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ON APPROXIMATE CONTINUITY AND THE SUPPORT OF REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS[1]

BY JIAGANG REN AND JING WU

Sun Yat-sen University

In this paper we prove an approximate continuity result for stochastic differential equations with normal reflections in domains satisfying Saisho's conditions, which together with the Wong–Zakai approximation result completes the support theorem for such diffusions in the uniform convergence topology. Also by adapting Millet and Sanz-Solé's idea, we characterize in Hölder norm the support of diffusions reflected in domains satisfying the Lions–Sznitman conditions by proving limit theorems of adapted interpolations. Finally we apply the support theorem to establish a boundary-interior maximum principle for subharmonic functions.

1. Introduction. The support theorem for diffusion processes defined by stochastic differential equations has been a much studied topic for probabilists and analysts since the seminal work of Stroock and Varadhan [13]. The typical approach to a support theorem in the norm of uniform convergence consists of two steps. One step is to establish a limit theorem for SDEs, meaning that the solution of an SDE can be approximated by a sequence of solutions of ODEs, obtained by regularizing the Brownian paths [15]; the other is to prove a Denjoy-type approximate continuity theorem, stating that the solution of an SDE is approximately continuous at points in a dense set of the Cameron–Martin space. Millet and Sanz-Solé [7, 8] proposed a simple approach to characterizing in Hölder spaces the support of diffusions described by general SDEs, obtained by approximating Brownian motions with linear adapted interpolations, and proved the two inclusions through approximation results.

In this work we are concerned with the support problem of diffusions constrained in a domain D with normal reflection boundary. Such diffusions have been constructed by Anderson and Orey [2] if D has smooth boundary and by Tanaka [14] if D is convex. Correspondingly the support theorem has been established by Doss and Priouret [3] if D has smooth boundary, and a limit theorem has been proved by Pettersson [9] when D is a convex domain and the diffusion coefficient is constant. Recently in [10], a support theorem was proved for stochastic

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variational inequalities; this means, in particular, that the support theorem holds true for diffusions normally reflected in convex domains.

However, normally reflected diffusions have been constructed for domains much wider than convex domains and smooth domains (see Lions and Sznitman [6] and Saisho [12]), so a natural (and application-motivated) question is whether or not the support theorem continues to hold true for such diffusions. The first step in this respect was taken by Evans and Stroock [4] who proved, under the set of conditions given by Lions and Sznitman, that a weak limit theorem holds. Very recently this result was improved by Aida and Sasaki [1], and independently by Zhang [16], who used an adapted version of the Wong–Zakai approximations rather than the usual one, by removing the admissibility condition from the set of conditions and proving that the convergence takes place, in fact, in L^p (and they obtained the convergence speed). Roughly speaking, they proved a strong limit theorem for the reflected diffusions studied by Saisho in [12]. To date, this was the widest, well-studied situation.

On the other hand, however, approximate continuity has not yet been touched in such situations. Our first result fills this gap, and it, together with the Wong–Zakai convergence result in [1] and [16], will yield the support theorem in the locally uniform convergence topology for normally reflected SDEs in domains, satisfying the conditions of Lions and Sznitman [6], except the admissibility. The second contribution of this paper is to present a characterization of the support for reflected diffusions in Hölder spaces in domains satisfying the conditions in [4], by extending the idea of Millet and Sanz-Solé [8] to SDEs with normal reflections.

We recall the Skorohod problem here. Let D be a domain in $\mathbb{R}^d$ and $w_\cdot \in \mathcal{C}([0,+\infty);\mathbb{R}^d)$ such that $w_0 \in \bar{D}$. A pair of continuous functions (x,k) is a solution of the Skorohod problem if:

- $x_t \in \bar{D}$ for all $t \geq 0$ and $x_0 = w_0$;
- for all $t \geq 0$, $x_t = w_t + k_t$;
- $k(0) = 0$, and k is of bounded variation on each finite interval and satisfies

$$k_t = \int_0^t n_s \, \mathrm{d}|k|_s, \qquad |k|_t = \int_0^t 1_{\partial D}(x_s) \, \mathrm{d}|k|_s,$$

  where $n_s \in \mathcal{N}_{x_s}$ and $\mathcal{N}_x$ is the set of inward normal unit vectors at $x \in \partial D$ defined by

$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r},$$

$$\mathcal{N}_{x,r} = \{n \in \mathbb{R}^d; |n| = 1, B(x - rn, r) \cap D = \varnothing\}.$$

Here and in what follows $B(a,r) = \{y \in \mathbb{R}^d; |y-a| < r\}$, $a \in \mathbb{R}^d$, $r > 0$ and $|k|_t$ denotes the total variation of k on $[0,t]$.

Let $\Omega = \mathcal{C}_0([0,\infty),\mathbb{R}^{d_1})$ be the space consisting of continuous functions from $[0,\infty)$ to $\mathbb{R}^{d_1}$ vanishing at 0. Let $\mathcal{F}$ be the completion of the Borel σ-algebra on

Ω associated with the locally uniform convergence topology and $\mathbf{P}$ the distribution of an d_1-dimensional Brownian motion. Then $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space, and the coordinate process

$$w_t(\omega) := \omega(t), \qquad t \geq 0$$

is a d_1-dimensional standard Brownian motion. The natural filtration generated by $(w_t)_{t\geq 0}$ is denoted by $(\mathscr{F}_t)_{t\geq 0}$.

We consider the following reflected SDE:

$$\begin{cases} X_t = X_0 + \int_0^t \sigma(X_s) \circ \mathrm{d}w_s + \int_0^t b(X_s)\,\mathrm{d}s + K_t, & X_0 = x \in \bar{D}, \\ |K|_t = \int_0^t 1_{\partial D}(X_s)\,\mathrm{d}|K|_s, & K_t = \int_0^t \xi_s\,\mathrm{d}|K|_s, \end{cases} \tag{1.1}$$

where $\xi_s \in \mathcal{N}_{X_s}$. In Itô's notation, it takes the following form:

$$\begin{cases} X_t = X_0 + \int_0^t \sigma(X_s)\,\mathrm{d}w_s + \int_0^t \tilde{b}(X_s)\,\mathrm{d}s + K_t, & X_0 = x \in \bar{D}, \\ |K|_t = \int_0^t 1_{\partial D}(X_s)\,\mathrm{d}|K|_s, & K_t = \int_0^t \xi_s\,\mathrm{d}|K|_s \end{cases}$$

with

$$\tilde{b}^i(x) := b^i(x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d_1}\big[\sigma_k^i(x)\big]_j \sigma_k^j(x).$$

Throughout the paper we will assume that $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^{d_1}$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ are $\mathcal{C}_b^2$ and $\mathcal{C}_b^1$ functions, respectively. Then by Saisho [12] this equation has a unique solution (X, K).

Let $\mathbb{W}^d$ (resp., $\mathbb{W}^{d_1}$) denote the space of all $\mathbb{R}^d$ (resp., $\mathbb{R}^{d_1}$)-valued continuous functions defined on $[0, \infty)$, and for each $\alpha \in (0, \frac{1}{2})$, $\mathbb{W}_\alpha^d$ denote the subspace of $\mathbb{W}^d$ consisting of locally α-Hölder continuous functions. Then for every $\alpha \in [0, \frac{1}{2})$, $\mathbb{W}_\alpha^d$ is a Fréchet space with the topology defined by the system of seminorms $\{\|\cdot\|_{T,\alpha}, T > 0\}$, where for $x \in \mathbb{W}^d$,

$$\|x\|_T := \sup_{0\leq t\leq T} |x_t|, \qquad \|x\|_{T,\alpha} := \|x\|_T + \sup_{0\leq s,t\leq T, s\neq t} \frac{|x(t) - x(s)|}{|t-s|^\alpha}.$$

Denote

$$\begin{aligned} \mathcal{H} := \big\{h : h \in \mathbb{W}^{d_1}; h(0) = 0, h(\cdot) &\text{ is absolutely continuous and} \\ &\dot{h} \in \mathbb{L}^2([0,\infty); \mathbb{R}^{d_1}), \forall T > 0\big\}, \\ \mathcal{S} := \big\{h \in \mathbb{W}^{d_1}; h(0) = 0, t \to h(t) &\text{ is smooth}\big\}, \\ \mathcal{S}_p := \big\{h \in \mathbb{W}^{d_1}; h(0) = 0, t \to h(t) &\text{ is piecewise smooth}\big\}. \end{aligned}$$

$\mathcal{H}$ will be endowed with the topology given by the family of seminorms $\{\|h\|_{\mathcal{H}_T} := (\int_0^T |\dot{h}_t|^2\,\mathrm{d}t)^{1/2}, T > 0\}$. Given $h \in \mathcal{H}$, denote by $(Z(h), \psi(h))$ the solution to the following deterministic Skorohod problem:

$$Z_t = x + \int_0^t \sigma(Z_s)\dot{h}_s\,\mathrm{d}s + \int_0^t b(Z_s)\,\mathrm{d}s + \psi_t. \tag{1.2}$$

Let

$$\mathscr{S}(\mathcal{H}) := \{Z(h), h \in \mathcal{H}\}; \qquad \mathscr{S} := \{Z(h), h \in \mathcal{S}\}; \qquad \mathscr{S}_p := \{Z(h), h \in \mathcal{S}_p\}.$$

Denote by $\overline{\mathscr{S}(\mathcal{H})}^{\alpha}$ the closure of $\mathscr{S}(\mathcal{H})$ in $\mathbb{W}^d_\alpha$, and $\overline{\mathscr{S}}$, $\overline{\mathscr{S}_p}$ and $\overline{\mathscr{S}(\mathcal{H})}$ the closures of $\mathscr{S}$, $\mathscr{S}_p$ and $\mathscr{S}(\mathcal{H})$ in $\mathbb{W}^d$, respectively. We are going to prove in Section 2 the approximate continuity theorem, which together with the result in [1] and [16] yields that the support of $\mathbf{P} \circ X^{-1}$ in $\mathbb{W}^d$ coincides with $\overline{\mathscr{S}}$. We also prove in Section 3 an enhanced version of the support theorem by showing that for every $\alpha \in (0, \frac{1}{2})$, the support of $\mathbf{P} \circ X^{-1}$ in $\mathbb{W}^d_\alpha$ coincides with $\overline{\mathscr{S}(\mathcal{H})}^{\alpha}$.

The paper is organized as follows: in Section 2 an approximate continuity theorem for normally reflected diffusions is proved, and this result combined with the main result in [1] and [16] implies, of course, the support theorem for such diffusions. Next, we provide in Section 3 an alternate approach to solving the support problem in Hölder spaces. Finally in Section 4, we give a first application of our support theorem to maximum principle for L-subharmonic functions in domains having nonsmooth boundaries and with possibly degenerate L.

Throughout the paper we use C to denote a generic constant which may be different in different places, and we use summation convention for repeated indices. Finally $A \lesssim B$ means that there exists a $C \geq 0$ such that $A \leq CB$.

2. Approximate continuity.

In this section we will work in the setup of [6]. But, as in [12], we will not need the admissibility condition on the domain. Precisely, we assume that we are given a domain $D \subset \mathbb{R}^d$ satisfying:

(H$_1$) There exists $c_0 > 0$ such that for any $x \in \partial D$, $y \in \bar{D}$ and $\xi \in \mathcal{N}_x$,

$$(y - x, \xi) + c_0|x - y|^2 \geq 0,$$

where $\mathcal{N}_x$ denotes the set of unit inward normals at x;

(H$_2$) There exist a function $\varphi \in \mathcal{C}_b^3(\mathbb{R}^d; \mathbb{R})$ and a constant $\alpha > 0$ such that

$$D\varphi(x) \cdot \xi \geq \alpha c_0 \qquad \forall x \in \partial D, \xi \in \mathcal{N}_x.$$

It is obvious that under the conditions(H$_1$)–(H$_2$), $\overline{\mathscr{S}} = \overline{\mathscr{S}_p} = \overline{\mathscr{S}(\mathcal{H})}$. To see this, we only need to show $\overline{\mathscr{S}} \supset \overline{\mathscr{S}(\mathcal{H})}$. In fact, for any $h \in \mathcal{H}$, we can take a sequence $h^n \in \mathcal{S}$ such that $h^n \to h$ in $\mathcal{H}$. Denote by (Z, Ψ) and (Z^n, Ψ^n) the corresponding solutions of the Skorohod problem (1.2). Set $\rho(t) :=$

$e^{-(2/\alpha)(\varphi(Z_t)+\varphi(Z_t^n))}$. Then for any $t \geq 0$, by (H_2) and the assumptions $b \in \mathcal{C}_b^1$ and $\sigma \in \mathcal{C}_b^2$, we have

$$\begin{aligned}&\big|Z_t^n - Z_t\big|^2 e^{-(2/\alpha)(\varphi(Z_t)+\varphi(Z_t^n))}\\&\qquad \leq C\int_0^t \rho(s)\big|Z_s^n - Z_s\big|^2\big(1 + |\dot{h}_s^n| + |\dot{h}_s|\big)\,\mathrm{d}s + C\int_0^t \big|\dot{h}_s^n - \dot{h}_s\big|^2\,\mathrm{d}s,\end{aligned}$$

which implies by Gronwall's lemma that $\sup_{0\leq t\leq T}|Z_t^n - Z_t|^2 \to 0$ as $n \to \infty$ and thus $Z \in \overline{\mathscr{S}}$, yielding that $\overline{\mathscr{S}} \supset \overline{\mathscr{S}(\mathcal{H})}$.

Before we proceed, a few words about these conditions are in order. The constant c_0 appearing in condition (H_1) is also allowed to equal to zero in [6]. Then the function φ in condition (H_2) can be taken to be identically zero, and it turns out that some arguments below will break down, and different treatments will be needed. But in this case D is a convex domain, and thus the equation is a special case of stochastic variational inequalities already treated in [10]. Hence we simply assume $c_0 > 0$ here.

For convenience we record here some basic facts which will be used below; see [5]. Set for $i, j = 1, \ldots, d_1$,

$$\kappa^{ij}(t) := \frac{1}{2}\int_0^t \big[w_s^i\,\mathrm{d}w_s^j - w_s^j\,\mathrm{d}w_s^i\big], \qquad \zeta^{ij}(t) := \int_0^t w_s^i \circ \mathrm{d}w_s^j.$$

Let $T > 0$ be arbitrarily fixed.

LEMMA 2.1. (i) *There exist two positive constants c_1 and c_2 such that*

$$\mathbf{P}\big(\|w\|_T < \delta\big) \sim c_1 \exp\left(-\frac{c_2}{\delta^2}\right) \qquad \textit{as } \delta \downarrow 0.$$

(ii) *For all $i, j = 1, \ldots, d_1$,*

$$\lim_{M\uparrow\infty}\sup_{0<\delta\leq 1}\mathbf{P}\big(\|\kappa^{ij}\|_T > M\delta \,|\, \|w\|_T < \delta\big) = 0.$$

(iii) *For all $i, j = 1, \ldots, d_1$, we have*

$$\lim_{M\uparrow\infty}\sup_{0<\delta\leq 1}\mathbf{P}\big(\|\zeta^{ij}\|_T > M\delta \,|\, \|w\|_T < \delta\big) = 0.$$

In particular, we deduce from this lemma that for every $\varepsilon > 0$ and $\alpha \in (0, 1)$,

$$\mathbf{P}\big(\|\zeta^{ij}\|_T > \varepsilon\delta^\alpha \,|\, \|w\|_T < \delta\big) \to 0 \qquad \text{as } \delta \downarrow 0. \tag{2.1}$$

In fact, for arbitrary $M > 0$, take $\delta_0 > 0$ such that $\varepsilon\delta_0^{\alpha-1} \geq M$. Then for any $0 < \delta < \delta_0$,

$$\mathbf{P}\big(\|\zeta^{ij}\|_T > \varepsilon\delta^\alpha \,|\, \|w\|_T < \delta\big) \leq \mathbf{P}\big(\|\zeta^{ij}\|_T > M\delta \,|\, \|w\|_T < \delta\big).$$

Thus

$$\limsup_{\delta\downarrow 0} \mathbf{P}\big(\|\zeta^{ij}\|_T > \varepsilon\delta^\alpha \,|\, \|w\|_T < \delta\big)$$
$$\le \sup_{0<\delta<1} \mathbf{P}\big(\|\zeta^{ij}\|_T > M\delta \,|\, \|w\|_T < \delta\big).$$

By letting $M \uparrow \infty$ we arrive at (2.1) according to (iii) in the above lemma. In the same way, we can also obtain

$$\mathbf{P}\big(\|\kappa^{ij}\|_T > \varepsilon\delta^\alpha \,|\, \|w\|_T < \delta\big) \to 0 \qquad \text{as } \delta \downarrow 0. \tag{2.2}$$

We have the following exponential integrability result.

PROPOSITION 2.1. *There exists $\beta > 0$ such that*

$$\mathbf{E}\big[e^{\beta(|K|_T)^2}\big] < \infty, \qquad \mathbf{E}\big[e^{\beta\|X\|_T^2}\big] < \infty.$$

PROOF. By Itô's formula and (H_2) we have

$$\begin{aligned}\alpha c_0 |K|_t \le \varphi(X_t) - \varphi(X_0) - \int_0^t (D\varphi)(X_s)\sigma(X_s)\,\mathrm{d}w_s - \int_0^t D\varphi(X_s)\tilde{b}(X_s)\,\mathrm{d}s \\ - \frac{1}{2}\int_0^t \mathrm{tr}\big[D^2\varphi(X_s)\big(\sigma\sigma^*\big)(X_s)\big]\,\mathrm{d}s.\end{aligned} \tag{2.3}$$

Since $\varphi \in \mathcal{C}_b^2$, there exists a $\beta' > 0$ such that

$$\mathbf{E}\bigg[\exp\bigg\{\beta'\bigg\|\int_0^\cdot (D\varphi)(X_s)\sigma(X_s)\,\mathrm{d}w_s\bigg\|_T^2\bigg\}\bigg] < \infty.$$

From this the first inequality follows immediately, and the second follows from the first together with equation (1.1). □

LEMMA 2.2. $\lim_{\delta\downarrow 0} \mathbf{P}(|K|_T \ge \varepsilon\delta^{-1/2} \,|\, \|w\|_T < \delta) = 0.$

PROOF. We have by Lemma 2.1 and Proposition 2.1 that

$$\lim_{\delta\downarrow 0} \mathbf{P}\big(|K|_T \ge \varepsilon\delta^{-3/2} \,|\, \|w\|_T < \delta\big) \lesssim \lim_{\delta\downarrow 0} \frac{\exp\{-\varepsilon^2\delta^{-3}\beta\}}{\exp\{-C\delta^{-2}\}} = 0. \tag{2.4}$$

Next we prove that for $f \in \mathcal{C}_b^2(\mathbb{R}^d;\mathbb{R})$ and $1 \le k \le d_1$ we have

$$\lim_{\delta\downarrow 0} \mathbf{P}\bigg(\bigg\|\int_0^\cdot f(X_s)\circ \mathrm{d}w_s^k\bigg\|_T \ge \varepsilon\delta^{-1/2}\bigg|\|w\|_T < \delta\bigg) = 0. \tag{2.5}$$

Set $f_i(x) := \frac{\partial f}{\partial x_i}(x)$. By Itô's formula we have

$$\begin{aligned}\int_0^t f(X_s)\circ \mathrm{d}w_s^k &= f(X_t)w_t^k - \int_0^t [f_i\sigma_j^i](X_s)w_s^k \circ \mathrm{d}w_s^j \\ &\quad - \int_0^t [f_i b^i](X_s)w^k \,\mathrm{d}s - \int_0^t f_i(X_s)w_s^k \,\mathrm{d}K_s^i \\ &=: I_1(t) - I_2(t) - I_3(t) + I_4(t).\end{aligned}$$

We need to prove

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_i\|_T \ge \varepsilon\delta^{-1/2} | \|w\|_T < \delta\big) = 0, \qquad i = 1,2,3,4.$$

This is obvious for I_1 and I_3. To show this for I_2 we notice that

$$\begin{aligned}I_2(t) &= \int_0^t [f_i\sigma_j^i](X_s)w_s^k \,\mathrm{d}w_s^j + \frac{1}{2}\int_0^t [f_i\sigma_j^i](X_s)\delta^{kj}\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t [f_i\sigma_j^i]_q \sigma_l^q(X_s)w_s^k\delta^{lj}\,\mathrm{d}s \\ &:= I_{21}(t) + I_{22}(t) + I_{23}(t).\end{aligned}$$

Noticing that f and σ are bounded, the sets $\{\|I_{2i}\|_T > \varepsilon\delta^{-1/2}\} \cap \{\|w\|_T < \delta\}$, $i = 2, 3$ will be empty for small δ and thus

$$\lim_{\delta\downarrow 0}\big\{\mathbf{P}\big(\|I_{22}\|_T \ge \varepsilon\delta^{-1/2} | \|w\|_T < \delta\big) + \mathbf{P}\big(\|I_{23}\|_T \ge \varepsilon\delta^{-1/2} | \|w\|_T < \delta\big)\big\} = 0.$$

Since for $t \in [0, T]$,

$$\langle I_{21}, I_{21}\rangle(t) = \sum_{j=1}^{d_1}\int_0^t [f_i\sigma_j^i]^2(X_s)\big(w_s^k\big)^2\,\mathrm{d}s \lesssim \|w\|_t^2.$$

By the exponential inequality (cf. [11], Exercise IV.3.16) we have

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_{21}\|_T \ge \varepsilon\delta^{-1/2} | \|w\|_T < \delta\big) \lesssim \lim_{\delta\downarrow 0}\frac{\exp\{-\varepsilon^2\delta^{-3}\}}{\exp\{-C\delta^{-2}\}} = 0.$$

Hence

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_2\|_T \ge \varepsilon\delta^{-1/2} | \|w\|_T \le \delta\big) = 0.$$

Finally, since

$$\|I_4\|_T \lesssim \|w\|_T |K|_T,$$

we have by using (2.4) that

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_4\|_T \ge \varepsilon\delta^{-1/2} | \|w\|_T < \delta\big) = 0.$$

Thus (2.5) has been proved. Now the result follows from (2.3) and (2.5). $\square$

COROLLARY 2.1. *For every* $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0} \mathbf{P}\big(\|\zeta^{ij}\|_T |K|_T > \varepsilon \big| \|w\|_T < \delta\big) = 0, \tag{2.6}$$

$$\lim_{\delta\downarrow 0} \mathbf{P}\left(\left\|\int_0^\cdot \zeta^{ij}(s)\,\mathrm{d}K_s\right\|_T > \varepsilon \middle| \|w\|_T < \delta\right) \to 0. \tag{2.7}$$

PROOF. It suffices to prove (2.6). Using (2.1) with $\alpha = \frac{1}{2}$ and the above lemma we have

$$\begin{aligned}
&\mathbf{P}\big(\|\zeta^{ij}\|_T |K|_T > \varepsilon \big| \|w\|_T < \delta\big) \\
&\quad \le \mathbf{P}\big(\|\zeta^{ij}\|_T > \delta^{1/2} \big| \|w\|_T < \delta\big) + \mathbf{P}\big(|K|_T > \varepsilon\delta^{-1/2} \big| \|w\|_T < \delta\big) \\
&\quad \to 0, \qquad \delta \downarrow 0.
\end{aligned}$$

□

Now we can prove the following:

LEMMA 2.3. *Suppose* $f \in \mathcal{C}_b(\mathbb{R}^d; \mathbb{R})$ *is uniformly continuous. Then for all* $\varepsilon > 0$ *and* $i, j = 1, 2, \dots, d_1$,

$$\lim_{\delta\downarrow 0} \mathbf{P}\left(\left\|\int_0^\cdot f(X_s)\,\mathrm{d}\zeta^{ij}(s)\right\|_T > \varepsilon \middle| \|w\|_T < \delta\right) \to 0. \tag{2.8}$$

PROOF. First we assume that $f \in \mathcal{C}_b^2(\mathbb{R}^d; \mathbb{R})$. Itô's formula gives us

$$\begin{aligned}
\int_0^t f(X_s)\,\mathrm{d}\zeta^{ij}(s) &= f(X_t)\zeta^{ij}(t) - \int_0^t \zeta^{ij}(s) f_l(X_s)\sigma_k^l(X_s)\,\mathrm{d}w_s^k \\
&\quad - \int_0^t (Lf)(X_s)\zeta^{ij}(s)\,\mathrm{d}s - \int_0^t f_l(X_s)\sigma_j^l(X_s)w_s^i\,\mathrm{d}s \\
&\quad - \int_0^t f_l(X_s)\zeta^{ij}(s)\,\mathrm{d}K_s^l \\
&=: \sum_{q=1}^5 I_{2q},
\end{aligned}$$

where $L := \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j + \sum_i \tilde{b}_i\partial_i$.

