

SOCIAL CONTACT PROCESSES AND THE PARTNER MODEL

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We consider a stochastic model of infection spread on the complete graph on N vertices incorporating dynamic partnerships, which we assume to be monogamous. This can be seen as a variation on the contact process in which some form of edge dynamics determines the set of contacts at each moment in time. We identify a basic reproduction number R_0 with the property that if $R_0 < 1$ the infection dies out by time $O(\log N)$, while if $R_0 > 1$ the infection survives for an amount of time $e^{\gamma N}$ for some $\gamma > 0$ and hovers around a uniquely determined metastable proportion of infectious individuals. The proof in both cases relies on comparison to a set of mean-field equations when the infection is widespread, and to a branching process when the infection is sparse.

1. Introduction. The contact process is a well-studied model of the spread of an infection, in which an undirected graph $G = (V, E)$ determines a collection of sites V and edges E which we can think of as individuals and as links between individuals along which the infection can be transmitted. Each site is either healthy or infectious; infectious sites recover at a certain fixed rate, which is usually normalized to 1, and transmit the infection to each of their neighbours at rate λ .

The contact process has been studied in a variety of different settings, including lattices [1, 4, 8, 9] (to cite just a few), infinite trees [11], power law graphs [3, 10] and complete graphs [12]. In each case, there is a critical value λ_c below which the infection quickly vanishes from the graph, and above which the infection has a positive probability of surviving either for all time (if the graph is infinite), or for an amount of time that grows quickly (either exponentially or at least faster than polynomially) with the size of the graph; in the power law case $\lambda_c = 0$ so long-time survival is possible whenever $\lambda > 0$.

In a social context, G might describe a contact network in which an edge connects sites x and y if and only if the corresponding individuals have sufficiently frequent interactions that infection can be spread from one to the other. In the contact process, the contact network is fixed, that is, a given pair of individuals is either connected or not connected for all time. However, we can easily imagine

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a scenario in which connections form and break up dynamically, which we can model by having edges open and close according to certain rules; here, we use the convention of percolation theory, in which “open” means there is a connection across the edge; note this is the opposite of the convention for electric circuits. In this case, the edges E represent *possible* connections and we have a process $E_t \subseteq E$ that describes the set of open edges as a function of time. This type of process we will call a *social contact process*, since it involves some form of social dynamics.

In the simplest case, edges open and close independently at some fixed rates r_+ and r_- . In this case, the distribution of open edges at a given time converges to the product measure on $\{0, 1\}^E$ with density $r_+/(r_- + r_+)$. Estimates on the survival region can then be obtained using the results of [2] and following the pattern of [13]. On the other hand, edge dynamics could depend on the state of the infection; for example, site x might be less likely to connect with site y , if y is infected. If we then relax the tendency to avoid infected sites, then for a given value of λ , we might ask at what point does the infection start to spread, if it does.

Here, we consider edges opening and closing independently as described above but with the added restriction of *monogamy*, that is, if two sites are connected (i.e., linked by an edge) then so long as they remain connected, they cannot connect to other sites. In this model, we think of connected pairs as partners, so we call it the *partner model*. For simplicity, we study the model on the sequence of complete graphs K_N on N vertices, where N will tend to ∞ ; this is a reasonable model for, say, the spread of a sexually transmitted infection through a population of monogamous homosexual individuals in a big city. We rescale the partner formation rate per edge to r_+/N to ensure that a given individual in a pool of entirely singles finds a partner at total rate approximately r_+ . For future reference, we use interchangeably both the words healthy and susceptible, and the words unpartnered and single, to describe respectively an individual that is not infectious, or an individual that does not have a partner. Even in this simple model, as described below, there is a phase transition between extinction and spread of the infection.

2. Statement of main results. In order to analyze the partner model, we should first ensure that it is well defined, so following [6] we give a graphical construction which makes it easy to visualize its evolution in time and space. We write the model as (V_t, E_t) where $V_t \subseteq V$ is the set of infectious sites at time t and $E_t \subseteq E$ is the set of open edges at time t . In general, we assume $\min(r_+, r_-, \lambda) > 0$ since if any of the parameters is equal to zero the dynamics are trivial.

The complete graph $K_N = (V, E)$ has sites $V = \{1, \dots, N\}$ and edges $E = \{\{x, y\} : x, y \in \{1, \dots, N\}, x \neq y\}$. On the spacetime set $K_N \times [0, \infty)$, place independent Poisson point processes (p.p.s) along the fibers $\{\cdot\} \times [0, \infty)$ as follows:

- for recovery, at each site with intensity 1 and label \times ,
- for transmission, along each edge $xy \in E$ with intensity λ and label \leftrightarrow ,

- for partnership formation, along each edge with intensity r_+/N and label \uparrow ,
- for partnership breakup, along each edge with intensity r_- and label \downarrow .

These define the probability space Ω , whose realizations $\omega \in \Omega$ consist of collections of labelled points on $K_N \times [0, \infty)$. Since the graph is finite, the total intensity of p.p.s is finite, thus with probability 1 events are well ordered in time. Fixing an admissible initial configuration (V_0, E_0) , that is, such that no two edges xy and yz are both open, we determine (V_t, E_t) as follows. For a well-ordered realization with event times $t_1 < t_2 < t_3 < \dots$, suppose (V_{t_i}, E_{t_i}) is known. If the event at time t_{i+1} is:

- an \times at site x and $x \in V_{t_i}$ then $V_{t_{i+1}} = V_{t_i} \setminus \{x\}$,
- a \leftrightarrow along edge xy , $xy \in E_{t_i}$, $x \in V_{t_i}$ and $y \notin V_{t_i}$ then $V_{t_{i+1}} = V_{t_i} \cup \{y\}$,
- a \uparrow along edge xy and $xz, zy \notin E_{t_i}$ for all z then $E_{t_{i+1}} = E_{t_i} \cup \{xy\}$,
- a \downarrow along edge xy and $xy \in E_{t_i}$ then $E_{t_{i+1}} = E_{t_i} \setminus \{xy\}$.

Otherwise the configuration is unchanged. This gives (V_t, E_t) at times $t_0 := 0, t_1, t_2, \dots$; for $t \in (t_i, t_{i+1})$ set $V_t = V_{t_i}$ and $E_t = E_{t_i}$.

For the partner model, we are mostly concerned not with the exact values of V_t and E_t but with the total number of susceptible and infectious singles S_t and I_t and the total number of partnered pairs SS_t, SI_t, II_t of the three possible types; as shown in Section 5, for each N , $(S_t, I_t, SS_t, SI_t, II_t)$ is a continuous time Markov chain. In general, it will be more convenient to work with the rescaled quantities $s_t = S_t/N$, $i_t = I_t/N$, $ss_t = SS_t/N$, $si_t = SI_t/N$ and $ii_t = II_t/N$.

Starting from any configuration, as shown in Section 6, after a short time the proportion of singles $y_t := s_t + i_t$ approaches and remains close to a certain fixed value $y^* \in (0, 1)$. The computation of y^* is given in Section 3: setting $\alpha = r_+/r_-$, we find that

$$(2.1) \quad y^* = 1/(2\alpha)[-1 + \sqrt{1 + 4\alpha}].$$

To determine the conditions under which the infection can spread, we use a heuristic argument. Once we know the correct values, we can then worry about proving they are correct. Suppose we start with $V_0 = \{x\}$ for some $x \in V$ with x single and $y_0 \approx y^*$, and keep track of x until the first moment when x either:

- recovers without finding a partner, or
- if it finds a partner before recovering, breaks up from that partnership.

