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A new stochastic model and its diffusion approximation

Shai Covo and Amir Elalouf

Bar Ilan University

Abstract. This paper considers a kind of queueing problem with a Poisson number of customers or, more generally, objects which may arrive in groups of random size. The focus is on the total quantity over time, called system size. The main result is that the process representing the system size, properly normalized, converges in finite-dimensional distributions to a centered Gaussian process (the diffusion approximation) with several attractive properties. Comparison with existing works (where the number of objects is assumed nonrandom) highlights the contribution of the present paper.

1 Introduction

Consider the following stochastic model (formally described in the next section). A system starts empty at time 0. N_n objects, where N_n is Poisson distributed with mean $n \in \mathbb{N}$, enter and leave the system, independently of each other; the arrival and departure times of the N_n objects are sampled from a common absolutely continuous bivariate distribution. Associated with each object is its "size;" the sizes are i.i.d. (not necessarily nonnegative) random variables from an arbitrary distribution with finite second moment. This paper considers the stochastic process representing the system size at time t. The key point in the present model is the assumption that the number of objects is Poisson distributed. Indeed, if the number of objects were to be *nonrandom*, that is, $N_n = n$, where n is an arbitrary but fixed integer, then the process representing the system size at time t would be of the form $\sum_{i=1}^{n} f_i(t)$ considered in Steinsaltz (1996), as indicated shortly below. While, by the central limit theorem, a Poisson(n) random variable, N_n, can be approximated, for large n, by $n + \sqrt{n}Z$, where $Z \sim \mathrm{N}(0, 1)$, it turns out that the difference $N_n - n$, though asymptotically negligible relative to n, results in an essential difference between the two models. As shown in this paper, the Poisson case leads to considerably more elegant results. Moreover, there might be scenarios where the assumption of a Poisson distributed number of objects is also more realistic. This is because the objects may be associated with a prior system where arrivals are modeled by a (not necessarily homogeneous) Poisson process. For example, people reserve tickets to a show/event; requests can be made within a given

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time period. Under the assumption of a Poisson process arrival of requests, the total number of reservations follows a Poisson distribution, with approximated mean $n \in \mathbb{N}$. Then, the number of audience physically present at the venue (unconditionally on the actual number of reservations, and assuming no cancellations) may represent a stochastic process conforming with the paper's setting.

Having briefly indicated two major advantages of the new model, it is instructive to consider the standard one (where the number of objects is fixed). Two key references are Louchard (1988) and Steinsaltz (1996).

A special but particularly important case arises when the object size is 1 and the amount of time it spends in the system ("service time") is independent of its arrival time. This case was studied in Louchard (1988), where the following description is given: "We analyse in this paper the large *finite* population infinite server model: a total population of n customers applies for some facility; each customer applies for service only once; the time of applications of the n customers are independent, identically distributed random variables, with distribution function $F(\cdot)$; customers are served in parallel." As in the general case, the results obtained for the new model (with Poisson distributed number of arrivals) are considerably more elegant.

Not only that the assumption made in the previous paragraph that arrival and service times are independent is not needed in the present setting, but the object size distribution can be assumed arbitrary with finite second moment (thus, for example, customers may arrive in groups of random size). With τ_1^i and τ_2^i denoting the arrival and departure times of the ith object, respectively, and Y_i its size, the system size at time t can be expressed $\sum_{i=1}^{N_n} Y_i \mathbf{1}_{[\tau_1^i,\tau_2^i)}(t)$, where $N_n \sim \text{Poisson}(n)$ and $\mathbf{1}$ is the indicator function. The random functions f_i defined by $f_i(t) = Y_i \mathbf{1}_{[\tau_1^i,\tau_2^i)}(t)$ are, in particular, *independent càdlàg functions*. For such functions, the process $\sum_{i=1}^{n} f_i$, where n is an arbitrary but fixed integer, was considered in Steinsaltz (1996); its relation to queueing problems is clearly indicated there.

The pertinent works Louchard (1988) and Steinsaltz (1996) will be considered in more detail in Section 5, in order to highlight the advantages of the new model.

Further motivation of the new model is provided in the following practical example, which represents a common lending process.

A lending entity begins with an initial quantity of funds. Borrowers enter the system at arbitrary times. Upon entering the system, each borrower requests a sum of money (unknown in advance), receives it, and returns it later on, in one payment (without interest). The process can either continue forever or terminate at time T, after all loans are returned.

This description fits, for example, the case of a bridging loan, a short-term loan that a bank provides a customer to "bridge a gap" until he or she obtains longer-term financing. (The model does not take into account the interest, which may be paid each period (monthly/weekly) and can be investigated and calculated separately as the organization's profit. Rather, we deal with the processes related to the capital.)

Such loans are common in real-estate purchases: A borrower might take out such a loan pending the sale of a property or when waiting to be approved for a mortgage. An additional example of an entity who might seek out such a loan is a corporation that needs to secure working capital until a round of equity financing goes through. Once a borrower obtains longer-term financing (usually within a period of up to 3 years), the bridging loan is repaid to the bank. In many cases, the financing that has been secured is used to repay the loan in one lump sum, although lending entities do allow customers to repay the loan in multiple installments. Bridging loans can range from several thousands of dollars to several millions.

The analysis presented in the paper can assist the creditor in making decisions before and during the lending process, for example, to determine the initial capital required to satisfy borrowers' requests. We assume that the borrowed quantities are independent and identically distributed (i.i.d.); this assumption is reasonable under the assumption that the lender is dealing with a single type of loan, such as bridging loans for real-estate. Moreover, the customers' arrival at the creditor is assumed to be random. As the time at which a loan is returned clearly depends on the time at which the loan is given, we use a continuous bivariate distribution to capture the relationship between these two variables. Finally, the number of customers is assumed to be large (and clearly random), and hence naturally Poisson distributed.

The rest of this paper is organized as follows. Section 2 is devoted to basic notation. The main result, namely Theorem 3.1, is presented in Section 3. It gives the diffusion approximation, X, to the process representing the system size, S_{N_n}, by showing that, as $n (= \mathrm{E}(N_n))$ tends to ∞, the properly normalized process, X_n, converges in finite-dimensional distributions to X, where X is a centered Gaussian process whose covariance function is simply expressed in terms of the common joint density function of the arrival and departure times. Continuity of X is briefly considered at the end of Section 3. Section 4 is devoted to examples. The important case where the arrival and departure times are distributed as order statistics is considered in Example 4.1, while the general Example 4.3 corresponds to the infinite server model of Louchard (1988) (the particular case of exponential service times is considered in Example 4.4). Section 5 is devoted to comparison with Louchard (1988) and Steinsaltz (1996). Section 6 presents key features of the diffusion approximation X. This process has nonpositively correlated increments and, moreover, a *biconvex covariance*. The main results of Section 6 are Propositions 6.1 and 6.2, which give simple (analogous) representations of X, the first as a stochastic integral with respect to Brownian sheet, the second in terms of a "modified" inhomogeneous Brownian sheet. Based on the second representation, a very simple algorithm is presented in Section 7 for generating an (essentially) exact discrete-time realization of X. Additional insights are provided in the Appendix.

2 Basic notation

The following notations and abbreviations will be used frequently in the sequel.

- RV, DF, FDD, BM, and BB: abbreviations of "random variable," "distribution function," "finite-dimensional distributions," "Brownian motion," and "Brownian bridge," respectively.
- T: fixed positive number or infinity.
- $N_n \sim \text{Poisson}(n)$, where n is a positive integer: total number of objects that enter the system. The system starts empty at time 0.
- (τ_1^i, τ_2^i), $i = 1, 2, \ldots$: sequence of i.i.d. absolutely continuous bivariate RVs, independent of N_n, such that, for each i, $0 < \tau_1^i < \tau_2^i < T$. The RVs τ_1^i and τ_2^i represent the arrival and departure times of the ith object.
- $E_T := \{(x, y) \in (0, T)^2 : y > x\}$.
- $f(\cdot, \cdot)$: common joint density function of the (τ_1^i, τ_2^i)'s, vanishing on the complement of E_T.
- Y_i, $i = 1, 2, \ldots$: sequence of i.i.d. (not necessarily nonnegative) RVs with common mean $\mu_Y \in \mathbb{R}$ and finite variance $\sigma_Y^2 \geq 0$, independent of N_n and the (τ_1^i, τ_2^i)'s. The trivial case $\mu_Y^2 + \sigma_Y^2 = 0$ is excluded. Y_i represents the size of the ith object.
- $a_i = \{a_i(t) : t \in [0, T]\}$, $i = 1, 2, \ldots$: sequence of i.i.d. stochastic processes defined by
$$a_i(t) = Y_i \mathbf{1}_{[\tau_1^i, \tau_2^i)}(t), \tag{2.1}$$
where $\mathbf{1}$ is the indicator function.
- $S_{N_n} = \{S_{N_n}(t) : t \in [0, T]\}$: stochastic process defined by
$$S_{N_n}(t) = \sum_{i=1}^{N_n} a_i(t). \tag{2.2}$$
Thus, $S_{N_n}(t)$ represents the system size at time t. (Obviously, $S_{N_n}(0) = S_{N_n}(T) = 0$.)
- $R(\cdot, \cdot)$: function defined for $s, t \in [0, T]$ with $s \leq t$ by
$$R(s, t) = \int_0^s \int_t^T f(x, y)\, \mathrm{d}y\, \mathrm{d}x. \tag{2.3}$$
Alternatively (and sometimes more usefully),
$$R(s, t) = \mathrm{P}\big(\tau_1^i \leq s, \tau_2^i > t\big). \tag{2.4}$$
- $V(\cdot)$: function defined for $t \in [0, T]$ by
$$V(t) = R(t, t) = \int_0^t \int_t^T f(x, y)\, \mathrm{d}y\, \mathrm{d}x.$$
Thus, $V(0) = V(T) = 0$.

