathbb{P}\left(\mathcal{S}_{km_n} \text{ contains a } \left(0, km_n, \frac{\rho}{2}, K'\right)\text{-hierarchically clustered tree}\right)$$

$$\geq 1 - 3\exp\left(-\frac{\rho}{2^{1+\varepsilon}}\frac{m^\xi}{2^\xi}\right) \geq 1 - \exp\left(-\frac{\rho}{2^{d+1}}m^\xi\right),$$

which completes the proof. $\quad\square$

The last step is to extend the claim to hold for all $\lambda > 0$ and weights following a power-law given by (3.2).

PROOF OF THEOREM 2.6. Recall that we consider SFP models with $1 < \gamma < 2$ and any $\lambda > 0$, and that in this setting, $\lambda_c = 0$ [9]. Let μ_1 and λ_1 be the values that we obtain from Lemma 6.3. To apply Lemma 6.3, we partition $\mathbb{Z}^d$ into N-boxes. In every N-box, we only consider the vertex with maximum weight and call it the dominant vertex. Choose β large enough, using Lemma 3.6, such that two N-boxes that are k-boxes apart, with dominant vertices u_1 and u_2 having weight at least $\beta N^{\alpha/2}$, are connected by an open edge between u_1 and u_2 with probability at least

$$\mathbb{P}_{(\lambda_1, W'')}(\{v_1, v_2\} \text{ is open}),$$

for $v_1, v_2 \in \mathbb{Z}^d$ such that $|v_1 - v_2| = k$ and weights W'' with law given by (3.2). Define the N-boxes that contain a vertex with weight at least $\beta N^{\alpha/2}$ to be the good boxes. Choose N large enough so that the probability that an N-box is good is larger than μ_1 using Lemma 3.4. Thus, the status of the edges between dominant vertices in good N-boxes in the SFP model with parameters α, λ and weight-law W

stochastically dominates an SFP model on $\mathbb{Z}^d$ with parameters α, λ_1 and weight-law W'' combined with a site percolation of intensity μ_1, exactly as described in Lemma 6.3. Let K' and ρ' be the constants we obtain from Lemma 6.3. Observe that

$$\frac{\lfloor \frac{m}{N} \rfloor N}{m} \geq \frac{1}{2}.$$

Then the assertions of Theorem 2.6 follows if we set $K = K'$ and $\rho = \frac{\rho'}{2^d N^d}$. $\square$

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