It is easy to see that for $q = 1, 3, 4$,

$$\lim_{\delta\downarrow 0} \mathbf{P}\big(\|I_{2q}\|_T > \varepsilon \big| \|w\|_T < \delta\big) = 0. \tag{2.9}$$

Since

$$\left\|\int_0^\cdot f_l(X_s)\zeta_s^{ij}\,\mathrm{d}K_s^l\right\|_T \lesssim \|\zeta\|_T |K|_T,$$

we have

$$\mathbf{P}\big(\|I_{25}\|_T > \varepsilon \,|\, \|w\|_T < \delta\big) \leq P\big(\|\zeta\|_T |K|_T > \varepsilon \,|\, \|w\|_T < \delta\big).$$

Consequently by (2.6),

$$\lim_{\delta \downarrow 0} \mathbf{P}\big(\|I_{25}\|_T > \varepsilon \,|\, \|w\|_T < \delta\big) = 0.$$

Now we deal with I_{22}. Set $g_k(x) := -f_l(x)\sigma_k^l(x)$, $g_{k,l} := \frac{\partial}{\partial x^l} g_k(x)$. We have by Itô's formula,

$$\begin{aligned} I_{22} &= \int_0^t g_k(X_s)\zeta^{ij}(s)\,\mathrm{d}w_s^k \\ &= g_k(X_t)\zeta^{ij}(t)w_t^k - \int_0^t g_{k,l}(X_s)\sigma_q^l(X_s)\zeta^{ij}(s)w_s^k\,\mathrm{d}w_s^q \\ &\quad - \int_0^t (Lg_k)(X_s)\zeta^{ij}(s)w_s^k\,\mathrm{d}s - \int_0^t g_k(X_s)w_s^k\,\mathrm{d}\zeta^{ij}(s) \\ &\quad - \int_0^t g_j(X_s)w_s^i\,\mathrm{d}s - \int_0^t \zeta^{ij}(s)g_{k,l}(X_s)\sigma_q^l(X_s)\delta^{kq}\,\mathrm{d}s \\ &\quad - \int_0^t g_{k,l}(X_s)\sigma_j^l(X_s)w^k(s)w_s^i\,\mathrm{d}s - \int_0^t g_{k,l}(X_s)\zeta^{ij}(s)w_s^k\,\mathrm{d}K_s^l \\ &:= \sum_{i=1}^8 I_{22i}. \end{aligned}$$

Obviously

$$\lim_{\delta \downarrow 0} \mathbf{P}\big(\|I_{22i}\|_T > \varepsilon \,|\, \|w\| < \delta\big) = 0, \qquad i = 1, 3, 4, 5, 7,$$

and it is clear from Corollary 2.1 that it holds also for $i = 8$. For I_{222} we notice

$$I_{222}(t) = M_t,$$

where

$$M_t = \int_0^t g_{k,l}(X_s)\sigma_q^l(X_s)\zeta^{ij}(s)w_s^k\,\mathrm{d}w_s^q.$$

It suffices to prove

$$\lim_{\delta \downarrow 0} \mathbf{P}\big(\|M\|_T > \varepsilon \,|\, \|w\|_T < \delta\big) = 0. \tag{2.10}$$

Since

$$\langle M\rangle(t) \lesssim \int_0^t \|\zeta^{ij}\|_s^2 \|w\|_s^2\,\mathrm{d}s,$$

we have by exponential inequality

$$\begin{aligned}&\mathbf{P}\big(\|M\|_T > \varepsilon, \|\zeta^{ij}\|_T < A\delta, \|w\|_T < \delta\big)\\ &\qquad \le \mathbf{P}\big(\|M\|_T > \varepsilon, \langle M\rangle(T) \le cA^2\delta^4\big) \le c\exp\{-cA^{-2}\delta^{-4}\} \to 0, \qquad \delta \downarrow 0.\end{aligned}$$

Since

$$\begin{aligned}\mathbf{P}\big(\|M\|_T > \varepsilon | \|w\|_T < \delta\big) &= \mathbf{P}\big(\|M\|_T > \varepsilon, \|\zeta^{ij}\|_T > A\delta | \|w\|_T < \delta\big)\\ &\qquad + \mathbf{P}\big(\|M\|_T > \varepsilon, \|\zeta^{ij}\|_T \le A\delta | \|w\|_T < \delta\big)\\ &\le \sup_{0\le\delta\le 1} \mathbf{P}\big(\|M\|_T > \varepsilon, \|\zeta^{ij}\|_T > A\delta | \|w\|_T < \delta\big)\\ &\qquad + \mathbf{P}\big(\|M\|_T > \varepsilon, \|\zeta^{ij}\|_T \le A\delta | \|w\| < \delta\big),\end{aligned}$$

we have

$$\lim_{\delta\downarrow 0} \mathbf{P}\big(\|M\|_T > \varepsilon | \|w\|_T < \delta\big) \le \sup_{0\le\delta\le 1} \mathbf{P}\big(\|M\|_T > \varepsilon, \|\zeta^{ij}\|_T > A\delta | \|w\|_T < \delta\big).$$

Hence by letting $A \to \infty$ we have

$$\lim_{\delta\downarrow 0} \mathbf{P}\big(\|M\|_T > \varepsilon | \|w\|_T < \delta\big) = 0.$$

Now we extend the result to $f \in \mathcal{C}_b$, which is uniformly continuous. Let $\varepsilon > 0$ be given. For any $\varepsilon' > 0$ choose an $\eta \in (0, \frac{\varepsilon}{2T})$ sufficiently small such that

$$c_2 - \frac{\varepsilon^2}{32\eta^2 T} < 0, \qquad \frac{4}{c_1}\exp\left\{c_2 - \frac{\varepsilon^2}{32\eta^2 T}\right\} < \varepsilon',$$

where c_1 and c_2 are constants appearing in Lemma 2.1. Then choose a $g \in \mathcal{C}_b^2$ such that $\|f - g\|_T < \eta$. Note that

$$\begin{aligned}&\int_0^t f(X_s)\,\mathrm{d}\zeta^{ij}(s) - \int_0^t g(X_s)\,\mathrm{d}\zeta^{ij}(s)\\ &\qquad = \int_0^t (f-g)(X_s) w_s^i\,\mathrm{d}w_s^j + \frac{\delta_{ij}}{2}\int_0^t (f-g)(X_s)\,\mathrm{d}s\\ &\qquad =: Y_1(t) + Y_2(t).\end{aligned}$$

It is easy to see $\|Y_2\|_T < \frac{\varepsilon}{4}$. Moreover, since $\langle Y_1\rangle(T) \le \eta^2\|w\|_T^2 T$, we have by exponential inequality and with arguments similar to the proof of (2.10) that if $\delta \in (0, 1]$,

$$\begin{aligned}&\mathbf{P}\left(\|Y_1\|_T \ge \frac{\varepsilon}{4}\Big| \|w\|_T < \delta\right)\\ &\qquad \le \mathbf{P}\left(\|Y_1\|_T \ge \frac{\varepsilon}{4}, \langle Y_1\rangle(T) \le \eta^2\delta^2 T\right) P\big(\|w\|_T < \delta\big)^{-1}\\ &\qquad \le \frac{4}{c_1}\exp\left\{\frac{1}{\delta^2}\left(c_2 - \frac{\varepsilon^2}{32\eta^2 T}\right)\right\} \le \varepsilon'.\end{aligned}$$

Thus for such δ,

$$\mathbf{P}\bigg(\bigg\|\int_0^\cdot f(X_s)\,\mathrm{d}\zeta^{ij}(s)\bigg\|_T \geq \varepsilon\bigg|\|w\|_T < \delta\bigg)$$
$$\leq \varepsilon' + \mathbf{P}\bigg(\bigg\|\int_0^\cdot g(X_s)\,\mathrm{d}\zeta^{ij}(s)\bigg\|_T \geq \varepsilon/2\bigg|\|w\| < \delta\bigg).$$

Now we conclude by letting $\delta \to 0$ and by the arbitrariness of ε'. $\square$

LEMMA 2.4. *We have*: (i) *For all* $f \in \mathcal{C}_b^2(\mathbb{R}^d;\mathbb{R})$, $\varepsilon > 0$ *and* $1 \leq k \leq d_1$,

$$\lim_{\delta\downarrow 0}\mathbf{P}\bigg(\bigg\|\int_0^\cdot f(X_s)\circ \mathrm{d}w_s^k\bigg\|_T \geq \varepsilon\bigg|\|w\|_T < \delta\bigg) = 0.$$

(ii) *There exists a constant* $c_3 > 0$ *such that*

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(|K|_T > c_3 | \|w\|_T < \delta\big) = 0.$$

PROOF. It suffices to prove (i), since then (ii) follows from (i) and (2.3). We have

$$\begin{aligned}\int_0^t f(X_s)\circ \mathrm{d}w_s^k &= f(X_t)w_t^k - \int_0^t [f_i\sigma_j^i](X_s)\,\mathrm{d}\zeta^{kj} \\ &\quad - \int_0^t [f_i b^i](X_s)w_s^k\,\mathrm{d}s - \int_0^t f_i(X_s)w_s^k\,\mathrm{d}K_s^i \\ &:= I_1(t) - I_2(t) - I_3(t) - I_4(t).\end{aligned}$$

Since

$$\|I_4\|_T \lesssim \|w\|_T |K|_T,$$

by Lemma 2.2 we have

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_4\|_T \geq \varepsilon | \|w\|_T < \delta\big) \leq \lim_{\delta\downarrow 0}\mathbf{P}\big(|K|_T \geq c\varepsilon\delta^{-1} | \|w\|_T < \delta\big) = 0,$$

while by Lemma 2.3 we have

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_2\|_T \geq \varepsilon | \|w\|_T < \delta\big) = 0.$$

Finally, it is trivial that

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_i\|_T \geq \varepsilon | \|w\|_T < \delta\big) = 0, \qquad i = 1, 3.$$

This completes the proof. $\square$

Now are ready to state our main result. Let (Y, l) denote the solution of the following deterministic Skorohod problem:

$$
(2.11)\quad \begin{cases} Y_t = Y_0 + \int_0^t \sigma(Y_s)\,\mathrm{d}h_s + \int_0^t b(Y_s)\,\mathrm{d}s + l_t, & Y_0 = x, \\ |l|_t = \int_0^t 1_{\partial D}(Y_s)\,\mathrm{d}|l|_s, & l_t = \int_0^t \eta(s)\,\mathrm{d}|l|_s, \end{cases}
$$

where $\eta(s) \in \mathcal{N}_{Y_s}$.

THEOREM 2.1. *For any $h \in \mathcal{S}$ and $\varepsilon > 0$,*

$$
\mathbf{P}\big(\|X - Y\|_T + \|K - l\|_T < \varepsilon \,|\, \|w - h\|_T < \delta\big) \to 1 \qquad \text{as } \delta \downarrow 0.
$$

PROOF. We first assume $h \equiv 0$. Since (X, K) and (Y, l) are solutions to equations (1.1) and (2.11), respectively, we have

$$
X_t - Y_t = \int_0^t \sigma(X_s) \circ \mathrm{d}w_s + \int_0^t \big(b(X_s) - b(Y_s)\big)\,\mathrm{d}s + \int_0^t (\mathrm{d}K_s - \mathrm{d}l_s).
$$

Set

$$
\Psi(x) := 1 - e^{-|x|^2/2};
$$

then

$$
\begin{aligned}
\Psi_i(x) &:= \frac{\partial}{\partial x_i}\Psi(x) = e^{-|x|^2/2} x_i;\\
\Psi_{i,j}(x) &:= \frac{\partial^2}{\partial x_i\,\partial x_j}\Psi(x) = -e^{-|x|^2/2}[x_i x_j + \delta_{ij}];\\
G(t) &:= X_t - Y_t, \qquad \varphi_i(x) := \frac{\partial}{\partial x_i}\varphi(x).
\end{aligned}
$$

By Itô's formula we have

$$
\begin{aligned}
&\exp\left\{\frac{2}{\alpha}\big(\varphi(X_t) + \varphi(Y_t)\big)\right\}\mathrm{d}\left[\exp\left\{-\frac{2}{\alpha}\big(\varphi(X_t) + \varphi(Y_t)\big)\right\}\Psi\big(G(t)\big)\right]\\
&\quad = \Psi_i\big(G(t)\big)\sigma_k^i(X_t) \circ \mathrm{d}w_t^k + \Psi_i\big(G(t)\big)\big(b^i(X_t) - b^i(Y_t)\big)\,\mathrm{d}t\\
&\qquad + \Psi_i\big(G(t)\big)\big(\mathrm{d}K_t^i - \mathrm{d}l_t^i\big)\\
&\qquad - \frac{2}{\alpha}\Psi\big(G(t)\big)\big[\varphi_i(X_t)\sigma_k^i(X_t) \circ \mathrm{d}w_t^k + \varphi_i(X_t)b^i(X_t)\,\mathrm{d}t + \varphi_i(X_t)\,\mathrm{d}K_t^i\\
&\qquad\qquad + \varphi_i(Y_t)b^i(Y_t)\,\mathrm{d}t + \varphi_i(Y_t)\,\mathrm{d}l_t^i\big]\\
&\qquad - \frac{2}{\alpha}\Psi\big(G(t)\big)\Psi_i\big(G(t)\big)\varphi_i(X_t)\sigma_k^i\sigma_k^i(X_t)\mathrm{d}t.
\end{aligned}
$$

Using the elementary inequality $1 - e^{-t} \geq te^{-t}$ for $t \geq 0$ and conditions (H$_1$)–(H$_2$), we have

$$\begin{aligned}
&\Psi_i\big(G(t)\big)\,\mathrm{d}K_t^i - \frac{2}{\alpha}\Psi\big(G(t)\big)\varphi_i(X_t)\,\mathrm{d}K_t^i \\
&\quad = \left[e^{-|X_t-Y_t|^2/2}(X_t - Y_t)^*\xi_t - \frac{2}{\alpha}\big(1 - e^{-|X_t-Y_t|^2/2}\big)\varphi_i(X_t)\xi_t^i\right]\mathrm{d}|K|_t \\
&\quad \leq e^{-|X_t-Y_t|^2/2}\big[(X_t - Y_t)^*\xi_t - c_0|X_t - Y_t|^2\big]\,\mathrm{d}|K|_t \leq 0, \\
&-\Psi_i\big(G(t)\big)\,\mathrm{d}l_t^i - \frac{2}{\alpha}\Psi\big(G(t)\big)\varphi_i(Y_t)\,\mathrm{d}l_t^i \leq 0.
\end{aligned}$$

Combining these with the fact $|\Psi_i(x)x_i| \lesssim \Psi(x)$, we have

$$\exp\left\{-\frac{2}{\alpha}\big(\varphi(X_t) + \varphi(Y_t)\big)\right\}\Psi\big(G(t)\big) \leq \int_0^t \rho_k(s) \circ \mathrm{d}w_s^k + C\int_0^t \Psi\big(G(s)\big)\,\mathrm{d}s,$$

where

$$\begin{aligned}
\rho_k(s) := \exp&\left\{-\frac{2}{\alpha}\big(\varphi(X_s) + \varphi(Y_s)\big)\right\} \\
&\times\left[\Psi_i\big(G(s)\big)\sigma_k^i(X_s) - \frac{2}{\alpha}\Psi\big(G(s)\big)\varphi_i(X_s)\sigma_k^i(X_s)\right].
\end{aligned}$$

By Itô's formula

$$\begin{aligned}
&\exp\left\{\frac{2}{\alpha}\big(\varphi(X_s) + \varphi(Y_s)\big)\right\} \circ \mathrm{d}\rho_k(s) \\
&\quad = \Bigg[\Psi_i\big(G(s)\big)\sigma_{kj}^i(X_s) + \sigma_k^i(X_s)\Psi_{ij}\big(G(s)\big) - \frac{2}{\alpha}\Big(\Psi\big(G(s)\big)\varphi_i(X_s)\sigma_{kj}^i(X_s) \\
&\qquad\qquad + \Psi\big(G(s)\big)\varphi_{ij}(X_s)\sigma_k^i(X_s) + \Psi_j\big(G(s)\big)\varphi_i(X_s)\sigma_k^i(X_s)\Big)\Bigg] \\
&\qquad \times\big[\sigma_l^j(X_s) \circ \mathrm{d}w_s^l + b^j(X_s)\,\mathrm{d}s + \mathrm{d}K_s^j\big] \\
&\qquad + \left[\frac{2}{\alpha}\Psi_j\big(G(s)\big)\varphi_i(X_s)\sigma_k^i(X_s) - \sigma_k^i(X_s)\Psi_{ij}\big(G(s)\big)\right]\big[b^j(Y_s)\,\mathrm{d}s + \mathrm{d}l_s^j\big] \\
&\qquad - \frac{2}{\alpha}\rho_k(s)\exp\left\{\frac{2}{\alpha}\big(\varphi(X_s) + \varphi(Y_s)\big)\right\} \\
&\qquad \times\big[\varphi_j(X_s)\big(\sigma_l^j(X_s) \circ \mathrm{d}w_s^l + b^j(X_s)\,\mathrm{d}s + \mathrm{d}K_s^j\big) \\
&\qquad\qquad + \varphi_j(Y_s)\big(b^j(Y_s)\,\mathrm{d}s + \mathrm{d}l_s^j\big)\big],
\end{aligned}$$

where $\sigma_{kl}^i(x) := \frac{\partial}{\partial x_l}\sigma_k^i(x)$. Rearranging, we write

$$\begin{aligned}
\mathrm{d}\rho_k(s) = F_{kl}(X_s, Y_s) \circ \mathrm{d}w_s^l &+ G_{kj}(X_s, Y_s)b^j(X_s)\,\mathrm{d}s + G_{kj}(X_s, Y_s)\,\mathrm{d}K_s^j \\
&+ H_{kj}(X_s, Y_s)b^j(Y_s)\,\mathrm{d}s + H_{kj}(X_s, Y_s)\,\mathrm{d}l_s^j,
\end{aligned}$$

where

$$
\begin{aligned}
G_{kj}(x,y) :={}& \exp\Big\{-\frac{2}{\alpha}\big(\varphi(x)+\varphi(y)\big)\Big\} \\
&\times\Big[\Psi_i(x-y)\sigma^i_{kj}(x)+\sigma^i_k(x)\Psi_{ij}(x-y) \\
&\quad-\frac{2}{\alpha}\big(\Psi(x-y)\varphi_i(x)\sigma^i_{kj}(x)+\Psi(x-y)\varphi_{ij}(x)\sigma^i_k(x) \\
&\qquad+\Psi_j(x-y)\varphi_i(x)\sigma^i_k(x)\big)\Big] \\
&-\frac{2}{\alpha}\exp\Big\{-\frac{2}{\alpha}\big(\varphi(x)+\varphi(y)\big)\Big\} \\
&\times\Big[\Psi_i(x-y)-\frac{2}{\alpha}\Psi(x-y)\varphi_i(x)\Big]\sigma^i_k(x)\varphi_j(x), \\
F_{kl}(x,y) :={}& G_{kj}(x,y)\sigma^j_l(x), \\
H_{kj}(x,y) :={}& \Big[\frac{2}{\alpha}\Psi_j(x-y)\varphi_i(x)\sigma^i_k(x)-\sigma^i_k(x)\Psi_{ij}(x-y)\Big] \\
&\times\exp\Big\{-\frac{2}{\alpha}\big(\varphi(x)+\varphi(y)\big)\Big\} \\
&-\frac{2}{\alpha}\exp\Big\{-\frac{2}{\alpha}\big(\varphi(x)+\varphi(y)\big)\Big\} \\
&\times\Big[\Psi_i(x-y)-\frac{2}{\alpha}\Psi(x-y)\varphi_i(x)\Big]\sigma^i_k(x)\varphi_j(y).
\end{aligned}
$$