- $X_n = \{X_n(t) : t \in [0, T]\}$: normalized process defined by

$$X_n(t) = \frac{S_{N_n}(t) - n\mu_Y V(t)}{\sqrt{n(\sigma_Y^2 + \mu_Y^2)}}. \tag{2.5}$$

  (Thus, $X_n(0) = X_n(T) = 0$.)
- $X = \{X(t) : t \in [0, T]\}$: centered Gaussian process (vanishing at the endpoints 0 and T) with covariance function given, for $0 \le s \le t \le T$, by $\mathrm{E}(X(s)X(t)) = R(s,t)$.
- $\mathbb{G}_H = \{\mathbb{G}_H(t) : t \in [0, T]\}$, where H is a DF such that $H(0) = 0$ and $H(T) = 1$: H-BB on $[0, T]$, that is, a centered Gaussian process with covariance function given, for $0 \le s \le t \le T$, by

$$\mathrm{E}\big(\mathbb{G}_H(s)\mathbb{G}_H(t)\big) = H(s)\big(1 - H(t)\big).$$

  When $T = 1$ and H is the uniform(0, 1) DF, $\mathbb{G}_H$ is a standard BB on [0, 1], and is denoted by B^0.
- $\stackrel{\mathrm{d}}{\to}$ and $\stackrel{\mathrm{d}}{=}$: notations for convergence and equality in distribution, respectively.

3 The main result

In order to study the behavior of the process S_{N_n} as n (the mean number of objects) tends to infinity, it should be properly normalized. As shown below (Theorem 3.1), the normalized process X_n converges in FDD to the centered Gaussian process X. The following lemma accounts for the definition of X_n.

Lemma 3.1. *For any* $0 \le t \le T$, *it holds that*

$$\mathrm{E}\big(S_{N_n}(t)\big) = n\mu_Y V(t) \tag{3.1}$$

and

$$\mathrm{Var}\big(S_{N_n}(t)\big) = n\big(\sigma_Y^2 + \mu_Y^2\big)V(t). \tag{3.2}$$

Moreover, for any $0 \le s \le t \le T$, *it holds that*

$$\mathrm{Cov}\big(S_{N_n}(s), S_{N_n}(t)\big) = n\big(\sigma_Y^2 + \mu_Y^2\big)R(s,t). \tag{3.3}$$

Proof. Since, for fixed $0 \le t \le T$, the $a_i(t)$ are i.i.d. RVs, $S_{N_n}(t)$ is a compound Poisson RV with mean $n\mathrm{E}(a_i(t))$ and variance $n\mathrm{E}(a_i^2(t))$. It follows directly from the definition of $a_i(t)$ that $\mathrm{E}(a_i(t)) = \mu_Y V(t)$ and $\mathrm{E}(a_i^2(t)) = (\sigma_Y^2 + \mu_Y^2)V(t)$, thus proving (3.1) and (3.2). To prove (3.3), first note that, for any $0 \le s \le t \le T$,

$$\mathrm{E}\big(a_i(s)a_j(t)\big) = \begin{cases} (\sigma_Y^2 + \mu_Y^2)R(s,t), & i = j, \\ \mu_Y^2 V(s)V(t), & i \ne j. \end{cases} \tag{3.4}$$

Then,

$$\begin{aligned}
\mathrm{E}\big(S_{N_n}(s)S_{N_n}(t)\big) &= \mathrm{E}\left(\sum_{i=1}^{N_n} a_i(s)a_i(t)\right) + \mathrm{E}\left(\sum_{i,j=1,i\neq j}^{N_n} a_i(s)a_j(t)\right) \\
&= n\big(\sigma_Y^2 + \mu_Y^2\big)R(s,t) + \sum_{k=0}^{\infty} \mathrm{E}\left(\sum_{i,j=1,i\neq j}^{k} a_i(s)a_j(t)\right)\frac{\mathrm{e}^{-n}n^k}{k!} \\
&= n\big(\sigma_Y^2 + \mu_Y^2\big)R(s,t) + \sum_{k=0}^{\infty} (k^2-k)\mu_Y^2 V(s)V(t)\frac{\mathrm{e}^{-n}n^k}{k!} \\
&= n\big(\sigma_Y^2 + \mu_Y^2\big)R(s,t) + n^2\mu_Y^2 V(s)V(t),
\end{aligned}$$

from which (3.3) follows using (3.1). □

The following proposition is an immediate corollary of Lemma 3.1.

Proposition 3.1. *For any* $0 \le t \le T$, *it holds that*

$$\mathrm{E}\big(X_n(t)\big) = 0 \tag{3.5}$$

and

$$\mathrm{Var}\big(X_n(t)\big) = V(t). \tag{3.6}$$

Moreover, for any $0 \le s \le t \le T$, *it holds that*

$$\mathrm{Cov}\big(X_n(s), X_n(t)\big) = R(s,t). \tag{3.7}$$

Thus, the covariance function of X_n is fully determined by the joint density $f(\cdot,\cdot)$ of (τ_1^i, τ_2^i). In particular, it follows from (3.7) that

$$\mathrm{Cov}\big(X_n(v) - X_n(u), X_n(t) - X_n(s)\big) = \int_s^t \int_u^v -f(x,y)\,\mathrm{d}y\,\mathrm{d}x, \tag{3.8}$$

for any $0 \le s < t \le u < v \le T$, and thus X_n has nonpositively correlated increments, characterized by f. Obviously, though, the process behavior depends heavily on n. The diffusion approximation of S_{N_n} is given in the following theorem, the paper's main result.

Theorem 3.1. *As* $n \to \infty$, X_n *converges in FDD to* X.

Proof. It is required to show that, as $n \to \infty$,

$$\big(X_n(t_1), \ldots, X_n(t_m)\big) \xrightarrow{\mathrm{d}} \big(X(t_1), \ldots, X(t_m)\big),$$

for any m times $0 < t_1 < \cdots < t_m < T$. (Note that X_n and X vanish at the endpoints 0 and T.) Let $J_1, \ldots, J_n$ be i.i.d. Poisson(1) RVs, independent of the Y_i's and (τ_1^i, τ_2^i)'s, and represent N_n as $N_n = J_1 + \cdots + J_n$. Let $Z^i = (Z_1^i, \ldots, Z_m^i)$, $i = 1, \ldots, n$, be n random vectors, with the kth component ($k = 1, \ldots, m$) given by

$$Z_k^i = \frac{\sum_{j=J_1+\cdots+J_{i-1}+1}^{J_1+\cdots+J_i} a_j(t_k) - \mu_Y V(t_k)}{\sqrt{\sigma_Y^2 + \mu_Y^2}}.$$

The Z^i are thus i.i.d. centered vectors, with

$$\mathrm{Cov}\big(Z_{k_1}^i, Z_{k_2}^i\big) = R(t_{k_1}, t_{k_2}),$$

for all $1 \le k_1 \le k_2 \le m$ (cf. Proposition 3.1 in the case $n = 1$). Hence, by the multivariate central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_1^i, \ldots, Z_m^i) \xrightarrow{\mathrm{d}} \big(X(t_1), \ldots, X(t_m)\big).$$

Since the left-hand side equals $(X_n(t_1), \ldots, X_n(t_m))$, the proof is completed. □

Remark 3.1. Addressing the question of weak convergence of X_n is far beyond the scope of this paper. On the other hand, continuity of X will be considered and exemplified shortly.

Since each process a_i is right-continuous with left limits, so is X_n. Moreover, for any *fixed* $t \in [0, T]$, X_n is continuous at t with probability 1 (obviously, though, it has $2N_n$ jump discontinuities a.s. if $\mathrm{P}(Y_i \neq 0) = 1$). A simple upper bound for $\mathrm{P}(|X_n(t) - X_n(s)| \ge \varepsilon)$, independent of n, is established next. By Chebyshev's inequality, for any $\varepsilon > 0$,

$$\mathrm{P}\big(\big|X_n(t) - X_n(s)\big| \ge \varepsilon\big) \le \frac{\mathrm{E}(X_n(t) - X_n(s))^2}{\varepsilon^2}.$$

Assuming without loss of generality that $0 \le s < t \le T$, it thus holds that

$$\begin{aligned}\mathrm{P}\big(\big|X_n(t) - X_n(s)\big| \ge \varepsilon\big) &\le \frac{1}{\varepsilon^2}\big[V(t) + V(s) - 2R(s,t)\big] \\ &= \frac{1}{\varepsilon^2}\left[\int_0^s \int_s^t f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_s^t \int_t^T f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right].\end{aligned}$$

(Note that this upper bound tends to 0 as $s \uparrow t$, for fixed t.) Of course, this last result also holds for the limit process X. Moreover, the equality

$$\mathrm{E}\big(X(t) - X(s)\big)^2 = \int_0^s \int_s^t f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_s^t \int_t^T f(x,y)\,\mathrm{d}y\,\mathrm{d}x, \tag{3.9}$$

where $0 \le s < t \le T$, is important in view of the following standard lemma. (See, e.g., Covo and Elalouf (2015); the unnecessary restriction of r below to $(0, 1]$ naturally corresponds to (3.9).)