This leads to the continuous time Markov chain shown in Figure 1. Each of A, B, \dots, G represents a state for the chain, and arrows show possible transitions, with the arrow labelled by the transition rate. Shaded circles represent infectious individuals and unshaded circles, healthy individuals. A pair of circles connected by a line represents a partnered pair. Starting from A , a single infectious site either recovers (goes to D) at rate 1, or finds a healthy partner at rate r_+y^* . Infection takes place at rate λ . If only one individual in a partnership is infectious (state B),

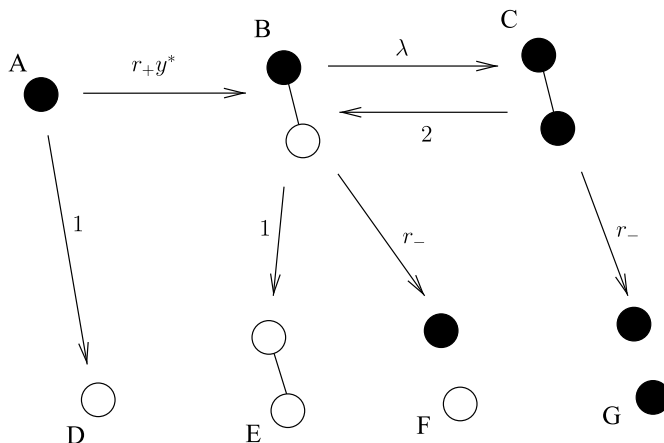


FIG. 1. Markov chain used to compute R_0 , with transition rates indicated; infectious sites are shaded.

then it recovers at rate 1 (state E), and we do not need to worry about them any more, since neither is infectious. If both are infectious (state C), recovery of one or the other occurs at rate 2. While in a partnership, breakup occurs at rate r_- .

Define the *basic reproduction number*

$$(2.2) \quad R_0 = \mathbb{P}(A \rightarrow F) + 2\mathbb{P}(A \rightarrow G)$$

which is the expected number of infectious singles upon absorption of the above Markov chain, starting from state A . As intuition suggests, and Theorem 2.2 confirms, the infection can spread if $R_0 > 1$, and cannot spread if $R_0 \leq 1$.

If the dynamics is in equilibrium, that is, $(s_t, i_t, ss_t, si_t, ii_t)$ hovers around a fixed value $(s^*, i^*, ss^*, si^*, ii^*)$, then in particular the proportion of infectious singles is roughly constant. To compute this proportion, we again use a heuristic argument. Three events affect infectious singles:

- $I \rightarrow S$, which occurs at rate $I_t = i_t N$,
- $I + I \rightarrow II$, which occurs at rate $(r_+/N) \binom{I_t}{2} \approx r_+(i_t^2/2)N$, and
- $S + I \rightarrow SI$, which occurs at rate $(r_+/N) I_t S_t = r_+ i_t s_t N$.

If a partnership is formed, then using these rates and Figure 1, we can compute the expected number of infectious singles upon breakup. Fixing $i_t = i$ for some $i \in [0, y^*]$ and $s_t + i_t = y^*$, define the normalizing constant $z = 1 + r_+ i/2 + r_+(y^* - i) = 1 + r_+(y^* - i/2)$ and the probabilities $p_S = 1/z$, $p_{II} = r_+ i/(2z)$ and $p_{SI} = r_+(y^* - i)/z$ and let

$$(2.3) \quad \Delta(i) = p_S \Delta_S + p_{II} \Delta_{II} + p_{SI} \Delta_{SI},$$

where $\Delta_S = -1$, $\Delta_{II} = -2 + \mathbb{P}(C \rightarrow F) + 2\mathbb{P}(C \rightarrow G)$ and $\Delta_{SI} = -1 + \mathbb{P}(B \rightarrow F) + 2\mathbb{P}(B \rightarrow G)$. The function $\Delta(i)$ tracks the expected change in the number of

infectious singles, per event affecting one or more infectious singles. Thus, for an equilibrium solution we should have $\Delta(i^*) = 0$. As shown in Lemma 4.2, to have a solution with $i^* > 0$, we need $R_0 > 1$.

As shown in Lemma 4.1, for fixed r_+ , r_- , R_0 is continuous and increasing in λ . Defining

$$(2.4) \quad \lambda_c = \sup\{\lambda \geq 0 : R_0 \leq 1\}$$

with $\sup \mathbb{R}_+ := \infty$, it follows that if $\lambda_c = \infty$ then $R_0 < 1$ for all λ , and if $\lambda_c < \infty$ then $R_0 < 1$ if $\lambda < \lambda_c$, $R_0 = 1$ if $\lambda = \lambda_c$ and $R_0 > 1$ if $\lambda > \lambda_c$. In models exhibiting a phase transition, one often seeks a *critical exponent* γ such that for an observable $F(\lambda)$ it holds that $F(\lambda) \sim C(\lambda - \lambda_c)^\gamma$. As we see in the statement of the upcoming Theorem 2.1, here the critical exponent for i^* is equal to 1.

The following two theorems are the main results of this paper. The first result tells us where and when we should expect a phase transition to occur. In particular, it gives a formula for λ_c and describes the behaviour of i^* near λ_c .

THEOREM 2.1. *Let y^* , R_0 , $\Delta(i)$ and λ_c be as in (2.1), (2.2), (2.3) and (2.4) and let r_+ , r_- be fixed. Then $\lambda_c < \infty \Leftrightarrow r_+ y^* > 1 \Leftrightarrow r_+ > 1 + 1/r_-$ and in this case*

$$\lambda_c = \frac{2}{r_-} \frac{2}{(r_+ y^* - 1)} + \frac{2}{r_-} + \frac{4}{r_+ y^* - 1} + 1 + \frac{r_-}{r_+ y^* - 1}.$$

If $R_0 = R_0(\lambda) > 1$, there is a unique solution $i^(\lambda) \in (0, y^*)$ to the equation $\Delta(i^*) = 0$ and $i^*(\lambda) \sim C(\lambda - \lambda_c)$ as $\lambda \downarrow \lambda_c$, for some constant $C > 0$.*

The second result shows that our heuristics are correct. More precisely, $R_0 > 1$ is a necessary and sufficient condition for spread and long-time survival of the infection. Moreover, when $R_0 > 1$ there is a unique and globally stable endemic equilibrium with $i^* > 0$ given by $\Delta(i^*) = 0$.

THEOREM 2.2. *Fix λ , r_+ , r_- and let y^* , R_0 and $\Delta(i)$ be as defined in (2.1), (2.2) and (2.3).*

- *If $R_0 \leq 1$, for each $\varepsilon > 0$ there are constants $C, T, \gamma > 0$ so that, from any initial configuration, with probability $\geq 1 - Ce^{-\gamma N}$, $|V_T| \leq \varepsilon N$.*
- *If $R_0 < 1$ there are constants $C, T, \gamma > 0$ so that, from any initial configuration, with probability tending to 1 as $N \rightarrow \infty$ all sites are healthy by time $T + C \log N$.*
- *If $R_0 > 1$, there is a unique vector $(s^*, i^*, ss^*, si^*, ii^*)$, satisfying $i^* > 0$, $s^* + i^* = y^*$ and $\Delta(i^*) = 0$, such that*
 - *for each $\varepsilon > 0$, there are constants $C, T, \gamma > 0$ so that, from any initial configuration with $|V_0| \geq \varepsilon N$, with probability $\geq 1 - Ce^{-\gamma N}$, $|(s_t, i_t, ss_t, si_t, ii_t) - (s^*, i^*, ss^*, si^*, ii^*)| \leq \varepsilon$ for $T \leq t \leq e^{\gamma N}$, and*

- there are constants $\delta, p, C, T > 0$ so that, from any initial configuration with $|V_0| > 0$, with probability $\geq p$, $|V_{T+C \log N}| \geq \delta N$.