Thus we have by Itô's formula,

$$
\begin{aligned}
\int_0^t \rho_k(s)\circ \mathrm{d}w^k_s ={}& \rho_k(t)w^k_t-\int_0^t F_{kl}(X_s,Y_s)w^k_s\circ \mathrm{d}w^l_s \\
&-\int_0^t G_{kj}(X_s,Y_s)b^j(X_s)w^k_s\,\mathrm{d}s-\int_0^t G_{kj}(X_s,Y_s)w^k_s\,\mathrm{d}K^j_s \\
&-\int_0^t H_{kj}(X_s,Y_s)b^j(Y_s)w^k_s\,\mathrm{d}s-\int_0^t H_{kj}(X_s,Y_s)w^k_s\,\mathrm{d}l^j_s \\
=:{}& I_1(t)-I_2(t)-I_3(t)-I_4(t)-I_5(t)-I_6(t).
\end{aligned}
$$

Obviously,

$$
\sum_{i\neq 2}\|I_i\|_T \lesssim \big(1+|K|_T\big)\|w\|_T.
$$

Thus

$$\lim_{\delta\downarrow 0}\mathbf{P}\Big(\sum_{i\neq 2}\|I_i\|_T \geq \varepsilon\Big|\|w\|_T<\delta\Big) \leq \lim_{\delta\downarrow 0}\mathbf{P}\big((1+|K|_T)\|w\|_T \gtrsim \varepsilon|\|w\|_T<\delta\big)=0.$$

As for I_2 we have

$$\begin{aligned} I_2(t) &= \int_0^t F_{kl}(X_s,Y_s)\,\mathrm{d}\zeta_s^{kl} + \frac{1}{2}\int_0^t \frac{\partial}{\partial x_j}F_{kl}(X_s,Y_s)\sigma_p^j(X_s)w_s^k\delta^{pl}\,\mathrm{d}s \\ &=: I_{21}+I_{22}. \end{aligned}$$

It is easily seen that

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_{22}\|_T\geq\varepsilon|\|w\|_T<\delta\big)=0,$$

and by applying Lemma 2.3 to the functions F_{kl} (in place of f there) and the system satisfied by (X,Y) (in place of X there), we have that

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_{21}\|_T\geq\varepsilon|\|w\|_T<\delta\big)=0.$$

Consequently

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|I_{2}\|_T\geq\varepsilon|\|w\|_T<\delta\big)=0.$$

Combining all the above and the fact that φ is bounded, we have

$$\Psi\big(G(t)\big)\leq C\int_0^t\Psi\big(G(s)\big)\,\mathrm{d}s+A(t),$$

where $A(t)$ satisfies that for every $\varepsilon>0$,

$$\lim_{\delta\downarrow 0}\mathbf{P}\big(\|A\|_T>\varepsilon|\|w\|_T<\delta\big)=0.$$

On the set $\{\omega;\|A\|_T<\varepsilon\}$, we have

$$\Psi\big(G(t)\big)\leq\varepsilon e^C\leq C\varepsilon,$$

that is,

$$\|X-Y\|_T\leq\sqrt{-2\ln(1-C\varepsilon)}.$$

Since ε is arbitrarily small,

$$\mathbf{P}\big(\|X-Y\|_T>\varepsilon|\|w\|_T<\delta\big)\to 0\qquad\text{as }\delta\downarrow 0.$$

Finally, to see

$$\mathbf{P}\big(\|K-l\|_T<\varepsilon|\|w\|_T<\delta\big)\to 1\qquad\text{as }\delta\downarrow 0,$$

it suffices to notice that

$$K_t-l_t=X_t-Y_t-\int_0^t\sigma(X_s)\circ\mathrm{d}w_s-\int_0^t\big(b(X_s)-b(Y_s)\big)\,\mathrm{d}s$$

and use Lemma 2.4.

For general $h \in \mathcal{S}$, just as in the proof of [5], Theorem 8.2, pages 527–528, we set

$$M_1(w) := \exp\left\{\int_0^T \dot{h}_s \,\mathrm{d}w_s - \frac{1}{2}\int_0^T |\dot{h}_s|^2 \,\mathrm{d}s\right\}, \qquad \mathrm{d}\mathbf{P}' = M_1 \,\mathrm{d}\mathbf{P}.$$

Then $w'_t := w_t - h_t$ is a Brownian motion under $\mathbf{P}'$, and (X, K), (Y, l) satisfy the following equations, respectively:

$$X_t = x + \int_0^t b'(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(X_t) \circ \mathrm{d}w_s + K_t,$$
$$Y_t = x + \int_0^t b'(s, Y_s)\,\mathrm{d}s + l_t,$$

where $b'(s, x) := b(x) + \sigma(x)\dot{h}_s$.

Therefore according to the case of $h \equiv 0$ we have for every $\varepsilon > 0$,

$$\mathbf{P}'\big(\|X - Y\|_T > \varepsilon \big| \|w'\|_T < \delta\big) \to 0 \qquad \text{as } \delta \downarrow 0,$$
$$\mathbf{P}'\big(\|K - l\|_T > \varepsilon \big| \|w'\|_T < \delta\big) \to 0 \qquad \text{as } \delta \downarrow 0,$$

which, together with the fact that M_1 is a continuous functional of w, yields that

$$\begin{aligned}&\lim_{\delta\downarrow 0} \mathbf{P}\big(\|X - Y\|_T < \varepsilon | \|w'\|_T < \delta\big)\\ &\quad = \lim_{\delta\downarrow 0} \frac{\mathbf{P}(\|X - Y\|_T < \varepsilon, \|w'\|_T < \delta)}{\mathbf{E}(M_1 \mathbb{1}_{\{\|X-Y\|_T<\varepsilon, \|w-h\|_T<\delta\}})} \times \frac{\mathbf{E}(M_1 \mathbb{1}_{\{\|w-h\|_T<\delta\}})}{\mathbf{P}(\|w - h\|_T < \delta)} \to 1. \qquad \square\end{aligned}$$

REMARK 2.1. In the last step of the proof above, we encounter the situation that the drift b' depends also on time t. But as in [5], Theorem 8.2, everything still works with trivial modifications.

3. The support problem.

3.1. *Conditions and useful estimates*. The approximate continuity theorem proved in the above section together with the Wong–Zakai approximation theorem proved in [1] gives, in a similar way paved in [13], the support theorem for reflected diffusions under the conditions (H$_1$)–(H$_2$). In this section we will prove the support theorem based upon the idea in [8] when the domain D is supposed to satisfy the following conditions:

(A) There exists a constant $r_0 > 0$ such that for any $x \in \partial D$,

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \varnothing.$$

(B) There exist constants $\delta > 0$ and $\beta \geq 1$ satisfying that for any $x \in \partial D$, there exists a unit vector l_x such that

$$\langle l_x, n\rangle \geq 1/\beta \qquad \text{for any } n \in \bigcup_{y \in B(x,\delta)\cap\partial D} \mathcal{N}_y,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$.

(C) There exists a function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$ and a positive constant γ such that for any $x \in \partial D$, $y \in \bar{D}$ and $n \in \mathcal{N}_x$,

$$\langle y - x, n\rangle + \frac{1}{\gamma}\langle D\varphi(x), n\rangle |y - x|^2 \geq 0.$$

(D) There exist $m \geq 1$, $\lambda > 0$, $R > 0$, $a_1, \dots, a_m \in \mathbb{S}^{d-1}$ and $x_1, \dots, x_m \in \partial D$ such that $\partial D \subset \bigcup_{i=1}^m B(x_i, R)$ and $x \in \partial D \cap B(x_i, 2R) \Rightarrow n \cdot a_i \geq \lambda$, $\forall n \in \mathcal{N}_x$.

We will need some results from [1].

LEMMA 3.1 ([1], Lemma 2.3). *Assume* (A)–(B) *hold*, *and* (x,k) *is the solution to the Skorohod problem associated with a continuous function* w *such that* $x_0 = w_0 \in \bar{D}$. *Then for* $\theta \in (0,1]$, *there exist constants* c_1, c_2, C *dependent on* $\theta, \delta, \beta, \gamma_0$ *such that for all* $0 \leq s \leq t \leq T$,

$$|k|_t^s \leq C\big(1 + \|w\|_{[s,t],\theta}^{c_1}(t - s)\big)e^{c_2\|w\|_{[s,t]}}\|w\|_{[s,t]},$$

where (*and throughout*) $|k|_t^s$ *denotes the total variation of* k *on* $[s,t]$ *and*

$$\|w\|_{[s,t],\theta} := \sup_{u,v\in[s,t]} \frac{|w_u - w_v|}{|u - v|^\theta}, \qquad \|w\|_{[s,t]} := \sup_{u,v\in[s,t]} |w_u - w_v|.$$

LEMMA 3.2 ([1], Lemma 2.4). *Assume* (A) *holds*, *and* (x,k) *is the solution to the Skorohod problem associated with a function* w *having continuous bounded variation path. Then*

$$|x|_t^s \leq 2(\sqrt{2} + 1)|w|_t^s.$$

LEMMA 3.3 ([1], Lemma 2.8). *Assume* D *satisfies conditions* (A)–(B), *and* b, σ *are bounded*, *Lipschitz continuous functions*. *Then there exists a unique solution* (X, K) *to equation* (1.1). *Moreover, for all* $0 \leq s < t < \infty$,

$$\mathbf{E}\big(\|X\|_{[s,t]}\big)^{2p} \leq C_p|t - s|^p, \qquad \mathbf{E}\big(|K|_t^s\big)^{2p} \leq C_p|t - s|^p.$$

Let $n \in \mathbb{N}$ and $t_i = iT2^{-n}$ (here we should have used t_i^n instead of t_i to indicate the dependence on n, but in order to not surcharge the notation, we omit the superscript n), $\Delta = 2^{-n}T$, and for $t \in [t_i, t_{i+1})$ set

$$\bar{t}_n := t_{i-1} \vee 0, \qquad \hat{t}_n := t_i, \qquad \Delta w_i := w_{t_i} - w_{t_{i-1}\vee 0},$$

$$w_t^n := w_{\bar{t}_n} + \frac{w_{\hat{t}_n} - w_{\bar{t}_n}}{\Delta}(t - \hat{t}_n).$$

Consider the following reflected equation:

$$X^n(t) = x + \int_0^t b\big(X^n(s)\big)\,\mathrm{d}s + \int_0^t \sigma\big(X^n(s)\big)\,\mathrm{d}w_s^n + K^n(t).$$

Denote the solution by (X^n, K^n).

3.2. *Support theorem.* We first state our main theorem.

THEOREM 3.1. *Suppose conditions* (A)–(D) *hold and* $\sigma \in \mathcal{C}_b^2, b \in \mathcal{C}_b^1$. *Then for the solution* X *to equation* (1.1) *we have*

$$\text{the support of } \big(\mathbf{P} \circ X^{-1}\big) \text{ in } \mathbb{W}_\alpha = \overline{\mathscr{S}(\mathcal{H})}^{\alpha} \qquad \forall \alpha \in \big[0, \tfrac{1}{2}\big).$$

To prove the theorem, we will apply the following results; cf. [8].

PROPOSITION 3.1. *Let* F *be a measurable map from* Ω *to a Banach space* $(\mathbb{X}, \|\cdot\|)$:

(1) *Let* $Z_1^{\mathbb{X}} : \mathcal{H} \to \mathbb{X}$ *be measurable and* $H_n : \Omega \to \mathcal{H}$ *be a sequence of random variables such that for any* $\varepsilon > 0$,

$$\lim_n \mathbf{P}\big(\big\| Z_1^{\mathbb{X}}\big(H_n(\omega)\big) - F(\omega)\big\| > \varepsilon\big) = 0.$$

Then $\operatorname{supp}(\mathbf{P} \circ F^{-1}) \subset \overline{Z_1^{\mathbb{X}}(\mathcal{H})}$.

(2) *Let* $Z_2^{\mathbb{X}} : \mathcal{H} \to \mathbb{X}$ *be measurable and for fixed* h, $T_n^h : \Omega \to \Omega$ *be a sequence of measurable transformations such that* $\mathbf{P} \circ (T_n^h)^{-1} \ll \mathbf{P}$, *and for any* $\varepsilon > 0$,

$$\limsup_n \mathbf{P}\big(\big\| F\big(T_n^h(\omega)\big) - Z_2^{\mathbb{X}}(h)\big\| < \varepsilon\big) > 0.$$

Then $\operatorname{supp}(\mathbf{P} \circ F^{-1}) \supset Z_2^{\mathbb{X}}(\mathcal{H})$.

PROPOSITION 3.2. *Suppose* $\{X_t^n\}$ *is a sequence of finite dimensional processes satisfying that for every* $p \geq 1$ *and* $s, t \in [0, T]$, *there exists a constant* $C > 0$,

$$\sup_n \mathbf{E}\big|X_t^n - X_s^n\big|^{2p} \leq C|t - s|^p. \tag{3.1}$$

Then for any $\varepsilon > 0$ *and* $\theta < \frac{1}{2} - \frac{1}{2p}$, *there exists a constant* $C > 0$ *such that*

$$\sup_n \mathbf{P}\big(\|X^n\|_{T,\theta} > \varepsilon\big) \leq C\varepsilon^{-2p}.$$

Moreover, besides (3.1), *if for any* $\varepsilon > 0$,

$$\lim_n \mathbf{P}\Big(\sup_{1 \leq i \leq 2^n} \big|X_{t_i}^n\big| > \varepsilon\Big) = 0$$

holds as well, then for any $\theta \in [0, 1/2)$,

$$\lim_n \mathbf{P}(\|X^n\|_{T,\theta} > \varepsilon) = 0,$$

where $\|\cdot\|_{T,\theta}$ *is defined in the Introduction.*

Following the idea in [8], take

$$Z_1^{\mathbb{X}} = Z_2^{\mathbb{X}} = Z(\cdot), \qquad H_n(\omega) = w^n(\omega), \qquad T_n^h(\omega) = w - w^n + h.$$

Then by Girsanov's theorem, $\mathbf{P} \circ (T_n^h)^{-1} \ll \mathbf{P}$.

To prove Theorem 3.1, by Proposition 3.1, it suffices to prove that for every $\varepsilon > 0$,

$$\lim_n \mathbf{P}(\|X - X^n\|_{T,\theta} > \varepsilon) = 0 \tag{3.2}$$

and

$$\lim_n \mathbf{P}(\|X(w - w^n + h) - Z(h)\|_{T,\theta} > \varepsilon) = 0, \tag{3.3}$$

where $Z(h)$ solves the following deterministic Skorohod problem:

$$Z(h)_t = x + \int_0^t \sigma(Z(h)_s)\dot{h}_s \, \mathrm{d}s + \int_0^t b(Z(h)_s) \, \mathrm{d}s + \psi_t.$$

In what follows we will use Z instead of $Z(h)$ if no confusion is possible. (3.2) is proved in [16], so we only need to prove (3.3).

Using the Riemannian sum approximation of stochastic integrals, it is easy to see that $Y^n := X(w - w^n + h)$ solves the following RSDE:

$$Y_t^n = x + \int_0^t \sigma(Y_s^n) \, \mathrm{d}w_s - \int_0^t \sigma(Y_s^n)\dot{w}_s^n \, \mathrm{d}s + \int_0^t \sigma(Y_s^n)\dot{h}_s \, \mathrm{d}s + \int_0^t \tilde{b}(Y_s^n) \, \mathrm{d}s + \phi_t^n,$$

where $\tilde{b} := b + \frac{1}{2}(\nabla\sigma)\sigma$ and $\phi^n(w) = K(w - w^n + h)$.

We first prepare some auxiliary results.

LEMMA 3.4. *For* $0 \le s \le t \le T$, $|Z_t - Z_s|^{2p} \le C_p|t - s|^p$.

PROOF. By Lemma 3.2,

$$\begin{aligned} |Z_t - Z_s|^{2p} &\le [2(\sqrt{2} + 1)]^{2p} \left(\int_s^t |\sigma(Z_u)\dot{h}_u + \tilde{b}(Z_u)| \, \mathrm{d}u \right)^{2p} \\ &\le C_p|t - s|^p. \end{aligned}$$

□

PROPOSITION 3.3. *Let* $p \ge 1$. *Then there exists a constant* $C_p > 0$ *independent of n such that for all* $0 \le s \le t \le T$,

$$\mathbf{E}(|Y_t^n - Y_s^n|^{4p}) \le C_p|t - s|^p, \qquad \mathbf{E}(|\phi_t^n - \phi_s^n|^{4p}) \le C_p|t - s|^p. \tag{3.4}$$

Moreover, for all $0 \le s \le t \le T$ *and for any* $\theta \in (0, \frac{1}{4})$,

$$\mathbf{E}(\|Y^n\|_{[s,t],\theta}^p) \le C_{p,\theta}, \qquad \mathbf{E}(\|\phi^n\|_{[s,t],\theta}^p) \le C_{p,\theta}. \tag{3.5}$$

To prove this proposition, we need some lemmas, and without loss of generality we take $T = 1$.

LEMMA 3.5. *Let $\lambda, t > 0$. Then there exists a constant $C > 0$ independent of λ and t such that*

$$\mathbf{E}\big(e^{\lambda\|w\|_t}\big) \le (1 + C\lambda\sqrt{t})^{d_1} C e^{\lambda^2 d_1 t/2}.$$

PROOF. Set $\xi = \max_{0\le s\le t} |w_s|$. Note that $\mathbf{P}(|w_t^i| \in \mathrm{d}x) = \sqrt{\frac{2}{\pi t}} e^{-x^2/(2t)}\,\mathrm{d}x$, $i = 1, \dots, d_1$ and thus

$$\begin{aligned}\mathbf{E}\big(e^{\lambda\xi}\big) &= \int_0^\infty e^s \mathbf{P}(\lambda\xi > s)\,\mathrm{d}s + 1 \le 2\int_0^\infty e^s \mathbf{P}\big(\lambda|w_t| > s\big)\,\mathrm{d}s + 1\\ &= 2\mathbf{E}\big(e^{\lambda|w_t|}\big) - 1 \le 2\prod_{i=1}^{d_1}\sqrt{\frac{2}{\pi t}}\int_0^\infty e^{\lambda x} e^{-x^2/(2t)}\,\mathrm{d}x\\ &\le (1 + C\lambda\sqrt{t})^{d_1} C e^{\lambda^2 d_1 t/2}. \qquad \square\end{aligned}$$

LEMMA 3.6. *Let $M_t := \int_0^t f_s\,\mathrm{d}w_s$ and $|f_s| \le c$ for some constant c. Then there exists a constant $C > 0$ such that for any integer m,*

$$\mathbf{E}\|M\|_{[s,t]}^m \le C^m (m/2)^{m/2} (t-s)^{m/2}.$$

PROOF. It suffices to prove the result for $s = 0$. Then $M_t = B_{\langle M\rangle_t}$ where B is the DDS-Brownian motion of M. Note that

$$\langle M\rangle_t \le c^2 t.$$

The result follows from Doob's maximal inequality and that

$$\mathbf{E}\big(|B_t|^{2m}\big) \le (2d_1)^m m^m t^m. \tag{3.6}$$

$\square$

Set

$$L_t^n := x + \int_0^t \sigma\big(Y_s^n\big)\,\mathrm{d}w_s - \int_0^t \sigma\big(Y_s^n\big)\dot{w}_s^n\,\mathrm{d}s + \int_0^t \sigma\big(Y_s^n\big)\dot{h}_s\,\mathrm{d}s + \int_0^t \tilde{b}\big(Y_s^n\big)\,\mathrm{d}s.$$

LEMMA 3.7. *There exists a constant C_p such that for any $t \in [0,1]$ and any $p \ge 1$,*

$$\mathbf{E}\big(\|Y^n\|_{[\bar{t}_n,t]}^{2p}\big) \le C_p\Delta^p, \qquad \mathbf{E}\big(\big|\phi^n\big|_t^{\bar{t}_n}\big)^{2p} \le C_p\Delta^p.$$

PROOF. By Lemma 3.1, for any $\theta \in (0,1]$,

$$\big|\phi^n\big|_t^{\bar{t}_n} \le C\big(1 + \|L^n\|_{[\bar{t}_n,t],\theta}^{c_1}(t - \bar{t}_n)\big)e^{c_2\|L^n\|_{[\bar{t}_n,t]}}\|L^n\|_{[\bar{t}_n,t]}.$$

Note that for any $p \geq 1$,

$$\mathbf{E}(\|L^n\|_{[\bar{t}_n,t]}^{2p}) \leq C_p\biggl[\mathbf{E}\biggl(\int_{\bar{t}_n}^t \|\sigma(Y_r^n)\|^2\,\mathrm{d}r\biggr)^p + \mathbf{E}\biggl(\int_{\bar{t}_n}^t \|\sigma(Y_r^n)\||\dot{w}_r^n|\,\mathrm{d}r\biggr)^{2p} + \mathbf{E}\biggl(\int_{\bar{t}_n}^t \|\sigma(Y_r^n)\||\dot{h}_r|\,\mathrm{d}r\biggr)^{2p} + (t-\bar{t}_n)^{2p}\biggr] \leq C_p(t-\bar{t}_n)^p.$$