Lemma 3.2. *Let $\xi = \{\xi(t) : t \geq 0\}$ be a centered Gaussian process such that*

$$\mathrm{E}\big(\xi(t) - \xi(s)\big)^2 \leq c|t - s|^r,$$

for all $s, t \geq 0$ and some constants $c > 0$ and $r \in (0, 1]$. Then ξ has a continuous modification $\tilde{\xi}$, which is locally Hölder continuous of any order $\gamma \in (0, r/2)$, that is, for any given $M > 0$,

$$\sup\left\{\frac{|\tilde{\xi}(t) - \tilde{\xi}(s)|}{|t - s|^\gamma} : s, t \in [0, M], s \neq t\right\} < \infty.$$

Thus, for example, if $T < \infty$ and $f(\cdot, \cdot)$ is bounded on E_T, then, by (3.9), X has a continuous modification which is Hölder continuous of any order $\gamma \in (0, 1/2)$ (as is the case, e.g., for standard BM). Some other cases are considered in the next section.

4 Examples

The first example below involves the H-BB on $[0, T]$, $\mathbb{G}_H$. This process can be expressed in terms of the standard BB on [0, 1], B^0, as follows:

$$\mathbb{G}_H(t) = B^0\big(H(t)\big). \tag{4.1}$$

Since B^0 is simply expressed in terms of standard BM, so is $\mathbb{G}_H$. In particular, simulation of $\mathbb{G}_H$ is straightforward.

Example 4.1. Let ξ_1 and ξ_2 be i.i.d. RVs from an absolutely continuous distribution on $(0, T)$, say with density h and DF H. If

$$\big(\tau_1^i, \tau_2^i\big) \stackrel{\mathrm{d}}{=} \big(\min(\xi_1, \xi_2), \max(\xi_1, \xi_2)\big),$$

then $f(\cdot, \cdot)$ is given by $f(x, y) = 2h(x)h(y)$, $(x, y) \in E_T$, and, hence,

$$R(s, t) = 2H(s)\big[1 - H(t)\big], \tag{4.2}$$

for all $0 \leq s \leq t \leq T$. It thus follows from Theorem 3.1 that X_n converges in FDD to the process $X = \sqrt{2}\mathbb{G}_H$. Clearly, in view of (4.1), X can be assumed continuous. Moreover, by (3.9),

$$\begin{aligned}
\mathrm{E}\big(X(t) - X(s)\big)^2 &= 2H(s)\big[H(t) - H(s)\big] + 2\big[H(t) - H(s)\big]\big[1 - H(t)\big] \\
&= 2\big[H(t) - H(s)\big]\big[H(s) + 1 - H(t)\big] \\
&\leq 2\big[H(t) - H(s)\big],
\end{aligned}$$

for all $0 \leq s < t \leq T$. Hence, by Lemma 3.2, if it holds that

$$H(t) - H(s) \leq c(t - s)^r,$$

for all $0 \le s < t \le T$ and some constants $c > 0$ and $r \in (0, 1]$, X can be assumed to be (locally, if $T = \infty$) Hölder continuous of any order $\gamma \in (0, r/2)$. In the particular case when $\xi_1, \xi_2 \sim \text{uniform}(0, T)$, $T < \infty$, X_n converges in FDD to the process $\{\sqrt{2}B^0(t/T) : t \in [0, T]\}$. An interesting generalization is given in the next example.

Example 4.2. For fixed $m \ge 2$, let $\xi_1, \ldots, \xi_m$ be i.i.d. uniform RVs on $(0, T)$, $T < \infty$. If

$$\bigl(\tau_1^i, \tau_2^i\bigr) \stackrel{\mathrm{d}}{=} \bigl(\min(\xi_1, \ldots, \xi_m), \max(\xi_1, \ldots, \xi_m)\bigr),$$

then

$$f(x, y) = \frac{m(m-1)}{T^m}(y - x)^{m-2}, \tag{4.3}$$

for $(x, y) \in E_T$, and, hence,

$$\begin{aligned} R(s, t) &= \int_0^s \int_t^T \frac{m(m-1)}{T^m}(y - x)^{m-2}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{T^m}\bigl[T^m - (T - s)^m + (t - s)^m - t^m\bigr], \end{aligned} \tag{4.4}$$

for all $0 \le s \le t \le T$. For $m = 2$, $R(s, t)$ can be written, correspondingly to (4.2), as $R(s, t) = 2(s/T)(1 - t/T)$; however, it is readily concluded that $R(s, t)$ cannot be factorized analogously for $m \ge 3$. [The importance of this issue is discussed in Section 6; cf. equation (6.2).] Since $f(\cdot, \cdot)$ in (4.3) is bounded, the diffusion approximation X has a continuous modification which is Hölder continuous of any order $\gamma \in (0, 1/2)$.

The next example corresponds to the infinite server model of Louchard (1988). (Comparison will be made in the next section.)

Example 4.3. Let Q and W be independent, positive, absolutely continuous RVs with densities f_Q and g_W, respectively; let F_Q and G_W denote the corresponding DFs. Suppose that $Q + W < T$ (for all ω in the sample space), so that $(Q, Q + W) \in E_T$. If

$$\bigl(\tau_1^i, \tau_2^i\bigr) \stackrel{\mathrm{d}}{=} (Q, Q + W), \tag{4.5}$$

then

$$f(x, y) = f_Q(x) g_W(y - x), \tag{4.6}$$

for $(x, y) \in E_T$. However, in this case, $R(s, t)$ is more readily obtained using the alternative definition (2.4): for all $0 \le s \le t \le T$,

$$\begin{aligned} R(s, t) &= \mathrm{P}(Q \le s, Q + W > t) \\ &= \int_0^s \mathrm{P}(W > t - x) f_Q(x)\,\mathrm{d}x \\ &= \int_0^s \bigl[1 - G_W(t - x)\bigr] f_Q(x)\,\mathrm{d}x. \end{aligned} \tag{4.7}$$

Suppose that, for some constants $c_1, c_2 > 0$ and $r_1, r_2 \in (0, 1]$, the following two conditions are satisfied:

$$F_Q(t) - F_Q(s) \le c_1(t-s)^{r_1}, \tag{4.8}$$

$$G_W(t) - G_W(s) \le c_2(t-s)^{r_2}, \tag{4.9}$$

for all $0 \le s < t \le T$. Then, the diffusion approximation X has a modification which is (locally, if $T = \infty$) Hölder continuous of any order $\gamma \in (0, r'/2)$, where $r' = \min(r_1, r_2)$. Indeed, by (3.9), it holds that

$$\begin{aligned} \mathrm{E}\big(X(t) - X(s)\big)^2 &= \int_0^s f_Q(x) \int_{s-x}^{t-x} g_W(y)\,\mathrm{d}y\,\mathrm{d}x + \int_s^t f_Q(x) \int_{t-x}^{T-x} g_W(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\le \int_0^s f_Q(x) \min\big(1, c_2(t-s)^{r_2}\big)\,\mathrm{d}x + \int_s^t f_Q(x)\,\mathrm{d}x \\ &\le \min\big(1, c_2(t-s)^{r_2}\big) + \min\big(1, c_1(t-s)^{r_1}\big) \\ &\le (2 + c_1 + c_2)(t-s)^{r'}, \end{aligned}$$

for all $0 \le s < t \le T$. Hence, the assertion follows from Lemma 3.2. A special important case is given in the next example.