To obtain the value of the endemic equilibrium and the behaviour when $|V_0| \geq \varepsilon N$, which we call the *macroscopic* regime, we use the *mean-field equations* (MFE) introduced in Section 5, which are a set of differential equations that give a good approximation to the evolution of $(s_t, i_t, ss_t, si_t, ii_t)$ when N is large. To describe the behaviour when $1 \leq |V_0| \leq \varepsilon N$ for small $\varepsilon > 0$, which we call the *microscopic* regime, we use comparison to a branching process; if $R_0 < 1$ we bound above and if $R_0 > 1$ we bound below.

The paper is laid out as follows. Sections 3 and 4 contain the heuristic calculations that allow us to determine y^* , R_0 , λ_c , $\Delta(i)$ and prove Theorem 2.1. In Section 3, we give an informal description of the edge dynamics and compute y^* . In Section 4, we analyze R_0 , λ_c , $\Delta(i)$ and prove Theorem 2.1, in two parts: Propositions 4.1 and 4.2. In Section 5, we introduce the mean-field equations and characterize their dynamics. In Sections 6, 7 and 8, we consider the stochastic process and prove Theorem 2.2. In Section 6, we develop the tools needed to relate the stochastic model to the mean-field equations. In Section 7, we prove the macroscopic part of Theorem 2.2, and in Section 8 we prove the microscopic part.

3. Proportion of singles. Starting from the total number of singles $Y_t = S_t + I_t$, the transitions are:

- $Y \rightarrow Y - 2$ at rate $(r_+/N)Y(Y-1)/2$,
- $Y \rightarrow Y + 2$ at rate $(N-Y)r_-/2$,

which for $y_t := Y_t/N$ gives:

- $y \rightarrow y - 2/N$ at rate $[r_+y(y-1/N)/2]N = (r_+y^2/2)N - r_+y/2$,
- $y \rightarrow y + 2/N$ at rate $[(1-y)r_-/2]N$.

Combining these transitions gives

$$\frac{d}{dt}\mathbb{E}(y_t|y_t=y) = -r_+y^2 + r_-(1-y) + \frac{r_+y}{N}.$$

In Lemma 6.4, we make a rigorous statement about the behaviour of y_t . For now, though, some heuristics are helpful. Letting $y = Y/N$ and Δy denote the increment in y over a time step of size $1/N$, we find $\mathbb{E}\Delta y = O(1/N)$ while $\mathbb{E}(\Delta y)^2 = O(1/N^2)$, which means $\text{Var}(\Delta y) = O(1/N^2)$. This suggests that as $N \rightarrow \infty$ we should expect the sample paths of y to approach solutions to the differential equation

$$(3.1) \quad y' = -r_+y^2 + r_-(1-y).$$

Notice the right-hand side is positive at $y = 0$, negative at $y = 1$ and strictly decreases with y , so there is a unique and globally stable equilibrium for $y \in [0, 1]$,

that lies in $(0, 1)$. Setting $y' = 0$ and letting $\alpha = r_+/r_-$ gives the equation $\alpha y^2 + y - 1 = 0$ which has the unique solution $y^* = 1/(2\alpha)[-1 + \sqrt{1 + 4\alpha}]$ in $[0, 1]$. Notice that $y^* \sim 1 - \alpha$ as $\alpha \rightarrow 0^+$ and $y^* \sim 1/\sqrt{\alpha}$ as $\alpha \rightarrow \infty$.

4. Survival analysis. In this section, we analyze R_0 , λ_c and $\Delta(i)$ which are defined in Section 2. We begin with R_0 defined in (2.2). Define the recruitment probability $p_r = r_+ y^*/(1 + r_+ y^*) = \mathbb{P}(A \rightarrow E \cup F \cup G)$ which is the probability of finding a partner before recovering and depends only on r_+, r_- . Define $a = 1 + \lambda + r_-$, $b = 2 + r_-$ which are the rates at which the Markov chain of Figure 1 jumps away from states B and C , respectively. Also, let

$$\sigma = \sum_{k=0}^{\infty} \left(\frac{\lambda}{a} \frac{2}{b} \right)^k = \frac{ab}{ab - 2\lambda}.$$

It is easy to check that $ab > 2\lambda$. Notice that any path from A to $E \cup F \cup G$ must go to B and then goes around the B, C loop some number of times before being absorbed at E, F or G , and σ accounts for this looping. Summing probabilities over all possible paths we find

$$\mathbb{P}(A \rightarrow F) = p_r \sigma \frac{r_-}{a} \quad \text{and} \quad \mathbb{P}(A \rightarrow G) = p_r \sigma \frac{\lambda}{a} \frac{r_-}{b}$$

so we obtain the explicit expression

$$R_0 = p_r \sigma r_- (1 + 2\lambda/b)/a$$

which after re-substituting and a bit of algebra gives

$$(4.1) \quad R_0 = p_r r_- \frac{b + 2\lambda}{ab - 2\lambda} = p_r r_- \frac{2 + r_- + 2\lambda}{2 + 3r_- + \lambda r_- + r_-^2}.$$

LEMMA 4.1. *Fixing r_+ and r_- , R_0 is continuous and increasing with respect to λ .*

PROOF. Continuity is obvious from the formula above. We write $R_0(\lambda)$ and compute the derivative $R'_0(\lambda)$, noting that p_r is fixed. Letting $c_1 = 2 + r_-$, $c_2 = 2$, $c_3 = 2 + 3r_- + r_-^2$ and $c_4 = r_-$, $R_0(\lambda) = p_r r_- (c_1 + c_2 \lambda)/(c_3 + c_4 \lambda)$ so $R'_0(\lambda) = p_r r_- (c_2 c_3 - c_1 c_4)/(c_3 + c_4 \lambda)^2$ and $c_2 c_3 - c_1 c_4 = 4 + 4r_- + r_-^2 > 0$ so $R'_0(\lambda) > 0$. \square

From this, it follows that for fixed r_+, r_- , if $R_0(\lambda) = 1$ has a solution then it is unique and is equal to λ_c . So, setting $R_0 = 1$ gives

$$(4.2) \quad p_r r_- (2 + r_- + 2\lambda_c) = 2 + 3r_- + \lambda_c r_- + r_-^2.$$

To get a handle on this equation, we first examine the limit of large r_+ , that is, quick formation of partnerships. As noted in Section 3, $y^* \sim 1/\sqrt{\alpha} = \sqrt{r_-}/\sqrt{r_+}$

as $\alpha = r_+/r_- \rightarrow \infty$, so for fixed r_- , $r_+y^* \sim \sqrt{r_-r_+} \rightarrow \infty$, and so $p_r \rightarrow 1$, as $r_+ \rightarrow \infty$. Setting $p_r = 1$ in the equation above, after cancelling like terms and dividing both sides by r_- gives

$$\lambda_c = 1 + 2/r_-$$

for fixed r_- , when $r_+ = \infty$. For the contact process on a large complete graph $\lambda_c = 1$, so here the only difference is the term $2/r_-$, which makes it harder for the infection to spread when partnerships last a long time.

Accounting for p_r , we still get a fairly nice expression. From (4.2), putting all terms involving λ_c on the left and all other terms on the right gives

$$\lambda_c r_- (2p_r - 1) = 2 + (3 - 2p_r)r_- + r_-^2(1 - p_r).$$

Letting $\beta = 2p_r - 1$ then substituting for β and dividing by r_- gives

$$(4.3) \quad \lambda_c \beta = 2/r_- + (2 - \beta) + (1/2)r_-(1 - \beta).$$

This equation suggests that we view $\lambda\beta$ as a sort of force of infection, which makes sense as λ is the transmission rate and $\beta = 2p_r - 1$ measures the chance of finding a partner before recovering. Although β depends on r_- , $-1 \leq \beta \leq 1$ regardless, so we see from (4.3) that for fixed λ , if r_- is either too small or too large, the infection cannot spread. The reason for this can be understood as follows: if r_- is too small, partners tend both to recover before breaking up and transmitting the infection to anyone else, whereas if r_- is too large, partnerships do not last long enough for transmission to occur.

