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MODEL-FREE SUPERHEDGING DUALITY

BY MATTEO BURZONI*, MARCO FRITTELLI† AND MARCO MAGGIS†

ETH Zurich and University of Milan†*

In a model-free discrete time financial market, we prove the superhedging duality theorem, where trading is allowed with dynamic and semistatic strategies. We also show that the initial cost of the cheapest portfolio that dominates a contingent claim on every possible path $\omega \in \Omega$, might be strictly greater than the upper bound of the no-arbitrage prices. We therefore characterize the subset of trajectories on which this duality gap disappears and prove that it is an analytic set.

1. Introduction.

The aim of this article is the proof of the following discrete time, model independent version of the superhedging theorem.

THEOREM 1.1 (Superhedging). *Let $g : \Omega \mapsto \mathbb{R}$ be an $\mathcal{F}$-measurable random variable. Then*

$$\begin{aligned}
&\inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g\ \mathcal{M}\text{-q.s.}\} \\
&\quad = \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_*\} \\
&\quad = \sup_{Q \in \mathcal{M}_f} E_Q[g] = \sup_{Q \in \mathcal{M}} E_Q[g],
\end{aligned}$$

where

$$\Omega_* := \{\omega \in \Omega \mid \exists Q \in \mathcal{M} \text{ such that } Q(\omega) > 0\} \tag{1.1}$$

and the inf *is attained by a strategy $H \in \mathcal{H}$ whenever it is finite.*

We adopt the following setting and notation: let Ω be a Polish space and $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel sigma-algebra; $T \in \mathbb{N}$, $I := \{0, \dots, T\}$, $S = (S_t)_{t \in I}$ be an $\mathbb{R}^d$-valued stochastic process on $(\Omega, \mathcal{F})$ representing the price process of $d \in \mathbb{N}$ assets; $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$;$\mathbb{F}^S := \{\mathcal{F}_t^S\}_{t \in I}$ be the natural filtration and $\mathbb{F} := \{\mathcal{F}_t\}_{t \in I}$ be the universal filtration, namely,

$$\mathcal{F}_t := \bigcap_{P \in \mathcal{P}} \mathcal{F}_t^S \vee \mathcal{N}_t^P, \qquad \text{where } \mathcal{N}_t^P = \{N \subseteq A \in \mathcal{F}_t^S \mid P(A) = 0\};$$

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$\mathcal{H}$ be the class of $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable stochastic processes, representing the family of admissible trading strategies; $(H \cdot S)_T := \sum_{t=1}^T \sum_{j=1}^d H_t^j (S_t^j - S_{t-1}^j) = \sum_{t=1}^T H_t \cdot \Delta S_t$ be the gain up to time T from investing in S adopting the strategy H. We denote

$$\mathcal{M} := \{Q \in \mathcal{P} | S \text{ is an } \mathbb{F}\text{-martingale under } Q\},$$
$$\mathcal{P}_f := \{Q \in \mathcal{P} | \operatorname{supp}(Q) \text{ is finite}\},$$
$$\mathcal{M}_f := \mathcal{M} \cap \mathcal{P}_f,$$

where the support of $P \in \mathcal{P}$ is defined by $\operatorname{supp}(P) = \bigcap\{C \in \mathcal{F} | C \text{ closed}, P(C) = 1\}$. The family of $\mathcal{M}$-polar sets is given by $\mathcal{N} := \{N \subseteq A \in \mathcal{F} | Q(A) = 0\ \forall Q \in \mathcal{M}\}$ and a property is said to hold *quasi surely* (q.s.) if it holds outside a polar set. We adopt the convention $\infty - \infty = -\infty$ for those random variables g whose positive and negative part is not integrable. For random variables X and Y, we write $X > Y$ if $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$. When we specify $X > Y$ on a set $A \subset \Omega$, it means that $X(\omega) > Y(\omega)$ holds for all $\omega \in A$, and similarly for $X \geq Y$ and $X = Y$. We are also assuming the existence of a numeraire asset $S_t^0 = 1$ for all $t \in I$.

Probability-free set up. In the statement of the superhedging theorem, there is no reference to any a priori assigned probability measure and the notions of $\mathcal{M}$, $\mathcal{H}$ and Ω_* only depend on the measurable space $(\Omega, \mathcal{F})$ and the price process S. In general, the class $\mathcal{M}$ is not dominated.

We are not imposing any restriction on S so that it may describe generic financial securities (e.g., stocks and/or options). However, in the framework of Theorem 1.1 the class $\mathcal{H}$ of admissible trading strategies requires dynamic trading in all assets. In Theorem 1.2 below, we extend this setup to the case of semistatic trading on a finite number of options.

As illustrated in Section 4, we explicitly show that the initial cost of the cheapest portfolio that dominates a contingent claim g on *every possible path*, namely,

$$\inf\{x \in \mathbb{R} | \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g \text{ on } \Omega\} \tag{1.2}$$

can be strictly greater than $\sup_{Q \in \mathcal{M}} E_Q[g]$, unless some artificial assumptions are imposed on g or on the market. In order to avoid these restrictions on the class of derivatives, it is crucial to select the correct set of paths (namely, Ω_*) where the superhedging strategy can be efficiently employed.

On the set Ω_*. In Theorem 1.1, the pathwise model independent inequality in (1.2) is replaced with an inequality involving only those $\omega \in \Omega$, which are weighted by at least one martingale measure $Q \in \mathcal{M}$. In [7] (see also Proposition 3.1), it is shown the existence of the maximal $\mathcal{M}$-polar set N_*, namely, a set $N_* \in \mathcal{N}$ containing any other set $N \in \mathcal{N}$. Moreover,

$$\Omega_* = (N_*)^C. \tag{1.3}$$

The inequality $x + (H \cdot S)_T \geq g$ $\mathcal{M}$-q.s. holds by definition outside any $\mathcal{M}$-polar set and therefore it is equivalent, thanks to (1.3), to the inequality $x + (H \cdot S)_T \geq g$ on Ω_*, which justifies the first equality in Theorem 1.1. The set Ω_* can be equivalently determined (see Proposition 3.1) via the set $\mathcal{M}_f$ of martingale measures with finite support, a property that turns out to be crucial in several proofs.

We stress that we do not make any ad hoc assumptions on the discrete time financial model and notice that Ω_* is determined only by S: indeed the set $\mathcal{M}$ can be written also as $\mathcal{M} = \{Q \in \mathcal{P} | S \text{ is an } \mathbb{F}^S\text{-martingale under } Q\}$. One of the main technical results of the paper is the proof that the set Ω_* is an *analytic set* (Proposition 5.5), and so our findings show that the natural setup for studying this problem is $(\Omega, S, \mathbb{F}, \mathcal{H})$ with $\mathbb{F}$ the universal filtration (which contains the analytic sets) and $\mathcal{H}$ the class of $\mathbb{F}$-predictable processes. We also point out that we could replace any sigma-algebra $\mathcal{F}_t$ with the sub sigma-algebra generated by the analytic sets of $\mathcal{F}_t^S$.

On Model Independent Arbitrage and the condition $\mathcal{M} \neq \varnothing$. In the case $\mathcal{M} = \varnothing$, then $\Omega_* = \varnothing$ and the theorem is trivial, as each term in the equalities of Theorem 1.1 is equal to $-\infty$, provided we convene that any $\mathcal{M}$-q.s. inequalities hold true when $\mathcal{M} = \varnothing$.

For this reason, we will assume without loss of generality $\mathcal{M} \neq \varnothing$, and recall that this condition can be reformulated in terms of absence of Model Independent Arbitrages. A Model Independent $\mathcal{H}$-Arbitrage consists of a trading strategy $H \in \mathcal{H}$ which satisfies $(H \cdot S)_T(\omega) > 0$ $\forall \omega \in \Omega$. However, as shown in [7], the absence of Model Independent $\mathcal{H}$-Arbitrage is not sufficient to guarantee $\mathcal{M} \neq \varnothing$. Indeed, we need the stronger condition of No Model Independent $\widetilde{\mathcal{H}}$-Arbitrage to hold, where $\widetilde{\mathcal{H}}$ is a wider class of $\widetilde{\mathbb{F}}$-predictable stochastic processes for a suitable enlarged filtration $\widetilde{\mathbb{F}}$. Hence, the nontrivial statement in Theorem 1.1 (namely, when $\mathcal{M} \neq \varnothing$) regards the superhedging duality under No Model Independent $\widetilde{\mathcal{H}}$-Arbitrage.

1.1. *Superhedging with semistatic strategies on options and stocks.* We now allow for the possibility of static trading in a finite number of options. Let us add to the previous market k options $\Phi = (\phi^1, \dots, \phi^k)$, which expires at time T and assume without loss of generality that they have zero initial cost. We assume that each ϕ^j is an $\mathcal{F}$-measurable random variable. Define $h\Phi := \sum_{j=1}^k h^j \phi^j$, $h \in \mathbb{R}^k$, and

$$\begin{aligned} \mathcal{M}_\Phi &:= \{Q \in \mathcal{M}_f | E_Q[\phi^j] = 0 \ \forall j = 1, \dots, k\} \\ &= \{Q \in \mathcal{M}_f | E_Q[h\Phi] = 0 \ \forall h \in \mathbb{R}^k\}, \end{aligned} \tag{1.4}$$

which are the options-adjusted martingale measures, and

$$\Omega_\Phi := \{\omega \in \Omega | \exists Q \in \mathcal{M}_\Phi \text{ such that } Q(\omega) > 0\} \subseteq \Omega_*. \tag{1.5}$$

We have by definition that for every $Q \in \mathcal{M}_\Phi$ the support satisfies $\mathrm{supp}(Q) \subseteq \Omega_\Phi$. We define the superhedging price when semistatic strategies are allowed by

$$\begin{aligned} \pi_\Phi(g) := \inf\{x \in \mathbb{R} | \exists (H, h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that} \\ x + (H \cdot S)_T + h\Phi \geq g \text{ on } \Omega_\Phi\}. \end{aligned} \tag{1.6}$$

With the same methodology used for the proof of Theorem 1.1, in Section 5.3 we will obtain the superhedging duality when semistatic trading is allowed, under the assumption $\mathcal{M}_\Phi = \{Q \in \mathcal{M}_f \,|\, \mathrm{supp}(Q) \subseteq \Omega_\Phi\}$:[1]

THEOREM 1.2 (Superhedging with options). *Let $g : \Omega \mapsto \mathbb{R}$ and $\phi^j : \Omega \mapsto \mathbb{R}$ for $j = 1, \ldots, k$, be $\mathcal{F}$-measurable random variables. Then*

$$\pi_\Phi(g) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g].$$

1.2. *Comparison with the related literature*. In the classical case when a reference probability is fixed, this subject was originally studied by El Karoui and Quenez [13]; see also [19] and [9] and the references cited therein.

In [5], a superhedging theorem is proven in the case of a nondominated class of priors $\mathcal{P}' \subseteq \mathcal{P}$. The result strongly relies on two technical hypotheses: (i) The state space Ω has a product structure, $\Omega = \Omega_1^T$, where Ω_1 is a certain fixed Polish space and Ω_1^t is the t-fold product space; (ii) the set of priors $\mathcal{P}'$ is also obtained as a collection of product measures $P := P_0 \otimes \cdots \otimes P_T$ where every P_t is a measurable selector of a certain random class $\mathcal{P}'_t \subseteq \mathcal{P}(\Omega_1)$. $\mathcal{P}'_t(\omega)$ represents the set of possible models for the tth period, given state ω at time t. An essential requirement on $\mathcal{P}'_t$ is that the graph$(\mathcal{P}'_t)$ must be an analytic subset of $\Omega_1^t \times \mathcal{P}(\Omega_1)$. These assumptions are crucial in order to apply the measurable selection and stochastic control arguments which lead to the proof of the superhedging theorem. In our setting, we do not impose restrictions on the state space Ω so the result cannot be deduced from [5] for $\mathcal{P}' = \mathcal{M}$. Moreover, even in the case of $\Omega = \Omega_1^T$, the class of martingale probability measures $\mathcal{M}$ is endogenously determined by the market and we do not require that it satisfies any additional restrictions. Furthermore, the techniques employed to deduce our version of the superhedging duality theorem are completely different, as they rely on the results of [7]. Note that in the particular simple case of $\Omega := (\mathbb{R}^d)^T$ with S the canonical process, from [7], we have that $\Omega_* = \Omega$ and there are no $\mathcal{M}$-polar sets. We thus have the equivalence between $\mathcal{P}$-q.s. and $\mathcal{M}$-q.s. equalities. The superhedging theorem of [5] can be therefore applied with $\mathcal{P}' = \mathcal{P}$ and the two results coincide.