Example 4.4. When W above is Exponential(λ_W), so that $T = \infty$ and $G_W(x) = 1 - \mathrm{e}^{-\lambda_W x}$, $x \ge 0$, (4.7) yields

$$R(s,t) = \left(\int_0^s \mathrm{e}^{\lambda_W x} f_Q(x)\,\mathrm{d}x\right)\mathrm{e}^{-\lambda_W t}, \tag{4.10}$$

for all $0 \le s \le t \le \infty$, where, by definition, $R(s,t) = 0$ if $s = t = \infty$. Thus, the diffusion approximation X can be represented in terms of standard BM B as

$$X(t) = b(t)B\big(a(t)/b(t)\big), \tag{4.11}$$

for $t \ge 0$, with $X(\infty) := 0$, where $a(t) = \int_0^t \mathrm{e}^{\lambda_W x} f_Q(x)\,\mathrm{d}x$ and $b(t) = \mathrm{e}^{-\lambda_W t}$. (A general discussion is provided in Section 6.) In particular, X can be assumed continuous on $[0, \infty)$ (note that its variance function, $V(t) = a(t)b(t)$, tends to 0 as $t \to \infty$). Moreover, since the condition (4.9) is satisfied with $r_2 = 1$ (and $c_2 = \lambda_W$), if it holds that

$$F_Q(t) - F_Q(s) \le c(t-s)^r, \tag{4.12}$$

for all $0 \le s < t < \infty$ and some constants $c > 0$ and $r \in (0, 1]$ (i.e., the condition (4.8) is satisfied), X can be assumed to be locally Hölder continuous of any order $\gamma \in (0, r/2)$. In the particular case when Q is Exponential(λ_Q), (4.10) yields

$$R(s,t) = \begin{cases} \left(\lambda_Q \dfrac{\exp[(\lambda_W - \lambda_Q)s] - 1}{\lambda_W - \lambda_Q}\right)\mathrm{e}^{-\lambda_W t}, & \lambda_Q \ne \lambda_W, \\ (\lambda_Q s)\mathrm{e}^{-\lambda_W t}, & \lambda_Q = \lambda_W, \end{cases} \tag{4.13}$$

for all $0 \le s \le t \le \infty$, where $R(\infty,\infty) := 0$. The correspondence with (4.11) is obvious. Moreover, here (4.12) holds with $r = 1$ (and $c = \lambda_Q$), and so X can be assumed to be locally Hölder continuous of any order $\gamma \in (0, 1/2)$.

Remark 4.1. It should be stressed that the general case considered in Example 4.3 is just a special case of the general setting. That is, (τ_1^i, τ_2^i) cannot in general be represented in the form (4.5) (indeed, $\tau_2^i - \tau_1^i$ is generally not independent of τ_1^i).

5 Comparison with existing works

This section emphasizes the attractiveness of the new model by considering the pertinent works Louchard (1988) and Steinsaltz (1996). The issue will be further developed in the subsequent sections.

The work Steinsaltz (1996) is considered first. Keeping the notation of Section 2, define, for $n \in \mathbb{N}$ arbitrary but *fixed*, the process $S_n = \{S_n(t) : t \in [0, T]\}$ by

$$S_n(t) = \sum_{i=1}^{n} a_i(t). \tag{5.1}$$

This is the same process as the process S_{N_n} conditioned on $N_n = n$. Thus, $S_n(t)$ represents the system size at time t in the case of n arrivals. The random, independent càdlàg functions a_i play the same role as the functions f_i in Steinsaltz (1996) (as a special case). Identifying the functions f_i with the functions a_i, define, as in Steinsaltz (1996), the stochastic processes F_n and $\tilde{F}_n$ by

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} a_i(t),$$

$$\tilde{F}_n(t) = \sqrt{n}\big(F_n(t) - \bar{F}_n(t)\big),$$

where $\bar{F}_n(t) := \mathrm{E}(F_n(t))$. By (5.1) and $\mathrm{E}(a_i(t)) = \mu_Y V(t)$,

$$\tilde{F}_n(t) = \frac{S_n(t) - n\mu_Y V(t)}{\sqrt{n}}. \tag{5.2}$$

Thus, the stochastic process $\tilde{F}_n$ of Steinsaltz (1996) plays the same role as the process X_n in the present paper (cf. (2.5)). The following lemma, analogous to Lemma 3.1, is readily verified using (3.4).

Lemma 5.1. *For any* $0 \le t \le T$, *it holds that*

$$\mathrm{E}\big(S_n(t)\big) = n\mu_Y V(t)$$

and

$$\mathrm{Var}\big(S_n(t)\big) = n\big[(\sigma_Y^2 + \mu_Y^2)V(t) - \mu_Y^2 V^2(t)\big].$$

Moreover, for any $0 \le s \le t \le T$, *it holds that*

$$\mathrm{Cov}\big(S_n(s), S_n(t)\big) = n\big[(\sigma_Y^2 + \mu_Y^2)R(s,t) - \mu_Y^2 V(s)V(t)\big].$$

Thus, fixing the number of arrivals results in a considerably less elegant expression for the covariance function (cf. Lemma 3.1).

Remark 5.1. When $\mu_Y = 0$ (and $\sigma_Y^2 > 0$), the processes S_{N_n} and S_n have zero mean and the same covariance function; hence, in this special case only, there is no essential difference between the models (for large n).

The following proposition, analogous to Proposition 3.1, is an immediate corollary of Lemma 5.1.

Proposition 5.1. *For any* $0 \le t \le T$, *it holds that*

$$\mathrm{E}\big(\tilde{F}_n(t)\big) = 0$$

and

$$\mathrm{Var}\big(\tilde{F}_n(t)\big) = (\sigma_Y^2 + \mu_Y^2)V(t) - \mu_Y^2 V^2(t).$$

Moreover, for any $0 \le s \le t \le T$, *it holds that*

$$\mathrm{Cov}\big(\tilde{F}_n(s), \tilde{F}_n(t)\big) = (\sigma_Y^2 + \mu_Y^2)R(s,t) - \mu_Y^2 V(s)V(t).$$

The effect of fixing the number of arrivals is even more pronounced here (cf. Proposition 3.1).

The diffusion approximation of S_n is given in the following proposition.

Proposition 5.2. *As* $n \to \infty$, $\tilde{F}_n$ *converges in FDD to the centered Gaussian process* $\tilde{X}$ *with covariance function given, for* $0 \le s \le t \le T$, *by*

$$\mathrm{E}\big(\tilde{X}(s)\tilde{X}(t)\big) = (\sigma_Y^2 + \mu_Y^2)R(s,t) - \mu_Y^2 V(s)V(t).$$

Proof. It is required to show that, as $n \to \infty$,

$$\big(\tilde{F}_n(t_1), \ldots, \tilde{F}_n(t_m)\big) \xrightarrow{\mathrm{d}} \big(\tilde{X}(t_1), \ldots, \tilde{X}(t_m)\big),$$

for any m times $0 < t_1 < \cdots < t_m < T$. (Note that $\tilde{F}_n$ and $\tilde{X}$ vanish at the endpoints 0 and T.) Let the random vectors $Z^i = (Z_1^i, \ldots, Z_m^i)$, $i = 1, \ldots, n$, be defined by

$$Z^i = \big(a_i(t_1) - \mathrm{E}\big(a_i(t_1)\big), \ldots, a_i(t_m) - \mathrm{E}\big(a_i(t_m)\big)\big).$$

The Z^i are thus i.i.d. centered vectors, with

$$\mathrm{Cov}\big(Z_{k_1}^i, Z_{k_2}^i\big) = (\sigma_Y^2 + \mu_Y^2)R(t_{k_1}, t_{k_2}) - \mu_Y^2 V(t_{k_1})V(t_{k_2}),$$

for all $1 \le k_1 \le k_2 \le m$. Hence, by the multivariate central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_1^i, \ldots, Z_m^i) \xrightarrow{\mathrm{d}} (\tilde{X}(t_1), \ldots, \tilde{X}(t_m)).$$

Since the left-hand side equals $(\tilde{F}_n(t_1), \ldots, \tilde{F}_n(t_m))$, the proof is completed. □

Remark 5.2. It should be stressed that the setting of Steinsaltz (1996) is much more general than that considered here; see, in particular, Corollary 5.7 of that paper. Thus, for example, Steinsaltz (1996) does not contain any result similar to Proposition 5.2 above.

Remark 5.3. It may be easily concluded from the forms of the respective covariance functions that the diffusion approximation of S_{N_n}, that is, X, is fundamentally simpler than that of S_n, that is, $\tilde{X}$. This point will be made clearer in the sequel.

The infinite server model of Louchard (1988) is considered next, following the brief description in the above introduction (third paragraph) but using the notation of the present paper. Let Q, W, f_Q, g_W, F_Q, and G_W be as in Example 4.3, and assume that (4.5) is satisfied, that is, $(\tau_1^i, \tau_2^i) \stackrel{\mathrm{d}}{=} (Q, Q + W)$. Thus, F_Q represents the common DF of customers' arrival time, with density f_Q, and G_W that of customers' service time, with density g_W. (In Louchard (1988), these functions are denoted F, f, G, and g, respectively; however, G is not necessarily assumed to be absolutely continuous.) In this special case, $\mu_Y = 1$ and $\sigma_Y^2 = 0$ (object size is 1). Thus, with the process $\tilde{F}_n$ as given by (5.2) above, the diffusion approximation of S_n, that is, the limit, in the sense of FDD, of $\tilde{F}_n$, is the centered Gaussian process, $\tilde{X}$, with covariance function

$$\mathrm{E}\big(\tilde{X}(s)\tilde{X}(t)\big) = R(s,t) - V(s)V(t), \tag{5.3}$$

for $0 \le s \le t \le T$ (Proposition 5.2), where, by (4.7),

$$R(s,t) = \int_0^s [1 - G_W(t-x)] f_Q(x)\,\mathrm{d}x.$$

Thus, $V(t)$ is given by

$$V(t) = \int_0^t [1 - G_W(t-x)] f_Q(x)\,\mathrm{d}x.$$

With $f := f_Q$ and $G := G_W$, the above expressions of $R(s,t)$ and $V(t)$ match exactly those of $z^L(s,t)$ and $z(t)$, respectively, as defined in Louchard (1988), p. 478. Further, with $\tilde{Q} := \tilde{X}$, (5.3) can be written

$$\mathrm{E}\big(\tilde{Q}(s)\tilde{Q}(t)\big) = z^L(s,t) - z(s)z(t). \tag{5.4}$$

Indeed, the process denoted $\tilde{Q}$ in Louchard (1988) plays the same role as $\tilde{X}$ above, i.e., the diffusion approximation of S_n (in Louchard (1988), S_n is denoted by Q_n, and its normalization, $\tilde{F}_n$, by $\tilde{Q}_n$; the convergence in FDD of $\tilde{Q}_n$ to $\tilde{Q}$ is established in Louchard (1988), Theorem 1).