Using (4.3), we can now prove the first assertion of Theorem 2.1.

PROPOSITION 4.1. *For fixed r_+ , r_- and λ_c given by (2.4), $\lambda_c < \infty$ if and only if $r_+y^* > 1$, if and only if $r_+ > 1 + 1/r_-$ and in this case*

$$\lambda_c = \frac{2}{r_-} \frac{2}{(r_+y^* - 1)} + \frac{2}{r_-} + \frac{4}{r_+y^* - 1} + 1 + \frac{r_-}{r_+y^* - 1}.$$

PROOF. It is easy to check, using the formula $y^* = (r_-/(2r_+))(-1 + (1 + 4r_+/r_-)^{1/2})$, that $r_+y^* > 1$ if and only if $r_+ > 1 + 1/r_-$. Since $\beta \in [-1, 1]$, the right-hand side of (4.3) is positive, so to have a solution it is necessary that $\beta > 0$; dividing by β on both sides shows that it is also sufficient. Then observe that $\beta > 0$ if and only if $r_+y^* > 1$. To get the formula for λ_c , divide by β in (4.3) and observe that $\beta^{-1} - 1 = 2/(r_+y^* - 1)$. \square

Figure 2 shows level curves of λ_c in the r_+, r_- plane. Using the formula for λ_c , we can see how it scales in various limits of r_+ , r_- and α . First, we see what happens when we speed up and slow down the partnership dynamics. Let α be fixed (and by extension, y^*) and let r_-^* denote the unique value of r_- such that $r_+y^* = 1$. We find that:

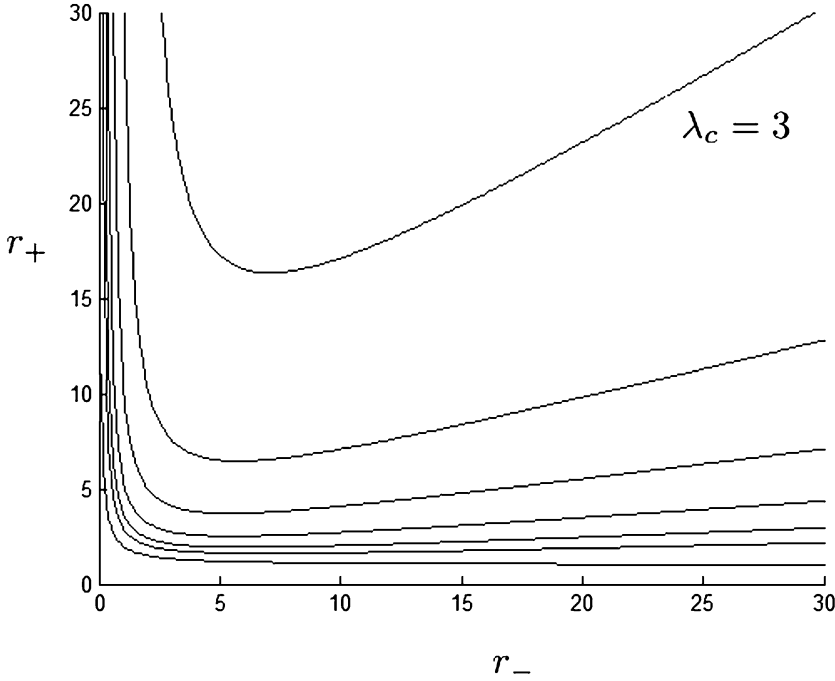


FIG. 2. Level curves of λ_c depicted in the r_+, r_- plane. Starting from the top curve and going down, $\lambda_c = 3, 5, 8, 13, 21, 34, \infty$.

- $\lambda_c \downarrow 1 + 1/(\alpha y^*)$ as $r_+ \uparrow \infty$ (fast partner dynamics),
- $\lambda_c(r_+ y^* - 1) \downarrow 4/r_-^* + 4 + r_-^*$ as $r_+ y^* \downarrow 1$ (slow partner dynamics).

In particular, in the limit of fast partner dynamics λ_c approaches its value for the contact process on a complete graph, plus a correction for the proportion of available singles. In the slow limit, that is, as the recruitment probability approaches $1/2$, λ_c diverges like $1/(r_+ y^* - 1)$, with a proportionality that itself diverges as r_-^* approaches either 0 or ∞ . Now we fix $r_+ > 1$ and vary r_- . Note that $y^* \downarrow 0$ as $r_- \downarrow 0$:

- as $r_- \uparrow \infty$, $y^* \uparrow 1$, $\alpha \downarrow 0$ and $\lambda_c/r_- \downarrow 1/(r_+ - 1)$, and
- as $r_+ y^* \downarrow 1$, $\lambda_c(r_+ y^* - 1) \downarrow 4/r_-^* + 4 + r_-^*$.

Here, in both limits λ_c diverges, in the first case like r_- and in the second case like $1/(r_+ y^* - 1)$. Finally, we fix r_- and vary r_+ , and we find that:

- as $r_+ \uparrow \infty$, $y^* \sim 1/\sqrt{\alpha} = \sqrt{r_-^*/r_+}$ and $\lambda_c \rightarrow 1 + 2/r_-$, and
- as $r_+ y^* \downarrow 1$, $\lambda_c(r_+ y^* - 1) \downarrow 4/r_- + 4 + r_-$.

The first limit agrees with the previous large r_+ approximation, and the second limit shows that when $r_+ y^*$ is close to 1, $\lambda(r_+ y^* - 1)/2 \approx \lambda(r_+ y^* - 1)/(r_+ y^* +$

$1) = \lambda\beta$ behaves like the force of infection and we require again that r_- be neither too small nor too large in order for the infection to be able to spread.

We now examine $\Delta(i)$, defined in (2.3).

LEMMA 4.2. $\Delta(0) = R_0 - 1$, and:

- if $R_0 < 1$ the equation $\Delta(i) = 0$ has no solution $i \in [0, y^*]$,
- if $R_0 = 1$ the equation $\Delta(i) = 0$ has the unique solution $i = 0$ and
- if $R_0 > 1$ the equation $\Delta(i^*) = 0$ has a unique solution $i^* \in (0, y^*)$.

PROOF. Letting $z = 1 + r_+(y^* - i/2)$ we recall the definition:

$$(4.4) \quad \Delta(i) = p_S \Delta_S + p_H \Delta_H + p_{SI} \Delta_{SI}$$

with $p_S = 1/z$, $p_{SI} = r_+(y^* - i)/z$, $p_H = r_+i/(2z)$, $\Delta_S = -1$, $\Delta_H = -2 + \mathbb{P}(C \rightarrow F) + 2\mathbb{P}(C \rightarrow G)$ and $\Delta_{SI} = -1 + \mathbb{P}(B \rightarrow F) + 2\mathbb{P}(B \rightarrow G)$, where probabilities are with respect to the Markov chain in Figure 1.