For any c, by Lemmas 3.5 and 3.6,

$$\begin{aligned}
&\mathbf{E}(e^{cp\|L^n\|_{[\bar{t}_n,t]}}) \\
&\quad\leq \mathbf{E}\bigl(e^{cp\max_{u,v\in[\bar{t}_n,t]}|\int_u^v \sigma(Y_r^n)\,\mathrm{d}w_r + cp\int_u^v \sigma(Y_r^n)\dot{w}_r^n\,\mathrm{d}r|} \\
&\qquad\qquad \times e^{cp|\int_{\bar{t}_n}^t \|\sigma(Y_r^n)\||\dot{h}_r|\,\mathrm{d}r + cp\int_{\bar{t}_n}^t \tilde{b}(Y_r^n)\,\mathrm{d}r|}\bigr) \\
&\quad\leq (1+Cp\Delta^{1/2})^{d_1} e^{Cd_1p^2\Delta + Cp\int_{\bar{t}_n}^t (1+|\dot{h}_r|)\,\mathrm{d}r} \leq C_p < \infty.
\end{aligned}$$

Now combining these two estimates gives

$$\mathbf{E}(|\phi^n|_t^{\bar{t}_n})^{2p} \leq C_p\Delta^p.$$

The other result follows from $Y_t^n = L_t^n + \phi_t^n$ and the above estimate. $\square$

LEMMA 3.8. *For any $s,t \in [0,1]$,*

$$\mathbf{E}\sup_{u,v\in[s,t]}\biggl|\int_u^v \sigma(Y_r^n)\,\mathrm{d}w_r^n\biggr|^{2p} \leq C_p|t-s|^p, \qquad \mathbf{E}(\|L^n\|_{[s,t]}^{2p}) \leq C_p|t-s|^p.$$

PROOF. When $t_{i-1} \leq s \leq t_i \leq t \leq t_{i+1}$ for some $1 \leq i \leq 2^n$, the result is trivial. For general s,t, choose $1 \leq l < m-1 < m \leq 2^n$ such that $t_{l-1} \leq s \leq t_l < t_{m-1} \leq t \leq t_m$. Note that

$$\int_s^t \sigma(Y_r^n)\,\mathrm{d}w_r^n = \int_s^t (\sigma(Y_r^n) - \sigma(Y_{\bar{r}_n}^n))\,\mathrm{d}w_r^n + \int_s^t \sigma(Y_{\bar{r}_n}^n)\,\mathrm{d}w_r^n$$

and

$$\begin{aligned}
&\int_s^t \sigma(Y_{\bar{r}_n}^n)\,\mathrm{d}w_r^n \\
&\quad= \int_s^{t_l} \sigma(Y_{\bar{r}_n}^n)\,\mathrm{d}w_r^n + \sum_{j=l+1}^{m-1}\int_{t_{j-1}}^{t_j} \sigma(Y_{\bar{r}_n}^n)\,\mathrm{d}w_r^n + \int_{t_{m-1}}^t \sigma(Y_{\bar{r}_n}^n)\,\mathrm{d}w_r^n \\
&\quad= \sigma(Y_{t_{l-2}\vee 0}^n)\frac{w_{t_{l-1}} - w_{t_{l-2}\vee 0}}{\Delta}(t_l - s) \\
&\qquad+ \sum_{j=l+1}^{m-1} \sigma(Y_{t_{j-2}\vee 0}^n)(w_{t_{j-1}} - w_{t_{j-2}\vee 0}) \\
&\qquad+ \sigma(Y_{t_{m-2}\vee 0}^n)\frac{w_{t_{m-1}} - w_{t_{m-2}\vee 0}}{\Delta}(t - t_{m-1}),
\end{aligned}$$

$\int_0^\cdot \sigma(Y^n_{\bar{r}_n})\,\mathrm{d}w^n_r$ is the piecewise linear interpolation of

$$M^n_\cdot := \int_0^{\cdot-\Delta} \sigma\big(Y^n\big(\pi_n(r)\big)\big)\,\mathrm{d}w_r$$

with $\pi_n(r) := \max\{t_k; t_k \le r\}$, at $\{t_k\}_{k=0,1,\dots,2^n-1}$. Thus

$$\begin{aligned}\sup_{u,v\in[s,t]}\left|\int_u^v \sigma\big(Y^n_{\bar{r}_n}\big)\,\mathrm{d}w^n_r\right| &\le \sup_{l-2\le k,k'\le m-1}\big|M^n_{t_k} - M^n_{t_{k'}}\big| \\ &\le 2\sup_{t_{l-2}\le r\le t_{m-1}}\big|M^n_r - M^n_{t_l}\big|.\end{aligned}$$

Using Doob's inequality we get

$$\begin{aligned}\mathbf{E}\sup_{u,v\in[s,t]}\left|\int_u^v \sigma\big(Y^n_{\bar{r}_n}\big)\,\mathrm{d}w^n_r\right|^{2p} &\le C_p\mathbf{E}\sup_{t_{l-2}\le r\le t_{m-1}}\big|M^n_r - M^n_{t_l}\big|^{2p} \\ &\le C_p\mathbf{E}\bigg(\int_{t_{l-2}}^{t_{m-1}} \big\|\sigma\big(Y^n_{\bar{r}_n}\big)\big\|^2\,\mathrm{d}r\bigg)^p \\ &\le C_p|t_{m-1} - t_{l-2}|^p \\ &\le C_p|t-s|^p.\end{aligned}$$

By Hölder's inequalities and Lemma 3.7,

$$\begin{aligned}&\mathbf{E}\sup_{u,v\in[s,t]}\left|\int_u^v \big(\sigma\big(Y^n_r\big) - \sigma\big(Y^n_{\bar{r}_n}\big)\big)\,\mathrm{d}w^n_r\right|^{2p} \\ &\quad\le \mathbf{E}\int_s^t \big\|\sigma\big(Y^n_r\big) - \sigma\big(Y^n_{\bar{r}_n}\big)\big\|^{2p}\big|\dot{w}^n_r\big|^{2p}\,\mathrm{d}r\,(t-s)^{2p-1} \\ &\quad\le (t-s)^{2p-1}\int_s^t \big(\mathbf{E}\big\|\sigma\big(Y^n_r\big) - \sigma\big(Y^n_{\bar{r}_n}\big)\big\|^{4p}\big)^{1/2}\big(\mathbf{E}\big|\dot{w}^n_r\big|^{4p}\big)^{1/2}\,\mathrm{d}r \\ &\quad\le C_p(t-s)^{2p}.\end{aligned}$$

Now note that

$$L^n_t - L^n_s = \int_s^t \sigma\big(Y^n_r\big)\big(\mathrm{d}w_r - \mathrm{d}w^n_r\big) + \int_s^t \sigma\big(Y^n_r\big)\dot{h}_r\,\mathrm{d}r + \int_s^t \tilde{b}\big(Y^n_r\big)\,\mathrm{d}r.$$

Trivially by the Burkholder and Hölder inequalities we have

$$\mathbf{E}\sup_{u,v\in[s,t]}\left|\int_u^v \sigma\big(Y^n_r\big)\,\mathrm{d}w_r + \int_u^v \sigma\big(Y^n_r\big)\dot{h}_r\,\mathrm{d}r + \int_u^v \tilde{b}\big(Y^n_r\big)\,\mathrm{d}r\right|^{2p} \le C|t-s|^p.$$

From the estimates above we deduce

$$\mathbf{E}\sup_{u,v\in[s,t]}\big|L^n_u - L^n_v\big|^{2p} \le C|t-s|^p. \tag{3.7}$$

□

Now we are ready to prove Proposition 3.3.

PROOF OF PROPOSITION 3.3. For cases of $s,t\in[t_{i-1},t_i]$ and $t_{i-1}\le s\le t_i<t\le t_{i+1}$ for some $1\le i\le 2^n$, it follows from Lemmas 3.7–3.8 that

$$\mathbf{E}\|Y^n\|_{[s,t]}^{2p}\le C_p|t-s|^p,\qquad \mathbf{E}\big[\big(|\phi^n|_t^s\big)^{2p}\big]\le C_p|t-s|^p. \tag{3.8}$$

For general cases, choose $1\le l<m-1<m\le 2^n$ such that $t_{l-1}\le s\le t_l<t_{m-1}\le t\le t_m$. We get by Itô's formula,

$$\begin{aligned}&\mathrm{d}\big(e^{-(2/\gamma)\varphi(Y_t^n)}|Y_t^n-Y_s^n|^2\big)\\&\qquad=: U_s^n(t)\,\mathrm{d}w_t+U_s^n(t)\,\mathrm{d}w_t^n+V_s^n(t)\,\mathrm{d}t+Z_s^n(t)\,\mathrm{d}t+A_t^5,\end{aligned}$$

where according to (C),

$$A_t^5:=e^{-(2/\gamma)\varphi(Y_t^n)}\left[2\langle Y_t^n-Y_s^n,\mathrm{d}\phi_t^n\rangle-\frac{2}{\gamma}|Y_t^n-Y_s^n|^2\langle D\varphi(Y_t^n),\mathrm{d}\phi_t^n\rangle\right]\le 0$$

and

$$\begin{aligned}U_s^n(t)&:=e^{-(2/\gamma)\varphi(Y_t^n)}\Big(2(Y_t^n-Y_s^n)-\frac{2}{\gamma}|Y_t^n-Y_s^n|^2D\varphi(Y_t^n)\Big)\sigma(Y_t^n),\\
V_s^n(t)&:=e^{-(2/\gamma)\varphi(Y_t^n)}\Big(2(Y_t^n-Y_s^n)-\frac{2}{\gamma}|Y_t^n-Y_s^n|^2D\varphi(Y_t^n)\Big)\big(\sigma(Y_t^n)\dot h_t+\tilde b(Y_t^n)\big),\\
Z_s^n(t)&:=e^{-(2/\gamma)\varphi(Y_t^n)}\Big[\mathrm{tr}(\sigma\sigma^*)(Y_t^n)-\frac{1}{\gamma}|Y_t^n-Y_s^n|^2\,\mathrm{tr}(D\varphi\sigma\sigma^*)(Y_t^n)\\
&\qquad-\frac{4}{\gamma}(Y_t^n-Y_s^n)\sigma(Y_t^n)D\varphi(Y_t^n)\sigma(Y_t^n)\\
&\qquad\qquad+\frac{2}{\gamma^2}|Y_t^n-Y_s^n|^2|D\varphi(Y_t^n)\sigma(Y_t^n)|^2\Big].\end{aligned}$$

By the conditions on σ,b,φ,

$$\begin{aligned}|U_s^n(t)|&\le C\big(|Y_t^n-Y_s^n|+|Y_t^n-Y_s^n|^2\big),\\
|U_s^n(t)-U_s^n(t')|&\le C|Y_t^n-Y_{t'}^n|(1+|Y_t^n-Y_s^n|)\\
&\qquad+C|Y_t^n-Y_{t'}^n|\big(|Y_t^n-Y_s^n|+|Y_{t'}^n-Y_s^n|+|Y_{t'}^n-Y_s^n|^2\big)\\
&\le C|Y_t^n-Y_{t'}^n|(1+|Y_t^n-Y_s^n|)+C|Y_t^n-Y_{t'}^n|^2(1+|Y_{t'}^n-Y_s^n|^2),\\
|V_s^n(t)|&\le C\big(|Y_t^n-Y_s^n|+|Y_t^n-Y_s^n|^2\big)(1+|\dot h_t|),\\
|Z_s^n(t)|&\le C\big(1+|Y_t^n-Y_s^n|+|Y_t^n-Y_s^n|^2\big).\end{aligned}$$

Thus

$$
\begin{aligned}
&\mathbf{E}\big|Y_t^n - Y_s^n\big|^{4p} \\
&\quad \le C_p \mathbf{E}\biggl(\biggl|\int_s^t U_s^n(r)\,\mathrm{d}w_r\biggr| + \int_s^t \big|U_s^n(r)\big|\big|\dot{w}_r^n\big|\,\mathrm{d}r + \biggl|\int_s^t V_s^n(r)\,\mathrm{d}r\biggr| \\
&\qquad\qquad + \int_s^t \big|Z_s^n(r)\big|\,\mathrm{d}r\biggr)^{2p}.
\end{aligned}
$$

Using the BDG inequality we get

$$
\begin{aligned}
&\mathbf{E}\biggl(\int_s^t U_s^n(r)\,\mathrm{d}w_r\biggr)^{2p} \\
&\quad \le C_p \mathbf{E}\biggl(\int_s^t \big|U_s^n(r)\big|^2\,\mathrm{d}r\biggr)^p \\
&\quad \le C_p (t-s)^{p-1} \mathbf{E}\biggl(\int_s^t \bigl(\big|Y_r^n - Y_s^n\big|^{2p} + \big|Y_r^n - Y_s^n\big|^{4p}\bigr)\,\mathrm{d}r\biggr) \\
&\quad \le C_p (t-s)^p + C_p \mathbf{E}\biggl(\int_s^t \big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r\biggr), \\
&\mathbf{E}\biggl(\int_s^t U_s^n(r)\,\mathrm{d}w_r^n\biggr)^{2p} \\
&\quad \le C_p \mathbf{E}\biggl[\biggl(\int_s^t \bigl(U_s^n(r) - U_s^n(\bar{r}_n \vee s)\bigr)\,\mathrm{d}w_r^n + \int_s^t U_s^n(\bar{r}_n \vee s)\,\mathrm{d}w_r^n\biggr)^{2p}\biggr].
\end{aligned}
$$

Note that $\int_0^{\cdot} U_s^n(\bar{r}_n)\,\mathrm{d}w_r^n$ is the piecewise linear interpolation of $M_\cdot^n := \int_0^{\cdot-\Delta} U_s^n(\pi_n(r))\,\mathrm{d}w_r$ with

$$
\pi_n(r) := \max\{t_k;\, t_k \le r\}.
$$

Thus by Doob's inequality and Lemma 3.7 we get

$$
\begin{aligned}
&\mathbf{E}\biggl|\int_s^t U_s^n(\bar{r}_n \vee s)\,\mathrm{d}w_r^n\biggr|^{2p} \\
&\quad \le C_p \mathbf{E}\biggl(\int_s^t \big|U_s^n(\bar{r}_n \vee s)\big|^2\,\mathrm{d}r\biggr)^p \\
&\quad \le C_p |t-s|^{p-1} \mathbf{E}\biggl(\int_s^t \bigl(\big|Y_{\bar{r}_n \vee s}^n - Y_s^n\big|^{2p} + \big|Y_{\bar{r}_n \vee s}^n - Y_s^n\big|^{4p}\bigr)\,\mathrm{d}r\biggr) \\
&\quad \le C_p |t-s|^{p-1} \mathbf{E}\biggl(\int_s^t \bigl(\big|Y_r^n - Y_{\bar{r}_n \vee s}^n\big|^{2p} + \big|Y_r^n - Y_s^n\big|^{2p}\bigr)\,\mathrm{d}r\biggr) \\
&\qquad + C_p |t-s|^{p-1} \mathbf{E}\biggl(\int_s^t \bigl(\big|Y_r^n - Y_{\bar{r}_n \vee s}^n\big|^{4p} + \big|Y_r^n - Y_s^n\big|^{4p}\bigr)\,\mathrm{d}r\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&\le C_p|t-s|^p + C_p|t-s|^{p-1}\int_s^t \mathbf{E}\big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r \\
&\le C_p|t-s|^p + C_p\int_s^t \mathbf{E}\big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}&\biggl[\biggl|\int_s^t \bigl(U_s^n(r) - U_s^n(\bar{r}_n \vee s)\bigr)\,\mathrm{d}w_r^n\biggr|^{2p}\biggr] \\
&\le C_p\mathbf{E}\biggl[\biggl(\int_s^t \bigl|Y_r^n - Y_{\bar{r}_n\vee s}^n\bigr|\bigl(1+\bigl|Y_r^n - Y_s^n\bigr|\bigr)\frac{|w_{\hat{r}_n} - w_{\bar{r}_n\vee s}|}{\Delta}\,\mathrm{d}r\biggr)^{2p}\biggr] \\
&\quad + C_p\mathbf{E}\biggl[\biggl(\int_s^t \bigl|Y_r^n - Y_{\bar{r}_n\vee s}^n\bigr|^2\bigl(1+\bigl|Y_r^n - Y_s^n\bigr|^2\bigr)\frac{|w_{\hat{r}_n} - w_{\bar{r}_n\vee s}|}{\Delta}\,\mathrm{d}r\biggr)^{2p}\biggr] \\
&\le C_p\mathbf{E}\biggl[\biggl(\int_s^t \bigl|Y_r^n - Y_{\bar{r}_n\vee s}^n\bigr|\frac{|w_{\hat{r}_n} - w_{\bar{r}_n\vee s}|}{\Delta}\,\mathrm{d}r\biggr)^{2p}\biggr] \\
&\quad + C_p\mathbf{E}\biggl[\biggl(\int_s^t \biggl(\bigl|Y_r^n - Y_s^n\bigr|^2 + \bigl|Y_r^n - Y_{\bar{r}_n\vee s}^n\bigr|^2\frac{|w_{\hat{r}_n} - w_{\bar{r}_n\vee s}|^2}{\Delta^2}\biggr)\mathrm{d}r\biggr)^{2p}\biggr] \\
&\quad + C_p\mathbf{E}\biggl[\biggl(\int_s^t \bigl|Y_r^n - Y_{\bar{r}_n\vee s}^n\bigr|^2\bigl(1+\bigl|Y_r^n - Y_s^n\bigr|^2\bigr)\frac{|w_{\hat{r}_n} - w_{\bar{r}_n\vee s}|}{\Delta}\,\mathrm{d}r\biggr)^{2p}\biggr] \\
&\le C_p|t-s|^{2p} + C_p(t-s)^{2p-1}\int_s^t \mathbf{E}\big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r \\
&\le C_p|t-s|^p + C_p\int_s^t \mathbf{E}\big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mathbf{E}&\biggl(\biggl|\int_s^t V_s^n(r)\,\mathrm{d}r\biggr|^{2p} + \biggl|\int_s^t Z_s^n(r)\,\mathrm{d}r\biggr|^{2p}\biggr) \\
&\le \mathbf{E}\biggl[\biggl(\int_s^t \bigl(\bigl|Y_r^n - Y_s^n\bigr| + \bigl|Y_r^n - Y_s^n\bigr|^2\bigr)\bigl(1+|\dot{h}_r|\bigr)\,\mathrm{d}r\biggr)^{2p}\biggr] + (t-s)^{2p} \\
&\le C_p|t-s|^p\biggl(1+\int_s^t |\dot{h}_r|^2\mathrm{d}r\biggr)^p \\
&\qquad + C_p\int_s^t \mathbf{E}\big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r\biggl(1+\int_s^t |\dot{h}_r|^2\,\mathrm{d}r\biggr)^p.
\end{aligned}
$$

Summing up we have

$$
\mathbf{E}\big|Y_t^n - Y_s^n\big|^{4p} \le C_p|t-s|^p + C_p\int_s^t \mathbf{E}\big|Y_r^n - Y_s^n\big|^{4p}\,\mathrm{d}r\biggl(1+\int_s^t |\dot{h}_r|^2\,\mathrm{d}r\biggr)^p,
$$

which together with Gronwall's lemma yields

$$
\mathbf{E}\big|Y_t^n - Y_s^n\big|^{4p} \le C_p|t-s|^p.
$$

It follows from this estimate and Lemma 3.8 that

$$\mathbf{E}|\phi_t^n - \phi_s^n|^{4p} \le C_p|t-s|^p.$$

Now (3.5) holds due to Kolmogorov's continuity criterion. □

PROPOSITION 3.4.

$$\mathbf{E} \sup_{t\in[0,1]} |Y_t^n|^{2p} < C_p(1+|x|^{2p}), \qquad \sup_n \mathbf{E}[(|\phi^n|_1)^{2p}] < C_p.$$

PROOF. Using Proposition 3.3, choose a $\theta \in [0, \frac{1}{4})$, and we get

$$\begin{aligned}
\mathbf{E} \sup_{t\in[0,1]} |Y_t^n|^{2p} &\le 2^{2p-1}\mathbf{E} \sup_{t\in[0,1]} |Y_t^n - x|^{2p} + 2^{2p-1}|x|^{2p} \\
&\le 2^{2p-1}\mathbf{E}\left(\sup_{t\in[0,1]} \frac{|Y_t^n - x|}{|t|^\theta}|t|^\theta\right)^{2p} + 2^{2p-1}|x|^{2p} \\
&\le 2^{2p-1}\mathbf{E}\|Y^n\|_{[0,1],\theta}^{2p} + 2^{2p-1}|x|^{2p} \\
&\le C_p(1+|x|^{2p}).
\end{aligned}$$

Similar to [4], Theorem 3.6, by (D) we get for all $0 \le s < t \le 1$,

$$|\phi^n|_t^s \le C(|t-s|R^{-4}\|Y^n\|_{[s,t],\theta}^4 + 1)\|\phi^n\|_{[s,t]}.$$

From this and Proposition 3.3,

$$\begin{aligned}
\mathbf{E}[(|\phi^n|_1)^{2p}] &\le C_p\mathbf{E}[(R^{-4}\|Y^n\|_{[0,1],\theta}^4 + 1)^{2p}\|\phi^n\|_{[0,1]}^{2p}] \\
&\le C_{R,p} < \infty.
\end{aligned}$$

□

PROPOSITION 3.5.

$$\sup_{1\le k\le 2^n} \mathbf{E}(|Y_{t_k}^n - Z_{t_k}|^2) \le C\left[\Delta^{\theta/2} + \sup_{2\le k\le 2^n}\left(\int_{t_{k-2}}^{t_k} |\dot{h}_s|^2\,\mathrm{d}s\right)^{1/2}\right], \qquad \theta \in (0,1).$$