Whereas, as will be demonstrated in Sections 6 and 7, the diffusion approximation X can be very easily understood and constructed, its counterpart $\tilde{X}$ is quite involved, even in the special setting of Louchard (1988), as shown (somewhat informally) next. In order to understand the process $\tilde{Q}(=\tilde{X})$, it is decomposed in Louchard (1988) (Theorem 3) into a sum of two independent, centered Gaussian processes: $\tilde{Q}=\tilde{Q}_1+\tilde{Q}_2$. A stochastic integral representation of $\tilde{Q}_1$ is presented in Louchard (1988), equation (9); with f and G as above, and with $\gamma := 1-G$ and $\tau := F/(1-F)$, where $F := F_Q$, it is

$$\tilde{Q}_1(t)=\int_0^t \bigl[\gamma(t-u)\sqrt{f(u)}-\sqrt{\tau'(u)}z^R(u,t)\bigr]B_1(\mathrm{d}u), \tag{5.5}$$

where $z^R(u,t) := \int_u^t \gamma(t-v)f(v)\,\mathrm{d}v$ and B_1 is a standard BM. It is further stated that "From (9), we derive after some tedious but simple manipulations:"

$$\mathrm{E}\bigl(\tilde{Q}_1(s)\tilde{Q}_1(t)\bigr)=\int_0^s \gamma(s-u)\gamma(t-u)f(u)\,\mathrm{d}u - z(s)z(t) \tag{5.6}$$

(for $0\le s\le t\le T$). Turning to the second term in the decomposition of $\tilde{Q}$, that is, $\tilde{Q}_2$, according to Louchard (1988), equation (12) (and with the same notation) it can be represented

$$\tilde{Q}_2(t)=\int_0^t \sqrt{f(u)}\gamma(t-u)\int_0^t \sqrt{\rho'(v-u)}BT_0(\mathrm{d}u,\mathrm{d}v), \tag{5.7}$$

where $\rho := (1-\gamma)/\gamma$ and BT_0 is a standard two-parameter BM (Brownian sheet), independent of B_1, from which it is readily concluded (Louchard (1988), equation (13)) that

$$\mathrm{E}\bigl(\tilde{Q}_2(s)\tilde{Q}_2(t)\bigr)=\int_0^s f(u)\gamma(t-u)\bigl[1-\gamma(s-u)\bigr]\,\mathrm{d}u \tag{5.8}$$

(for $0\le s\le t\le T$). From (5.6) and (5.8), it follows (by independence) that

$$\begin{aligned}\mathrm{E}\bigl[\bigl(\tilde{Q}_1(s)+\tilde{Q}_2(s)\bigr)\bigl(\tilde{Q}_1(t)+\tilde{Q}_2(t)\bigr)\bigr]&=\int_0^s f(u)\gamma(t-u)\,\mathrm{d}u - z(s)z(t)\\&=z^L(s,t)-z(s)z(t),\end{aligned}$$

as required. While the decomposition $\tilde{Q}=\tilde{Q}_1+\tilde{Q}_2$ is useful in view of representations (5.5) and (5.7), it is evidently quite involved. Returning to the general setting of the present paper, where σ_Y^2 may be strictly positive and (4.5) is not necessarily satisfied, the above suggests that the diffusion approximation $\tilde{X}$ of S_n is not very attractive.

The general form of the covariance function of X, that is, $R(s,t)$ for $0\le s\le t\le T$, along with Examples 4.1–4.4 already indicate the simplicity, elegance and importance of this process, that is, the diffusion approximation of S_{N_n}. As demonstrated in the next sections, the special form of $R(s,t)$, as defined in (2.3), provides a fundamental merit to the process X.

6 General representations

As shown in (3.8), the process X_n (the normalization of S_{N_n}) has nonpositively correlated increments. Since X_n and X have the same covariance function, it also holds that

$$\mathrm{E}\big[\big(X(v)-X(u)\big)\big(X(t)-X(s)\big)\big]=\int_s^t\int_u^v -f(x,y)\,\mathrm{d}y\,\mathrm{d}x, \tag{6.1}$$

for any $0\le s<t\le u<v\le T$. Thus, X belongs to the class, henceforth denoted $\mathcal{NPCI}$, of centered Gaussian processes with nonpositively correlated increments. As demonstrated below, the class $\mathcal{NPCI}$ is of fundamental importance.

A particularly simple but important subclass of $\mathcal{NPCI}$ is the class of centered Gaussian processes $\xi=\{\xi(t):t\in I\}$ (where I is an arbitrary interval of the extended real line) with covariance function $R^{\xi}(\cdot,\cdot)$ of the form

$$R^{\xi}(s,t)=a(s)b(t), \tag{6.2}$$

for all $s,t\in I$ such that $s\le t$, where $a(\cdot)$ and $b(\cdot)$ are, respectively, nondecreasing and nonincreasing nonnegative continuous functions on I. For any such process ξ, it holds that

$$\mathrm{E}\big[\big(\xi(v)-\xi(u)\big)\big(\xi(t)-\xi(s)\big)\big]=\big[a(t)-a(s)\big]\big[b(v)-b(u)\big]\le 0, \tag{6.3}$$

for any $s,t,u,v\in I$ such that $s<t\le u<v$. So, indeed, $\xi\in\mathcal{NPCI}$. It may be worth noting here that, under mild regularity conditions (and with s,t,u,v as above),

$$\begin{aligned}\mathrm{E}\big[\big(\xi(v)-\xi(u)\big)\big(\xi(t)-\xi(s)\big)\big]&=\int_s^t\int_u^v a'(x)b'(y)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_s^t\int_u^v \frac{\partial^2}{\partial x\,\partial y}R^{\xi}(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Examples of diffusion approximations X with covariance function as in (6.2) appeared in Examples 4.1 and 4.4. The process $X=\sqrt{2}\mathbb{G}_H$ of Example 4.1 (where $\mathbb{G}_H$ is the H-BB on $[0,T]$) has covariance function of the form (6.2) with, say, $a(t)=2H(t)$ and $b(t)=1-H(t)$, $t\in[0,T]$ (cf. (4.2)). The factorization corresponding to the diffusion approximation of Example 4.4 was partially indicated in that example; the functions $a(\cdot)$ and $b(\cdot)$ defined below (4.11) indeed satisfy the required properties stated below (6.2). Another example (which may be somehow related to (4.13)) is provided by the stationary OU process, that is, the centered Gaussian process ξ with covariance function $R^{\xi}(s,t)=\alpha\mathrm{e}^{-\beta(t-s)}$ for $s\le t$ ($\alpha,\beta>0$ fixed); here, say, $a(t)=\alpha\mathrm{e}^{\beta t}$ and $b(t)=\mathrm{e}^{-\beta t}$.

A centered Gaussian process with covariance of the form (6.2) can be simply expressed in terms of standard BM, B. Indeed, letting

$$\xi(t)=\begin{cases}b(t)B\big(a(t)/b(t)\big), & b(t)>0,\\ 0, & b(t)=0,\end{cases} \tag{6.4}$$

it is readily verified that $\mathrm{E}(\xi(s)\xi(t)) = a(s)b(t)$ for all $s, t \in I$ such that $s \le t$. A simple implication of representation (6.4) is that the process ξ is Markovian. Further discussion is provided in the Appendix.

In general, however, processes in $\mathcal{NPCI}$ do not have covariance as in (6.2). This is easily concluded for the diffusion approximations X of this paper. Indeed, formally differentiating $R(s,t)$, for $0 < s < t < T$, gives

$$\frac{\partial^2}{\partial s\,\partial t} R(s,t) = -f(s,t). \tag{6.5}$$

Then, in view of (6.2), writing

$$-f(s,t) = a'(s)b'(t)$$

confirms the above assertion about X. [In the “exceptional” Examples 4.1 and 4.4, f takes the special form $f(x,y) = 2h(x)h(y)$ and $f(x,y) = \lambda_W f_Q(x)\mathrm{e}^{\lambda_W x}\mathrm{e}^{-\lambda_W y}$, $(x,y) \in E_T$, respectively.]

Despite the previous paragraph, the general form of $R(s,t)$ is simple enough. In particular, it corresponds to a *biconvex covariance*, as it satisfies the following defining conditions (Berman (1978), equations (1.1)–(1.4)):

- $R(s,t) \ge 0$ for all s and t in its domain.
- $R(s,t)$ is, for fixed t, nondecreasing in s for $s < t$.
- $R(s,t)$ is, for fixed s, nonincreasing in t for $t > s$.
- $R(s+h, t+k) - R(s, t+k) - R(s+h, t) + R(s,t) \le 0$ for $s < s+h \le t < t+k$.