First, we show $\Delta(0) = R_0 - 1$. If $i = 0$ then $p_S = 1/(1 + r_+y^*) = \mathbb{P}(A \rightarrow D)$, $p_H = 0$ and $p_{SI} = r_+y^*/(1 + r_+y^*) = \mathbb{P}(A \rightarrow B)$ so

$$\begin{aligned} \Delta(0) &= -\mathbb{P}(A \rightarrow D) + \mathbb{P}(A \rightarrow B)(-1 + \mathbb{P}(B \rightarrow F) + 2\mathbb{P}(B \rightarrow G)) \\ &= -\mathbb{P}(A \rightarrow D \cup B) + \mathbb{P}(A \rightarrow F) + 2\mathbb{P}(A \rightarrow G) \\ &= -1 + R_0. \end{aligned}$$

It is easy to check that $\Delta_H \leq 0$, so if $\Delta_{SI} \leq 0$ then $\Delta(i) < 0$ for $i \in [0, y^*]$, since $p_S > 0$ and $\Delta_S < 0$, and the other terms are ≤ 0 . Since $\partial_i z = -r_+/2$, we find

$$\partial_i p_S = r_+/2z^2 > 0 \quad \text{and} \quad \partial_i p_H = r_+/(2z) + r_+^2 i/(4z^2) > 0$$

and since $p_{SI} = 1 - (p_S + p_H)$, $\partial_i p_{SI} = -\partial_i p_S - \partial_i p_H < 0$. If $\Delta_{SI} > 0$ it follows that $\partial_i \Delta(i) < 0$ so if $R_0 < 1$ but $\Delta_{SI} > 0$ then $\Delta(i) \leq \Delta(0) < 0$ for $i \in [0, y^*]$. If $R_0 \geq 1$, then since $0 \leq \Delta(0) = p_S \Delta_S + p_{SI} \Delta_{SI}$ and $\Delta_S < 0$, it follows that $\Delta_{SI} > 0$ and so $\partial_i \Delta(i) < 0$. If $R_0 = 1$, then since $\Delta(0) = 0$ it follows that $i = 0$ is the only solution in $[0, y^*]$ to the equation $\Delta(i) = 0$. If $i = y^*$ then $p_{SI} = 0$ so $\Delta(y^*) < 0$, and clearly $\Delta(i)$ is continuous on $[0, y^*]$. Therefore, if $R_0 > 1$ then since $\Delta(0) > 0$, by the intermediate value theorem the equation $\Delta(i^*) = 0$ has a solution $i^* \in (0, y^*)$, and since $\partial_i \Delta(i) < 0$ the solution is unique. \square

Write $\Delta(i)$ as $\Delta(\lambda, i)$ to emphasize the λ dependence. By Lemma 4.2 and since $R_0 = 1 \Leftrightarrow \lambda = \lambda_c$ and $R_0 > 1 \Leftrightarrow \lambda > \lambda_c$, for fixed r_+, r_- such that $r_+y^* > 1$, we have a function $i^*(\lambda)$ defined for $\lambda \geq \lambda_c$ satisfying $\Delta(\lambda, i^*(\lambda)) = 0$ such that $i^*(\lambda_c) = 0$ and $i^*(\lambda) > 0$ for $\lambda > \lambda_c$. Next, we see how i^* behaves for $\lambda > \lambda_c$ near λ_c . As usual, C^1 means continuously differentiable.

PROPOSITION 4.2. For fixed r_+, r_- such that $r_+y^* > 1$, $i^* \sim C(\lambda - \lambda_c)$ as $\lambda \downarrow \lambda_c$ for some constant $C > 0$.

PROOF. Clearly, p_S , p_{SI} and p_{II} depend only on i and are C^1 in a neighbourhood of 0. Also, Δ_S is fixed and Δ_{SI} and Δ_{II} depend only on λ and are rational functions of λ whose range lies in a bounded interval, thus are C^1 in a neighbourhood of λ_c . Glancing at (4.4), this means that $\Delta(\lambda, i)$ is C^1 in a neighbourhood of $(\lambda_c, 0)$. If $\lambda \geq \lambda_c$ then $R_0 \geq 1$, so as shown in the proof of Lemma 4.2, $\partial_i \Delta(\lambda, i) < 0$ and in particular, $\partial_i \Delta(\lambda_c, 0) \neq 0$. Applying the implicit function theorem, there is a unique C^1 function $i^*(\lambda)$ defined in a neighbourhood of λ_c (and thus coinciding with the previous definition of $i^*(\lambda)$ when $\lambda \geq \lambda_c$) satisfying $\Delta(\lambda, i^*(\lambda)) = 0$, and noting that $i^*(\lambda_c) = 0$,

$$i^*(\lambda) \sim -(\lambda - \lambda_c) \frac{\partial_\lambda \Delta(\lambda_c, 0)}{\partial_i \Delta(\lambda_c, 0)}$$

as $\lambda \downarrow \lambda_c$. A straightforward Markov chain coupling argument shows that $\partial_\lambda \Delta_{SI}, \partial_\lambda \Delta_{II} > 0$, which implies $\partial_\lambda \Delta(\lambda, i) > 0$. Since $\partial_i \Delta(\lambda, i) < 0$, the result follows. \square

5. Mean-field equations. A set of differential equations defined below are indispensable to our analysis of the partner model as they enable a (better and better as N increases) approximate description of the model, when N is large. First, we write down the transitions for the variables introduced in Section 2 that track the total number of singles and pairs of various types; there are ten such transitions. The existence of well-defined transitions shows that $(S_t, I_t, SS_t, SI_t, II_t)$ is a continuous time Markov chain:

- $I \rightarrow I - 1$ and $S \rightarrow S + 1$ at rate I ,
- $S \rightarrow S - 2$ and $SS \rightarrow SS + 1$ at rate $(r_+/N)S(S - 1)/2$,
- $S \rightarrow S - 1$, $I \rightarrow I - 1$ and $SI \rightarrow SI + 1$ at rate $(r_+/N) \cdot S \cdot I$,
- $I \rightarrow I - 2$ and $II \rightarrow II + 1$ at rate $(r_+/N)I(I - 1)/2$,
- $SI \rightarrow SI - 1$ and $SS \rightarrow SS + 1$ at rate SI ,
- $II \rightarrow II - 1$ and $SI \rightarrow SI + 1$ at rate $2II$,
- $SI \rightarrow SI - 1$ and $II \rightarrow II + 1$ at rate λSI ,
- $SS \rightarrow SS - 1$ and $S \rightarrow S + 2$ at rate $r_- SS$,
- $SI \rightarrow SI - 1$, $S \rightarrow S + 1$ and $I \rightarrow I + 1$ at rate $r_- SI$, and
- $II \rightarrow II - 1$ and $I \rightarrow I + 2$ at rate $r_- II$.

Focusing now on the rescaled quantities $(s_t, i_t, ss_t, si_t, ii_t) = (S_t, I_t, SS_t, SI_t, II_t)/N$ and noting the relation $s_t + i_t + 2(ss_t + si_t + ii_t) = 1$, we shall ignore ss_t since it plays no role in the calculations that follow. Also, it will be convenient to use $y_t := s_t + i_t$ instead of s_t . Doing so, the above transitions become:

- $i \rightarrow i - 1/N$ at rate iN ,
- $y \rightarrow y - 2/N$ at rate $[r_+(y - i)(y - i - 1/N)/2]N = [r_+(y - i)^2/2]N - r_+(y - i)/2$,
- $y \rightarrow y - 2/N$, $i \rightarrow i - 1/N$ and $si \rightarrow si + 1/N$ at rate $r_+(y - i)iN$,

- $y \rightarrow y - 2/N$, $i \rightarrow i - 2/N$ and $ii \rightarrow ii + 1/N$ at rate $[r_+i(i - 1/N)/2]N = (r_+i^2/2)N - r_+i/2$,
- $si \rightarrow si - 1/N$ at rate siN ,
- $ii \rightarrow ii - 1/N$ and $si \rightarrow si + 1/N$ at rate $2iiN$,
- $si \rightarrow si - 1/N$ and $ii \rightarrow ii + 1/N$ at rate λsiN ,
- $y \rightarrow y + 2/N$ at rate $[r_-((1 - y)/2 - (si + ii))]N$,
- $si \rightarrow si - 1/N$, $y \rightarrow y + 2/N$ and $i \rightarrow i + 1/N$ at rate r_-siN , and
- $ii \rightarrow ii - 1/N$, $y \rightarrow y + 2/N$ and $i \rightarrow i + 2/N$ at rate r_-iiN .