[1] We wish to thank J. Obłoj and Z. Hou for pointing out that this hypothesis is necessary for the argument used in the proof of Theorem 1.2. We will show in a forthcoming paper (joint with J. Obłoj and Z. Hou) that the result holds in full generality dropping this hypothesis.

The relevance of the superhedging problem without any a priori specified set of probability measures is revealed by the increasing amount of literature on this topic. The problem has been studied as a particular case of a Skorokhod embedding problem (see [6, 8, 16]), following the pioneering work [17] on robust hedging. The reformulation of the superhedging duality in the framework of optimal mass transport led to important results both in discrete and continuous time as in [3, 11, 12, 14, 15, 20, 23].

Different approaches are taken in [1, 21]. In [21], the continuity assumptions on the assets allow to embed the problem in the linear programming framework and to obtain the desired equality in a one period market. In [1] from a model independent version of the fundamental theorem of asset pricing, they deduce the following superhedging duality (Theorem 1.4 [1]):

$$\pi_\Omega(g) = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g], \tag{1.7}$$

where $\pi_\Omega(g) := \inf\{x \in \mathbb{R} | \exists (H,h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that } x + (H\cdot S)_T + h\Phi \geq g \text{ on } \Omega\}$. They assume a discrete time market, with one dimensional canonical process S on the path space $\Omega = [0,\infty)^T$ and an arbitrary (but nonempty) set of options on S available for static trading. Theorem 1.4 in [1] relies on two additional technical assumptions: (i) The existence of an option with superlinearly growing and convex payoff; (ii) the upper semicontinuity of the claim g.

The example in Section 4 shows that without the upper semicontinuity of the claim g the duality in (1.7) fails and it also points out that the reason for this is the insistence of superhedging over the whole space Ω, instead of over the relevant set of paths Ω_*. Our result holds for a d-dimensional (not necessarily canonical) process S and does not necessitate the existence of any options.

2. Aggregation results. In this section we investigate when certain conditions (like superhedging or hedging) which hold Q-a.s. for all $Q \in \mathcal{M}$, ensure the validity of the correspondent pathwise conditions on Ω_*. We recall that absence of classical arbitrage opportunities, with respect to a probability $P \in \mathcal{P}$, is denoted by NA(P). We set

$$\mathcal{L}(\Omega,\mathcal{G}) := \{f : \Omega \to \mathbb{R} | \mathcal{G}\text{-measurable}\},$$
$$\mathcal{L}(\Omega,\mathcal{G})_+ := \{f \in \mathcal{L}(\Omega,\mathcal{G}) | f \geq 0\}.$$

The linear space of attainable random payoffs with zero initial cost is given by

$$\mathcal{K} := \{(H\cdot S)_T \in \mathcal{L}(\Omega,\mathcal{F}) | H \in \mathcal{H}\}.$$

Recall that the set Ω_* of events supporting martingale measures is defined in (1.1) and observe that the convex cones

$$\mathcal{C} := \{f \in \mathcal{L}(\Omega,\mathcal{F}) | f \leq k \text{ on } \Omega_* \text{ for some } k \in \mathcal{K}\}, \tag{2.1}$$

$$\mathcal{C}(Q) := \{f \in \mathcal{L}(\Omega,\mathcal{F}) | f \leq k \ Q\text{-a.s. for some } k \in \mathcal{K}\} \tag{2.2}$$

are related by $\mathcal{C} \subseteq \mathcal{C}(Q)$, if $Q \in \mathcal{M}$.

The main Theorem 1.1 relies on the following cornerstone proposition that will be proved in Section 5, as its proof requires several technical arguments.

PROPOSITION 2.1. *Let $g \in \mathcal{L}(\Omega, \mathcal{F})$ and define*

$$\pi_*(g) := \inf\{x \in \mathbb{R} | \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g \text{ on } \Omega_*\}, \tag{2.3}$$

$$\pi_Q(g) := \inf\{x \in \mathbb{R} | \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g\ Q\text{-a.s.}\}. \tag{2.4}$$

Then

$$\pi_*(g) = \sup_{Q \in \mathcal{M}_f} \pi_Q(g), \tag{2.5}$$

$$\mathcal{C} = \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q). \tag{2.6}$$

In particular, if $\pi_*(g) < +\infty$ the infimum in (2.3) is a minimum.

COROLLARY 2.2. *Let $g \in \mathcal{L}(\Omega, \mathcal{F})$ and $x \in \mathbb{R}$. If for every $Q \in \mathcal{M}_f$, there exists $H^Q \in \mathcal{H}$ such that $x + (H^Q \cdot S)_T \geq g$ Q-a.s. then there exists $H \in \mathcal{H}$ which satisfies $x + (H \cdot S)_T(\omega) \geq g(\omega)$ for every $\omega \in \Omega_*$.*

PROOF. By assumption, $g - x \in \mathcal{C}(Q)$ for every $Q \in \mathcal{M}_f$. From $\mathcal{C} = \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q)$, we obtain $g - x \in \mathcal{C}$. □

COROLLARY 2.3 (Perfect hedge). *Let $g \in \mathcal{L}(\Omega, \mathcal{F})$. If for every $Q \in \mathcal{M}_f$ there exists $H^Q \in \mathcal{H}, x^Q \in \mathbb{R}$ such that $x^Q + (H^Q \cdot S)_T = g$ Q-a.s. then there exists $H \in \mathcal{H}, x \in \mathbb{R}$ such that $x + (H \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in \Omega_*$, and $x^Q = x$ for every $Q \in \mathcal{M}_f$.*

PROOF. Note first that, from the hypothesis, for every $Q \in \mathcal{M}_f$ there exists $H^Q \in \mathcal{H}, x^Q \in \mathbb{R}$ such that $x^Q + (H^Q \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in \text{supp}(Q)$. We first show that x^Q does not depend on Q. Assume there exist $Q_1, Q_2 \in \mathcal{M}_f$ such that $x^{Q_1} < x^{Q_2}$. For every $\lambda \in (0, 1)$, set $Q_\lambda := \lambda Q_1 + (1 - \lambda) Q_2 \in \mathcal{M}_f$. Then there exist $H^{Q_\lambda} \in \mathcal{H}$ and $x^{Q_\lambda} \in \mathbb{R}$ such that $x^{Q_\lambda} + (H^{Q_\lambda} \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in \text{supp}(Q_\lambda) = \text{supp}(Q_1) \cup \text{supp}(Q_2)$. Therefore, $x^{Q_\lambda} + (H^{Q_\lambda} \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in \text{supp}(Q_i)$, for any $i = 1, 2$, and from $\text{NA}(Q_i)$ we necessarily have that $x^{Q_\lambda} = x^i$.

Since $x + (H^Q \cdot S)_T(\omega) = g(\omega)$ for every $\omega \in \text{supp}(Q)$ we can apply Corollary 2.2 which implies the existence of $H \in \mathcal{H}$ such that $x + (H \cdot S)_T(\omega) \geq g(\omega)$ on Ω_*. Moreover $x - x + ((H - H^Q) \cdot S)_T(\omega) \geq g(\omega) - g(\omega)$ $\forall \omega \in \text{supp}(Q)$ implies $((H - H^Q) \cdot S)_T(\omega) \geq 0$ $\forall \omega \in \text{supp}(Q)$. Since $\text{NA}(Q)$ holds, we conclude $((H - H^Q) \cdot S)_T(\omega) = 0$ $\forall \omega \in \text{supp}(Q)$. Thus, for every $Q \in \mathcal{M}_f$ we have $x + (H \cdot S)_T(\omega) = g(\omega)$ on $\text{supp}(Q)$, and hence the thesis follows from Proposition 4.18 [7] (or Proposition 3.1). □

COROLLARY 2.4 (Bipolar representation). *Let $\mathcal{C}$ be defined in* (2.1). *Then*

$$\mathcal{C} = \{g \in \mathcal{L}(\Omega, \mathcal{F}) | E_Q[g] \le 0 \; \forall Q \in \mathcal{M}_f\}. \tag{2.7}$$

PROOF. Clearly, $\mathcal{C} \subseteq \{g \in \mathcal{L}(\Omega, \mathcal{F}) | E_R[g] \le 0 \; \forall R \in \mathcal{M}_f\} =: \widetilde{\mathcal{C}}$. Fix $Q \in \mathcal{M}_f$ and observe that $L^0(\Omega, \mathcal{F}, Q) \equiv L^1(\Omega, \mathcal{F}, Q) \equiv L^\infty(\Omega, \mathcal{F}, Q)$, which denote, respectively, the space of equivalent classes of Q-a.s. finite, Q-integrable and Q-a.s. bounded $\mathcal{F}$-measurable random variables on Ω. For $g \in \mathcal{L}(\Omega, \mathcal{F})$, we denote with the capital letter G the corresponding equivalence class $G \in L^0(\Omega, \mathcal{F}, Q)$. Denote also by $L^0_+(\Omega, \mathcal{F}, Q)$ the Q-a.s. nonnegative elements of $L^0(\Omega, \mathcal{F}, Q)$. The quotient of $\mathcal{K}$ and $\mathcal{C}(Q)$ with respect to the Q-a.s. identification $\sim_Q$ are denoted respectively by

$$\begin{aligned}
\mathcal{K}_Q &:= \{K \in L^0(\Omega, \mathcal{F}, Q) | K = (H \cdot S)_T \; Q\text{-a.s.}, H \in \mathcal{H}\}, \\
\mathcal{C}_Q &:= \{G \in L^0(\Omega, \mathcal{F}, Q) | \exists K \in \mathcal{K}_Q \text{ such that } G \le K \; Q\text{-a.s.}\} \\
&= \mathcal{K}_Q - L^0_+(\Omega, \mathcal{F}, Q).
\end{aligned}$$

Now we may follow the classical arguments: the convex cone $\mathcal{C}_Q$ is closed in probability with respect to Q (see, e.g., [18] Theorem 1). As $Q \in \mathcal{M}_f$, $\mathcal{C}_Q$ is also closed in $L^1(\Omega, \mathcal{F}, Q)$ and, therefore,

$$\begin{aligned}
(\mathcal{C}_Q)^0 &= \{Z \in L^\infty(\Omega, \mathcal{F}, Q) | E[ZG] \le 0 \; \forall G \in \mathcal{C}_Q\} \\
&\subseteq L^\infty(\Omega, \mathcal{F}, Q) \cap L^0_+(\Omega, \mathcal{F}, Q).
\end{aligned}$$

Notice that $R \ll Q$ and $R \in \mathcal{M}_f$ if and only if $R \ll Q$ and $\frac{dR}{dQ} \in (\mathcal{C}_Q)^0$. Hence,

$$\begin{aligned}
(\mathcal{C}_Q)^{00} &= \{G \in L^1(\Omega, \mathcal{F}, Q) | E[ZG] \le 0 \; \forall Z \in (\mathcal{C}_Q)^0\} \\
&= \left\{G \in L^1(\Omega, \mathcal{F}, Q) | E_R[G] \le 0 \; \forall R \ll Q \text{ such that } \frac{dR}{dQ} \in (\mathcal{C}_Q)^0\right\} \qquad (2.8) \\
&= \{G \in L^1(\Omega, \mathcal{F}, Q) | E_R[G] \le 0 \; \forall R \ll Q \text{ such that } R \in \mathcal{M}_f\}.
\end{aligned}$$

Let $g \in \widetilde{\mathcal{C}}$. By the characterization in (2.8) the corresponding G belongs to $(\mathcal{C}_Q)^{00}$. By the bipolar theorem $\mathcal{C}_Q = (\mathcal{C}_Q)^{00}$ and, therefore, $G \in \mathcal{C}_Q$ and $g \in \mathcal{C}(Q)$ [as defined in (2.2)]. Since this holds for any $Q \in \mathcal{M}_f$, from $\mathcal{C} = \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q)$ (Proposition 2.1) we conclude that $g \in \mathcal{C}$. $\square$