The first three conditions are trivially satisfied; the fourth condition is satisfied by (6.1). Suppose, as in Berman (1978), Section 2 (with $a = 0$ and $b = T$), that the function $R(s,t)$ (continuous, defined for $0 \le s \le t \le T$) has the following derivatives for $0 < s < t < T$:

$$R_1 := \frac{\partial R}{\partial s}, \qquad R_2 := \frac{\partial R}{\partial t}, \qquad R_{12} := \frac{\partial^2 R}{\partial s \partial t} = \frac{\partial^2 R}{\partial t \partial s}. \tag{6.6}$$

Then, according to Berman (1978), p. 32, the preceding set of conditions on $R(s,t)$ is equivalent to the following one:

- $R(0,T) \ge 0$.
- $R_1(s,T) \ge 0$ for all s.
- $R_2(0,t) \le 0$ for all t.
- $R_{12}(s,t) \le 0$ for all $s < t$.

In the above,

$$R_1(s,T) := \lim_{t \to T} R_1(s,t), \qquad R_2(0,t) := \lim_{s \to 0} R_2(s,t). \tag{6.7}$$

For $R(s,t)$ as defined in (2.3), it is readily seen that, under mild regularity conditions,

$$
\begin{aligned}
R(0,T) &= 0,\\
R_1(s,T) &= \lim_{t\to T}\int_t^T f(s,y)\,\mathrm{d}y = 0,\\
R_2(0,t) &= \lim_{s\to 0} -\int_0^s f(x,t)\,\mathrm{d}x = 0,\\
R_{12}(s,t) &= -f(s,t) \le 0,
\end{aligned}
\tag{6.8}
$$

holds for all $s < t$, in agreement with the preceding set of conditions.

The key result relating to biconvex covariances is Theorem 2.1 of Berman (1978). Let Z be a N(0, 1) RV, B_1 and B_2 standard BMs on $[0,T]$, and W a standard two-parameter BM (Brownian sheet) on $[0,T]\times[0,T]$, that is, a centered Gaussian process $\{W(t_1,t_2): t_1,t_2 \in [0,T]\}$ with covariance given by

$$
\mathrm{E}\big[W(s_1,s_2)W(t_1,t_2)\big] = (s_1\wedge t_1)(s_2\wedge t_2). \tag{6.9}
$$

It is assumed that Z, B_1, B_2, and W are mutually independent. By Theorem 2.1 of Berman (1978), in the special case where $a = 0$ and $b = T$, and where $R(s,t)$ corresponds to a general biconvex covariance of a centered Gaussian process X, the process X admits the representation

$$
\begin{aligned}
X(t) &= Z\sqrt{R(0,T)} + \int_0^t \sqrt{R_1(u,T)}\,B_1(\mathrm{d}u)\\
&\quad + \int_t^T \sqrt{-R_2(0,v)}\,B_2(\mathrm{d}v) + \int_0^t\int_t^T \sqrt{-R_{12}(u,v)}\,W(\mathrm{d}v,\mathrm{d}u),
\end{aligned}
\tag{6.10}
$$

for $0\le t\le T$. For $R(s,t)$ as defined in (2.3), where, by (6.8), the first three terms on the right-hand side of (6.10) are all zero and $-R_{12}(s,t) = f(s,t)$, Theorem 2.1 of Berman (1978) yields as a corollary the following result, which highlights the general simplicity of the diffusion approximation X considered in this paper.

Proposition 6.1. *The diffusion approximation X admits the representation*

$$
X(t) = \int_0^t\int_t^T \sqrt{f(u,v)}\,W(\mathrm{d}v,\mathrm{d}u), \tag{6.11}
$$

for $0\le t\le T$, where W is a Brownian sheet on $[0,T]\times[0,T]$.

Actually, the proof of this proposition can be established, indirectly but instructively, following the proof of Berman (1978), Theorem 2.1, as follows. The covariance function corresponding to

$$
\int_0^t\int_t^T \sqrt{-R_{12}(u,v)}\,W(\mathrm{d}v,\mathrm{d}u)
$$

is given by

$$-\int_0^s \int_t^T R_{12}(u, v)\, \mathrm{d}v\, \mathrm{d}u,$$

or

$$R(s, t) + R(0, T) - R(s, T) - R(0, t).$$

Since the last three terms are all zero, it thus follows, using that $-R_{12}(s, t) = f(s, t)$, that the covariance function corresponding to the right-hand side of (6.11) is given by $R(s, t)$, as required.

Remark 6.1. The importance of the class of Gaussian processes with biconvex covariances is highlighted in the thorough paper Berman (1978). Particularly important there is Example 2.2 (for its relation to fractional BM with parameter $H < 1/2$), to be considered in the Appendix.

Alternatively to representation (6.11), and somewhat more instructively, the diffusion approximation X can be represented in terms of a "modified" inhomogeneous Brownian sheet, as described next. Let

$$C_T = \{(x, y) \in [0, T)^2 : y \geq x\}. \tag{6.12}$$

Define a measure ν^f on C_T by

$$\nu^f(\mathrm{d}x, \mathrm{d}y) = f(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{6.13}$$

Then, $(C_T, \mathcal{B}(C_T), \nu^f)$, where $\mathcal{B}(C_T)$ is the Borel σ-algebra on C_T, is a measure space with total measure 1 (in particular, finite). Thus, there exists a random set function $W^f : \mathcal{B}(C_T) \to \mathbb{R}$ such that, for all $A, B \in \mathcal{B}(C_T)$,

(i) $W^f(A) \sim \mathrm{N}(0, \nu^f(A))$.
(ii) If $A \cap B = \varnothing$, then $W^f(A \cup B) = W^f(A) + W^f(B)$ a.s.
(iii) If $A \cap B = \varnothing$, then $W^f(A)$ and $W^f(B)$ are independent.

Indeed, by definition (see, e.g., Adler and Taylor (2007), p. 24), W^f is a *Gaussian white noise* (GWN) on C_T based on ν^f. It is directly verified from the defining conditions that

$$\mathrm{E}\big(W^f(A) W^f(B)\big) = \nu^f(A \cap B), \tag{6.14}$$

for all $A, B \in \mathcal{B}(C_T)$. In fact, the symmetric function R_f on $\mathcal{B}(C_T) \times \mathcal{B}(C_T)$ defined by $R_f(A, B) = \nu^f(A \cap B)$ is nonnegative definite, and hence there exists a centered Gaussian process on $\mathcal{B}(C_T)$ with covariance function R_f; this process satisfies (i)–(iii) above, and so can (and will) be identified with the GWN W^f (see, e.g., Adler and Taylor (2007), Theorem 1.4.3, for the general GWN case).

Now, a two-parameter "modified" inhomogeneous Brownian sheet

$$W^f = \{W^f(t_1, t_2) : (t_1, t_2) \in C_T\} \tag{6.15}$$

can be defined by means of the GWN $\{W^f(A) : A \in \mathcal{B}(C_T)\}$ by letting

$$W^f(t_1, t_2) = W^f\big([0, t_1] \times [t_2, T)\big), \tag{6.16}$$

for $(t_1, t_2) \in C_T$. (The term "modified" refers to the "non-typical" rectangles $[0, t_1] \times [t_2, T)$.) This defines a centered Gaussian process with covariance

$$\mathrm{E}\big[W^f(s_1, s_2)W^f(t_1, t_2)\big] = \nu^f\big([0, s_1 \wedge t_1] \times [s_2 \vee t_2, T)\big), \tag{6.17}$$

for $(s_1, s_2), (t_1, t_2) \in C_T$. (Note that it can be assumed to vanish on the y-axis.) Finally, the following counterpart of Proposition 6.1 holds.

Proposition 6.2. *The diffusion approximation X can be represented in terms of the "modified" inhomogeneous Brownian sheet W^f, as follows*:

$$X(t) = \begin{cases} W^f(t, t), & t \in [0, T), \\ 0, & t = T. \end{cases} \tag{6.18}$$

Indeed, the right-hand side defines a centered Gaussian process on $[0, T]$ with covariance

$$\begin{aligned} \mathrm{E}\big[W^f(s, s)W^f(t, t)\big] &= \nu^f\big([0, s] \times [t, T)\big) \\ &= \int_0^s \int_t^T f(x, y)\,\mathrm{d}y\,\mathrm{d}x \\ &= R(s, t), \end{aligned} \tag{6.19}$$

for $0 \le s \le t < T$, and, trivially, $\mathrm{E}(X(s)X(t)) = R(s, t)(= 0)$ if $t = T$.

7 Discrete-time construction

The "white noise" representation of Proposition 6.2 gives rise to a very simple algorithm for generating an exact (up to numerical precision) discrete-time realization of X. Given times

$$0 = t_0 < t_1 < t_2 < \cdots < t_m < T,$$

the aim of this section is to construct a random vector $(X_1, \ldots, X_m)$ such that

$$(X_1, \ldots, X_m) \stackrel{\mathrm{d}}{=} \big(X(t_1), \ldots, X(t_m)\big). \tag{7.1}$$

This will be done in Algorithm 1, provided that the integrals defining $\sigma_{i,j}^2$ in (7.4) below can be evaluated exactly. [Of course, in special cases (as those of Examples 4.1 and 4.4) where X can be simply expressed in terms of standard BM, the construction is straightforward.]