As we did for y_t in Section 3, we derive some differential equations that approximate the evolution of (y_t, i_t, si_t, ii_t) ; since we already have an equation for y_t we focus on i_t, si_t, ii_t . We have

$$\begin{aligned}\frac{d}{dt}\mathbb{E}(i_t|i_t = i) &= -(1 + r_+(y - i) + 2r_+(i - 1/N)/2)i + r_-(si + 2ii), \\ \frac{d}{dt}\mathbb{E}(si_t|si_t = si) &= r_+(y - i)i + 2ii - (1 + \lambda + r_-)si, \\ \frac{d}{dt}\mathbb{E}(ii_t|ii_t = ii) &= r_+i(i - 1/N)/2 + \lambda si - (2 + r_-)ii\end{aligned}$$

and as before, in a time step of size $1/N$ the increment in each variable has expected value $O(1/N)$ while its square has expected value $O(1/N^2)$. Adding in the y' equation (3.1), this suggests again that in the limit as $N \rightarrow \infty$ we should expect the sample paths of (y_t, i_t, si_t, ii_t) to approach solutions to the *mean-field equations*

$$\begin{aligned}y' &= -r_+y^2 + r_-(1 - y), \\ i' &= -(1 + r_+y)i + r_-(si + 2ii), \\ si' &= r_+(y - i)i - (1 + \lambda + r_-)si + 2ii, \\ ii' &= r_+i^2/2 + \lambda si - (2 + r_-)ii.\end{aligned}\tag{5.1}$$

It is sometimes convenient to replace si with $ip := si + ii$, where the ip stands for “infected partnership”. Since $si = ip - ii$, both forms lead to the same solutions. After the change of variables, we have

$$\begin{aligned}y' &= -r_+y^2 + r_-(1 - y), \\ i' &= -(1 + r_+y)i + r_-(ip + ii), \\ ip' &= r_+(y - i/2)i - (1 + r_-)ip + ii, \\ ii' &= r_+i^2/2 + \lambda ip - (2 + r_- + \lambda)ii.\end{aligned}\tag{5.2}$$

We will often use the shorthand $u' = F(u)$ for the MFE (5.1) or (5.2), where $u \in \mathbb{R}^4$. In both cases the MFE have the form $y' = f(y)$, $u' = G(y, u)$, where

$u \in \mathbb{R}^3$, that is, the y dynamics does not depend on the other 3 variables, but it does influence them; systems of this form are often referred to as *skew product*. The next three results have natural analogues for the stochastic model, and in fact the analogue of Lemma 5.2 shows up in Section 7 as Lemma 7.1. First, we show the domain of interest is an invariant set.

LEMMA 5.1. *The following set is invariant for the MFE:*

$$\Lambda := \{(y, i, ip, ii) \in \mathbb{R}_+^4 : i \leq y \leq 1, ii \leq ip \leq (1 - y)/2\}.$$

PROOF. We examine the boundary and use the form (5.2) of the MFE. If $y = 0$ then $y' > 0$ and if $y = 1$ then $y' < 0$, so $[0, 1]$ is invariant for y . Let $u = (i, ip, ii)$. If $u = (0, 0, 0)$, then $u' = (0, 0, 0)$, so $(0, 0, 0)$ is invariant for u . If $u \neq (0, 0, 0)$ and $u_j = 0$ for coordinate j , then $u'_j > 0$ (note for ip' that since $i \leq y$, if $i > 0$ then $y - i/2 > 0$). If $i = y \neq 0$, then since $ip + ii \leq (1 - y)$, $i' \leq -y - r_+ y^2 + r_-(1 - y) = -y + y' < y'$. If $i = y = 0$ then $i' \leq -y + y' = y'$ and since $y' > 0$, $i'' \leq -y' + y'' < y''$. For the remainder, we may assume $i < y$. If $ii = ip \neq 0$, then $ii' = r_+ i^2/2 - (2 + r_-)ip \leq r_+(y - i/2)i - (2 + r_-)ip < ip'$ while if $ii = ip = 0$ then we may assume $i > 0$ in which case $ii' = r_+ i^2/2 < r_+(y - i/2)i = ip'$. \square

Written in the form (5.2), the MFE have a useful monotonicity property which is described in the following lemma.

LEMMA 5.2. *Let $(y(t), u(t))$ and $(y(t), v(t))$ be solutions to the MFE written in (y, i, ip, ii) coordinates, and say that $u \leq v \Leftrightarrow u_j \leq v_j \forall j \in \{1, 2, 3\}$. If $u(0) \leq v(0)$, then $u(t) \leq v(t)$ for $t > 0$.*

PROOF. Since trajectories are continuous it suffices to check that if $u \leq v$, $u \neq v$ and $u_j = v_j$ then $u'_j < v'_j$. Referring to (5.2), i' increases with ip and ii , ip' increases with i and ii [note $\partial_i(y - i/2)i = y - i$ and $i \leq y$] and ii' increases with i and ip . \square

For what follows, we set $y = y^*$ in which case the MFE are three-dimensional. Since Λ is invariant,

$$\Lambda^* := \{(y, u) \in \Lambda : y = y^*\}$$

is also invariant. Since $\Lambda^* \cong \{(i, ip, ii) \in \mathbb{R}_+^3 : i \leq y^*, ii \leq ip \leq (1 - y^*)/2\}$ is three-dimensional, elements of Λ^* are usually written as a three-vector in either (i, si, ii) or (i, ip, ii) coordinates.

LEMMA 5.3. *Say that $u = (i, ip, ii)$ is increasing if $u'_j > 0$ in each coordinate. For the MFE with $y = y^*$ and any solution $u(t)$:*

- if $(0, 0, 0)$ is the only equilibrium then $u(t) \rightarrow (0, 0, 0)$ as $t \rightarrow \infty$, and

- if there is a unique equilibrium $u^* \neq (0, 0, 0)$ and a sequence of nonzero increasing states tending to $(0, 0, 0)$, then for $u(0) \neq (0, 0, 0)$, $u(t) \rightarrow u^*$ as $t \rightarrow \infty$.

PROOF. Defining $\bar{u} := (y^*, (1 - y^*)/2, (1 - y^*)/2)$, $\bar{u} \geq v$ for all $v \in \Lambda^*$, so letting $\bar{u}(t)$ be the solution to the MFE with $\bar{u}(0) = \bar{u}$, for $s \geq 0$, $\bar{u}(0) \geq \bar{u}(s)$. Since $y = y^*$, by monotonicity (Lemma 5.2) $\bar{u}(t) \geq \bar{u}(t + s)$ for $t > 0$, so $\bar{u}(t)$ is nonincreasing in t . Since Λ^* is compact, $\lim_{t \rightarrow \infty} \bar{u}(t)$ exists and by continuity of the MFE is an equilibrium. If $(0, 0, 0)$ is the only equilibrium, then since $\bar{u}(t) \geq (0, 0, 0)$, $\bar{u}(t) \rightarrow (0, 0, 0)$ as $t \rightarrow \infty$, so for any solution $v(t)$, since $\bar{u}(0) \geq v(0)$, $\bar{u}(t) \geq v(t)$ for $t > 0$, and since $v(t) \geq (0, 0, 0)$, $v(t) \rightarrow (0, 0, 0)$.