PROOF. Set

$$\begin{aligned}
&\mu_n(t) := e^{-(2/\gamma)(\varphi(Y_t^n)+\varphi(Z_t))}, \qquad m_n(t) := \mu_n(t)|Y_t^n - Z_t|^2, \\
&a(t) := \mathbf{E}(m_n(t)).
\end{aligned}$$

Using the condition $\varphi \in \mathcal{C}_b^2$, Lemma 3.4 and (3.8) it is trivial to prove the following:

LEMMA 3.9.

$$\mathbf{E}\Big(\sup_{t,t'\in[t_{k-2},t_k]} |\mu_n(t) - \mu_n(t')|^2\Big) \le C\Delta,$$

$$\mathbf{E}\Big(\sup_{t,t'\in[t_{k-2},t_k]} |m_n(t) - m_n(t')|\Big) \le C\Delta^{1/2}.$$

For all $t_{k-1} \le t \le t_k$, $2 \le k \le 2^n$,

$$\mathrm{d}\mu_n(t)\big|Y_t^n - Z_t\big|^2 = \sum_{i=1}^{11} \mathrm{d}I_i(t) + 2\mu_n(t)\langle Y_t^n - Z_t, \mathrm{d}\phi_t^n - \mathrm{d}\psi_t\rangle$$
$$- \frac{2}{\gamma}\mu_n(t)\big|Y_t^n - Z_t\big|^2\big(\langle D\varphi(Y_t^n), \mathrm{d}\phi_t^n\rangle + \langle D\varphi(Z_t), \mathrm{d}\psi_t\rangle\big),$$

where

$$I_1(s) := 2\int_{t_{k-1}}^s \mu_n(t)\langle Y_t^n - Z_t, \sigma(Y_t^n) - \sigma(Z_t)\rangle \dot{h}_t\,\mathrm{d}t,$$

$$I_2(s) := 2\int_{t_{k-1}}^s \mu_n(t)\langle Y_t^n - Z_t, \tilde{b}(Y_t^n) - b(Z_t)\rangle\,\mathrm{d}t,$$

$$I_3(s) := 2\int_{t_{k-1}}^s \mu_n(t)\langle Y_t^n - Z_t, \sigma(Y_t^n) - \sigma(Y_{\bar{t}_n})\rangle\,\mathrm{d}w_t,$$

$$I_4(s) := 2\int_{t_{k-1}}^s \mu_n(t)\big(\mathrm{tr}(\sigma\sigma^*)(Y_t^n) - \mathrm{tr}(\sigma\sigma^*)(Y_{\bar{t}_n}^n)\big)\,\mathrm{d}t$$
$$+ \int_{t_{k-1}}^s \big(\mu_n(t) - \mu_n(\bar{t}_n)\big)\,\mathrm{tr}(\sigma\sigma^*)(Y_{\bar{t}_n}^n)\,\mathrm{d}t,$$

$$I_5(s) := 2\int_{t_{k-1}}^s \mu_n(t)\Big(\langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle$$
$$+ \int_{t_{k-1}}^s \mu_n(\bar{t}_n)\,\mathrm{tr}(\sigma\sigma^*)(Y_{\bar{t}_n}^n)\Big)\,\mathrm{d}t,$$

$$I_6(s) := -2\int_{t_{k-1}}^s \mu_n(t)\langle Y_t^n - Z_t, \big(\sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\big)\dot{w}_t^n\rangle\,\mathrm{d}t,$$

$$I_7(s) := -\frac{2}{\gamma}\int_{t_{k-1}}^s \mu_n(t)\big|Y_t^n - Z_t\big|^2\langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle,$$

$$I_8(s) := -\frac{2}{\gamma}\int_{t_{k-1}}^s \mu_n(t)\big|Y_t^n - Z_t\big|^2$$
$$\times \big(\langle D\varphi(Y_t^n), \sigma(Y_t^n)\dot{h}_t\rangle + \langle D\varphi(Z_t), \sigma(Z_t)\dot{h}_t\rangle\big)\,\mathrm{d}t,$$

$$I_9(s) := -\frac{2}{\gamma}\int_{t_{k-1}}^s \mu_n(t)\big|Y_t^n - Z_t\big|^2$$
$$\times \Big(\langle D\varphi(Y_t^n), \tilde{b}(Y_t^n)\rangle + \langle D\varphi(Z_t), b(Z_t)\rangle$$
$$+ \frac{1}{2}\,\mathrm{tr}\big(D^2\varphi(Y_t^n)\sigma\sigma^*(Y_t^n)\big)\Big)\,\mathrm{d}t,$$

$$I_{10}(s) := -\frac{4}{\gamma}\int_{t_{k-1}}^{s} \mu_n(t)\Big(\sum_i \langle D\varphi(Y_t^n), \sigma(Y_t^n)e_i\rangle\langle Y_t^n - Z_t, \sigma(Y_t^n)e_i\rangle\Big)\,\mathrm{d}t,$$

$$I_{11}(s) := \frac{2}{\gamma^2}\int_{t_{k-1}}^{s} \mu_n(t)\big|Y_t^n - Z_t\big|^2\big|D\varphi(Y_t^n)\sigma(Y_t^n)\big|^2\,\mathrm{d}t.$$

By (C),

$$\langle Y_t^n - Z_t,\, \mathrm{d}\phi_t^n - \mathrm{d}\psi_t\rangle - \frac{1}{\gamma}\big|Y_t^n - Z_t\big|^2\big(\langle D\varphi(Y_t^n), \mathrm{d}\phi_t^n\rangle + \langle D\varphi(Z_t), \mathrm{d}\psi_t\rangle\big) \le 0.$$

Thus

$$\mu_n(t_k)\big|Y_{t_k}^n - Z_{t_k}\big|^2 \le \mu_n(t_{k-1})\big|Y_{t_{k-1}}^n - Z_{t_{k-1}}\big|^2 + \sum_{i=1}^{11} I_i(t_k). \tag{3.9}$$

By the hypotheses $\sigma \in \mathcal{C}_b^2, \varphi \in \mathcal{C}_b^2$,

$$\begin{aligned}
&\big|I_1(t_k) + I_8(t_k)\big| \\
&\quad\le \bigg|\int_{t_{k-1}}^{t_k} 2\mu_n(s)\langle Y_s^n - Z_s, \sigma(Y_s^n) - \sigma(Z_s)\rangle\dot{h}_s\,\mathrm{d}s\bigg| \\
&\qquad + \bigg|\frac{2}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(s)\big|Y_s^n - Z_s\big|^2\big(\langle D\varphi(Y_s^n), \sigma(Y_s^n)\rangle \\
&\qquad\qquad + \langle D\varphi(Z_s), \sigma(Z_s)\rangle\big)\dot{h}_s\,\mathrm{d}s\bigg| \\
&\quad\le C\int_{t_{k-1}}^{t_k} m_n(s)|\dot{h}_s|\,\mathrm{d}s.
\end{aligned}$$

For I_3, $\mathbf{E}(I_3(t_k)) = \mathbf{E}(\int_{t_{k-1}}^{t_k} 2\langle Y_t^n - Z_t, \sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\rangle\,\mathrm{d}w_t) = 0$.

Throughout the proof we need several lemmas which will be proved afterward. Now we deal with the terms I_2 and I_6. Note that for I_2,

$$\begin{aligned}
I_2 &= 2\int_{t_{k-1}}^{t_k} \mu_n(t)\langle Y_t^n - Z_t, \tilde{b}(Y_t^n) - b(Z_t)\rangle\,\mathrm{d}t \\
&= 2\int_{t_{k-1}}^{t_k} \mu_n(t)\langle Y_t^n - Z_t - \big(Y_{\bar{t}_n}^n - Z_{\bar{t}_n}\big), \tilde{b}(Y_t^n) - b(Z_t)\rangle\,\mathrm{d}t \\
&\quad + 2\int_{t_{k-1}}^{t_k} \big(\mu_n(t) - \mu_n(\bar{t}_n)\big)\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \tilde{b}(Y_t^n) - b(Z_t)\rangle\,\mathrm{d}t \\
&\quad + 2\int_{t_{k-1}}^{t_k} \mu_n(\bar{t}_n)\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \tilde{b}(Y_t^n) - \tilde{b}(Y_{\bar{t}_n}^n) - b(Z_t) + b(Z_{\bar{t}_n})\rangle\,\mathrm{d}t \\
&\quad + 2\int_{t_{k-1}}^{t_k} \mu_n(\bar{t}_n)\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, b(Y_{\bar{t}_n}^n) - b(Z_{\bar{t}_n})\rangle\,\mathrm{d}t
\end{aligned}$$

$$+2\int_{t_{k-1}}^{t_k}\mu_n(t_{k-2})\langle Y^n_{t_{k-2}}-Z_{t_{k-2}},\tilde b(Y^n_{t_{k-2}})-b(Y^n_{t_{k-2}})\rangle\,\mathrm{d}t$$
$$=:\sum_{i=1}^5 I_{2,i}.$$

Taking expectations and applying Lemmas 3.4, 3.7 and 3.9, we get

$$\begin{aligned}
&\left|\mathbf{E}\sum_{i=1}^4 I_{2,i}\right|\\
&\quad\le C\Delta^{3/2}+\left|\mathbf{E}\int_{t_{k-1}}^{t_k}\big(\mu_n(t)-\mu_n(\bar t_n)\big)\big|Y^n_{\bar t_n}-Z_{\bar t_n}\big|\big|\tilde b(Y^n_t)-b(Z_t)\big|\,\mathrm{d}t\right|\\
&\qquad+C\left|\mathbf{E}\int_{t_{k-1}}^{t_k}\mu_n(\bar t_n)\big|Y^n_{\bar t_n}-Z_{\bar t_n}\big|\big|Y^n_t-Z_t-\big(Y^n_{\bar t_n}-Z_{\bar t_n}\big)\big|\,\mathrm{d}t\right|+Ca(t_{k-2})\Delta\\
&\quad\le C\Delta^{3/2}+Ca(t_{k-2})\Delta.
\end{aligned}$$

Note that

$$\begin{aligned}
I_6&=-2\int_{t_{k-1}}^{t_k}\mu_n(t)\langle Y^n_t-Z_t-\big(Y^n_{\bar t_n}-Z_{\bar t_n}\big),\sigma\big(Y^n_t\big)-\sigma\big(Y^n_{\bar t_n}\big)\rangle\,\mathrm{d}w^n_t\\
&\quad-2\int_{t_{k-1}}^{t_k}\big(\mu_n(t)-\mu_n(\bar t_n)\big)\langle Y^n_{\bar t_n}-Z_{\bar t_n},\sigma\big(Y^n_t\big)-\sigma\big(Y^n_{\bar t_n}\big)\rangle\,\mathrm{d}w^n_t\\
&\quad-2\int_{t_{k-1}}^{t_k}\mu_n(\bar t_n)\langle Y^n_{\bar t_n}-Z_{\bar t_n},\sigma\big(Y^n_t\big)-\sigma\big(Y^n_{\bar t_n}\big)\rangle\,\mathrm{d}w^n_t=:\sum_i I_{6,i}
\end{aligned}$$

and

$$\big|\mathbf{E}(I_{6,1}+I_{6,2})\big|\le C\Delta^{3/2}.$$

As for $I_{6,3}+I_{2,5}$, note that $I_{6,3}+I_{2,5}=-2A^n_k$, where

$$A^n_k:=\int_{t_{k-1}}^{t_k}\mu_n(\bar t_n)\left\langle Y^n_{\bar t_n}-Z_{\bar t_n},\big(\sigma\big(Y^n_t\big)-\sigma\big(Y^n_{\bar t_n}\big)\big)\dot w^n_t-\frac12(\nabla\sigma)\sigma\big(Y^n_{\bar t_n}\big)\right\rangle\mathrm{d}t, \tag{3.10}$$

and by Lemma 3.10,

$$\left|\mathbf{E}\left(\sum_{i=1}^{2^n}A^n_i\right)\right|\le C\left[\Delta^{1/2}+\sup_{2\le k\le 2^n}\left(\int_{t_{k-2}}^{t_k}|\dot h_s|^2\,\mathrm{d}s\right)^{1/2}\right].$$

By Lemmas 3.7 and 3.9,

$$\begin{aligned}
&\big|\mathbf{E}I_4(t_k)\big|\\
&\quad\le\left|\mathbf{E}\int_{t_{k-1}}^{t_k}\mu_n(t)\big(\mathrm{tr}\big(\sigma\sigma^*\big)\big(Y^n_t\big)-\mathrm{tr}\big(\sigma\sigma^*\big)\big(Y^n_{\bar t_n}\big)\big)\,\mathrm{d}t\right|
\end{aligned}$$

$$+\left|\mathbf{E}\int_{t_{k-1}}^{t_k}(\mu_n(t)-\mu_n(\bar{t}_n))\operatorname{tr}(\sigma\sigma^*)(Y^n_{\bar{t}_n})\,\mathrm{d}t\right|\le C\Delta^{3/2},$$

$$\begin{aligned}
&\left|\mathbf{E}[I_9(t_k)+I_{11}(t_k)]\right|\\
&\quad\le\frac{2}{\gamma}\left|\mathbf{E}\int_{t_{k-1}}^{t_k}(m_n(s)-m_n(\bar{s}_n)+m_n(\bar{s}_n))\right.\\
&\qquad\qquad\times\Big(\left|D\varphi(Y^n_s)\right|\left|\tilde{b}(Y^n_s)\right|+\left|D\varphi(Z_s)\right|\left|b(Z_s)\right|\\
&\qquad\qquad\qquad\left.+\frac{1}{2}\operatorname{tr}(D^2\varphi\sigma\sigma^*(Y^n_s))\Big)\,\mathrm{d}s\right|\\
&\qquad+\frac{2}{\gamma^2}\left|\mathbf{E}\int_{t_{k-1}}^{t_k}(m_n(s)-m_n(\bar{s}_n)+m_n(\bar{s}_n))\left|D\varphi(Y^n_s)\sigma(Y^n_s)\right|^2\mathrm{d}s\right|\\
&\quad\le C\Delta^{3/2}+Ca(t_{k-2})\Delta.
\end{aligned}$$

For I_5, we have

$$\begin{aligned}
I_5(t_k)&=2\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\langle Y^n_t-Z_t,\sigma(Y^n_{\bar{t}_n})\rangle(\mathrm{d}w_t-\mathrm{d}w^n_t)\\
&\quad+\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\operatorname{tr}(\sigma\sigma^*)(Y^n_{\bar{t}_n})\,\mathrm{d}t\\
&\quad+2\int_{t_{k-1}}^{t_k}(\mu_n(t)-\mu_n(\bar{t}_n))\langle Y^n_t-Z_t,\sigma(Y^n_{\bar{t}_n})\rangle(\mathrm{d}w_t-\mathrm{d}w^n_t)\\
&=:I_{5,1}+I_{5,2}.
\end{aligned}$$

However, by Lemma 3.11,

$$\left|\mathbf{E}\left(\sum_{i=0}^{2^n}I_{5,1}(t_i)\right)\right|\le C\Delta^{\theta/2}\qquad\forall\theta\in(0,1).$$

With respect to $I_{5,2}$,

$$\begin{aligned}
I_{5,2}&=-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t}\mu_n(s)\langle D\varphi(Y^n_s),\sigma(Y^n_s)(\mathrm{d}w_s-\mathrm{d}w^n_s)\rangle\\
&\qquad\qquad\times\langle Y^n_t-Z_t,\sigma(Y^n_{\bar{t}_n})\rangle(\mathrm{d}w_t-\mathrm{d}w^n_t)\\
&\quad-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t}\mu_n(s)(\langle D\varphi(Y^n_s),\tilde{b}(Y^n_s)\rangle\,\mathrm{d}s+\langle D\varphi(Z_s),b(Z_s)\rangle\,\mathrm{d}s)\\
&\qquad\qquad\times\langle Y^n_t-Z_t,\sigma(Y^n_{\bar{t}_n})\rangle(\mathrm{d}w_t-\mathrm{d}w^n_t)\\
&\quad-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t}\mu_n(s)(\langle D\varphi(Y^n_s),\sigma(Y^n_s)\rangle\dot{h}_s\,\mathrm{d}s+\langle D\varphi(Z_s),\sigma(Z_s)\rangle\dot{h}_s\,\mathrm{d}s)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&\quad - \frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(s)\big(\langle D\varphi(Y_s^n), \mathrm{d}\phi_s^n\rangle + \langle D\varphi(Z_s), \mathrm{d}\psi_s\rangle\big) \\
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&\quad + \frac{2}{\gamma^2}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(s)\big|D\varphi(Y_s^n)\sigma(Y_s^n)\big|^2\,\mathrm{d}s \\
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&\quad - \frac{1}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(s)\,\mathrm{tr}\big(D^2\varphi(Y_s^n)\sigma(Y_s^n)\big)\,\mathrm{d}s \\
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&=: \sum_{i=1}^{6} I_{5,2,i}.
\end{aligned}
$$

Applying the BDG inequality, the conditions $\sigma \in \mathcal{C}_b^2$, $b \in \mathcal{C}_b^1$, $\varphi \in \mathcal{C}_b^2$, Lemmas 3.4, 3.7 and Proposition 3.4, we get

$$
\begin{aligned}
|\mathbf{E}I_{5,2,2}| &\le C\mathbf{E}\Big(\int_{t_{k-1}}^{t_k} C\Delta^2\big|Y_t^n - Z_t\big|^2\big\|\sigma(Y_{\bar{t}_n}^n)\big\|^2\,\mathrm{d}t\Big)^{1/2} \\
&\quad + \Big|\mathbf{E}\int_{t_{k-1}}^{t_k} C\Delta\big|Y_t^n - Z_t\big|\big\|\sigma(Y_{\bar{t}_n}^n)\big\|\Big|\mathrm{d}w_t^n\Big| \\
&\le C\Delta^{3/2}, \\
|\mathbf{E}I_{5,2,3}| &\le C\mathbf{E}\Big(\int_{t_{k-1}}^{t_k}\big|Y_t^n - Z_t\big|^2\big\|\sigma(Y_{\bar{t}_n}^n)\big\|^2\Big|\int_{\bar{t}_n}^{t}\dot{h}_s\,\mathrm{d}s\Big|^2\,\mathrm{d}t\Big)^{1/2} \\
&\quad + C\mathbf{E}\int_{t_{k-1}}^{t_k}\big|Y_t^n - Z_t\big|\big\|\sigma(Y_{\bar{t}_n}^n)\big\|\Big|\int_{\bar{t}_n}^{t}\dot{h}_s\,\mathrm{d}s\Big||\mathrm{d}w_t^n| \\
&\le C\Delta^{1/2}\int_{t_{k-2}}^{t_k}|\dot{h}_s|\,\mathrm{d}s, \\
\big|\mathbf{E}(I_{5,2,5} + I_{5,2,6})\big| &\le C\Delta^{3/2}, \\
|\mathbf{E}I_{5,2,4}| &\le \frac{4}{\gamma}\Big|\mathbf{E}\Big(\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t}\mu_n(s)\langle D\varphi(Y_s^n) - D\varphi(Y_{\bar{t}_n}^n), \mathrm{d}\phi_s^n\rangle \\
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n)\Big)\Big| \\
&\quad + \Big|\mathbf{E}\Big(\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t}\mu_n(s)\langle D\varphi(Z_s) - D\varphi(Z_{\bar{t}_n}), \mathrm{d}\psi_s\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n)\Big)\Big| \\
&\quad + \Big|\mathbf{E}\Big(\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(s)\big(\langle D\varphi(Y_{\bar{t}_n}^n), \mathrm{d}\phi_s^n\rangle + \langle D\varphi(Z_{\bar{t}_n}), \mathrm{d}\psi_s\rangle\big) \\
&\qquad\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n)\Big)\Big| \\
&\le C\Delta^{3/2} + C\mathbf{E}G_k,
\end{aligned}
$$

where

$$
G_k := \max_{t\in[t_{k-1},t_k]} \big|Y_t^n - Z_t\big||\Delta w_{k-1}| \times \big(\big|\phi^n\big|_{t_k}^{t_{k-2}} + |\psi|_{t_k}^{t_{k-2}}\big). \tag{3.11}
$$

Again by Lemma 3.4 and Proposition 3.4,

$$
\begin{aligned}
\sum_{k=1}^{2^n} \mathbf{E}G_k &\le \mathbf{E}\Big(\max_{1\le k\le 2^n}\max_{t\in[t_{k-1},t_k]} \big|Y_t^n - Z_t\big||\Delta w_{k-1}| \times \big(\big|\phi^n\big|_1 + |\psi|_1\big)\Big) \\
&\le \Big[\mathbf{E}\Big(\sup_{0\le t\le 1}\big|Y_t^n - Z_t\big|^{2p}\Big)\mathbf{E}\Big(\sup_{1\le k\le 2^n}|\Delta w_k|^{2p}\Big)\Big]^{1/2p} \\
&\quad \times \big[\mathbf{E}\big(\big|\phi^n\big|_1 + |\psi|_1\big)^q\big]^{1/q} \\
&\le C\Delta^{(p-1)/2p}, \qquad p,q>1,\ 1/p + 1/q = 1
\end{aligned}
$$

and

$$
\begin{aligned}
I_{5,2,1} = &-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \big(\mu_n(s) - \mu_n(\bar{t}_n)\big)\langle D\varphi(Y_s^n), \sigma(Y_s^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\rangle \\
&\qquad \times \langle Y_t^n - Z_t, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(\bar{t}_n)\langle D\varphi(Y_s^n), \sigma(Y_s^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\rangle \\
&\qquad \times \langle Y_t^n - Z_t - \big(Y_{\bar{t}_n}^n - Z_{\bar{t}_n}\big), \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(\bar{t}_n)\langle D\varphi(Y_s^n), \big(\sigma(Y_s^n) - \sigma(Y_{\bar{t}_n}^n)\big)(\mathrm{d}w_s - \mathrm{d}w_s^n)\rangle \\
&\qquad \times \langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t} \mu_n(\bar{t}_n)\langle D\varphi(Y_s^n) - D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\rangle \\
&\qquad \times \langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \sigma(Y_{\bar{t}_n}^n)\rangle(\mathrm{d}w_t - \mathrm{d}w_t^n) \\
&-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(\bar{t}_n)\langle D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_{\bar{t}_n}^n)(w_t - w_{\bar{t}_n} - (w_t^n - w_{\bar{t}_n}^n))\rangle
\end{aligned}
$$

$$\times \langle Y^n_{\bar{t}_n} - Z_{\bar{t}_n}, \sigma(Y^n_{\bar{t}_n})\rangle(\mathrm{d}w_t - \mathrm{d}w^n_t)$$

$$=: \sum_{j=1}^{5} I^j_{5,2,1}.$$

Using the BDG inequality, the fact that $\sigma \in \mathcal{C}^2_b, \varphi \in \mathcal{C}^2_b$, Lemmas 3.4, 3.7, 3.9 and Proposition 3.4, we get

$$\begin{aligned}
|\mathbf{E}I^1_{5,2,1}| &\le C\Delta(\mathbf{E}|\Delta w_{k-1}|^2)^{1/2} \le C\Delta^{3/2},\\
|\mathbf{E}I^2_{5,2,1}| &\le C\Delta^{1/2}\left(\mathbf{E}\left(\int_{t_{k-1}}^{t_k} |Y^n_t - Z_t - (Y^n_{\bar{t}_n} - Z_{\bar{t}_n})|^2 |\Delta w_{k-1}|^2 \Delta^{-1}\,\mathrm{d}t\right)\right)^{1/2}\\
&\le C\Delta^{3/2},\\
|\mathbf{E}I^j_{5,2,1}| &\le C\Delta^{3/2}, \qquad j = 3, 4
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}(I^5_{5,2,1}|\mathscr{F}_{t_{k-2}})\\
&\quad = \frac{2\Delta}{\gamma}\mu_n(t_{k-2})\sum_i \langle D\varphi(Y^n_{t_{k-2}}), \sigma(Y^n_{t_{k-2}})e_i\rangle\langle Y^n_{t_{k-2}} - Z_{t_{k-2}}, \sigma(Y^n_{t_{k-2}})e_i\rangle.
\end{aligned} \tag{3.12}$$

This estimate will be used in Lemma 3.12.