Let, for any $i = 1, \ldots, m$ and $j = 1, \ldots, m + 1 - i$,

$$A_{i,j} = \begin{cases} W^f\big([t_{i-1}, t_i] \times [t_{m+1-j}, t_{m+2-j}]\big), & j > 1, \\ W^f\big([t_{i-1}, t_i] \times [t_m, T)\big), & j = 1, \end{cases} \tag{7.2}$$

where W^f is the GWN from which X is constructed in Proposition 6.2. Note that, by (6.14),

$$\mathrm{E}(A_{i,j}A_{i',j'}) = 0,$$

for any distinct pairs (i, j) and (i', j'). Being pairwise uncorrelated and jointly normal, the $A_{i,j}$ are mutually independent $\mathrm{N}(0, \sigma^2_{i,j})$ RVs, where

$$\sigma^2_{i,j} = \begin{cases} \nu^f\big([t_{i-1}, t_i] \times [t_{m+1-j}, t_{m+2-j}]\big), & j > 1, \\ \nu^f\big([t_{i-1}, t_i] \times [t_m, T)\big), & j = 1, \end{cases} \tag{7.3}$$

that is,

$$\sigma^2_{i,j} = \begin{cases} \displaystyle\int_{t_{i-1}}^{t_i} \int_{t_{m+1-j}}^{t_{m+2-j}} f(x, y)\,\mathrm{d}y\,\mathrm{d}x, & j > 1, \\ \displaystyle\int_{t_{i-1}}^{t_i} \int_{t_m}^{T} f(x, y)\,\mathrm{d}y\,\mathrm{d}x, & j = 1. \end{cases} \tag{7.4}$$

Next, let, for any $i = 1, \dots, m$ and $j = 1, \dots, m + 1 - i$,

$$S_{i,j} = W^f\big([0, t_i] \times [t_{m+1-j}, T)\big), \tag{7.5}$$

and set $S_{0,j} = S_{i,0} = S_{0,0} = 0$. Then, it holds that

$$S_{i,j} \overset{\text{a.s.}}{=} S_{i-1,j} + S_{i,j-1} - S_{i-1,j-1} + A_{i,j}. \tag{7.6}$$

With X as defined in (6.18), it thus follows from

$$S_{i,m+1-i} = W^f\big([0, t_i] \times [t_i, T)\big) = X(t_i) \tag{7.7}$$

that $(X(t_1), \dots, X(t_m))$ admits an $O(m^2)$ construction. Moreover, by (7.5), the $(S_{i,j})$ can be used for constructing the "modified" inhomogeneous Brownian sheet $W^f(\cdot, \cdot)$ at the points (t_i, t_{m+1-j}). These results are described conveniently in Algorithm 1.

Appendix

This Appendix contains some brief discussions pertinent to the overall paper.

Consider the stochastic processes a_i, defined in (2.1). By the definition of (τ^i_1, τ^i_2), it holds that

$$\mathbf{1}_{[0,t]\times[t,T)}\big(\tau^i_1, \tau^i_2\big) \equiv \mathbf{1}_{[\tau^i_1, \tau^i_2]}(t),$$

for $t \in [0, T]$. Thus, alternatively to (2.1), a_i could be defined by

$$a_i(t) = Y_i \mathbf{1}_{[0,t]\times[t,T)}\big(\tau^i_1, \tau^i_2\big). \tag{A.1}$$

While (A.1) is not identical to (2.1) (since $\mathbf{1}_{[\tau^i_1, \tau^i_2]}(t) \not\equiv \mathbf{1}_{[\tau^i_1, \tau^i_2)}(t)$), the two definitions are essentially the same (the only difference being the value of the process at

Algorithm 1: Discrete realization of X and the associated "modified" inhomogeneous Brownian sheet W^f

Data: Times $0 = t_0 < t_1 < t_2 < \cdots < t_m < T$.

Result: Arrays $X[1..m]$ and $S[0..m, 0..m]$ such that $(X[i]) \stackrel{\mathrm{d}}{=} (X(t_i))$ and $(S[i, j]) \stackrel{\mathrm{d}}{=} (W^f(t_i, t_{m+1-j}))$, $i = 1, \ldots, m$, $j = 1, \ldots, m + 1 - i$.

Initialize arrays std$[1..m, 1..m]$ and $S[0..m, 0..m]$ by setting std$[i, j]$ equal to $\sqrt{\sigma^2_{i,j}}$, $i = 1, \ldots, m$, $j = 1, \ldots, m + 1 - i$, and $S[i, 0] = S[0, j] = S[0, 0] = 0$.

```
for i = 1 to m do
    for j = 1 to m + 1 − i do
        S[i, j] = S[i − 1, j] + S[i, j − 1] − S[i − 1, j − 1] + std[i, j] ∗ randn();
        /* randn() generates standard normal variates */
    end
    X[i] = S[i, m + 1 − i];
end
```

the random time τ_2^i, i.e. $a_i(\tau_2^i) = Y_i$ or $a_i(\tau_2^i) = 0$, respectively). Accordingly, the process representing the system size at time t, that is, S_{N_n}, could be represented

$$S_{N_n}(t) = \sum_{i=1}^{N_n} Y_i \mathbf{1}_{[0,t]\times[t,T)}\big(\tau_1^i, \tau_2^i\big). \tag{A.2}$$

With this representation, the process S_{N_n} can be described as follows. Suppose that N_n points are placed independently in the set E_T, with the ith point ($i = 1, \ldots, N_n$) being placed at (τ_1^i, τ_2^i). Associate with each point (τ_1^i, τ_2^i) the size Y_i. Then the total size in the rectangle $[0, t] \times [t, T)$ is given by $S_{N_n}(t)$. Normalizing (A.2) as in the definition (2.5) of X_n, the connection to the "white noise" representation of X in Proposition 6.2 can be easily seen. To emphasize this point, note that S_{N_n}, while being a one-parameter process, can be seen as a two-parameter process along the diagonal (t, t), $t \in [0, T)$ (with $S_{N_n}(T) = 0$, by definition); indeed, with C_T as defined in (6.12), the corresponding two-parameter process is the process (cf. (6.15)–(6.16))

$$Z_{N_n} = \big\{Z_{N_n}(t_1, t_2) : (t_1, t_2) \in C_T\big\}$$

defined by

$$Z_{N_n}(t_1, t_2) = \sum_{i=1}^{N_n} Y_i \mathbf{1}_{[0,t_1]\times[t_2,T)}\big(\tau_1^i, \tau_2^i\big),$$

for $(t_1, t_2) \in C_T$. (Thus, $Z_{N_n}(t, t) = S_{N_n}(t)$ for $t \in [0, T)$.)

The usefulness of considering two-parameter processes along paths (parameterized curves) has been demonstrated in Covo (2011). In fact, Section 5.2 of Covo

(2011) has inspired the present work. Moreover, that paper has also inspired the work Covo and Elalouf (2015) (which is most pertinent to Sections 6 and 7 above) as described briefly below (and as described in Covo and Elalouf (2015) itself, where further details are given).

As in Covo (2011) (but using different notation), define a *decreasing path* in $\mathbb{R}_+^2$ as a parameterized curve $(a(t), b(t))$, $t \in I$, where I is a given interval of the real line and $a(\cdot)$ and $b(\cdot)$ are, respectively, nondecreasing and nonincreasing (continuous) functions on I, both strictly positive in the interior of I, and at least one is not identically constant. Now, let W be a standard two-parameter Brownian sheet, that is, a centered Gaussian process $\{W(t_1, t_2) : t_1, t_2 \geq 0\}$ with covariance as given in (6.9). Then, the one-parameter process $\{W(a(t), b(t)) : t \in I\}$ is referred to as the *Brownian sheet W along the path $(a(t), b(t))$, $t \in I$*. This process is centered Gaussian with covariance function given, for $s, t \in I$ with $s \leq t$, by

$$\mathrm{E}\big[W\big(a(s), b(s)\big)W\big(a(t), b(t)\big)\big] = a(s)b(t). \tag{A.3}$$

(Note that it can be assumed to have continuous paths.)

Brownian sheets along decreasing paths form an important class of centered Gaussian processes, as indicated by the second and third paragraphs of Section 6 above. (A detailed account is given in Covo (2011), Section 3.) Denote this class by $\mathcal{W}_{\mathrm{DP}}$. By Covo (2011), Theorem 4.1, processes in $\mathcal{W}_{\mathrm{DP}}$ typically have nonstationary increments. Particularly notable exceptions are the BB, which, in standard form, can be represented $\{W(t, 1-t) : t \in [0, 1]\}$, and the stationary OU process, which can be represented $\{W(\alpha_1 \mathrm{e}^{\beta t}, \alpha_2 \mathrm{e}^{-\beta t}) : t \in I\}$ (say $I = \mathbb{R}$ or $I = \mathbb{R}_+$). Being in the class $\mathcal{W}_{\mathrm{DP}}$, both processes have nonpositively correlated increments, corresponding to (6.3) (thus negatively correlated here). Further, they are not martingales (as is the case in general for processes in $\mathcal{W}_{\mathrm{DP}}$ when the component $b(\cdot)$ is not a constant function; cf. Covo (2011), Corollary 2.2). Now, fractional BM (FBM) with parameter $H < 1/2$ is also a continuous centered Gaussian process having stationary and negatively correlated increments, and is not a martingale. However, it does not belong to the class $\mathcal{W}_{\mathrm{DP}}$ (indeed, it is not Markovian).