REMARK 2.5. One may ask whether the bipolar duality (2.7) implies that $\mathcal{C}$ is closed with respect to some topology. To answer this question let us introduce on $\mathcal{L}(\Omega, \mathcal{F})$ the following equivalence relation: for any $X, Y \in \mathcal{L}(\Omega, \mathcal{F})$

$$X \sim Y \quad \text{if and only if} \quad X(\omega) - Y(\omega) = k(\omega)$$
$$\text{for some } k \in \mathcal{K} \text{ and for every } \omega \in \Omega_*.$$

Consider the quotient space $\mathbf{L}(\Omega,\mathcal{F}) = \mathcal{L}(\Omega,\mathcal{F})/_\sim$, denote with $[X]$ the equivalent class in $\mathbf{L}(\Omega,\mathcal{F})$ having X as a representative and let V_f be the vector space generated by $\mathcal{M}_f$. We first claim that the couple $(\mathbf{L}(\Omega,\mathcal{F}), V_f)$ is a separated dual pair under the bilinear form $\langle\cdot,\cdot\rangle : \mathbf{L}(\Omega,\mathcal{F}) \times V_f \to \mathbb{R}$ defined by: $\langle [X],\mu\rangle \mapsto E_\mu[X]$, for any $X \in [X]$. Notice that the form $\langle [X],\mu\rangle \mapsto E_\mu[X]$ is well-posed as $E_\mu[k] = 0$ for all $k \in \mathcal{K}$ and the pairing is obviously bilinear. Clearly, if $\mu \neq 0$ then there exists $\omega \in \Omega_*$ such that $\mu(\{\omega\}) \neq 0$ and $E_\mu[\mathbf{1}_\omega] \neq 0$. Thus, we have showed that $\langle [X],\mu\rangle = 0$, for every $[X]$, implies $\mu = 0$.

We now prove that $\langle [X],\mu\rangle = 0$ for every μ implies $[X] = [0]$. By contradiction, assume $[X] \neq [0]$. By assumption, X can not be replicable at a nonzero cost. Observe that if $X \in [X]$ is replicable at zero cost in any market $(\Omega,\mathcal{F},\mathbb{F},S;Q)$ for any possible choice $Q \in \mathcal{M}_f$ then by Corollary 2.3 X is pathwise replicable for every $\omega \in \Omega_*$, or in other words: $[X] = [0]$.

Hence, our assumption $[X] \neq [0]$ implies that there exists a $Q \in \mathcal{M}_f$ such that the market $(\Omega,\mathcal{F},\mathbb{F},S;Q)$ is not complete, so that $\mathcal{M}_e(Q) := \{Q^* \sim Q \mid Q^* \in \mathcal{M}\}\} \neq \{Q\}$, and $X \in [X]$ is not replicable in such market. Then

$$\inf_{Q^* \in \mathcal{M}_e(Q)} E_{Q^*}[X] < \sup_{Q^* \in \mathcal{M}_e(Q)} E_{Q^*}[X].$$

As $Q \in \mathcal{M}_f$ has finite support, $\mathcal{M}_e(Q) \subset \mathcal{M}_f$ and there exists a $\mu \in \mathcal{M}_e(Q) \subset V_f$ such that $E_\mu[X] \neq 0$, which is a contradiction.

Now we conclude that the cone $\mathcal{C}/_\sim$ is closed with respect to the weak topology $\sigma(\mathbf{L}(\Omega,\mathcal{F}), V_f)$. Indeed, from (2.7) we obtain that

$$\begin{aligned}\mathcal{C}/_\sim &= \{[g] \in \mathbf{L}(\Omega,\mathcal{F}) \mid E_Q[g] \le 0\ \forall Q \in \mathcal{M}_f\} \\ &= \bigcap_{Q \in \mathcal{M}_f} \{[g] \in \mathbf{L}(\Omega,\mathcal{F}) \mid E_Q[g] \le 0\}\end{aligned}$$

is the intersection of $\sigma(\mathbf{L}(\Omega,\mathcal{F}), V_f)$-closed sets.

3. Proof of Theorem 1.1. We first recall from [7] the relevant properties of the set Ω_* that will be needed several times in the proofs.

PROPOSITION 3.1 (Proposition 4.18 [7]). *In the setting described in Section 1, we have*

$$\begin{aligned}&\mathcal{M} \neq \varnothing \iff \Omega_* \neq \varnothing \iff \mathcal{M}_f \neq \varnothing, \\ &\Omega_* = \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_f \text{ such that } Q(\omega) > 0\}.\end{aligned} \tag{3.1}$$

The complement of Ω_* is the maximal $\mathcal{M}$-polar set.

PROOF OF THEOREM 1.1. As already stated in the Introduction, we may assume w.l.o.g. that $\mathcal{M} \neq \varnothing$, or equivalently $\mathcal{M}_f \neq \varnothing$. The first equality of the theorem holds because of the definition of $\mathcal{M}$-q.s. inequality and the fact that Ω_* is the maximal $\mathcal{M}$-polar set.

Step 1: Here, we show that

$$\inf\{x \in \mathbb{R} | \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_*\} = \sup_{Q \in \mathcal{M}_f} E_Q[g],$$

and recall, from Proposition 2.1, that the inf is attained whenever finite. Note first that the left-hand side of the previous equation can be rewritten as $\inf\{x \in \mathbb{R} | g - x \in \mathcal{C}\}$. From Corollary 2.4 it follows

$$\begin{aligned}\inf\{x \in \mathbb{R} | g - x \in \mathcal{C}\} &= \inf\{x \in \mathbb{R} | E_Q[g - x] \leq 0\ \forall Q \in \mathcal{M}_f\} \\ &= \inf\{x \in \mathbb{R} | x \geq E_Q[g]\ \forall Q \in \mathcal{M}_f\} \\ &= \sup\{E_Q[g] | Q \in \mathcal{M}_f\}.\end{aligned}$$

Step 2: We complete the proof by showing that, for any $g \in \mathcal{L}(\Omega, \mathcal{F})$,

$$\sup_{Q \in \mathcal{M}} E_Q[g] = \sup_{Q \in \mathcal{M}_f} E_Q[g], \tag{3.2}$$

where we adopt the convention $\infty - \infty = -\infty$ for those random variables g whose positive and negative part is not integrable. Set

$$m := \sup_{Q \in \mathcal{M}} E_Q[g], \qquad l := \sup_{Q \in \mathcal{M}_f} E_Q[g].$$

We obviously have that $l \leq m$ so that we only have to prove the converse inequality. If $l = \infty$, there is nothing to prove. Suppose then $l < \infty$. We first show that

$$\text{if } Q \in \mathcal{M} \text{ satisfy } E_Q[g] > l \quad \Rightarrow \quad E_Q[g] = \infty. \tag{3.3}$$

Suppose indeed by contradiction that there exists $Q \in \mathcal{M} \setminus \mathcal{M}_f$ such that $l < E_Q[g] < \infty$. Consider now an arbitrary version of the process $g_t := E_Q[g | \mathcal{F}_t]$ and extend the original market with the asset $S_t^{d+1} := g_t$ for $t \in I$. We obviously have that Q is a martingale measure for the extended market and from Proposition 3.1 this implies the existence of a finite support martingale measure Q_f which, by construction, belongs to $\mathcal{M}_f$. Since $E_{Q_f}[g] = g_0 > l$, which is the supremum of the expectations of g over $\mathcal{M}_f$, we have a contradiction.

From (3.3), we readily infer that if $m < \infty$ then $l = m$. We are only left to study the case of $m = \infty$ and we show that this is not possible under the hypothesis $l < \infty$. Consider first the class of martingale measures $\mathcal{Q}(g) \subset \mathcal{M}$ such that $E_Q[g^-] = \infty$. We obviously have that $\mathcal{Q}(g) \cap \mathcal{M}_f = \varnothing$, moreover, since $l < m = \infty$ from (3.3) and from $\infty - \infty = -\infty$, there exists $\widetilde{Q} \in \mathcal{M} \setminus \mathcal{Q}(g)$ such that $E_{\widetilde{Q}}[g] = \infty$ and $E_{\widetilde{Q}}[g^-] < \infty$. Consider now the sequence of claims $g_n := g \wedge n$ for any $n \in \mathbb{N}$. From $E_{\widetilde{Q}}[g^-] < \infty$ and monotone convergence theorem, we have $E_{\widetilde{Q}}[g \wedge n] \uparrow E_{\widetilde{Q}}[g] = \infty$; hence, there exists $\overline{n} \in \mathbb{N}$ such that $\overline{n} \geq E_{\widetilde{Q}}[g \wedge \overline{n}] > l$. Note now that

$$\sup_{Q \in \mathcal{M}_f} E_Q[g \wedge \overline{n}] \leq \sup_{Q \in \mathcal{M}_f} E_Q[g] = l < E_{\widetilde{Q}}[g \wedge \overline{n}]. \tag{3.4}$$

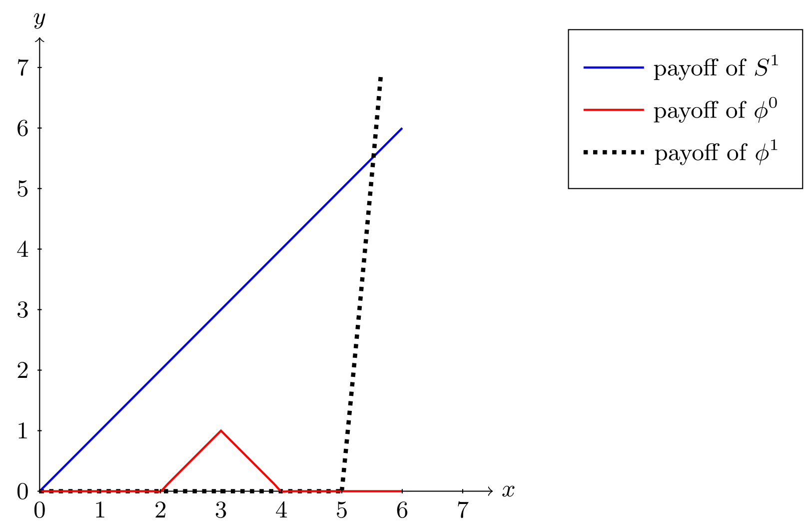

FIG. 1. *Payoffs.*

Applying (3.3) to $g \wedge \overline{n}$, we get $E_{\tilde{Q}}[g \wedge \overline{n}] = +\infty$, which is a contradiction since the contingent claim $g \wedge \overline{n}$ is bounded. $\square$

4. Example: Forget about superhedging everywhere. Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Consider a one period market ($T = 1$) defined by a nonrisky asset $S_t^0 \equiv 1$ for $t = 0, 1$ (interest rate is zero) and a single risky asset $S_T^1(\omega) = \omega$ with initial price $S_0^1 := s_0 > 0$. In this market, we also have two options $\Phi = (\phi^0, \phi^1)$, where $\phi^0 := f^0(S_T)$ is a butterfly spread option and $\phi^1 := f^1(S_T)$ is a power option, namely,

$$f^0(x) := (x - K_0)^+ - 2\big(x - (K_0 + 1)\big)^+ + \big(x - (K_0 + 2)\big)^+,$$

$$f^1(x) := \big(x^2 - K_1\big)^+.$$

Assume $K_0 > s_0$, $K_1 > (K_0 + 2)^2$ and that these options are traded at prices $c_0 = 0$ and $c_1 > 0$, respectively. Set $c = (c_0, c_1)$. The payoffs of these financial instruments are shown in Figure 1 for $K_0 = 2$, $K_1 = 25$:

DEFINITION 4.1. (1) There exists a *Model Independent Arbitrage* (in the sense of Acciaio et al. [1]) if $\exists (H, h) \in \mathcal{H} \times \mathbb{R}^2$ such that $(H \cdot S)_T(\omega) + h(\Phi(\omega) - c) > 0\ \forall \omega \in \Omega$.