If $u(0)$ is increasing, then $u(0) \neq (0, 0, 0)$ and by continuity of the MFE there is $\varepsilon > 0$ so that $u(s) \geq u(0)$ for $0 \leq s \leq \varepsilon$. By monotonicity $u(t + s) \geq u(t)$ for $0 \leq s \leq \varepsilon$ and if $(k - 1)\varepsilon \leq s \leq k\varepsilon$, by iterating at most k times $u(t + s) \geq u(t)$, so $u(t)$ is increasing for all time. As in the previous case, $\lim_{t \rightarrow \infty} u(t)$ exists and is an equilibrium which in this case is not $(0, 0, 0)$. If there is a unique equilibrium $u^* \neq (0, 0, 0)$, and if for any nonzero solution $v(t)$ there is $T > 0$ so that $v(T) \geq u$ for some increasing u , then setting $u(T) = u$, since $\bar{u}(t) \geq v(t) \geq u(t)$ for $t \geq T$ and $\lim_{t \rightarrow \infty} \bar{u}(t) = \lim_{t \rightarrow \infty} u(t) = u^*$ it follows that $\lim_{t \rightarrow \infty} v_t = u^*$. If $v(0) \neq (0, 0, 0)$, then for $t > 0$, $v_j(t) > 0$ in each coordinate j ; this follows from the fact that for $j = 1, 2, 3$, $v'_j \geq -Cv_j$ for some C , and if $v_j = 0$ but $v_k > 0$ for some $k \neq j$ then $v'_j > 0$. Thus, fixing $T > 0$, if $v(0) \neq (0, 0, 0)$ then since $\varepsilon := \min_j v_j(T) > 0$, if there is a sequence of increasing states tending to $(0, 0, 0)$ there is an increasing state u with $\max_j u_j \leq \varepsilon$, and thus $v(T) \geq u$, as desired. \square

As the next result shows, on Λ^* the MFE have a simple dynamics with a bifurcation at $R_0 = 1$. Since we refer back to quantities from Section 4, in this proof we mostly use (i, si, ii) coordinates.

THEOREM 5.1. *For the MFE:*

- if $R_0 \leq 1$ there is the unique equilibrium $(0, 0, 0)$ which is attracting on Λ^* and
- if $R_0 > 1$ there is a unique positive equilibrium (i^*, s^*, ii^*) satisfying $\Delta(i^*) = 0$ which is attracting on $\Lambda^* \setminus \{(0, 0, 0)\}$.

PROOF. By Lemma 5.3 it is enough to show that if $R_0 \leq 1$ then $(0, 0, 0)$ is the only equilibrium, and that if $R_0 > 1$ there is a unique equilibrium $(i^*, si^*, ii^*) \neq (0, 0, 0)$ satisfying $\Delta(i^*) = 0$, and a sequence of increasing states converging to $(0, 0, 0)$. Treating si, ii as a separate system with input function i , we have the nonhomogenous linear system

$$\begin{pmatrix} si' \\ ii' \end{pmatrix} = \begin{pmatrix} -a & 2 \\ \lambda & -b \end{pmatrix} \begin{pmatrix} si \\ ii \end{pmatrix} + r+i \begin{pmatrix} (y^* - i) \\ i/2 \end{pmatrix}$$

or, in matrix form, $v' = Kv + Li$, with $v = (si, ii)^\top$, $K = \begin{pmatrix} -a & 2 \\ \lambda & -b \end{pmatrix}$ and $L = r_+((y^* - i), i/2)^\top$, whose solution is given by

$$(5.3) \quad v(t) = \Phi(t)v(0) + \int_0^t \Phi(t-s)L(s)i(s)ds,$$

where $\Phi(t) = \exp(Kt)$ is the solution of the associated homogenous system—note that $\Phi(t)$ is the restriction of the transition semigroup for the continuous-time Markov chain from Figure 1 to the states B and C . Substituting the solution for the si, ii system into the equation for i , we have

$$(5.4) \quad i'(t) = -(1 + r_+y^*)i(t) + r_-(1, 2) \left[\Phi(t)v(0) + \int_0^t \Phi(t-s)L(s)i(s)ds \right],$$

where $(1, 2)$ is a row vector that multiplies the column vector in the square brackets. This equation depends only on i , the initial values $v(0) = (si(0), ii(0))^\top$ and the solution matrix $\Phi(t)$.

Linearizing (5.4) around $(i, si, ii) = (0, 0, 0)$ and using the ansatz $i(t) = \exp(\mu t)$, we obtain

$$\mu e^{\mu t} = -(1 + r_+y^*)e^{\mu t} + r_-(1, 2) \left[\Phi(t)v_0 + \int_0^t \Phi(t-s)e^{\mu s}ds \right] L_0,$$

where $L_0 = r_+(y^*, 0)^\top$, and using $\Phi(t) = \exp(Kt)$ the integral in the square brackets is

$$e^{Kt} \int_0^t e^{(\mu I - K)s} ds = e^{Kt} (\mu I - K)^{-1} (e^{(\mu I - K)t} - I) = (\mu I - K)^{-1} (e^{\mu t} - e^{Kt}),$$

where I is the identity matrix. Letting $t \rightarrow \infty$ and noting $\Phi(t) = e^{Kt} \rightarrow 0$ since K is a stable matrix, we obtain the eigenvalue equation

$$\mu = -(1 + r_+y^*) + r_-(1, 2)(\mu I - K)^{-1}L_0$$

which, expanding, is

$$\mu = -(1 + r_+y^*) + r_- \frac{\mu + b + 2\lambda}{(\mu + b)(\mu + a) - 2\lambda} r_+y^*$$

and setting $\mu = 0$ gives the equation

$$1 = \frac{r_+y^*}{1 + r_+y^*} \frac{r_-}{ab - 2\lambda} (b + 2\lambda)$$

which, comparing to (4.1), is exactly $R_0 = 1$. Recalling that $ab - 2\lambda > 0$,

$$\begin{aligned} & \frac{d}{d\mu} \left(\frac{\mu + b + 2\lambda}{(\mu + b)(\mu + a) - 2\lambda} \right) \\ &= \frac{(\mu + b)(\mu + a) - 2\lambda - (\mu + b + 2\lambda)(2\mu + b + a)}{[(\mu + b)(\mu + a) - 2\lambda]^2} \\ &= \frac{-2\lambda - [(\mu + b)^2 + 2\lambda(2\mu + b + a)]}{[(\mu + b)(\mu + a) - 2\lambda]^2} \end{aligned}$$

is negative when $\mu \geq 0$. Setting $\mu = 0$ in (5.5), the right-hand side is positive if $R_0 > 1$, so since both sides are continuous in μ , the left-hand side is equal to 0 at $\mu = 0$ and increases unboundedly as μ increases and the right-hand side decreases with μ it follows that (5.5) has a positive solution $\mu > 0$ when $R_0 > 1$.

To obtain the increasing states mentioned in Lemma 5.3, we show that for $R_0 > 1$ the unstable eigenvector of the linearized system near $(0, 0, 0)$ is strictly positive when viewed in (i, ip, ii) coordinates; we can then take for the initial states small multiples of the eigenvector. To show the eigenvector is strictly positive, linearize (5.3) around $(i, si, ii) = (0, 0, 0)$ with input $i(t) = \exp(\mu t)$, substitute the solution form $v(t) = v \exp(\mu t)$ and let $t \rightarrow \infty$ to obtain $v = (\mu I - K)^{-1} L_0$ which has positive entries, which implies that in (ip, ii) coordinates it also has positive entries.