Let $B^H = \{B^H(t) : t \geq 0\}$ denote a FBM with parameter $H \in (0, 1)$, that is, a centered Gaussian process with covariance function given, for $0 \leq s \leq t$, by

$$\mathrm{E}\big[B^H(s)B^H(t)\big] = \frac{1}{2}\big[s^{2H} + t^{2H} - (t-s)^{2H}\big]. \tag{A.4}$$

[For $H = 1/2$, B^H is a standard BM, and so has independent increments; for $H \in (1/2, 1)$, B^H has positively correlated increments. Hence, the focus will be on the case $H \in (0, 1/2)$, where B^H has negatively correlated increments.] While the covariance function of B^H cannot be factorized in the form (A.3), it is still relatively simple. Thus, in view of the previous paragraph, it is natural to check whether B^H, for $H < 1/2$, can be represented as an *inhomogeneous* Brownian sheet along a decreasing path. This is what inspired the paper Covo and Elalouf (2015), whose main results are considered briefly next.

First, observe that the term $s^{2H}/2$ on the right-hand side of (A.4) corresponds to the covariance function of the process $Y^H = \{Y^H(t) : t \geq 0\}$ defined by

$$Y^H(t) = B\big(t^{2H}/2\big), \tag{A.5}$$

where B is a standard BM. [Thus, Y^H is a (nonlinearly) scaled BM.] It is shown in Covo and Elalouf (2015) that the complementary term in (A.4), that is, $[t^{2H} - (t-s)^{2H}]/2$, corresponds to a covariance function (of a Gaussian process) for $H < 1/2$ but not for $H > 1/2$. Hence, the following decomposition in law holds true in the $H < 1/2$ case:

$$B^H \stackrel{\mathrm{d}}{=} Y^H + Z^H, \tag{A.6}$$

where $Z^H = \{Z^H(t) : t \geq 0\}$ is a centered Gaussian process with covariance function given, for $0 \leq s \leq t$, by

$$\mathrm{E}\big[Z^H(s)Z^H(t)\big] = \frac{1}{2}\big[t^{2H} - (t-s)^{2H}\big]. \tag{A.7}$$

[In (A.6), $\stackrel{\mathrm{d}}{=}$ means equality in FDD.] Clearly, the process Z^H can be considered as the fundamental part of FBM B^H.

Remark A.1. Despite its notable simplicity, decomposition (A.6) appears to be unrecognized in the literature. This point is highlighted in Covo and Elalouf (2015) by comparison with known decompositions of B^H (in particular the stochastic integral decomposition by which B^H is commonly defined). However, see Remark A.2 below.

Now, let $(t, \varphi(t))$, $t \geq 0$, be a decreasing path (consider e.g., $\varphi(t) = \mathrm{e}^{-t}$). Under mild assumptions on φ, it is shown in Covo and Elalouf (2015) that Z^H can be represented as

$$Z^H(t) = W^{\nu_\varphi^H}\big([0, t] \times [0, \varphi(t)]\big), \tag{A.8}$$

with $W^{\nu_\varphi^H}$ a GWN based on a specified measure ν_φ^H depending on H and φ. While this representation has the advantage of working with "typical" rectangles (i.e., of the form $[0, t_1] \times [0, t_2]$), the general form of ν_φ^H is rather cumbersome and not instructive. [For certain choices of φ, e.g., $\varphi(t) = \mathrm{e}^{-t}$, ν_φ^H has a relatively simple expression.] In any case, as demonstrated in Covo and Elalouf (2015), the underlying function φ has no intrinsic meaning; in particular, Algorithm 1 of Covo and Elalouf (2015) for generating discrete realization of Z^H *based on representation* (A.8) is invariant to the choice of φ.

The study related to representation (A.8) has led Covo and Elalouf (2015) to the counterpart representation

$$Z^H(t) = W^{\nu^H}\big([0, t] \times [t, \infty)\big), \tag{A.9}$$

with W^{ν^H} a GWN based on the measure ν^H concentrated on $\{(x, y) : y > x > 0\}$ and given there by

$$\nu^H(\mathrm{d}x, \mathrm{d}y) = \alpha_H (y - x)^{2H-2} \,\mathrm{d}x \,\mathrm{d}y, \tag{A.10}$$

where

$$\alpha_H = H(1 - 2H).$$

While this representation has the drawback of working with the "non-typical" rectangles $[0, t] \times [t, \infty)$, it is overall very simple and instructive. It leads to the same algorithm for generating discrete realization of Z^H as its counterpart (A.8).

Thus, with (A.9), Z^H admits the same type of "white noise" representation as does the diffusion approximation X in Proposition 6.2. There is, however, one essential difference between the two cases. Whereas the underlying measure ν^f in (6.13) is finite—with total measure 1, being simply the distribution of (τ_1^i, τ_2^i)—its counterpart ν^H in (A.10) is infinite on any neighborhood of each point of the diagonal (x, x), $x \geq 0$.

Not surprisingly in view of the first sentence of the previous paragraph, Algorithm 1 of the present paper is, up to the definition of the variance coefficients $\sigma_{i,j}^2$, the same as Algorithm 1 of Covo and Elalouf (2015). Moreover, the $\sigma_{i,j}^2$ as defined (explicitly) in Covo and Elalouf (2015) can be expressed implicitly (analogously to (7.3) above) as

$$\sigma_{i,j}^2 = \begin{cases} \nu^H\big([t_{i-1}, t_i] \times [t_{m+1-j}, t_{m+2-j}]\big), & j > 1, \\ \nu^H\big([t_{i-1}, t_i] \times [t_m, \infty)\big), & j = 1. \end{cases}$$

Thus, those algorithms are the same up to the underlying measures, ν^H and ν^f.

A key point indicated briefly in the Addendum of Covo and Elalouf (2015) is considered next. Denote by $\tilde{R}$ the covariance function of B^H. Suppose, as above, that $H < 1/2$. Using the partial derivatives notation of Section 6 ((6.6), (6.7)), it holds that

$$\begin{aligned} \tilde{R}(0, \infty) &:= \lim_{t \to \infty} \tilde{R}(0, t) = 0, \\ \tilde{R}_1(s, \infty) &= \lim_{t \to \infty} \big[H s^{2H-1} + H(t - s)^{2H-1}\big] = H s^{2H-1} \geq 0, \\ \tilde{R}_2(0, t) &= \lim_{s \to 0} H\big[t^{2H-1} - (t - s)^{2H-1}\big] = 0, \\ \tilde{R}_{12}(s, t) &= -\alpha_H (t - s)^{2H-2} \leq 0, \end{aligned} \tag{A.11}$$

for all $s < t$, thus showing that $\tilde{R}$ is biconvex. Then, by (6.10) (i.e., Theorem 2.1 of Berman (1978)), B^H admits the representation

$$\begin{aligned} B^H(t) &= \int_0^t \sqrt{\tilde{R}_1(u, \infty)} B(\mathrm{d}u) + \int_0^t \int_t^\infty \sqrt{-\tilde{R}_{12}(u, v)} W(\mathrm{d}v, \mathrm{d}u) \\ &= \int_0^t \sqrt{H u^{2H-1}} B(\mathrm{d}u) + \int_0^t \int_t^\infty \sqrt{\alpha_H (v - u)^{2H-2}} W(\mathrm{d}v, \mathrm{d}u), \end{aligned} \tag{A.12}$$

where W is a standard Brownian sheet independent of standard BM B. Note that decomposition (A.12) of B^H corresponds to the decomposition $B^H = Y^H + Z^H$, with Y^H the scaled BM defined in (A.5) and Z^H as represented in (A.9). Decomposition (A.12), it should be stressed, has not been noted explicitly in Covo and Elalouf (2015).

Remark A.2. Actually, the fact that the covariance function of FBM B^H, for $H < 1/2$, is biconvex implicitly appears in Example 2.2 of Berman (1978). Indeed, for this process, all the required assumptions in that example are satisfied, as follows. First, it is centered Gaussian with stationary increments, and $B^H(0) = 0$ almost surely. Second, define $g(t) = \mathrm{E}[(B^H(t))^2]$, so that, with $\tilde{R}$ as above,

$$\tilde{R}(s,t) = \frac{1}{2}\bigl[g(t) + g(s) - g(t-s)\bigr].$$

Finally, since $g(t) = t^{2H}$, g is concave and nondecreasing. Then, as shown in Berman (1978), Example 2.2, it follows that $\tilde{R}$ is biconvex.

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Department of Management
Bar Ilan University
Ramat-Gan
Israel
E-mail: green355@netvision.net.il
amir.elalouf@biu.ac.il