(2) There exists a *one point arbitrage* (in the sense of [7]) if $\exists (H, h) \in \mathcal{H} \times \mathbb{R}^2$ such that $(H \cdot S)_T(\omega) + h(\Phi(\omega) - c) \geq 0\ \forall \omega \in \Omega$ and $(H \cdot S)_T(\omega) + h(\Phi(\omega) - c) > 0$ for some $\omega \in \Omega$.

It is clear that any long position in the option ϕ^0 is a one point arbitrage but it is not a Model Independent Arbitrage. We have indeed that there are No Model

Independent Arbitrages as

$$\mathcal{M}_\Phi \neq \varnothing.$$

More precisely, any $Q \in \mathcal{M}_\Phi$ must satisfy $Q((K_0, K_0 + 2)) = 0$, so that $(K_0, K_0 + 2)$ is an $\mathcal{M}_\Phi$-polar set. Nevertheless,

$$\Omega_\Phi = \mathbb{R}^+ \setminus (K_0, K_0 + 2).$$

One possible way to see this is to observe that on $\Gamma := \mathbb{R}^+ \setminus (K_0, K_0 + 2)$ the option ϕ^0 has zero payoff and zero initial cost so that any probability P, with $\mathrm{supp}(P) \subseteq \Gamma$, that is, a martingale measure for S^1, ϕ^1, is also a martingale measure for S^0, S^1, ϕ^0, ϕ^1. Take now $\omega_1 = 0$, $\omega_2 \in (K_0 + 2, \sqrt{K_1})$, $\omega_3 > \sqrt{K_1 + c_1}$ and observe that the corresponding points $x_1 := (-s_0, -c_1)$, $x_2 := (\omega_2 - s_0, -c_1)$ and $x_3 := (\omega_3 - s_0, \phi^1(\omega_3) - c_1)$ clearly belong to $\mathrm{conv}(\Delta X(\omega) | \omega \in \Gamma)$ where ΔX is the random vector $[S_1^1 - s_0; \phi^1 - c_1]$. Consider now $\varepsilon := \frac{1}{2} \min\{c_1, s_0, |\omega_2 - s_0|\}$ so that for ω_3 sufficiently large we have

$$B_\varepsilon(0) \subseteq \mathrm{conv}\bigl(\Delta X(\omega) | \omega \in \{\omega_1, \omega_2, \omega_3\}\bigr) \subseteq \mathrm{conv}\bigl(\Delta X(\omega) | \omega \in \Gamma\bigr).$$

We have therefore that 0 is in the interior of $\mathrm{conv}(\Delta X(\omega) | \omega \in \Gamma)$ and from Corollary 4.11 item (1) in [7], $\Omega_\Phi = \Gamma = \mathbb{R}^+ \setminus (K_0, K_0 + 2)$. Note, moreover, that this is true for any value of the price $c_1 > 0$.

Consider now the digital options $g_i = F_i(S_T)$, $i = 1, 2$, with

$$F_1(x) = \mathbf{1}_{(K_0, K_0+2)}(x),$$
$$F_2(x) = \mathbf{1}_{[K_0, K_0+2]}(x)$$

which differ only at the extreme points of the interval $(K_0, K_0 + 2)$ and observe that F_2 is upper semicontinuous while F_1 is not. From the previous remark, g_1 has price zero under any martingale measure $Q \in \mathcal{M}_\Phi$, so that

$$\sup_{Q \in \mathcal{M}_\Phi} E_Q[g_1] = 0. \tag{4.1}$$

Recall that

$$\pi_\Omega(g) := \inf\bigl\{x \in \mathbb{R} | \exists (H, h) \in \mathcal{H} \times \mathbb{R}^2 \text{ such that } x + (H \cdot S)_T + h\Phi \geq g \text{ on } \Omega\bigr\}$$

and

$$\pi_\Phi(g) := \inf\bigl\{x \in \mathbb{R} | \exists (H, h) \in \mathcal{H} \times \mathbb{R}^2 \text{ such that } x + (H \cdot S)_T + h\Phi \geq g \text{ on } \Omega_\Phi\bigr\}.$$

CLAIM 4.2. *In this market*:

1. $\pi_\Phi(g_1) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_1] = 0$ *and* $\pi_\Phi(g_2) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_2]$;
2. $\pi_\Omega(g_1) = \min\{\frac{s_0}{K_0}, 1\} > \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_1] = 0$;
3. $\pi_\Omega(g_2) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g_2]$.

REMARK 4.3. (i) Item (1) is in agreement with the conclusion of Theorem 1.2.

(ii) Item (2) shows instead that the superhedging duality with respect to the whole Ω does not hold for the claim g_1 (which is even bounded). Note that in this example all the hypothesis of Theorem 1.4 in [1] are satisfied except for the upper semicontinuity of g_1.

As the comparison between g_1 and g_2 in items (2) and (3) shows, the assumption of upper semicontinuity of the claim seems artificial from the financial point of view, even though necessary for the validity of Theorem 1.4 in [1].

Our results demonstrates that it is possible to obtain a superhedging duality on the relevant set Ω_Φ (or Ω_* when there are no options) for *any measurable claim*, regardless of the continuity assumptions (as well as without the existence of an option with superlinear payoff).

PROOF OF THE CLAIM 4.2. Item (1) holds thanks to Theorem 1.1 since in the one-period model there is no difference between dynamic and static hedging. Notice also that the equalities $\pi_\Phi(g_1) = 0 = \sup_{Q\in\mathcal{M}_\Phi} E_Q[g_1]$ are consequences of (4.1) and the fact that $(H, h) = (0, 0)$ is a superhedging strategy for g_1 on Ω_Φ. As g_2 is upper semicontinuous, the superhedging duality in item (3) holds thanks to Theorem 1.4 in [1]; see (1.7). In the remainder of this section, we conclude the proof by showing $\pi_\Omega(g_1) = \min\{\frac{s_0}{K_0}, 1\} = \frac{s_0}{K_0}$ (by the assumption $K_0 > s_0$) and hence item (2).

Let us consider the model independent superhedging strategies, namely, the set of $(H, h) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g_1(\omega)$ for any $\omega \in \Omega$. Any admissible trading strategy is given by $(H, h) := [H^0, H^1, h^0, h^1] \in \mathbb{R}^4$ which correspond to positions in the securities $[S^0, S^1, \phi^0, \phi^1]$ so that

$$\begin{aligned} \text{price: } & V_0(H, h) := H^0 + H^1 s_0 + h^1 c_1, \\ \text{payoff: } & V_T(H, h) := H^0 + H^1\omega + h^0\phi^0(\omega) + h^1\phi^1(\omega). \end{aligned} \tag{4.2}$$

Trivial superhedges. There are two immediate strategies whose terminal payoff is a superhedge for g_1:

1. S^0 [namely, $H^0 = 1$ in (4.2) and $H^1 = h^0 = h^1 = 0$] with initial cost 1.
2. $\frac{1}{K_0}S^1$ [namely, $H^1 = \frac{1}{K_0}$ in (4.2) and $H^0 = h^0 = h^1 = 0$] with initial cost $\frac{s_0}{K_0}$.

Consider now a generic superhedging strategy (H, h) for the option g_1 and suppose first that $H^1 \geq 0$.

Observe that for every $\omega \in [0, K_0]$ we have: $V_T(H, h)(\omega) = H^0 + H^1\omega$ and $g_1(\omega) = 0$. If $H^0 < 0$, there exists $\widetilde{\omega} \in [0, K_0]$ such that $H^0 + H^1\widetilde{\omega} < 0 = g_1(\widetilde{\omega})$ so that the strategy does not dominate the payoff of g_1; necessarily $H^0 \geq 0$.

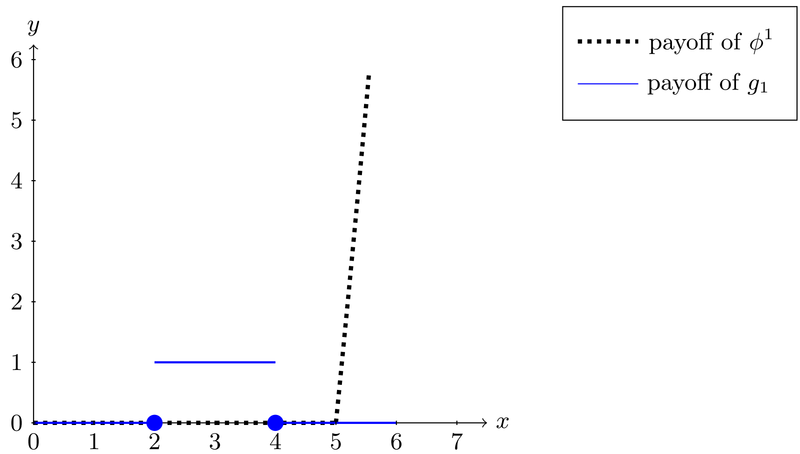

FIG. 2. *ϕ^1 has no positive wealth on $(K_0, K_0 + 2)$.*

$h^1 \neq 0$ is not optimal for superhedging g_1. If $h^1 \neq 0$, we necessarily have $h^1 \geq 0$, otherwise $V_T(H,h)(\omega) < 0$ for ω large enough (because of the superlinearity of f^1) and (H,h) is not a superhedge for g_1. Since $f^1(x) = 0$ on $(K_0, K_0 + 2)$ and $c_1 > 0$, the most convenient superhedge is with $h^1 = 0$ (cf. Figure 2).

From now on, with no loss of generality, $h^1 = 0$.

$h^0 \neq 0$ is not optimal for superhedging g_1. Since ϕ^0 has a positive payoff, if $h^0 \neq 0$, we might take $h^0 \geq 0$ otherwise we have a better superhedge (at the same cost) by replacing $h^0\phi^0$ with the zero portfolio. Suppose now $h^0 > 0$. By recalling that $H^0, H^1 \geq 0$, we note that $V_T(H,h)$ as in (4.2) satisfies

$$\inf_{\omega \in (K_0, K_0+2)} H^0 + H^1\omega + h^0\phi^0(\omega) = H^0 + H^1 K_0$$

so that the same superhedge is achieved by trading only in S^0 and S^1. In other words, with no loss of generality $h^0 = 0$ (cf. Figure 3).

We finally discuss the case $H^1 < 0$.

This is, in general, a more expensive choice for the strategy (H,h). Indeed, we have, for instance, that for $\widetilde{\omega} = K_0 + 1$, $H^1 S^1(\widetilde{\omega}) = H^1(K_0 + 1) < 0$ while $g_1(\widetilde{\omega}) = 1$. Since for any strategy $(H,h) \in \mathbb{R}^4$, $V_T(H,h)(\widetilde{\omega}) = H^0 + H^1\widetilde{\omega}$, we need $H^0 \geq 1 - H^1(K_0 + 1)$; hence, the initial price $V_0(H,h) \geq 1 - H^1(K_0 + 1 - s_0)$. By choosing the parameters s_0, K_0 such that $K_0 + 1 - s_0 < 0$ any superhedging strategy with $H^1 < 0$ is more expensive than the trivial superhedge given by $H^0 = 1, H^1 = h^0 = h^0 = 0$. Note moreover that in order to cover the losses in $H^1 S^1$ for large value of ω we would need to take a long position in the option ϕ^1 (whose payoff dominates S^1) for an additional cost of $h^1 c_1 > 0$ with $h^1 > -H^1 > 0$.