As for the term I_7,

$$\begin{aligned}
I_7 &= -\frac{2}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)|Y^n_t - Z_t|^2 \langle D\varphi(Y^n_s), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle\\
&= -\frac{2}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)|Y^n_{\bar{t}_n} - Z_{\bar{t}_n}|^2 \langle D\varphi(Y^n_s), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle\\
&\quad - \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)\int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s, \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle\\
&\qquad\qquad \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle\\
&\quad - \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)\int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s, (\sigma(Y^n_s) - \sigma(Z_s))\dot{h}_s\rangle\,\mathrm{d}s\\
&\qquad\qquad \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle\\
&\quad - \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)\int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s, \tilde{b}(Y^n_s) - b(Z_s)\rangle\,\mathrm{d}s\\
&\qquad\qquad \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle\\
&\quad - \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)\int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s, \mathrm{d}\phi^n_s - \mathrm{d}\psi_s\rangle
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad - \frac{2}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t)\int_{\bar{t}_n}^{t} \mathrm{tr}(\sigma\sigma^*(Y_s^n))\,\mathrm{d}s \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&=: \sum_{i=1}^{6} I_{7,i}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
I_{7,1} &= -\frac{2}{\gamma}\int_{t_{k-1}}^{t_k} (\mu_n(t) - \mu_n(\bar{t}_n))|Y_{\bar{t}_n}^n - Z_{\bar{t}_n}|^2 \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad - \frac{2}{\gamma}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&= \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} |Y_{\bar{t}_n}^n - Z_{\bar{t}_n}|^2 \int_{\bar{t}_n}^{t} \mu_n(s)\langle D\varphi(Y_s^n), \sigma(Y_s^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad + \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} |Y_{\bar{t}_n}^n - Z_{\bar{t}_n}|^2 \int_{\bar{t}_n}^{t} \mu_n(s)\bigl(\langle D\varphi(Y_s^n), \sigma(Y_s^n)\rangle \\
&\qquad\qquad + \langle D\varphi(Z_s), \sigma(Z_s)\rangle\bigr)\dot{h}_s\,\mathrm{d}s \\
&\qquad\qquad \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad + \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} |Y_{\bar{t}_n}^n - Z_{\bar{t}_n}|^2 \int_{\bar{t}_n}^{t} \mu_n(s)\bigl(\langle D\varphi(Y_s^n), \tilde{b}(Y_s^n)\rangle + \langle D\varphi(Z_s), b(Z_s)\rangle\bigr)\,\mathrm{d}s \\
&\qquad\qquad \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad + \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} |Y_{\bar{t}_n}^n - Z_{\bar{t}_n}|^2 \int_{\bar{t}_n}^{t} \mu_n(s)\bigl(\langle D\varphi(Y_s^n), \mathrm{d}\phi_s^n\rangle + \langle D\varphi(Z_s), \mathrm{d}\psi_s\rangle\bigr) \\
&\qquad\qquad \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad + \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} |Y_{\bar{t}_n}^n - Z_{\bar{t}_n}|^2 \int_{\bar{t}_n}^{t} \mu_n(s)\biggl(\frac{1}{2}\mathrm{tr}(D^2\varphi\sigma\sigma^*)(Y_s^n) - \frac{1}{\gamma}|D\varphi\sigma(Y_s^n)|^2\biggr)\,\mathrm{d}s \\
&\qquad\qquad \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\qquad\qquad - \frac{2}{\gamma}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&=: \sum_{i=1}^{6} I_{7,1}^{i}.
\end{aligned}
$$

For the first term $I^1_{7,1}$,

$$I^1_{7,1} = \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} |Y^n_{\bar{t}_n} - Z_{\bar{t}_n}|^2 \int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\ \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle \\ + \frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^t \langle D\varphi(Y^n_s), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\ \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle.$$

However, note that $\mathbf{E}\sup_{t\in[0,T]}|Y^n_t - Z_t|^4 < \infty$ by Lemma 3.4 and Proposition 3.3, and by Lemma 3.9,

$$\mathbf{E}\left|\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\ \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle\right|^2 \\ \le \mathbf{E}\left|\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\ \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w_t\rangle\right|^2 \\ + \mathbf{E}\left|\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)\,\mathrm{d}w_s\rangle \\ \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle\right|^2 \\ + \mathbf{E}\left|\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)\,\mathrm{d}w^n_s\rangle \\ \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle\right|^2 \\ \le C\Delta^3 + \left[\mathbf{E}\max_{t\in[t_{k-1},t_k]}\left(\int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)\,\mathrm{d}w_s\rangle\right)^4\right]^{1/2} \\ \times \left[\mathbf{E}\left(\int_{t_{k-1}}^{t_k}\langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle\right)^4\right]^{1/2}, \\ + \left[\mathbf{E}\max_{t\in[t_{k-1},t_k]}\left(\int_{\bar{t}_n}^t (\mu_n(s) - \mu_n(\bar{t}_n))\langle D\varphi(Y^n_s), \sigma(Y^n_s)\,\mathrm{d}w^n_s\rangle\right)^4\right]^{1/2} \\ \times \left[\mathbf{E}\left(\int_{t_{k-1}}^{t_k}\langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle\right)^4\right]^{1/2} \le C\Delta^3.$$

Similar to the term $I_{5,2,1}$,

$$\begin{aligned}
&\frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_s^n),\sigma(Y_s^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y_t^n),\sigma(Y_t^n)\,\mathrm{d}w_t^n\rangle \\
&\quad=\frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_s^n)-D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_s^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y_t^n),\sigma(Y_t^n)\,\mathrm{d}w_t^n\rangle \\
&\qquad+\frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_{\bar{t}_n}^n),(\sigma(Y_s^n)-\sigma(Y_{\bar{t}_n}^n))(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y_t^n),\sigma(Y_t^n)\,\mathrm{d}w_t^n\rangle \\
&\qquad+\frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y_t^n)-D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_t^n)\,\mathrm{d}w_t^n\rangle \\
&\qquad-\frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_t^n)-\sigma(Y_{\bar{t}_n}^n)\rangle\,\mathrm{d}w_t^n \\
&\qquad-\frac{4}{\gamma^2}\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)\rangle\,\mathrm{d}w_t^n
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbf{E}\left[\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle\right.\right. \\
&\qquad\qquad \left.\left.\times\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)\,\mathrm{d}w_t^n\rangle\right]\right| \\
&\quad=\left|\mathbf{E}\left[\mathbf{E}\left(\int_{t_{k-1}}^{t_k} m_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\times\langle D\varphi(Y_{\bar{t}_n}^n),\sigma(Y_{\bar{t}_n}^n)\,\mathrm{d}w_t^n\rangle\middle|\mathscr{F}_{t_{k-2}}\right)\right]\right| \\
&\quad\le Ca(t_{k-2})\Delta.
\end{aligned}$$

Summing up we get

$$|\mathbf{E}I_{7,1}^1|\le Ca(t_{k-2})\Delta+C\Delta^{3/2}.$$

Next, we have by Proposition 3.4 and Lemma 3.4,

$$\begin{aligned}
\left|\mathbf{E}I_{7,1}^2\right| &\le C\mathbf{E}\left(\left|Y_{t_{k-2}}^n - Z_{t_{k-2}}\right|^2 |\Delta w_{k-1}|\right) \int_{t_{k-2}}^{t_k} |\dot{h}_s|\,\mathrm{d}s \\
&\le C\Delta^{1/2} \int_{t_{k-2}}^{t_k} |\dot{h}_s|\,\mathrm{d}s, \\
\left|\mathbf{E}I_{7,1}^i\right| &\le C\Delta^{3/2}, \qquad i = 3, 5
\end{aligned}$$

and

$$\left|\mathbf{E}I_{7,1}^4\right| \le C\mathbf{E}(G_k^1),$$

where

$$\begin{aligned}
G_k^1 &:= \left|Y_{t_{k-2}}^n - Z_{t_{k-2}}\right|^2 \left(|\phi^n|_{t_k}^{t_{k-2}\vee 0} + |\psi|_{t_k}^{t_{k-2}\vee 0}\right)|\Delta w_{k-1}|, \\
\left|\mathbf{E}\left(\sum_k^{2^n} G_k^1\right)\right| &\le C\Delta^{\theta/2} \qquad \forall \theta \in (0,1).
\end{aligned} \tag{3.13}$$

For the term $I_{7,1}^6$,

$$\begin{aligned}
I_{7,1}^6 &= -\frac{2}{\gamma} m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&= -\frac{2}{\gamma} m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_t^n) - D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad - \frac{2}{\gamma} m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_{\bar{t}_n}^n), (\sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n))(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \\
&\quad - \frac{2}{\gamma} m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_{\bar{t}_n}^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_t^n) - D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_t^n)\,\mathrm{d}w_t^n\rangle\right]\right| \\
&\quad \le C\mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} |Y_t^n - Y_{\bar{t}_n}^n||\mathrm{d}w_t^n|\right] \\
&\quad = C\mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\int_{\bar{t}_n}^t \sigma(Y_s^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\right.\right. \\
&\qquad\qquad \left.\left. + \int_{\bar{t}_n}^t \sigma(Y_s^n)\dot{h}_s\,\mathrm{d}s + \int_{\bar{t}_n}^t \tilde{b}(Y_s^n)\,\mathrm{d}s + \phi_t^n - \phi_{\bar{t}_n}^n\right| |\mathrm{d}w_t^n|\right] \\
&\quad \le \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\int_{\bar{t}_n}^t (\sigma(Y_s^n) - \sigma(Y_{\bar{s}_n}^n) + \sigma(Y_{\bar{s}_n}^n))(\mathrm{d}w_s - \mathrm{d}w_s^n)\right| |\mathrm{d}w_t^n|\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|\mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\int_{\bar{t}_n}^{t} \sigma(Y_s^n)\dot{h}_s \,\mathrm{d}s + \int_{\bar{t}_n}^{t} \tilde{b}(Y_s^n)\,\mathrm{d}s + \phi_{\bar{t}_n}^n - \phi_t^n\right| |\mathrm{d}w_t^n|\right]\right| \\
&\le \mathbf{E}\left[m_n(t_{k-2}) \max_{t\in[t_{k-1},t_k]} \left|\int_{\bar{t}_n}^{t} (\sigma(Y_s^n) - \sigma(Y_{\bar{s}_n}^n))(\mathrm{d}w_s - \mathrm{d}w_s^n)\right| \int_{t_{k-1}}^{t_k} |\mathrm{d}w_t^n|\right] \\
&\quad + \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\int_{\bar{t}_n}^{t} \sigma(Y_{\bar{s}_n}^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\right| |\mathrm{d}w_t^n|\right] \\
&\quad + \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\int_{\bar{t}_n}^{t} \sigma(Y_s^n)\dot{h}_s \,\mathrm{d}s + \int_{\bar{t}_n}^{t} \tilde{b}(Y_s^n)\,\mathrm{d}s\right| |\mathrm{d}w_t^n|\right] \\
&\quad + \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \max_{t\in[t_{k-1},t_k]} |\phi_{\bar{t}_n}^n - \phi_t^n| |\mathrm{d}w_t^n|\right] \\
&\le \left[\mathbf{E}\big(m_n(t_{k-2})|\Delta w_{k-1}|\big)^2\right]^{1/2} \\
&\quad \times \left[\mathbf{E}\left(\max_{t\in[t_{k-1},t_k]} \int_{t_{k-2}}^{t} (\sigma(Y_s^n) - \sigma(Y_{\bar{s}_n}^n))(\mathrm{d}w_s - \mathrm{d}w_s^n)\right)^2\right]^{1/2} \\
&\quad + \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\left(\int_{t_{k-2}}^{t_{k-1}} + \int_{t_{k-1}}^{t}\right) \sigma(Y_{\bar{s}_n}^n)(\mathrm{d}w_s - \mathrm{d}w_s^n)\right| |\Delta w_{k-1}| \Delta^{-1}\,\mathrm{d}t\right] \\
&\quad + \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \left|\int_{\bar{t}_n}^{t} \sigma(Y_s^n)\dot{h}_s \,\mathrm{d}s + \int_{\bar{t}_n}^{t} \tilde{b}(Y_s^n)\,\mathrm{d}s\right| |\mathrm{d}w_t^n|\right] \\
&\quad + \mathbf{E}\left[m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \max_{t\in[t_{k-1},t_k]} |\phi_{\bar{t}_n}^n - \phi_t^n| |\mathrm{d}w_t^n|\right] \\
&\le C\Delta^{3/2} + Ca(t_{k-2})\Delta + C\mathbf{E}\left[|\Delta w_{k-1}| \int_{t_{k-2}}^{t_k} |\dot{h}_s|\,\mathrm{d}s\right] + C\mathbf{E}G_k^1,
\end{aligned}
$$

where G_k^1 is defined in (3.13) and

$$
\sum_{k=2}^{2^n} C\mathbf{E}\left[\int_{t_{k-2}}^{t_k} |\dot{h}_s|\,\mathrm{d}s |\Delta w_{k-1}|\right] \le C\Delta^{\theta/2} \int_0^1 |\dot{h}_s|\,\mathrm{d}s \qquad \forall \theta \in (0,1).
$$

Similarly,

$$
\begin{aligned}
&\left|\mathbf{E}\left(m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\rangle \mathrm{d}w_t^n\right)\right| \\
&\qquad \le Ca(t_{k-2})\Delta + C\Delta^{3/2} + C\mathbf{E}G_k^1, \\
&\mathbf{E}\left(m_n(t_{k-2}) \int_{t_{k-1}}^{t_k} \langle D\varphi(Y_{\bar{t}_n}^n), \sigma(Y_{\bar{t}_n}^n)\rangle (\mathrm{d}w_t - \mathrm{d}w_t^n)\right) \\
&\qquad =: \mathbf{E}G_k^2,
\end{aligned} \tag{3.14}
$$

while

$$
\begin{aligned}
\left|\sum_{k=2}^{2^n}\mathbf{E}G_k^2\right| &\le C\Big(\mathbf{E}\max_{1\le k\le 2^n} m_n^2(t_k)\Big)^{1/2}\\
&\quad\times\left(\mathbf{E}\max_{1\le k\le 2^n}\left|\int_0^{t_k}\langle D\varphi(Y^n_{\bar t_n}),\sigma(Y^n_{\bar t_n})\rangle(\mathrm{d}w_t-\mathrm{d}w_t^n)\right|^2\right)^{1/2}\\
&\le C\Delta^\theta \qquad \forall\theta\in(0,1).
\end{aligned}\tag{3.15}
$$

Here the last inequality follows since

$$
\begin{aligned}
&\mathbf{E}\max_{1\le k\le 2^n}\left|\int_0^{t_k}\langle D\varphi(Y^n_{\bar t_n}),\sigma(Y^n_{\bar t_n})\rangle(\mathrm{d}w_t-\mathrm{d}w_t^n)\right|^2\\
&\quad=\mathbf{E}\max_{1\le k\le 2^n}\left|\sum_{i=1}^{k-1}\langle D\varphi(Y^n_{t_{i-1}}),\sigma(Y^n_{t_{i-1}})\rangle(w_{t_{i+1}}-w_{t_i})\right.\\
&\qquad\qquad\left.-\sum_{i=1}^{k-1}\langle D\varphi(Y^n_{t_{i-1}}),\sigma(Y^n_{t_{i-1}})\rangle(w_{t_i}-w_{t_{i-1}})\right|^2\\
&\quad=\mathbf{E}\max_{1\le k\le 2^n}\left|\sum_{i=1}^{k-1}(\langle D\varphi(Y^n_{t_{i-1}}),\sigma(Y^n_{t_{i-1}})\rangle-\langle D\varphi(Y^n_{t_i}),\sigma(Y^n_{t_i})\rangle)(w_{t_{i+1}}-w_{t_i})\right.\\
&\qquad\qquad\left.-\langle D\varphi(x),\sigma(x)\rangle w_{t_1}+\langle D\varphi(Y^n_{t_{k-1}}),\sigma(Y^n_{t_{k-1}})\rangle(w_{t_k}-w_{t_{k-1}})\right|^2\\
&\quad\le C\sum_{i=1}^{3}J_{7,1}^{6,i},
\end{aligned}
$$

where by Lemma 3.7 and the conditions $\varphi,\sigma\in\mathcal{C}_b^2$,

$$
\begin{aligned}
J_{7,1}^{6,1}&:=\mathbf{E}\left(\max_{1\le k\le 2^n}\left|\int_0^{t_k}(\langle D\varphi(Y^n_{\bar t_n}),\sigma(Y^n_{\bar t_n})\rangle-\langle D\varphi(Y^n_{\hat t_n}),\sigma(Y^n_{\hat t_n})\rangle)\,\mathrm{d}w_t\right|^2\right)\le C\Delta,\\
J_{7,1}^{6,2}&:=\mathbf{E}(w_{t_1}^2)=\Delta,\\
J_{7,1}^{6,3}&:=\mathbf{E}\Big(\max_{1\le k\le 2^n}|\langle D\varphi(Y^n_{t_{k-1}}),\sigma(Y^n_{t_{k-1}})\rangle|^2|\Delta w_k|^2\Big)\le C\Delta^\theta \qquad \theta\in(0,1).
\end{aligned}
$$

We then consider $I_{7,2}$.

$$
\begin{aligned}
I_{7,2}=-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(t)\int_{\bar t_n}^{t}&\langle Y^n_s-Z_s-(Y^n_{\bar t_n}-Z_{\bar t_n}),\sigma(Y^n_s)(\mathrm{d}w_s-\mathrm{d}w_s^n)\rangle\\
&\times\langle D\varphi(Y^n_t),\sigma(Y^n_t)(\mathrm{d}w_t-\mathrm{d}w_t^n)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t) \int_{\bar{t}_n}^{t} \langle Y^n_{\bar{t}_n} - Z_{\bar{t}_n}, \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\
&\qquad\qquad\qquad \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)(\mathrm{d}w_t - \mathrm{d}w^n_t)\rangle \\
&=: I_{7,2,1} + I_{7,2,2}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{7,2,1} &= -\frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t) \int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s - (Y^n_{\bar{t}_n} - Z_{\bar{t}_n}), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\
&\qquad\qquad\qquad \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w_t\rangle \\
&\quad + \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \mu_n(t) \int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s - (Y^n_{\bar{t}_n} - Z_{\bar{t}_n}), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\
&\qquad\qquad\qquad \times \langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle \\
&=: I^{(1)}_{7,2,1} + I^{(2)}_{7,2,1}.
\end{aligned}
$$

However, note that $\mathbf{E}(I^{(1)}_{7,2,1}) = 0$, and by using the Schwartz and BDG inequalities, Lemmas 3.4 and 3.7,

$$
\begin{aligned}
&\big|\mathbf{E} I^{(2)}_{7,2,1}\big| \\
&\quad \le C\big(\mathbf{E}|\Delta w_{k-1}|^2\big)^{1/2} \\
&\qquad \times \left(\mathbf{E}\max_{t\in[t_{k-1},t_k]}\left|\int_{\bar{t}_n}^{t} \langle Y^n_s - Z_s - (Y^n_{\bar{t}_n} - Z_{\bar{t}_n}), \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle\right|^2\right)^{1/2} \\
&\quad \le C\Delta^{3/2}.
\end{aligned}
$$

Now according to Lemma 3.12,

$$
\mathbf{E}\big(I_{7,2,2}(t_k) + I^{5}_{5,2,1}(t_k) + I_{10}(t_k)\big) \le C\Delta^{3/2}.
$$

On the other hand, by the assumptions on f and σ, b, as well as Lemma 3.4 and Proposition 3.4,

$$
\begin{aligned}
|\mathbf{E} I_{7,3}| &\le C\mathbf{E}\int_{t_{k-1}}^{t_k} \mu_n(t) \int_{\bar{t}_n}^{t} |Y^n_s - Z_s|^2 |\dot{h}_s|\,\mathrm{d}s \\
&\qquad\qquad \times |D\varphi(Y^n_t)|\|\sigma(Y^n_t)\||\mathrm{d}w^n_t| \\
&\le C\mathbf{E}\Big(\sup_{t\in[t_{k-2},t_k]} |Y^n_t - Z_t|^2 |\Delta w_{k-1}|\Big)\int_{t_{k-2}}^{t_k} |\dot{h}_s|\,\mathrm{d}s \qquad (3.16)\\
&\le C\Delta^{1/2}\int_{t_{k-2}}^{t_k} |\dot{h}_s|\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathbf{E}I_{7,4}| &\le \frac{4}{\gamma}\bigg|\mathbf{E}\int_{t_{k-1}}^{t_k}\mu_n(t)\int_{\bar{t}_n}^{t}\big|Y_s^n-Z_s\big|\big|\tilde{b}\big(Y_s^n\big)-b(Z_s)\big|\,\mathrm{d}s \\
&\qquad \times\big\langle D\varphi\big(Y_t^n\big),\sigma\big(Y_t^n\big)\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)\big\rangle\bigg| \\
&\le C\bigg|\mathbf{E}\int_{t_{k-1}}^{t_k}\mu_n(t)\int_{\bar{t}_n}^{t}\big(\big|Y_s^n-Z_s\big|^2+1\big)\,\mathrm{d}s\,\mathrm{d}w_t^n\bigg| \\
&\le C\mathbf{E}\bigg(\int_{t_{k-1}}^{t_k}\int_{\bar{t}_n}^{t}m_n(s)\,\mathrm{d}s\big|\mathrm{d}w_t^n\big|\bigg) \\
&\quad + C\Delta\mathbf{E}\bigg(\int_{t_{k-2}}^{t_k}\big|\mathrm{d}w_t^n\big|\bigg) \\
&\le C\Delta^{3/2}, \\
I_{7,5} &= -\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(t)\int_{\bar{t}_n}^{t}\big\langle Y_s^n-Z_s-Y_{\bar{t}_n}^n+Z_{\bar{t}_n},\mathrm{d}\phi_s^n-\mathrm{d}\psi_s\big\rangle \\
&\qquad \times\big\langle D\varphi\big(Y_t^n\big),\sigma\big(Y_t^n\big)\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)\big\rangle \\
&\quad -\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(t)\big\langle Y_{\bar{t}_n}^n-Z_{\bar{t}_n},\phi_t^n-\phi_{\bar{t}_n}^n-(\psi_t-\psi_{\bar{t}_n})\big\rangle \\
&\qquad \times\big\langle D\varphi\big(Y_t^n\big),\sigma\big(Y_t^n\big)\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)\big\rangle \\
&=: I_{7,5,1}+I_{7,5,2}.
\end{aligned}
$$

By using (3.8),

$$
\begin{aligned}
&|\mathbf{E}I_{7,5,1}| \\
&\quad \le C\mathbf{E}\int_{t_{k-1}}^{t_k}\mu_n(t)\sup_{s\in[t_{k-2},t]}\big|Y_s^n-Z_s-Y_{t_{k-2}}^n+Z_{t_{k-2}}\big| \\
&\qquad\qquad \times\big(\big|\phi^n\big|_t^{t_{k-2}}+|\psi|_t^{t_{k-2}}\big)\big|\mathrm{d}w_t^n\big| \\
&\quad \le C\mathbf{E}\bigg(\big(\|Y^n\|_{[t_{k-2},t_k]}+\|Z\|_{[t_{k-2},t_k]}\big)\big(\big|\phi^n\big|_{t_k}^{t_{k-2}}+|\psi|_{t_k}^{t_{k-2}}\big)\int_{t_{k-1}}^{t_k}\big|\mathrm{d}w_t^n\big|\bigg) \\
&\quad \le C\Delta^{3/2}, \\
&|I_{7,5,2}| \le C\big|Y_{t_{k-2}}^n-Z_{t_{k-2}}\big| \\
&\qquad\qquad \times\bigg(\max_{t\in[t_{k-1},t_k]}\bigg|\int_{t_{k-1}}^{t}\big\langle D\varphi\big(Y_t^n\big),\sigma\big(Y_t^n\big)\,\mathrm{d}w_t\big\rangle\bigg|+|\Delta w_{k-1}|\bigg) \qquad (3.17)\\
&\qquad\qquad \times\big(\big|\phi^n\big|_{t_{k-2}}^{t_k}+|\psi|_{t_{k-2}}^{t_k}\big)=: G_k^3,
\end{aligned}
$$

while according to Lemma 3.4 and Proposition 3.4,

$$\begin{aligned}
&\sum_{k=1}^{2^n} \mathbf{E} G_k^3 \\
&\quad \le C \max_{1\le k\le 2^n} \Bigg(\mathbf{E}\Bigg[\big|Y_{t_{k-2}}^n - Z_{t_{k-2}}\big|^2 \\
&\qquad\qquad \times \Bigg(\max_{t\in[t_{k-1},t_k]} \bigg| \int_{t_{k-1}}^t \langle D\varphi(Y_t^n), \sigma(Y_t^n)\,\mathrm{d}w_t\rangle \bigg|^2 + |\Delta w_{k-1}|^2 \Bigg) \Bigg] \Bigg)^{1/2} \\
&\qquad \times \big(\mathbf{E}\big(|\phi^n|_1^0 + |\psi|_1^0 \big)^2 \big)^{1/2} \\
&\quad \le C\Delta^{1/2}.
\end{aligned}$$

Also by the boundedness of σ we have

$$\begin{aligned}
|\mathbf{E} I_{7,6}| &\le \frac{\gamma}{2} \bigg| \mathbf{E} \int_{t_{k-1}}^{t_k} \mu_n(t) \int_{\bar{t}_n}^t \operatorname{tr}(\sigma\sigma^*)(Y_s^n)\,\mathrm{d}s \times \langle D\varphi(Y_t^n), \sigma(Y_t^n)(\mathrm{d}w_t - \mathrm{d}w_t^n)\rangle \bigg| \\
&\le C\Delta^{3/2}.
\end{aligned}$$

Hence by applying all the above estimates to (3.9),

$$\begin{aligned}
a(t_k) - a(t_{k-1}) &\le Ca(t_{k-2})\Delta + C\Delta^{3/2} + b_k + \mathbf{E}\int_{t_{k-1}}^{t_k} m_n(t)|\dot{h}_t|\,\mathrm{d}t \\
&\le C\mathbf{E}\big(m_n(t_{k-2}) - m_n(t_{k-1}) + m_n(t_{k-1})\big)\Delta + C\Delta^{3/2} \\
&\quad + Cb_k + \mathbf{E}\int_{t_{k-1}}^{t_k} m_n(t)|\dot{h}_t|\,\mathrm{d}t \\
&\le Ca(t_{k-1})\Delta + C\Delta^{3/2} + Cb_k + \mathbf{E}\int_{t_{k-1}}^{t_k} m_n(t)|\dot{h}_t|\,\mathrm{d}t,
\end{aligned}$$

where

$$b_k := G_k + G_k^1 + G_k^2 + G_k^3 + A_k^n + C\Delta^{1/2}\int_{t_{k-2}}^{t_k} |\dot{h}_t|\,\mathrm{d}t,$$

and hence

$$\begin{aligned}
\sum_{k=1}^{2^n} b_k &= \mathbf{E}\Bigg[\sum_{k=1}^{2^n} \bigg(A_k^n + G_k + G_k^1 + G_k^2 + G_k^3 + C\Delta^{1/2}\int_{t_{k-2}}^{t_k} |\dot{h}_t|\,\mathrm{d}t \bigg) \Bigg] \\
&\le C\Bigg[\Delta^{\theta/2} + \sup_{2\le k\le 2^n} \bigg(\int_{t_{k-2}}^{t_k} |\dot{h}_s|^2\,\mathrm{d}s \bigg)^{1/2} \Bigg], \qquad \theta\in(0,1),
\end{aligned}$$

and $A_k^n, G_k, G_k^1, G_k^2, G_k^3$, are defined in (3.10), (3.11), (3.13), (3.14) and (3.17), respectively.

Therefore according to Bihari's inequality, by denoting $h_k := e^{C\int_{t_{k-1}}^{t_k} |\dot{h}_s|\,\mathrm{d}s}$, we have

$$\begin{aligned}
a(t_k) &\le \big[a(t_{k-1})(1+C\Delta) + C\Delta^{3/2} + b_k\big]h_k \\
&\le a(t_{k-1})h_k(1+C\Delta) + C\Delta^{3/2}h_k + b_k h_k \\
&\le \big[\big(a(t_{k-2})(1+C\Delta) + C\Delta^{3/2} + b_{k-1}\big)h_{k-1}\big]h_k(1+C\Delta) \\
&\quad + C\Delta^{3/2}h_k + b_k h_k \\
&\le \cdots \\
&\le (1+C\Delta)^k h_k \cdots h_1 a(t_0) + \sum_{i=0}^{k-1}(1+C\Delta)^i h_k \cdots h_{k-i}\big(C\Delta^{3/2} + b_{k-i}\big) \\
&\le e^{CT} e^{C\int_0^T |\dot{h}_t|\,\mathrm{d}t} C\bigg[\Delta^{\theta/2} + \sup_{1\le k\le 2^n}\bigg(\int_{t_{k-2}}^{t_k} |\dot{h}_s|^2\,\mathrm{d}s\bigg)^{1/2}\bigg], \qquad \theta\in(0,1),
\end{aligned}$$

and we obtain the desired result. □

LEMMA 3.10.

$$\bigg|\mathbf{E}\bigg(\sum_{i=1}^{2^n} A_i^n\bigg)\bigg| \le C\bigg[\Delta^{1/2} + \sup_{2\le k\le 2^n}\bigg(\int_{t_{k-2}}^{t_k} |\dot{h}_s|^2\,\mathrm{d}s\bigg)^{1/2}\bigg].$$

PROOF. Set

$$\zeta_t^n := \int_{\bar{t}_n}^t \big[\big(\sigma(Y_s^n) - \sigma(Y_{\bar{t}_n}^n)\big)(\mathrm{d}w_s - \mathrm{d}w_s^n) + \sigma(Y_s^n)\dot{h}_s\,\mathrm{d}s + \tilde{b}(Y_s^n)\,\mathrm{d}s\big].$$

We have for any $2\le k\le 2^n$,

$$\begin{aligned}
&\big|\mathbf{E}\big(A_k^n + A_{k+1}^n\big)\big| \\
&\quad = \bigg|\mathbf{E}\bigg(\int_{t_{k-1}}^{t_{k+1}} \mu_n(\bar{t}_n)\bigg\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \big(\sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\big)\dot{w}_t^n - \frac{1}{2}(\nabla\sigma)\sigma(Y_{\bar{t}_n}^n)\bigg\rangle\,\mathrm{d}t\bigg)\bigg| \\
&\quad = \bigg|\mathbf{E}\bigg(\int_{t_{k-1}}^{t_k} \mu_n(t_{k-2})\bigg\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \big(\sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\big)\dot{w}_t^n - \frac{1}{2}(\nabla\sigma)\sigma(Y_{\bar{t}_n}^n)\bigg\rangle\,\mathrm{d}t\bigg) \\
&\qquad + \mathbf{E}\bigg(\int_{t_k}^{t_{k+1}} \big(\mu_n(t_{k-1}) - \mu_n(t_{k-2})\big)\bigg\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \big(\sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\big)\dot{w}_t^n \\
&\qquad\qquad - \frac{1}{2}(\nabla\sigma)\sigma(Y_{\bar{t}_n}^n)\bigg\rangle\,\mathrm{d}t\bigg) \\
&\qquad + \mathbf{E}\bigg(\int_{t_k}^{t_{k+1}} \mu_n(t_{k-2})\bigg\langle Y_{\bar{t}_n}^n - Z_{\bar{t}_n}, \big(\sigma(Y_t^n) - \sigma(Y_{\bar{t}_n}^n)\big)\dot{w}_t^n
\end{aligned}$$

$$- \frac{1}{2}(\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big\rangle \mathrm{d}t\Big)\Big|$$

$$\leq \Big|\mathbf{E}\Big(\int_{t_{k-1}}^{t_{k+1}} \mu_n(t_{k-2})\Big\langle Y^n_{\bar{t}_n} - Z_{\bar{t}_n}, \big(\sigma\big(Y^n_t\big) - \sigma\big(Y^n_{\bar{t}_n}\big)\big)\dot{w}^n_t - \frac{1}{2}(\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big\rangle \mathrm{d}t\Big)\Big| + C\Delta^{3/2}.$$

Thus by continuing this procedure we get

$$\begin{aligned}
&\Big|\mathbf{E}\Big(\sum_{i=1}^{2^n} A^n_i\Big)\Big| \\
&\quad \leq C2^n\Delta^{3/2} + \Big|\mathbf{E}\Big(\int_0^1 \mu_n(t_1)\Big\langle Y^n_{\bar{t}_n} - Z_{\bar{t}_n}, \big(\sigma\big(Y^n_t\big) - \sigma\big(Y^n_{\bar{t}_n}\big)\big)\dot{w}^n_t - \frac{1}{2}(\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big\rangle \mathrm{d}t\Big)\Big| \\
&\quad \leq C\Delta^{1/2} + \Big[\mathbf{E}\Big(\mu_n^2(t_1)\sup_{0\leq t\leq 1}\big|Y^n_t - Z_t\big|^2\Big)\Big]^{1/2} \\
&\qquad \times\Big[\mathbf{E}\Big(\int_0^1\Big(\big(\sigma\big(Y^n_t\big) - \sigma\big(Y^n_{\bar{t}_n}\big)\big)\dot{w}^n_t - \frac{1}{2}(\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big)\mathrm{d}t\Big)^2\Big]^{1/2}.
\end{aligned}$$

Note that $\sigma \in \mathcal{C}^2_b$,

$$\begin{aligned}
&\big|\sigma\big(Y^n_t\big) - \sigma\big(Y^n_{\bar{t}_n}\big) - (\nabla\sigma)\big(Y^n_{\bar{t}_n}\big)\big(Y^n_t - Y^n_{\bar{t}_n}\big)\big| \leq C\big|Y^n_t - Y^n_{\bar{t}_n}\big|^2, \\
&Y^n_t - Y^n_{\bar{t}_n} = \int_{\bar{t}_n}^t \sigma\big(Y^n_s\big)\big(\mathrm{d}w_s - \mathrm{d}w^n_s\big) + \int_{\bar{t}_n}^t \sigma\big(Y^n_s\big)\dot{h}_s\,\mathrm{d}s \\
&\qquad + \int_{\bar{t}_n}^t \tilde{b}\big(Y^n_s\big)\,\mathrm{d}s + \phi^n_t - \phi^n_{\bar{t}_n}.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{2\leq k\leq 2^n}\Big|\int_0^{t_k}\Big(\big(\sigma\big(Y^n_t\big) - \sigma\big(Y^n_{\bar{t}_n}\big)\big)\dot{w}^n_t - \frac{1}{2}(\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big)\mathrm{d}t\Big|^2\Big] \\
&\quad \leq C\mathbf{E}\Big[\Big(\int_0^1 \big|Y^n_t - Y^n_{\bar{t}_n}\big|^2\big|\dot{w}^n_t\big|\,\mathrm{d}t\Big)^2\Big] \\
&\qquad + C\mathbf{E}\Big[\sup_{1\leq k\leq 2^n}\Big(\int_0^{t_k}\nabla\sigma\big(Y^n_{\bar{t}_n}\big)\big(\zeta_t + \phi^n_t - \phi^n_{\bar{t}_n}\big)\dot{w}^n_t\,\mathrm{d}t\Big)^2\Big] \\
&\qquad + C\mathbf{E}\Big[\sup_{1\leq k\leq 2^n}\Big(\int_0^{t_k}\Big((\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\int_{\bar{t}_n}^t \mathrm{d}w_s\,\dot{w}^n_t - (\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big)\mathrm{d}t\Big)^2\Big]
\end{aligned}$$

$$+ C\mathbf{E}\bigg[\sup_{1\le k\le 2^n}\bigg(\int_0^{t_k}\Big((\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\int_{\bar{t}_n}^t \mathrm{d}w^n_s\,\dot{w}^n_t - \frac{1}{2}(\nabla\sigma)\sigma\big(Y^n_{\bar{t}_n}\big)\Big)\,\mathrm{d}t\bigg)^2\bigg]$$

$$=: \sum_{\alpha=1}^4 T_\alpha.$$

Note that by (3.8),

$$\begin{aligned}
T_1 &= C\mathbf{E}\bigg[\bigg(\int_0^1 |Y^n_t - Y^n_{\bar{t}_n}|^2|\dot{w}^n_t|\,\mathrm{d}t\bigg)^2\bigg] \\
&\le C\mathbf{E}\bigg[\bigg(\sum_{i=1}^{2^n}\int_{t_{i-1}}^{t_i}|Y^n_t - Y^n_{\bar{t}_n}|^2|\dot{w}^n_t|\,\mathrm{d}t\bigg)^2\bigg] \\
&\le C2^{2n}\max_{1\le i\le 2^n}\mathbf{E}\Big(\sup_{t\in[t_{i-1},t_i]}|Y^n_t - Y^n_{t_{i-2}\vee 0}|^4|\Delta w_{i-1}|^2\Big) \le C\Delta.
\end{aligned}$$

By Lemma 3.7 and Proposition 3.4,

$$\begin{aligned}
T_2 &= C\mathbf{E}\bigg[\sup_{1\le k\le 2^n}\bigg(\int_0^{t_k}\nabla\sigma\big(Y^n_{\bar{t}_n}\big)(\zeta_t + \phi^n_t - \phi^n_{\bar{t}_n})\dot{w}^n_t\,\mathrm{d}t\bigg)^2\bigg] \\
&\le C\mathbf{E}\bigg[\bigg(\int_0^1|\nabla\sigma\big(Y^n_{\bar{t}_n}\big)\zeta_t\dot{w}^n_t|\,\mathrm{d}t\bigg)^2\bigg] \\
&\quad + C\mathbf{E}\bigg[\sup_{1\le k\le 2^n}\bigg(\int_0^{t_k}\nabla\sigma\big(Y^n_{\bar{t}_n}\big)(\phi^n_t - \phi^n_{\bar{t}_n})\dot{w}^n_t\,\mathrm{d}t\bigg)^2\bigg] \\
&\le C\bigg\{\mathbf{E}\int_0^1|\zeta_t|^4\,\mathrm{d}t\bigg\}^{1/2}\bigg\{\mathbf{E}\int_0^1|\dot{w}^n_t|^4\,\mathrm{d}t\bigg\}^{1/2} \\
&\quad + C\mathbf{E}\bigg[\sup_{1\le k\le 2^n}\bigg(\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\nabla\sigma\big(Y^n_{\bar{t}_n}\big)(\phi^n_t - \phi^n_{\bar{t}_n})\dot{w}^n_t\,\mathrm{d}t\bigg)^2\bigg] \\
&\le C\Delta\bigg(\Delta + \sup_{2\le k\le 2^n}\bigg(\int_{t_{k-2}}^{t_k}|\dot{h}_s|^2\,\mathrm{d}s\bigg)\bigg)\Delta^{-1} + C\mathbf{E}\Big[\Big(|\phi^n|_1\sup_{1\le i\le 2^n}|\Delta w_i|\Big)^2\Big] \\
&\le C\bigg(\Delta + \sup_{1\le k\le 2^n}\int_{t_{k-2}}^{t_k}|\dot{h}_s|^2\,\mathrm{d}s\bigg) \\
&\quad + \Big[\mathbf{E}\Big(\sup_{1\le k\le 2^n}|\Delta w_k|^{2p}\Big)\Big]^{1/p}\big[\mathbf{E}(|\phi^n|_1)^{2q}\big]^{1/q} \\
&\le C\bigg(\Delta^{1-1/p} + \sup_{1\le k\le 2^n}\int_{t_{k-2}}^{t_k}|\dot{h}_s|^2\,\mathrm{d}s\bigg) \qquad \forall p,q>1,\ 1/p+1/q=1.
\end{aligned}$$

Note that $T_3 = T_3^1 + T_3^2$, where

$$
\begin{aligned}
T_3^1 &:= C\mathbf{E} \sup_{1\le k\le 2^n} \biggl(\int_0^{t_k} \biggl((\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr) \int_{\bar t_n}^{\hat t_n} \mathrm{d}w_s\, \dot w^n_t - (\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\biggr)\,\mathrm{d}t\biggr)^2 \\
&\le C\mathbf{E}\biggl[\sum_{i=1}^{2^n}\bigl((\nabla\sigma)\sigma\bigl(Y^n_{t_{i-1}\vee 0}\bigr)\bigr)^2\bigl(|\Delta w_i|^2-\Delta\bigr)^2 \\
&\qquad + 2\sum_{i<j}(\nabla\sigma)\sigma\bigl(Y^n_{t_{i-1}\vee 0}\bigr)(\nabla\sigma)\sigma\bigl(Y^n_{t_{j-1}\vee 0}\bigr)\bigl(|\Delta w_i|^2-\Delta\bigr)\bigl(|\Delta w_j|^2-\Delta\bigr)\biggr] \\
&\le C\mathbf{E}\biggl[\sum_{i=1}^{2^n}\bigl((\nabla\sigma)\sigma\bigl(Y^n_{t_{i-1}\vee 0}\bigr)\bigr)^2\bigl(|\Delta w_i|^2-\Delta\bigr)^2\biggr] \\
&\le C2^n\Delta^2 \le C\Delta, \\
T_3^2 &:= C\mathbf{E} \sup_{1\le k\le 2^n} \biggl(\int_0^{t_k} (\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr) \int_{\hat t_n}^{t} \mathrm{d}w_s\, \dot w^n_t\,\mathrm{d}t\biggr)^2 \\
&\le C\mathbf{E}\biggl(\int_0^1 (\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\frac{\hat t_n+\Delta-t}{\Delta}(w_{\hat t_n}-w_{\bar t_n})\,\mathrm{d}w_t\biggr)^2 \\
&\le C\mathbf{E}\biggl(\int_0^1 \bigl|(\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\bigr|^2 |w_{\hat t_n}-w_{\bar t_n}|^2\,\mathrm{d}t\biggr) \\
&\le C\Delta.
\end{aligned}
$$

Also, $T_4 = T_4^1 + T_4^2$, where

$$
\begin{aligned}
T_4^1 &:= C\mathbf{E}\biggl[\sup_{1\le k\le 2^n}\biggl(\int_0^{t_k}(\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\int_{\bar t_n}^{\hat t_n}\mathrm{d}w^n_s\,\dot w^n_t\,\mathrm{d}t\biggr)^2\biggr] \\
&\le C\mathbf{E}\biggl(\int_0^1(\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\biggl(\int_{\bar t_n}^{\hat t_n}\mathrm{d}w^n_s\biggr)\mathrm{d}w_t\biggr)^2 \\
&\le C\mathbf{E}\biggl(\int_0^1\biggl|\int_{\bar t_n}^{\hat t_n}\mathrm{d}w^n_s\biggr|^2\mathrm{d}t\biggr) \\
&= C\sum_{i=0}^{2^n-1}\mathbf{E}\biggl(\int_{t_i}^{t_{i+1}}\biggl|\int_{\bar t_n}^{\hat t_n}\mathrm{d}w^n_s\biggr|^2\mathrm{d}t\biggr) \\
&\le C\Delta, \\
T_4^2 &:= C\mathbf{E}\sup_{1\le k\le 2^n}\biggl(\int_0^{t_k}\biggl((\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\int_{\hat t_n}^{t}\mathrm{d}w^n_s\,\dot w^n_t - \frac{1}{2}(\nabla\sigma)\sigma\bigl(Y^n_{\bar t_n}\bigr)\biggr)\mathrm{d}t\biggr)^2
\end{aligned}
$$

$$\leq C\mathbf{E}\sup_{1\leq k\leq 2^n}\Bigg(\sum_{i=0}^{k-2}(\nabla\sigma)\sigma\big(Y_{t_i}^n\big)\bigg(\bigg(\Delta^{-2}\int_{t_{i+1}}^{t_{i+2}}\int_{t_{i+1}}^{t}\mathrm{d}s\,\mathrm{d}t\bigg)(w_{t_{i+1}}-w_{t_i})^2 - \frac{1}{2}\Delta\bigg)\Bigg)^2$$

$$\leq C\Delta.$$

Summing these estimates we get

$$\left|\mathbf{E}\sup_{1\leq k\leq 2^n}\left(\sum_{i=1}^{k}A_i^n\right)\right|\leq C\left[\Delta^{1/2}+\sup_{1\leq k\leq 2^n}\left(\int_{t_{k-2}}^{t_k}|\dot{h}_s|^2\,\mathrm{d}s\right)^{1/2}\right]. \qquad \square$$

LEMMA 3.11. $|\mathbf{E}(\sum_{i=0}^{2^n}I_{5,1}(t_i))|\leq C\Delta^{\theta/2}, \forall\theta\in(0,1)$.