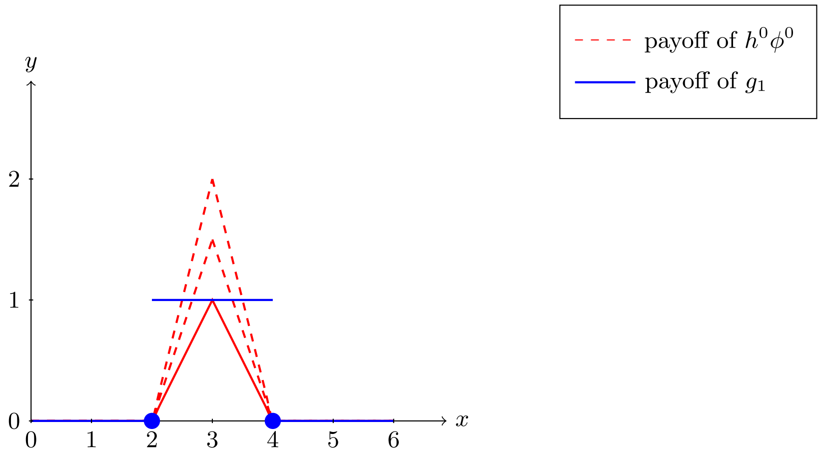

FIG. 3. *$h^0\phi^0$ does not dominate g_1 on $(K_0, K_0+\varepsilon)$ for any h^0 with $\varepsilon = \varepsilon(h^0)$.*

We can conclude that the cheapest super-replicating strategies are, in general, given by $H^0S^0 + H^1S^1$ with $H^0, H^1 \geq 0$ and it is easy to see that

$$\pi_\Omega(g_1) = \min\left\{\frac{s_0}{K_0}, 1\right\} = \frac{s_0}{K_0} > 0. \qquad \square$$

5. Technical results and proofs. Recall that $\{\mathcal{F}_t\}_{t\in I}$ is the universal filtration which satisfies in particular that $\mathcal{F}_t$ contains the family of analytic sets of $(\Omega, \mathcal{F}_t^S)$ for any $t \in I$.

We indicate by $\mathrm{Mat}(d \times (T+1); \mathbb{R})$ the space of $d \times (T+1)$ matrices with real entries representing the set of all the possible trajectories of the price process: for every $\omega \in \Omega$ we have $(S_0(\omega), S_1(\omega), \dots, S_T(\omega)) \in \mathrm{Mat}(d \times (T+1); \mathbb{R})$. Fix $t \leq T$: in the following, we indicate $S_{0:t} = (S_0, S_1, \dots, S_t)$ and recall that $S_{0:t}^{-1}(A) = \{\omega \in \Omega | S_{0:t}(\omega) \in A\}$ for $A \subset \mathrm{Mat}(d \times (t+1); \mathbb{R})$. We set $\Delta S_t := S_t - S_{t-1}$, $t = 1, \dots, T$.

5.1. *Ω_* and Ω_Φ are analytic sets.*

LEMMA 5.1. *The set $\mathcal{P}_f = \{P \in \mathcal{P} | P$ has finite support$\}$ is an analytic subset of $\mathcal{P}$ endowed with the sigma-algebra generated by the $\sigma(\mathcal{P}, C_b)$ topology.*

PROOF. Set $E = \{\delta_\omega | \omega \in \Omega\}$ which is $\sigma(\mathcal{P}, C_b)$ closed (Theorem 15.8 [2]) and observe that $\mathcal{P}_f$ is the convex hull of E. Consider for any $n \in \mathbb{N}$ the simplex $\Delta_n \subset \mathbb{R}^n$ and the map

$$\gamma_n : E^n \times \Delta_n \longrightarrow \mathcal{P}_f$$

defined by $\gamma_n(\delta_{\omega_1}, \dots, \delta_{\omega_n}, \lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i \delta_{\omega_i}$ which is a continuous function in the product topology. Since $E^n \times \Delta_n$ is closed in the product topology of the

Borel space $\mathcal{P}^n \times \mathbb{R}^n$, then the image $\gamma_n(E^n \times \Delta_n)$ is analytic (Proposition 7.40 [4]). Finally, we notice that $\mathcal{P}_f = \bigcup_n \gamma_n(E^n \times \Delta_n)$ which is therefore analytic, being countable union of analytic sets. $\square$

DEFINITION 5.2. Let $\mathcal{L}^\infty(\Omega, \mathcal{F}) := \{f \in \mathcal{L}(\Omega, \mathcal{F}) | f \text{ is bounded}\}$. A subset $\mathcal{U} \subset \mathcal{P}_f$ is countably determined if there exists a countable set $L \subseteq \mathcal{L}^\infty(\Omega, \mathcal{F})$ such that

$$\mathcal{U} := \{\mu \in \mathcal{P}_f | E_\mu[f] \leq 0 \ \forall f \in L\}.$$

LEMMA 5.3. *If $\mathcal{U} \subseteq \mathcal{P}_f$ is countably determined, then it is analytic.*

PROOF. For each $f_n \in L$, define

$$F_n : \mathcal{P} \to \mathbb{R} \qquad \text{such that } F_n(\mu) = \int_\Omega f_n \, d\mu.$$

From Theorem 15.13 in [2], F_n is Borel measurable so that

$$\mathcal{U} := \{\mu \in \mathcal{P}_f | E_\mu[f_n] \leq 0 \text{ for all } n \in \mathbb{N}\} = \bigcap_{n\in\mathbb{N}} (F_n)^{-1}(-\infty, 0] \cap \mathcal{P}_f$$

is analytic, being countable intersection of analytic sets. $\square$

LEMMA 5.4. *Let* $Z_1(\omega) := \max_{i=1,\dots,d} \max_{u=0,\dots,T} |S_u^i(\omega)|$, $Z_2(\omega) := \max_{j=1,\dots,k} |\phi^j(\omega)|$ *and* $Z = \max(Z_1, Z_2)$ *then*

$$\mathcal{P}_Z = \left\{\mu \in \mathcal{P}_f | \exists Q \in \mathcal{M}_f \text{ such that } \frac{dQ}{d\mu} = \frac{c(\mu)}{1+Z}\right\},$$
$$\mathcal{P}_{Z,\Phi} = \left\{\mu \in \mathcal{P}_f | \exists Q \in \mathcal{M}_\Phi \text{ such that } \frac{dQ}{d\mu} = \frac{c(\mu)}{1+Z}\right\}$$

are analytic subsets of $\mathcal{P}$ *where* $c(\mu) = E_\mu[(1+Z)^{-1}]^{-1}$.

PROOF. Assume $\mathcal{P}_Z \neq \varnothing$ (resp., $\mathcal{P}_{Z,\Phi} \neq \varnothing$) otherwise there is nothing to prove. Fix any $t \in \{1, \dots, T\}$. Let $\mathrm{Mat}(d \times t; \mathbb{Q})$ be the countable set of $d \times t$ matrices with rational entries and denote its elements by q_n, $n \in \mathbb{N}$. For $q_n \in \mathrm{Mat}(d \times t; \mathbb{Q})$, consider the set $\{A_{n,m}\}$ with $A_{n,m} = \{\omega \in \Omega | S_{0:t-1} \in B_{1/m}(q_n)\} \in \mathcal{F}_{t-1}$, where $B_{1/m}(q_n)$ denotes the ball [in the Euclidean norm of $\mathrm{Mat}(d \times t; \mathbb{R})$] with radius $1/m$ centered in q_n. Define

$$f_{n,m}^i := \left(\frac{S_t^i - S_{t-1}^i}{1+Z}\right)\mathbf{1}_{A_{n,m}} \in \mathcal{L}^\infty(\Omega, \mathcal{F}), \tag{5.1}$$
$$g^j := \left(\frac{\phi^j}{1+Z}\right) \in \mathcal{L}^\infty(\Omega, \mathcal{F}).$$

The following sets

$$\mathcal{U} := \{\mu \in \mathcal{P}_f \,|\, E_\mu[f^i_{n,m}] = 0 \ \forall i, n, m\},$$
$$\mathcal{U}_\Phi := \{\mu \in \mathcal{P}_f \,|\, E_\mu[f^i_{n,m}] = 0 \text{ and } E_\mu[g^j] = 0 \ \forall i, n, m, j\},$$

are analytic since they are countably determined. We now show $\mathcal{U} = \mathcal{P}_Z$ and $\mathcal{U}_\Phi = \mathcal{P}_{Z,\Phi}$ and this will complete the proof.

For any fixed $\mu \in \mathcal{U}$, we have by construction

$$\int_\Omega \frac{S^i_t}{1+Z}\mathbf{1}_{A_{n,m}}\,d\mu = \int_\Omega \frac{S^i_{t-1}}{1+Z}\mathbf{1}_{A_{n,m}}\,d\mu \qquad \text{for every } A_{n,m}. \tag{5.2}$$

Consider the finite set of matrices $\{s_j\}_{j=1}^h := \{S_{0:t-1}(\omega) \in \mathrm{Mat}(d \times t; \mathbb{R}) | \omega \in \mathrm{supp}(\mu)\}$ where $h = h(\mu)$ depends on μ. For every $j = 1, \dots, h$, there exists $q_{n(j)}, m(j)$ such that $s_j \in B_{1/m(j)}(q_{n(j)})$ and the balls $B_{1/m(j)}(q_{n(j)})$ are all disjoint. Therefore, $A_{n(j),m(j)}$ is such that

$$\mu(B_j) = \mu(A_{n(j),m(j)}),$$

where $B_j := \{S_{0:t-1} = s_j\}$. Since $\{B_j\}_{j=1}^h$ are atoms for μ in $\mathcal{F}_{t-1}$, we conclude that

$$\int_\Omega \frac{S^i_t}{1+Z}\mathbf{1}_{B_j}\,d\mu = \int_\Omega \frac{S^i_{t-1}}{1+Z}\mathbf{1}_{B_j}\,d\mu \qquad \text{for every } j = 1, \dots, h$$

and $E_\mu(\frac{S^i_t}{1+Z}|\mathcal{F}_{t-1}) = E_\mu(\frac{S^i_{t-1}}{1+Z}|\mathcal{F}_{t-1})$. Define Q by $\frac{dQ}{d\mu} := \frac{c}{1+Z}$ where $c := c(\mu) > 0$ is the normalization constant. Then $Q \sim \mu$, $Q \in \mathcal{P}_f$ and

$$\begin{aligned} &E_\mu\biggl(\frac{S^i_t}{1+Z}\bigg|\mathcal{F}_{t-1}\biggr) \\ &\quad = E_\mu\biggl(\frac{S^i_{t-1}}{1+Z}\bigg|\mathcal{F}_{t-1}\biggr) \quad \text{if and only if} \quad E_Q(S^i_t|\mathcal{F}_{t-1}) = S^i_{t-1}. \end{aligned} \tag{5.3}$$

Thus, we can conclude $Q \in \mathcal{M}_f$ and $\mathcal{U} \subseteq \mathcal{P}_Z$. Take now $\mu \in \mathcal{P}_Z$ then there exists Q such that $E_Q(S^i_t|\mathcal{F}_{t-1}) = S^i_{t-1}$ and $\frac{dQ}{d\mu} = \frac{c}{1+Z}$. From equation (5.3), we have that condition (5.2) holds, and hence $\mu \in \mathcal{U}$.

Recall that $\mathcal{M}_\Phi$ is defined in (1.4) and consider now $\mu \in \mathcal{U}_\Phi \subseteq \mathcal{U}$. Then there exists $Q \in \mathcal{M}_f$ such that $\frac{dQ}{d\mu} = \frac{c(\mu)}{1+Z}$. Moreover, $E_\mu[g^j] = 0$ for every $j = 1, \dots, k$ so that, by (5.1), $E_Q[\phi^j] = 0$. In this way, $\mathcal{U}_\Phi \subseteq \mathcal{P}_{Z,\Phi}$. Take now $\mu \in \mathcal{P}_{Z,\Phi}$, then $\mu \in \mathcal{P}_Z$ from the previous part of the proof. Moreover, there exists $Q \in \mathcal{M}_\Phi$ such that $E_Q[\phi^j] = 0$ and $\frac{dQ}{d\mu} = \frac{c}{1+Z}$. Again by (5.1), we have $E_\mu[g^j] = 0$ for every $j = 1, \dots, k$, and hence $\mu \in \mathcal{U}_\Phi$. □

PROPOSITION 5.5. *Ω_* and Ω_Φ are analytic subsets of* $(\Omega, \mathcal{F})$.

PROOF. Consider the Baire space $\mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers. In this proof, we denote by $B_\varepsilon(\omega)$ the closed ball of radius ε, centered in ω in (Ω, d).

Consider a dense subset $\{\omega_i\}_{i=1}^\infty$ of Ω. For any $\mathbf{n} = (n_1, \dots, n_k, \dots) \in \mathbb{N}^{\mathbb{N}}$, we denote by $\mathbf{n}(1), \dots, \mathbf{n}(k)$ the first k terms (namely, $n_1, \dots, n_k$). Define

$$A_{\mathbf{n}(1)} := B_1(\omega_{\mathbf{n}(1)}).$$

Let now $\{\omega_{\mathbf{n}(1),i}\}_{i=1}^\infty$ a dense subset of $A_{\mathbf{n}(1)}$ we define

$$A_{\mathbf{n}(1),\mathbf{n}(2)} := B_{\frac{1}{2}}(\omega_{\mathbf{n}(1),\mathbf{n}(2)}) \cap A_{\mathbf{n}(1)}.$$

At the kth step, we shall have $\{\omega_{\mathbf{n}(1),\dots,\mathbf{n}(k-1),i}\}_{i=1}^\infty$ a dense subset of $A_{\mathbf{n}(1),\dots,\mathbf{n}(k-1)}$ and we define the closed set

$$A_{\mathbf{n}(1),\dots,\mathbf{n}(k)} := B_{\frac{1}{k}}(\omega_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) \cap A_{\mathbf{n}(1),\dots,\mathbf{n}(k-1)}.$$

Notice that for any $\omega \in \Omega$ there will exist an $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\dots,\mathbf{n}(k)} = \{\omega\}. \tag{5.4}$$

We consider the *nucleus* of the Souslin scheme given by

$$\bigcup_{\mathbf{n}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\dots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) > 0\}. \tag{5.5}$$

Observe that $A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}$ closed in Ω implies $\{Q \in \mathcal{P} | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) \geq \frac{1}{m}\}$ is $\sigma(\mathcal{P}, C_b)$-closed from Corollary 15.6 in [2]. Therefore,

$$\{Q \in \mathcal{P} | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) > 0\} = \bigcup_m \left\{Q \in \mathcal{P} | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) \geq \frac{1}{m}\right\}$$

is Borel measurable in $(\mathcal{P}, \sigma(\mathcal{P}, C_b))$. Lemma 5.4 implies $\{Q \in \mathcal{P}_Z | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) > 0\}$ is analytic. We can thus conclude that $A_{\mathbf{n}(1),\dots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) > 0\}$ is an analytic subset of $\Omega \times \mathcal{P}$ (which is a Polish space).

From Lemma 5.4, we observe that any $\mu \in \mathcal{P}_Z$ admits an equivalent martingale measure with finite support. From $\Omega_* = \{\omega \in \Omega | \exists Q \in \mathcal{M}_f \text{ such that } Q(\omega) > 0\}$, if $\omega \notin \Omega_*$ we then have $\omega \notin \text{supp}(\mu)$ for any $\mu \in \mathcal{P}_Z$. Taking (5.4) into account, if $\omega \notin \Omega_*$ we can find a large enough $\bar{k}$ such that $A_{\mathbf{n}(1),\dots,\mathbf{n}(\bar{k})} \cap \text{supp}(\mu) = \varnothing$. We then have

$$\begin{aligned} &\bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\dots,\mathbf{n}(k)} \times \{Q \in \mathcal{P}_Z | Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)}) > 0\} \\ &\quad = \begin{cases} \{\omega\} \times \mathcal{P}_\omega, & \text{if } \omega \in \Omega_*, \\ \varnothing, & \text{if } \omega \notin \Omega_*, \end{cases} \end{aligned} \tag{5.6}$$

where $\mathcal{P}_\omega = \{Q \in \mathcal{P}_Z | Q(\{\omega\}) > 0\}$.

From Proposition 7.35 and Proposition 7.41 in [4], any kernel of a Souslin scheme of analytic sets is again an analytic set. Then

$$\bigcup_{\mathbf{n}\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\in\mathbb{N}} A_{\mathbf{n}(1),\dots,\mathbf{n}(k)} \times \{Q\in\mathcal{P}_Z \mid Q(A_{\mathbf{n}(1),\dots,\mathbf{n}(k)})>0\}$$

is an analytic set in $\Omega\times\mathcal{P}$ whose projection on Ω, thanks to (5.6), is equal to Ω_*. Since the projection $\Pi:\Omega\times\mathcal{P}\to\Omega$ is continuous, we finally deduce that Ω_* is analytic.

For Ω_Φ, repeat the same proof replacing $\mathcal{P}_Z$ with $\mathcal{P}_{Z,\Phi}$. $\square$

REMARK 5.6. Let $\widehat{\Omega}\subseteq\Omega$ be an analytic subset of $(\Omega,\mathcal{F})$. An inspection of the proof shows that

$$\begin{aligned}\widehat{\Omega}_* &:= \{\omega\in\widehat{\Omega} \mid \exists Q\in\mathcal{M}_f \text{ such that } Q(\widehat{\Omega})=1 \text{ and } Q(\omega)>0\},\\ \widehat{\Omega}_\Phi &:= \{\omega\in\widehat{\Omega} \mid \exists Q\in\mathcal{M}_\Phi \text{ such that } Q(\widehat{\Omega})=1 \text{ and } Q(\omega)>0\}\end{aligned} \tag{5.7}$$

are also analytic subsets of $(\Omega,\mathcal{F})$. Indeed, $\mathcal{P}_{\widehat{\Omega}}:=\{P\in\mathcal{P}\mid P(\widehat{\Omega})=1\}$ is an analytic subset of $\mathcal{P}$, by Proposition 7.43 in [4], therefore, $\mathcal{P}_Z\cap\mathcal{P}_{\widehat{\Omega}}$ is analytic and one may replace in the above proof $\mathcal{P}_Z$ with $\mathcal{P}_Z\cap\mathcal{P}_{\widehat{\Omega}}$ and Ω_* with $\widehat{\Omega}_*$ to obtain the conclusion.

REMARK 5.7. In one-period markets ($T=1$), Ω_* is a Borel measurable set. To see this, observe that if there are no one point arbitrages then $\Omega_*=\Omega\in\mathcal{B}(\Omega)$ by Corollary 4.11 in [7]. When this condition is violated, there exists a strategy $H^1\in\mathbb{R}^d$ such that $H^1\cdot(S_1-S_0)\geq 0$ and $B^1:=\{\omega\in\Omega\mid H^1\cdot(S_1(\omega)-S_0)>0\}$ is nonempty and Borel measurable. Indeed, $B^1=(f\circ S_1)^{-1}(0,\infty)$ with $f(x):=H^1\cdot(x-S_0)$ continuous and S_1 Borel measurable. Observe now that, restricted to the set $\Omega\setminus B^1$, one asset is redundant (say S^d) so that the market can be described by $(S^0,\dots,S^{d-1})$. If there is no one point arbitrage, we have $\Omega_*=\Omega\setminus B^1\in\mathcal{B}(\Omega)$. Otherwise, we can iteratively repeat the same argument to construct $B^i:=\{\omega\in\Omega\setminus\bigcup_{j=1}^{i-1}B^j\mid H^i\cdot(S_1(\omega)-S_0)>0\}\in\mathcal{B}(\Omega)$ and dropping iteratively one additional asset. Since the number of assets is finite, the procedure takes $\beta\leq d$ steps. On the resulting set, there are no one point arbitrages so that $\Omega_*=(\bigcup_{i=1}^{\beta}B^j)^C\in\mathcal{B}(\Omega)$.

5.2. *On the key Proposition* 2.1.

REMARK 5.8. We point out at this stage that Ω_* is not only analytic but also it belongs to $\mathcal{F}_T$ where $\mathcal{F}_T$ is the universal completion of $\sigma(S_t\mid t\leq T)$. Indeed, $\Omega_*\subseteq S_{0:T}^{-1}(S_{0:T}(\Omega_*))$. Moreover, for any $\omega_1\in S_{0:T}^{-1}(S_{0:T}(\Omega_*))$ there exists $\omega_2\in\Omega_*$ such that $S_{0:T}(\omega_1)=S_{0:T}(\omega_2)$. Therefore, for any $Q\in\mathcal{M}_f$ such that $Q(\{\omega_2\})>0$ and $Q(\{\omega_1\})=0$, the measure $\widetilde{Q}$ such that $\widetilde{Q}(\{\omega_1\}):=Q(\{\omega_2\})$, $\widetilde{Q}(\{\omega_2\}):=0$ and $\widetilde{Q}=Q$ elsewhere is a martingale measure; necessarily $\omega_1\in\Omega_*$.

In the proof of Proposition 2.1, we will make use of the following simple fact: first set $\Omega_*^T := \Omega_* \in \mathcal{F}_T$ then by backward recursion we have

$$\Omega_*^t := S_{0:t}^{-1}\big(S_{0:t}\big(\Omega_*^{t+1}\big)\big) \in \mathcal{F}_t,$$

$$\Omega_*^{t+1} \subseteq \Omega_*^t \qquad \text{for any } t = 0, \dots, T-1, \quad \text{and} \quad \Omega_* = \bigcap_{t=1}^{T} \Omega_*^t.$$

Notice that Ω_*^t can be interpreted as the $\mathcal{F}_t$-measurable projection of Ω_* since we have $\Omega_*^t = S_{0:t}^{-1}(S_{0:t}(\Omega_*))$.

We also recall that the condition no one point arbitrage holds true on Ω_*. If indeed there exists $H \in \mathcal{H}$ such that $(H \cdot S)_T \geq 0$ with $(H \cdot S)_T(\omega) > 0$ for some $\omega \in \Omega_*$, then any measure P such that $P(\omega) > 0$ cannot be a martingale measure, which contradicts (1.1).

5.2.1. *Proof of Proposition* 2.1. We show, in several steps, that $\pi_*(g) = \sup_{Q \in \mathcal{M}_f} \pi_Q(g)$ where π_* and π_Q are defined in (2.3) and (2.4) and $g \in \mathcal{L}(\Omega, \mathcal{F})$.

Step 1: The first step is to construct, for any $1 \leq t \leq T$, an $\mathcal{F}_{t-1}$-measurable random set $R_{t,X,D} \subseteq \mathbb{R}^{d+1}$ whose interpretation is the following: if ω occurs, any $H^1, \dots, H^d, H^{d+1} \in R_{t,X,D}(\omega)$ represents a strategy at time $t-1$ that allows to superhedge the random variable X at time t, for any trajectory in $D \subseteq \Omega$. Here, H^{d+1} represents the investment in the nonrisky asset. Note that we need to incorporate the additional feature given by the choice of the set D since we want to superhedge the random variable g only on $\Omega_* \subseteq \Omega$.

Recall $\Delta S_t = S_t - S_{t-1}$. Consider, for an arbitrary $1 \leq t \leq T$, $D \in \mathcal{F}_t$ and $X \in \mathcal{L}(\Omega, \mathcal{F})$, the multifunction

$$\psi_{t,X,D} : \omega \mapsto \big\{\big[\Delta S_t(\widetilde{\omega}); 1; X(\widetilde{\omega})\big]\mathbf{1}_D \,|\, \widetilde{\omega} \in \Sigma_{t-1}^{\omega}\big\} \subseteq \mathbb{R}^{d+2},$$

where $[\Delta S_t; 1; X]\mathbf{1}_D = [\Delta S_t^1 \mathbf{1}_D, \dots, \Delta S_t^d \mathbf{1}_D, \mathbf{1}_D, X\mathbf{1}_D]$ and Σ_{t-1}^{ω} is the level set of the trajectory ω up to time $t-1$ namely, $\Sigma_{t-1}^{\omega} = \{\widetilde{\omega} \in \Omega \,|\, S_{0:t-1}(\widetilde{\omega}) = S_{0:t-1}(\omega)\}$. We show that $\psi_{t,X,D}$ is an $\mathcal{F}_{t-1}$-measurable multifunction. Indeed, we need to show that, for any open set $O \subseteq \mathbb{R}^d \times \mathbb{R}^2$,

$$\big\{\omega \in \Omega \,|\, \psi_{t,X,D}(\omega) \cap O \neq \varnothing\big\} = S_{0:t-1}^{-1}\big(S_{0:t-1}(B)\big) \in \mathcal{F}_{t-1},$$

where $B := ([\Delta S_t; 1; X]\mathbf{1}_D)^{-1}(O)$. First $[\Delta S_t, 1, X]\mathbf{1}_D$ is an $\mathcal{F}$-measurable random vector then $B \in \mathcal{F}$. Second, S_u is a Borel measurable function for any $0 \leq u \leq t-1$ so that we have, as a consequence of Theorem III.18 in [10], that $S_{0:t-1}(B)$ belongs to the sigma-algebra generated by the analytic sets in $\mathrm{Mat}(d \times t; \mathbb{R})$ endowed with its Borel sigma-algebra. Applying now Theorem III.11 in [10], we deduce that $S_{0:t-1}^{-1}(S_{0:t-1}(B)) \in \mathcal{F}_{t-1}$ and hence the desired measurability for $\psi_{t,X,D}$.