PROOF. Since

$$I_{5,1}=2\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\langle Y_t^n-Z_t,\sigma\big(Y_{\bar{t}_n}^n\big)\rangle\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)+\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\operatorname{tr}\big(\sigma\sigma^*\big)\big(Y_{\bar{t}_n}^n\big)\,\mathrm{d}t,$$

it is trivial to see that $\mathbf{E}(I_{5,1}-\tilde{I}_{5,1})=0$, where

$$\tilde{I}_{5,1}(t_k):=2\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\langle Y_t^n-Z_t-\big(Y_{\bar{t}_n}^n-Z_{\bar{t}_n}\big),\sigma\big(Y_{\bar{t}_n}^n\big)\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)\rangle + \int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\operatorname{tr}\big(\sigma\sigma^*\big)\big(Y_{\bar{t}_n}^n\big)\,\mathrm{d}t.$$

Note that

$$\begin{aligned}&\big|\mathbf{E}\big(\tilde{I}_{5,1}(t_i)+\tilde{I}_{5,1}(t_{i+1})\big)\big|\\ &\quad\leq C\Delta^{3/2}\\ &\qquad+\bigg|\mathbf{E}\int_{t_i}^{t_{i+1}}\mu_n(t_{i-2})\big(2\langle Y_t^n-Z_t-\big(Y_{\bar{t}_n}^n-Z_{\bar{t}_n}\big),\sigma\big(Y_{\bar{t}_n}^n\big)\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)\rangle\\ &\qquad\qquad+\operatorname{tr}\big(\sigma\sigma^*\big)\big(Y_{\bar{t}_n}^n\big)\,\mathrm{d}t\big)\bigg|.\end{aligned}$$

Continuing this process and using arguments similar to those used in (3.15) above, we get

$$\begin{aligned}&\left|\mathbf{E}\left(\sum_{i=0}^{2^n}\tilde{I}_{5,1}(t_i)\right)\right|\\ &\quad\leq C\Delta^{1/2}\\ &\qquad+\bigg|\mathbf{E}\int_0^1\mu_n(0)\big(2\langle Y_t^n-Z_t-\big(Y_{\bar{t}_n}^n-Z_{\bar{t}_n}\big),\sigma\big(Y_{\bar{t}_n}^n\big)\big(\mathrm{d}w_t-\mathrm{d}w_t^n\big)\rangle\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \operatorname{tr}(\sigma\sigma^*)(Y^n_{\bar t_n})\,\mathrm{d}t)\Big| \\
&\le C\Delta^{1/2} \\
&\quad + \left|\mathbf{E}\int_0^1 \mu_n(0)\left(2\left\langle\int_0^t \sigma(Y^n_{\bar t_n})(\mathrm{d}w_s - \mathrm{d}w^n_s), \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right\rangle\right.\right. \\
&\qquad\qquad \left.\left. + \operatorname{tr}(\sigma\sigma^*)(Y^n_{\bar t_n})\,\mathrm{d}t\right)\right| \\
&\quad + \left|\mathbf{E}\int_0^1 \mu_n(0)\left(2\left\langle\int_0^{\bar t_n} \sigma(Y^n_{\bar t_n})(\mathrm{d}w_s - \mathrm{d}w^n_s), \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right\rangle\right)\right| \\
&\quad + \left|\mathbf{E}\int_0^1 \mu_n(0)\left(2\left\langle\int_{\bar t_n}^t (\sigma(Y^n_s) - \sigma(Y^n_{\bar t_n}))(\mathrm{d}w_s - \mathrm{d}w^n_s),\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right\rangle\right)\right| \\
&\quad + \left|\mathbf{E}\int_0^1 \mu_n(0)\left(2\left\langle\int_{\bar t_n}^t (\sigma(Y^n_s) - \sigma(Z_s))\dot h_s\,\mathrm{d}s, \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right\rangle\right)\right| \\
&\quad + \left|\mathbf{E}\int_0^1 \mu_n(0)\left(2\left\langle\int_{\bar t_n}^t (\tilde b(Y^n_s) - b(Z_s))\,\mathrm{d}s, \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right\rangle\right)\right| \\
&\quad + \left|\mathbf{E}\int_0^1 \mu_n(0)\left(2\left\langle\int_{\bar t_n}^t (\mathrm{d}\phi^n_s - \mathrm{d}\psi_s), \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right\rangle\right)\right| \\
&\le C\Delta^{1/2} + C\mathbf{E}\sup_{1\le k\le 2^n}\left(\int_0^{t_k} \sigma(Y^n_{\bar t_n})(\mathrm{d}w_t - \mathrm{d}w^n_t)\right)^2 + C\Delta^{\theta/2} \\
&\le C\Delta^{\theta/2} \qquad \forall\theta\in(0,1). \qquad \square
\end{aligned}
$$

LEMMA 3.12. $\mathbf{E}(I_{7,2,2}(t_k) + I^5_{5,2,1}(t_k) + I_{10}(t_k)) \le C\Delta^{3/2}$.

PROOF. Since $I_{7,2,2} = I^{(1)}_{7,2,2} + I^{(2)}_{7,2,2}$, where

$$
\begin{aligned}
\mathbf{E}[I^{(1)}_{7,2,2}] &= -\frac{4}{\gamma}\mathbf{E}\left(\int_{t_{k-1}}^{t_k} \mu_n(t)\int_{t_{k-2}}^t \langle Y^n_{\bar t_n} - Z_{\bar t_n}, \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle\right. \\
&\qquad\qquad \left. \times\langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w_t\rangle\right) \\
&= 0, \\
I^{(2)}_{7,2,2} &= \frac{4}{\gamma}\int_{t_{k-1}}^{t_k} (\mu_n(t) - \mu_n(\bar t_n))\int_{\bar t_n}^t \langle Y^n_{\bar t_n} - Z_{\bar t_n}, \sigma(Y^n_s)(\mathrm{d}w_s - \mathrm{d}w^n_s)\rangle \\
&\qquad\qquad \times\langle D\varphi(Y^n_t), \sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle Y^n_{\bar{t}_n}-Z_{\bar{t}_n},\big(\sigma(Y^n_s)-\sigma(Y^n_{\bar{t}_n})\big)(\mathrm{d}w_s-\mathrm{d}w^n_s)\rangle\\
&\qquad\qquad\times\langle D\varphi(Y^n_t),\sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle\\
&+\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle Y^n_{\bar{t}_n}-Z_{\bar{t}_n},\sigma(Y^n_{\bar{t}_n})(\mathrm{d}w_s-\mathrm{d}w^n_s)\rangle\\
&\qquad\qquad\times\langle D\varphi(Y^n_t)-D\varphi(Y^n_{\bar{t}_n}),\sigma(Y^n_t)\,\mathrm{d}w^n_t\rangle\\
&+\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\int_{\bar{t}_n}^{t}\langle Y^n_{\bar{t}_n}-Z_{\bar{t}_n},\sigma(Y^n_{\bar{t}_n})(\mathrm{d}w_s-\mathrm{d}w^n_s)\rangle\\
&\qquad\qquad\times\langle D\varphi(Y^n_{t_{k-2}}),\sigma(Y^n_t)-\sigma(Y^n_{\bar{t}_n})\rangle\,\mathrm{d}w^n_t\\
&+\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(t_{k-2})\int_{\bar{t}_n}^{t}\langle Y^n_{t_{k-2}}-Z_{t_{k-2}},\sigma(Y^n_{t_{k-2}})(\mathrm{d}w_s-\mathrm{d}w^n_s)\rangle\\
&\qquad\qquad\times\langle D\varphi(Y^n_{t_{k-2}}),\sigma(Y^n_{t_{k-2}})\rangle\,\mathrm{d}w^n_t\\
=:&\sum_{i=1}^{5}J^i_{7,2,2}.
\end{aligned}
$$

By applying Lemmas 3.4, 3.7, 3.9 and Proposition 3.4, we can get

$$
\big|\mathbf{E}(J^i_{7,2,2})\big|\le C\Delta^{3/2},\qquad i=1,2,3,4
$$

and

$$
\begin{aligned}
&\mathbf{E}(J^5_{7,2,2}|\mathscr{F}_{t_{k-2}})\\
&\quad=\frac{2\Delta}{\gamma}\mu_n(t_{k-2})\sum_i\langle D\varphi(Y^n_{t_{k-2}}),\sigma(Y^n_{t_{k-2}})e_i\rangle\langle Y^n_{t_{k-2}}-Z_{t_{k-2}},\sigma(Y^n_{t_{k-2}})e_i\rangle.
\end{aligned}\tag{3.18}
$$

Moreover, since

$$
\begin{aligned}
I_{10}&=-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(t)\sum_i\langle D\varphi(Y^n_t),\sigma(Y^n_t)e_i\rangle\langle Y^n_t-Z_t,\sigma(Y^n_t)e_i\rangle\,\mathrm{d}t\\
&=-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\big(\mu_n(t)-\mu_n(\bar{t}_n)\big)\sum_i\langle D\varphi(Y^n_t),\sigma(Y^n_t)e_i\rangle\langle Y^n_t-Z_t,\sigma(Y^n_t)e_i\rangle\,\mathrm{d}t\\
&\quad-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\sum_i\langle D\varphi(Y^n_t)-D\varphi(Y^n_{\bar{t}_n}),\sigma(Y^n_t)e_i\rangle\langle Y^n_t-Z_t,\sigma(Y^n_t)e_i\rangle\,\mathrm{d}t\\
&\quad-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\sum_i\langle D\varphi(Y^n_{\bar{t}_n}),\big(\sigma(Y^n_t)-\sigma(Y^n_{\bar{t}_n})\big)e_i\rangle\langle Y^n_t-Z_t,\sigma(Y^n_t)e_i\rangle\,\mathrm{d}t\\
&\quad-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\sum_i\langle D\varphi(Y^n_{\bar{t}_n}),\sigma(Y^n_{\bar{t}_n})e_i\rangle\langle Y^n_t-Z_t-(Y^n_{\bar{t}_n}-Z_{\bar{t}_n}),\sigma(Y^n_t)e_i\rangle\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\mu_n(\bar{t}_n)\sum_i\langle D\varphi(Y^n_{\bar{t}_n}),\sigma(Y^n_{\bar{t}_n})e_i\rangle\langle Y^n_{\bar{t}_n}-Z_{\bar{t}_n},(\sigma(Y^n_t)-\sigma(Y^n_{\bar{t}_n}))e_i\rangle\,\mathrm{d}t\\
&-\frac{4\Delta}{\gamma}\mu_n(t_{k-2})\sum_i\langle D\varphi(Y^n_{t_{k-2}}),\sigma(Y^n_{t_{k-2}})e_i\rangle\langle Y^n_{t_{k-2}}-Z_{t_{k-2}},\sigma(Y^n_{t_{k-2}})e_i\rangle\\
=:&\sum_{i=1}^{6}I_{10,i}.
\end{aligned}
$$

By Lemmas 3.4, 3.7 and 3.9 it is easy to get

$$
|\mathbf{E}(I_{10,i})|\le C\Delta^{3/2},\qquad i=1,\dots,5,
$$

and by using (3.12), (3.18),

$$
\mathbf{E}(I_{10,6}+J^5_{7,2,2}+I^5_{5,2,1})=\mathbf{E}[\mathbf{E}(I_{10,6}+J^5_{7,2,2}+I^5_{5,2,1}|\mathscr{F}_{t_{k-2}})]=0.
$$

Thus summing all the estimates above we have

$$
|\mathbf{E}(I_{7,2,2}+I^5_{5,2,1}+I_{10})|\le C\Delta^{3/2}. \qquad \square
$$

PROPOSITION 3.6. $\lim_n \mathbf{E}\sup_{t\in[0,T]}|Y^n_t-Z_t|^2=0$.

PROOF. Take two arbitrary integers $n>n_0$, and set $s_k=k2^{-n_0}$, $k=0,1,\dots,2^{n_0}-1$. Then $\{s_k\}_{k=1}^{2^{n_0}}\subset\{t^n_k\}_{k=1}^{2^n}$. Since for every $t\in[s_k,s_{k+1}]$

$$
|Y^n_t-Z_t|\le|Y^n_t-Y^n_{s_k}|+|Y^n_{s_k}-Z_{s_k}|+|Z_{s_k}-Z_t|,
$$

we have

$$
|Y^n_t-Z_t|\le\sup_{|s-t|\le 2^{-n_0}}(|Y^n_t-Y^n_s|+|Z_t-Z_s|)+\sup_{k=0,\dots,2^{n_0}-1}|Y^n_{s_k}-Z_{s_k}|.
$$

Consequently

$$
\begin{aligned}
\sup_{t\in[0,1]}|Y^n_t-Z_t|^2\le{}&3\sup_{t\in[0,1],|s-t|\le 2^{-n_0}}(|Y^n_t-Y^n_s|^2+|Z_t-Z_s|^2)\\
&+3\sup_{k=0,\dots,2^{n_0}-1}|Y^n_{s_k}-Z_{s_k}|^2.
\end{aligned}
$$

Thus by Lemma 3.4, Proposition 3.5 and (3.8), for any $\theta\in(0,1)$,

$$
\begin{aligned}
&\mathbf{E}\Big[\sup_{t\in[0,1]}|Y^n_t-Z_t|^2\Big]\\
&\quad\le C\mathbf{E}\Big[\sup_{t\in[0,1],|s-t|\le 2^{-n_0}}(|Y^n_t-Y^n_s|^2+|Z_t-Z_s|^2)\Big]\\
&\qquad+C\sum_{k=0}^{2^{n_0}-1}\mathbf{E}[|Y^n_{s_k}-Z_{s_k}|^2]
\end{aligned}
$$

$$
\begin{aligned}
&\le C\mathbf{E}\Big[\sup_{t\in[0,1],|s-t|\le 2^{-n_0}}\big(\big|Y_t^n - Y_s^n\big|^2 + |Z_t - Z_s|^2\big)\Big] \\
&\quad + C2^{n_0}\sup_{0\le k\le 2^{n_0}}\mathbf{E}\big[\big|Y_{s_k}^n - Z_{s_k}\big|^2\big] \\
&\le C2^{-n_0\theta/2} + C2^{n_0}\bigg[2^{-n\theta/2} + \sup_{2\le k\le 2^n}\bigg(\int_{(k-2)2^{-n}}^{k2^{-n}}|\dot h_s|^2\,\mathrm{d}s\bigg)^{1/2}\bigg].
\end{aligned}
$$

Letting first $n \to \infty$ and then $n_0 \to \infty$ gives the result. □

PROOF OF THEOREM 3.1. As we have already noticed, we only need to prove (3.3), and this follows from Proposition 3.2, Lemma 3.4, Propositions 3.3 and 3.6. We have completed the proof of Theorem 3.1. □

4. An elementary application. In this section we give an elementary application of the support theorem to maximum principle. Recall that a typical application of the support theorem for ordinary diffusions is the maximum principle for degenerate elliptic and parabolic operators of second order, so it comes with no surprise that the support theorem for reflected diffusions is applicable to obtain a boundary-interior maximum principle for the same operators.

DEFINITION 4.1. Let A be the generator of the Markov family $\{X_t(x)\}$, the solution to equation (1.1). A function u defined in $\bar D$ is said to be A-subharmonic in $\bar D$ if:

(i) it is locally bounded and upper semicontinuous;
(ii) $t \to u(X_t(x))$ is a local submartingale for every $x \in \bar D$.

For $x \in \bar D$, set

$$
\begin{aligned}
\mathscr{P} &:= \big\{h \in C\big([0,+\infty);\mathbb{R}^d\big); h_0 = 0, t \to h_t \text{ is smooth}\big\}, \\
D(x) &:= \overline{\big\{y \in \bar D, \exists h \in \mathscr{P}, t_0 > 0, \text{s.t. } y = Z_{t_0}(x,h)\big\}},
\end{aligned}
$$

where $Z_t(x,h)$ solves the following deterministic Skorohod problem in D:

$$
\begin{aligned}
Z_t(x,h) &= x + \int_0^t \sigma\big(Z_s(x,h)\big)\dot h_s\,\mathrm{d}s + \int_0^t b\big(Z_s(x,h)\big)\,\mathrm{d}s + \psi_t, \\
Z_0(x,h) &= x \in \bar D.
\end{aligned}
$$

We have the following theorem, the proof of which is a modification of [5], Theorem 6.8.3.

THEOREM 4.1. *Let u be an A-harmonic function on D and $x \in \bar D$. If $u(x) = \max_{y\in D(x)} u(y)$, then $u \equiv u(x)$ on $D(x)$.*

PROOF. Let $y \in D(x)$ and $\{\tau_n\}$ be a localization sequence of stopping times for $\{u(X_t(x))\}$. For every m we set

$$\varsigma_m := \inf\{t : |X_t(x) - y| \le 2^{-m}\}.$$

Then for every m there exists an N_m such that $\forall n > N_m$,

$$\mathbf{P}(\varsigma_m < \tau_n) > 0. \tag{4.1}$$

By the submartingale property we have

$$u(x) \le \mathbf{E}\big[u\big(X_{t\wedge\varsigma_m\wedge\tau_n}(x)\big)\big].$$

Note that since

$$\mathbf{P}\big(X_{t\wedge\varsigma_m\wedge\tau_n}(x) \in D(x)\big) = 1,$$

we have

$$\mathbf{P}\big(u\big(X_{t\wedge\varsigma_m\wedge\tau_n}(x)\big) = u(x)\big) = 1.$$

Therefore for $n > N_m$,

$$u\big(X_{t\wedge\varsigma_m}(x)\big)\mathbb{1}_{\{\varsigma_m<\tau_n\}} = u(x)\mathbb{1}_{\{\varsigma_m<\tau_n\}}.$$

This together with (4.1) implies that for every m there exists y_m such that

$$|y_m - y| = 2^{-m}, \qquad u(y_m) = u(x).$$

Hence by the upper semicontinuity property of u,

$$u(y) \ge \limsup_{m\to\infty} u(y_m) = u(x),$$

implying that $u(y) = u(x)$. $\square$

EXAMPLE. Suppose $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$, and let

$$Lu = \sum_{i,j} a^{ij}(x)\frac{\partial^2 u}{\partial x^i\,\partial x^j} + \sum_i b^i(x)\frac{\partial u}{\partial x^i}, \qquad x \in D.$$

If

$$Lu(x) \ge 0 \qquad \forall x \in D,$$
$$\frac{\partial u(x)}{\partial n} \ge 0, \qquad x \in \partial D$$

holds in the viscosity sense, then u is A-subharmonic.

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SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCE
SUN YAT-SEN UNIVERSITY
510275 GUANGZHOU
P.R. CHINA
E-MAIL: renjg@mail.sysu.edu.cn
wjjosie@hotmail.com