By preservation of measurability (see [22] for instance), the multifunction

$$\psi^*_{t,X,D}(\omega) := \{H \in \mathbb{R}^{d+2} | H \cdot y \le 0 \ \forall y \in \psi_{t,X,D}(\omega)\}$$

is also $\mathcal{F}_{t-1}$-measurable and thus, the same holds true for $-\psi^*_{t,X,D} \cap \{\mathbb{R}^{d+1} \times \{-1\}\}$. The projection on the first $d+1$ components, $R_{t,X,D} := \Pi_{x_1,\dots,x_{d+1}}(-\psi^*_{t,X,D} \cap \{\mathbb{R}^{d+1} \times \{-1\}\})$, provides the building blocks for the super-replicating strategy for X. By the previous construction, we have indeed that

$$\begin{aligned} &R_{t,X,D}(\omega) \\ &\quad = \left\{ H \in \mathbb{R}^{d+1} | H^{d+1}\mathbf{1}_D + \sum_{i=1}^d H^i \Delta S^i_t(\widetilde{\omega})\mathbf{1}_D \ge X(\widetilde{\omega})\mathbf{1}_D \ \forall \widetilde{\omega} \in \Sigma^\omega_{t-1} \right\}. \end{aligned} \tag{5.8}$$

Notice that if $D \cap \Sigma^\omega_{t-1} = \varnothing$ then $R_{t,X,D}(\omega) = \mathbb{R}^{d+1}$. Note also that $R_{t,X,D}$ is, by construction, a closed set.

Denote by $\Pi_{x_{d+1}}(R_{t,X,D})$ the projection on the $(d+1)$th component, which is a random interval in $\mathbb{R}$ with possible values $\{\varnothing\}, \{\mathbb{R}\}$. Observe now that the projection is continuous and that the infimum of a real-valued random set A preserve the measurability since

$$\{\omega \in \Omega | \inf\{a | a \in A(\omega)\} < y\} = \{\omega \in \Omega | A(\omega) \cap (-\infty, y) \neq \varnothing\}.$$

Conclude, therefore, that $X_{t-1} := \inf \Pi_{x_{d+1}}(R_{t,X,D})$ is an $\mathcal{F}_{t-1}$-measurable function with values in $\mathbb{R} \cup \{\pm\infty\}$.

Step 2. We prove that for every $\omega \in \{|X_{t-1}| < \infty\}$ the infimum in X_{t-1} is actually a minimum. To this aim, fix $\omega \in \{|X_{t-1}| < \infty\}$ and notice that there might exist $L \in \mathbb{R}^d \setminus \{0\}$ such that $L \cdot \Delta S_t = 0$ on $\Sigma^\omega_{t-1} \cap \Omega^t_*$, meaning that some assets are redundant on this level set. We can reduce the number of assets by selecting $i_1, \dots, i_k \in (1, \dots, d)$ such that $l_1 \Delta S^{i_1}_t + \dots + l_k \Delta S^{i_k}_t = 0$ implies $l_j = 0$ for every $j = 1, \dots, k$. Consider the closed set

$$\widetilde{R}(\omega) = \{H \in R_{t,X,D}(\omega) | H^{i_j} = 0 \text{ for every } j = 1, \dots, k\}$$

and observe that

$$\begin{aligned} X_{t-1}(\omega) &= \inf \Pi_{x_{d+1}}(R_{t,X,D}(\omega)) = \inf \Pi_{x_{d+1}}(\widetilde{R}(\omega)) \\ &= \inf \Pi_{x_{d+1}}(\widetilde{R}(\omega) \cap \{\mathbb{R}^d \times [X_{t-1}(\omega), X_{t-1}(\omega)+1]\}). \end{aligned}$$

The set $\mathrm{Ko}(\omega) := \widetilde{R}(\omega) \cap \{\mathbb{R}^d \times [X_{t-1}(\omega), X_{t-1}(\omega)+1]\}$ is closed being the intersection of closed sets. We claim that $\mathrm{Ko}(\omega)$ is bounded. By contradiction, suppose it is unbounded. Let $\hat{H}_n = (H_n, H^{d+1}_n) \in \mathrm{Ko}(\omega) \subset \mathbb{R}^d \times \mathbb{R}$, such that $\|H_n\| \to +\infty$. By definition, $H^{i_j}_n = 0$ for every $j = 1, \dots, k$ and H^{d+1}_n is bounded by $X_{t-1}(\omega) + 1$. For any $\widetilde{\omega} \in D \cap \Sigma^\omega_{t-1}$ and any n, we have

$$\frac{X_{t-1}(\omega)+1}{\|H_n\|} + \frac{H_n}{\|H_n\|} \cdot \Delta S_t(\widetilde{\omega}) \ge \frac{X_t(\widetilde{\omega})}{\|H_n\|}.$$

Since $\frac{H_n}{\|H_n\|}$ lies on the unit sphere of $\mathbb{R}^d$, we can extract a subsequence converging to H^* with $\|H^*\| = 1$. Thus, passing to the limit over this subsequence, we have $H^* \cdot \Delta S_t(\widetilde{\omega}) \geq 0$ for every $\widetilde{\omega} \in D \cap \Sigma_{t-1}^{\omega}$. From the No one point arbitrage condition, we deduce $H^* \cdot \Delta S_t = 0$ on $D \cap \Sigma_{t-1}^{\omega}$. Since $H_n \in \mathrm{Ko}(\omega)$ then $(H^*)^{i_j} = 0$ on the redundant assets, and thus $H^* = 0$ which is a contradiction.

The set $\mathrm{Ko}(\omega)$ is closed and bounded in $\mathbb{R}^{d+1}$, hence compact. From the continuity of the projection, $\Pi_{x_{d+1}}(\mathrm{Ko}(\omega))$ is compact, so that the infimum is attained.

Step 3: We now provide a backward procedure which yields the super-replication price and the corresponding optimal strategy. By classical arguments, when we fix a reference probability $Q \in \mathcal{M}_f$ this procedure yields two processes $X_t(Q)$ and $H_t(Q)$ such that

$$g \leq \sum_{u=t+1}^{T} H_u(Q) \cdot \Delta S_u + X_t(Q) = \sum_{t=1}^{T} H_t(Q) \cdot \Delta S_t + X_0(Q), \qquad Q\text{-a.s.}, \tag{5.9}$$

where $X_t(Q)$ represents the minimum amount of cash that we need at time t in order to superhedge g in the Q-a.s. sense. Recall that $\mathrm{NA}(Q)$ implies $X_t(Q) > -\infty$ on $\mathrm{supp}(Q)$. With no loss of generality, set $X_t(Q)(\omega) = -\infty$ for any $\omega \notin \mathrm{supp}(Q)$. Now we prove the pathwise counterpart of (5.9).

Set $X_T := g$ and $D_T := \Omega_*$ which belongs to $\mathcal{F}_T$ by Remark 5.8 and consider first the random set R_{T,X_T,D_T}. The random variable $X_{T-1} := \inf \Pi_{x_{d+1}}(R_{T,X_T,D_T})$ represents the minimum amount of cash that we need at time $T-1$ in order to superhedge g on Ω_*. X_{T-1} is therefore the $\mathcal{F}_{T-1}$-measurable random variable that needs to be super-replicated at time $T-2$.

For $t = T-1, \dots, 0$, we iterate the procedure by taking $X_t := \inf \Pi_{x_{d+1}}(R_{t+1,X_{t+1},D_{t+1}})$, $D_t = S_{0:t}^{-1}(S_{0:t}(D_{t+1})) \in \mathcal{F}_t$ and the random set $R_{t+1,X_{t+1},D_{t+1}}$ as defined before. We again have that X_t is an $\mathcal{F}_t$-measurable function with values in $\mathbb{R} \cup \{\pm\infty\}$.

This backward procedure yields the superhedging price X_0 on Ω_* but also provide the corresponding cheapest portfolio as follows: Note first that for every $\omega \in \Omega_*$, $X_t(\omega) > -\infty$. If this is not the case, there exists a sequence $(H_n, x_n)_{n\in\mathbb{N}} \in \mathbb{R}^d \times \mathbb{R}$ such that $x_n \downarrow -\infty$, $x_n + H_n \Delta S_{t+1}(\widetilde{\omega}) \geq X_{t+1}(\widetilde{\omega})$ for every $\widetilde{\omega} \in D_{t+1} \cap \Sigma_t^{\omega}$, and hence Q-a.s. for every $Q \in \mathcal{M}_f$ such that $Q(\Sigma_t^{\omega}) > 0$. This would lead to a contradiction with $X_t(Q) > -\infty$. From now on, we therefore assume that $X_t(\omega) > -\infty$. In the case $X_t(\omega) < \infty$ for every $t = 0, \dots, T-1$, Step 2 provides that X_t is actually a minimum. The $\mathcal{F}_t$-measurable multifunction given by $\Pi_{x_1,\dots,x_d}(R_{t+1,X_{t+1},D_{t+1}} \cap \{\mathbb{R}^d \times X_t\})$ is therefore nonempty for every $t = 0, \dots, T-1$ and thus admits a measurable selector H_{t+1}. The strategy $H_1, \dots, H_T$ satisfy the inequalities

$$\begin{aligned} g &\leq H_T \cdot \Delta S_T + X_{T-1} \qquad \text{on } D_T, \\ X_{T-1} &\leq H_{T-1} \cdot \Delta S_{T-1} + X_{T-2} \qquad \text{on } D_{T-1}, \\ &\dots \\ X_1 &\leq H_1 \cdot \Delta S_1 + X_0 \qquad \text{on } D_1 \end{aligned}$$

and it represents a superhedge on $\Omega_* = \bigcap_{t=1}^T D_t$ as

$$
\begin{aligned}
g &\le H_T \cdot \Delta S_T + X_{T-1} \le \sum_{t=T-1}^{T} H_t \cdot \Delta S_t + X_{T-2} \le \cdots \\
&\le \sum_{t=1}^{T} H_t \cdot \Delta S_t + X_0
\end{aligned}
\tag{5.10}
$$

holds true for any $\omega \in \Omega_*$. When instead $X_t(\omega) = \infty$ for some $\omega \in \Omega_*$ and for some $t \ge 0$ then by simply taking $X_u \equiv \infty$ and H_u arbitrary for every $u \le t$, the inequality (5.10) is trivially satisfied.

Step 4: In order to prove (2.5), we recursively show that $X_t(\omega) = \sup_{Q \in \mathcal{M}_f} X_t(Q)(\omega)$ for any $\omega \in \Omega_*$ which, in particular, implies $X_0 = \sup_{Q \in \mathcal{M}_f} X_0(Q)$. Obviously, we have $X_t(\omega) \ge X_t(Q)(\omega)$ for any $\omega \in \Omega_*$ so that $X_t \ge \sup_{Q \in \mathcal{M}_f} X_t(Q)$. Thus, we need only to prove the reverse inequality.

For $t = T$, the claim is obvious: $X_T = g$. By backward recursion, suppose now it holds true for any u with $t + 1 \le u \le T$, namely, $X_u(\omega) = \sup_{Q \in \mathcal{M}_f} X_u(Q)(\omega)$ for any $\omega \in \Omega_*$.

From the recursive hypothesis in order to find a super-replication strategy with the same price for any $Q \in \mathcal{M}_f$, we need to super-replicate X_{t+1}. We fix a level set Σ_t^ω and recall that X_t is $\mathcal{F}_t$-measurable, hence it is constant on Σ_t^ω. We first treat two trivial cases:

- If $X_{t+1}(\omega) = \infty$ for some $\omega \in \Omega_*$, then the claim is not super-replicable at a finite cost, hence the thesis follows with $X_0 = \sup_{Q \in \mathcal{M}_f} X_0(Q) = \infty$.
- If $\Sigma_t^\omega \cap \Omega_*^{t+1} = \varnothing$, we have two consequences: Σ_t^ω is an $\mathcal{M}_f$-polar set, hence by assumption, $X_t(Q) = -\infty$ on Σ_t^ω, for any $Q \in \mathcal{M}_f$. Moreover, as explained after equation (5.8), $\Pi_{x_{d+1}}(R_{t+1, X_{t+1}, D_{t+1}}) = \mathbb{R}$ so that $X_t(\omega) = -\infty$ and the desired equality follows.

From now on, we therefore assume $X_{t+1} < \infty$ and $\Sigma_t^\omega \cap \Omega_*^{t+1} \ne \varnothing$. Define, for any $y \in \mathbb{R}$, the set

$$\Gamma_y := \mathrm{co}\big(\mathrm{conv}\{[\Delta S_{t+1}(\widetilde{\omega}); y - X_{t+1}(\widetilde{\omega})] | \widetilde{\omega} \in \Sigma_t^\omega \cap \Omega_*^{t+1}\}\big).$$

We claim that

$$0 \in \mathrm{int}(\Gamma_y) \quad \Longrightarrow \quad X_t > y. \tag{5.11}$$

Indeed, from $0 \in \mathrm{int}(\Gamma_y)$ there is no $(H, h) \in \mathbb{R}^d \times \mathbb{R}$ (different from zero) such that either $h(y - X_{t+1}) + H \cdot \Delta S_{t+1} \ge 0$ or $h(y - X_{t+1}) + H \cdot \Delta S_{t+1} \le 0$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$. In particular, there is no $H \in \mathbb{R}^d$ such that

$$y + H \cdot \Delta S_{t+1} \ge X_{t+1} \qquad \text{on } \Sigma_t^\omega \cap \Omega_*^{t+1}. \tag{5.12}$$

Recalling that, by definition, X_t is the infimum of real numbers for which (5.12) is satisfied, we have $X_t \ge y$. Since, from Step 2, X_t, when finite, is actually a minimum, we have $X_t > y$ and (5.11) follows.

Premise: As in Step 1, we may suppose, without loss of generality, that if for some $H \in \mathbb{R}^d$, $H \cdot \Delta S_{t+1} = 0$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$ then $H = 0$. In fact, if this is not the case we can reduce, with an analogous procedure, the number of assets needed for super-replication on the level set.

We now distinguish two cases.

Case 1: Suppose there exist $(H, h, \alpha) \in \mathbb{R}^{d+2}$ with $(H, h, \alpha) \neq 0$ which satisfies the equality $h(y - X_{t+1}) + H \cdot \Delta S_{t+1} = \alpha$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$. We claim that $h \neq 0$. Indeed, if $h = 0$ then $\alpha \neq 0$, since $H \cdot \Delta S_{t+1} = 0$ implies $(H, h, \alpha) = 0$. However, $\alpha \neq 0$ implies $H \cdot \Delta S_{t+1} = \alpha$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$ which would yield a trivial one point arbitrage on Ω_*, hence a contradiction.

Since $h \neq 0$, we have $y - \frac{\alpha}{h} + \frac{H}{h} \cdot \Delta S_{t+1} = X_{t+1}$ on $\Sigma_t^\omega \cap \Omega_*^{t+1}$: this means that X_t from Step 3 coincides with $y - \frac{\alpha}{h}$ and X_{t+1} is replicable implementing the strategy $\bar{H} := \frac{H}{h}$ in the risky assets and $X_t = y - \frac{\alpha}{h}$ in the nonrisky asset. If now for some $Q \in \mathcal{M}_f$ such that $Q(\Sigma_t^\omega) > 0$, we have the existence of $x \leq X_t$ and $H_x \in \mathbb{R}^d$ such that $x + H_x \cdot \Delta S_{t+1} \geq X_{t+1}$ Q-a.s. then $x - X_t + (H_x - \bar{H})\Delta S_{t+1} \geq 0$ Q-a.s. hence, since NA(Q) holds true, $x \geq X_t$. Therefore, $X_t = X_t(Q)$ on Σ_{t-1}^ω.

Case 2: If a triplet $(H, h, \alpha) \in \mathbb{R}^{d+2}$ such as in Case 1 does not exist, then we define

$$\bar{y} = \sup\{y \in \mathbb{R} | \exists H \in \mathbb{R}^d : y + H \cdot \Delta S_{t+1} \leq X_{t+1} \text{ on } \Sigma_t^\omega \cap \Omega_*^{t+1}\}.$$

Obviously, $\bar{y} < X_t$ otherwise we are back to Case 1. For every $0 < \varepsilon < X_t - \bar{y}$ and for every $H \in \mathbb{R}^d$, neither $X_t - \varepsilon + H\Delta S_{t+1} \geq X_{t+1}$ nor $X_t - \varepsilon + H\Delta S_{t+1} \leq X_{t+1}$ holds true on $\Sigma_t^\omega \cap \Omega_*^{t+1}$. Moreover, if there exists $h \in \mathbb{R}$ such that $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \geq 0$ [or $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \leq 0$] on $\Sigma_t^\omega \cap \Omega_*^{t+1}$ necessarily h would be 0 (otherwise simply divide by h). In such a case, $H\Delta S_{t+1} \geq 0$ (or $H\Delta S_{t+1} \leq 0$) on $\Sigma_t^\omega \cap \Omega_*^{t+1}$ and by absence of one point arbitrage we get $H\Delta S_{t+1} = 0$, and hence $H = 0$. For this reason, neither $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \geq 0$ nor $h(X_t - \varepsilon - X_{t+1}) + H\Delta S_{t+1} \leq 0$ for any $(H, h) \in \mathbb{R}^{d+1} \setminus \{0\}$ so that $0 \in \operatorname{int}\Gamma_{X_t - \varepsilon}$. Take a finite set $\{\omega_i\}_{i=1}^k \subset \Sigma_t^\omega \cap \Omega_*$ (with $k \leq d$) with the following properties: $\{[\Delta S_{t+1}(\omega_i); X_t - \varepsilon - X_{t+1}(\omega_i)] | i = 1, \dots, k\}$ are linearly independent and generate the same linear space in $\mathbb{R}^{d+1}$ as $\Gamma_{X_t - \varepsilon}$. By Proposition 3.1, and the convexity of the set of martingale measures, there exists $Q \in \mathcal{M}_f$ such that $Q(\{\omega_i\}) > 0$ for any $i = 1, \dots, k$. For such a Q, we get

$$\Gamma_{X_t - \varepsilon} = \operatorname{co}\bigl(\operatorname{conv}\bigl\{\bigl[\Delta S_{t+1}(\widetilde{\omega}); X_t - \varepsilon - X_{t+1}(\widetilde{\omega})\bigr] | \widetilde{\omega} \in \operatorname{supp}(Q) \cap \Sigma_t^\omega\bigr\}\bigr).$$

From $0 \in \operatorname{int}\Gamma_{X_t - \varepsilon}$, there is no $H(Q) \in \mathbb{R}^d$ such that $X_t - \varepsilon + H(Q) \cdot \Delta S_{t+1} \geq X_{t+1}$ Q-a.s. We can conclude that $X_t \geq \sup_{Q \in \mathcal{M}_f} X_t(Q) \geq X_t - \varepsilon$. Letting $\varepsilon \downarrow 0$ we get $\sup_{Q \in \mathcal{M}_f} X_t(Q) = X_t$ as desired.

Step 5: Finally, we prove (2.6). Notice that obviously $\mathcal{C} \subseteq \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q)$. Moreover, if $g \in \bigcap_{Q \in \mathcal{M}_f} \mathcal{C}(Q)$, then (5.9) holds with $X_0(Q) \leq 0$ for every $Q \in \mathcal{M}_f$. Therefore, also in equation (5.10) we have $X_0 = \sup_{Q \in \mathcal{M}_f} X_0(Q) \leq 0$ and $g \leq \sum_{t=1}^T H_t \cdot \Delta S_t$ on Ω_*, namely, $g \in \mathcal{C}$.

REMARK 5.9. Note that the proof of Proposition 2.1 relies only on the fact that Ω_* is an analyitc set and that $(\Omega_*)^C$ is the maximal polar set for the class of finite support martingale measure. Given $\widehat{\Omega} \subseteq \Omega$, an analytic subset of $(\Omega, \mathcal{F})$, from Proposition 5.5 it also follows that

$$\widehat{\mathcal{C}} = \bigcap_{\{Q \in \mathcal{M}_f \mid Q(\widehat{\Omega})=1\}} \mathcal{C}(Q),$$

where $\widehat{\mathcal{C}} := \{f \in \mathcal{L}(\Omega, \mathcal{F}) \mid f \leq k \text{ on } \widehat{\Omega}_* \text{ for some } k \in \mathcal{K}\}$ and $\widehat{\Omega}_*$ as in (5.7).

5.3. *Proof of Theorem* 1.2. Recall that π_Φ is defined in (1.6) and $\mathcal{M}_\Phi$ in (1.4). Set

$$\widetilde{\pi}_\Phi(g) := \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega_\Phi\}.$$

LEMMA 5.10. *Let* $g : \Omega \mapsto \mathbb{R}$ *and* $\phi^j : \Omega \mapsto \mathbb{R}$, $j = 1, \dots, k$, *be* $\mathcal{F}$-*measurable random variables*. *Then*

$$\pi_\Phi(g) = \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi).$$

PROOF. Note that $\pi_\Phi(g) \leq \widetilde{\pi}_\Phi(g - h\Phi)$ $\forall h \in \mathbb{R}^k$, hence, $\pi_\Phi(g) \leq \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi)$. By contradiction, assume $\pi_\Phi(g) < \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi)$, then there exist $(\bar{x}, \bar{h}, \bar{H}) \in (\mathbb{R}, \mathbb{R}^k, \mathcal{H})$ such that

$$\bar{x} < \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi) \quad \text{and}$$
$$\bar{x} + (\bar{H} \cdot S)_T(\omega) + \bar{h}\Phi(\omega) \geq g(\omega) \qquad \text{for all } \omega \in \Omega_\Phi.$$

Clearly we have a contradiction since

$$\begin{aligned} \bar{x} &< \widetilde{\pi}_\Phi(g - \bar{h}\Phi) \\ &= \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega) - \bar{h}\Phi(\omega)\ \forall \omega \in \Omega_\Phi\} \\ &\leq \bar{x}. \end{aligned}$$

□

PROOF OF THEOREM 1.2. Since also Ω_Φ is analytic (Proposition 5.5), by comparing the definition of Ω_Φ in (1.5) with (3.1), we may repeat step by step the same arguments used in the proof of Theorem 1.1 and Proposition 2.1 replacing Ω_* with Ω_Φ. We then conclude that $\widetilde{\pi}_\Phi(g) = \sup_{\{Q \in \mathcal{M}_f \mid \text{supp}(Q) \subseteq \Omega_\phi\}} E_Q[g]$ for any $\mathcal{F}$-measurable random variable g. From the hypothesis, we also have $\widetilde{\pi}_\Phi(g) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g]$. Since $E_Q[h\Phi] = 0$ for all $Q \in \mathcal{M}_\Phi$ and $h \in \mathbb{R}^k$, for the $\mathcal{F}$-measurable random variable $g - h\Phi$ we have

$$\widetilde{\pi}_\Phi(g - h\Phi) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g - h\Phi] = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g], \qquad \forall h \in \mathbb{R}^k.$$

Lemma 5.10 then implies: $\pi_\Phi(g) = \inf_{h \in \mathbb{R}^k} \widetilde{\pi}_\Phi(g - h\Phi) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g]$. □

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M. BURZONI
DEPARTMENT OF MATHEMATICS
ETH ZURICH
RÄMISTRASSE 101
8092 ZURICH
SWITZERLAND
E-MAIL: matteo.burzoni@math.ethz.ch

M. FRITTELLI
M. MAGGIS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MILAN
VIA C. SALDINI 50
20122 MILAN
ITALY
E-MAIL: marco.frittelli@unimi.it
marco.maggis@unimi.it