It remains to look for nonzero equilibria. Focusing again on (5.4), as our steady state assumption we suppose the system was started in the distant past and has remained in equilibrium up to the present time. Since $\Phi(t) \rightarrow 0$ as $t \rightarrow \infty$ we ignore $\Phi(t)v(0)$, and letting $\Phi_\infty = \int_0^\infty \Phi(s) ds = -K^{-1}$, $\int_0^t \Phi(t-s)L(s)i(s) ds$ becomes $\int_{-\infty}^t \Phi(t-s)L^\dagger i^\dagger ds = \Phi_\infty L^\dagger i^\dagger$ where $L^\dagger = r_+((y^* - i^\dagger), i^\dagger/2)^\top$ and i^\dagger are the equilibrium values, and we obtain

$$(1 + r_+ y^*) = r_-(1, 2)\Phi_\infty L^\dagger.$$

Notice that $r_-(1, 2)\Phi_\infty$ returns the expected number of infectious singles that result from an SI or an II partnership upon breakup, so we have $r_-(1, 2)\Phi_\infty = (1 + \Delta_{SI}, 2 + \Delta_{II})$ and

$$\begin{aligned} (1 + r_+ y^*) &= r_+[(y^* - i^\dagger)(1 + \Delta_{SI}) + (i^\dagger/2)(2 + \Delta_{II})] \\ &= r_+ y^* + r_+[(y^* - i^\dagger)\Delta_{SI} + (i^\dagger/2)\Delta_{II}] \end{aligned}$$

and subtracting $r_+ y^*$, $1 = r_+(y^* - i^\dagger)\Delta_{SI} + r_+(i^\dagger/2)\Delta_{II}$ which comparing with (4.4) is exactly the equation $\Delta(i^\dagger) = 0$, as desired. By Lemma 4.2, we have the unique solution $i^\dagger = i^*$ if $R_0 > 1$, and there is no positive solution when $R_0 \leq 1$. Using the steady state assumption and (5.3) gives $(si^\dagger, ii^\dagger) = \Phi_\infty L^\dagger i^\dagger$, that is, si^\dagger, ii^\dagger are uniquely determined by i^\dagger . This proves uniqueness of the nonzero equilibrium when $R_0 > 1$ and uniqueness of $(0, 0, 0)$ as an equilibrium when $R_0 \leq 1$. \square

REMARK 5.1. Setting $y = y^*$ in (5.1) and writing the remaining equations in matrix form, we have $u' = Au$ with $u = (i, si, ii)^\top$ and

$$A = \begin{pmatrix} -(1 + r_+ y^*) & r_- & 2r_- \\ r_+(y^* - i) & -a & 2 \\ r_+ i & \lambda & -b \end{pmatrix}$$

that depends on u . Using the technique of [14], if we evaluate A at $i = 0$ and write it as $F - V$ with

$$F = \begin{pmatrix} 0 & 0 & 0 \\ r_+ y^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (1 + r_+ y^*) & -r_- & -2r_- \\ 0 & a & -2 \\ 0 & -\lambda & b \end{pmatrix}$$

and define $R_0 = \rho(FV^{-1})$ where ρ is the spectral radius, then it can be verified that this definition of R_0 coincides with the one given in (2.2). Then, according to Theorem 2 of [14], $R_0 < 1$ implies $(0, 0, 0)$ is locally asymptotically stable, while $R_0 > 1$ implies it is unstable.

6. Approximation by the mean-field equations. In this section, we show how to approximate the sample paths of (y_t, i_t, si_t, ii_t) with solutions to the MFE (5.1), and use this to get some control on y_t . Unless otherwise noted, for a vector, $|\cdot|$ denotes the ℓ^∞ norm, that is, $|u| = \max_i |u_i|$. We begin with a useful definition.

DEFINITION 6.1. An event A depending on a parameter n is said to hold *with high probability* or w.h.p. in n if there exists $\gamma > 0$ and n_0 so that $\mathbb{P}(A) \geq 1 - e^{-\gamma^n}$ when $n \geq n_0$.

When possible, probability estimates are given more or less explicitly, but we will occasionally use this definition to reduce clutter, especially in Section 7. We begin with a well-known large deviations result for Poisson random variables; since it is not hard to prove, we supply the proof. For a reference to large deviations theory, see Section 1.9 in [5].

LEMMA 6.1. Let X be Poisson distributed with mean μ , then

$$\mathbb{P}(X > (1 + \delta)\mu) \leq e^{-\delta^2\mu/4} \quad \text{for } 0 < \delta \leq 1/2,$$

$$\mathbb{P}(X < (1 - \delta)\mu) \leq e^{-\delta^2\mu/2} \quad \text{for } \delta > 0.$$

PROOF. We deal separately with $X > (1 + \delta)\mu$ and $X < (1 - \delta)\mu$. For $t > 0$ and using Markov's inequality we have

$$\mathbb{P}(X > (1 + \delta)\mu) = \mathbb{P}(e^{tX} > e^{(1+\delta)t\mu}) \leq \mathbb{E}e^{tX} e^{-(1+\delta)t\mu}.$$

Notice that

$$\mathbb{E}e^{tX} = \sum_{k \geq 0} e^{tk} e^{-\mu} \frac{\mu^k}{k!} = e^{-\mu} \sum_{k \geq 0} \frac{(e^t \mu)^k}{k!} = e^{-\mu} e^{e^t \mu} = \exp((e^t - 1)\mu)$$

so $\mathbb{E}e^{tX} e^{-(1+\delta)t\mu} = \exp(\mu(e^t - 1 - (1 + \delta)t))$. Minimizing $e^t - 1 - (1 + \delta)t$ gives $t = \log(1 + \delta)$, and thus $(1 + \delta) - 1 - (1 + \delta) \log(1 + \delta) = \delta - (1 + \delta) \log(1 + \delta)$.

Since $\log(1 + \delta) \geq \delta - \delta^2/2$ this is at most $\delta - (1 + \delta)(\delta - \delta^2/2) = -\delta^2/2 + \delta^3/2$ which is $\leq -\delta^2/4$ for $0 < \delta \leq 1/2$.

For the other direction we take a similar approach. For $t > 0$ and using Markov's inequality we have

$$\mathbb{P}(X < (1 - \delta)\mu) = \mathbb{P}(e^{-tX} > e^{-(1-\delta)t\mu}) \leq \mathbb{E}e^{-tX}e^{(1-\delta)t\mu}$$

and using $\mathbb{E}e^{-tX} = \exp((e^{-t} - 1)\mu)$ the right-hand side above is $\exp(\mu(e^{-t} - 1 + (1 - \delta)t))$. Minimizing $e^{-t} - 1 + (1 - \delta)t$ gives $-t = \log(1 - \delta)$, and thus $(1 - \delta) - 1 - (1 - \delta)\log(1 - \delta) = -\delta - (1 - \delta)\log(1 - \delta)$. Since $\log(1 - \delta) \geq -\delta - \delta^2/2$, this is at most $-\delta + (1 - \delta)(\delta + \delta^2/2) = -\delta^2/2 - \delta^3/2 \leq -\delta^2/2$. \square

For the next three results, we use the notation $u_t = (y_t, i, si_t, ii_t)$. First, we give an a priori bound on the change in u_t over a short period of time.

LEMMA 6.2. *Let $u_t = (y_t, i_t, si_t, ii_t)$. There are constants $C, \gamma > 0$ so that for all $h > 0$ and fixed t ,*

$$\mathbb{P}\left(\sup_{t \leq s \leq t+h} |u_s - u_t| \leq Ch\right) \geq 1 - e^{-\gamma Nh}.$$

PROOF. Looking to the transitions listed in Section 5, jumps in u_t are of size $\leq 2/N$ and occur at total rate $\leq MN$ for some $M > 0$ that depends only on parameters. Thus, in a time step $h > 0$ the number of events affecting u_t is stochastically bounded above by a Poisson random variable X with mean MNh , so if $X \leq x$ then $|u_s - u_t| \leq 2x/N$ for all $s \in [t, t + h]$. By Lemma 6.1, $\mathbb{P}(X > (1 + \delta)MNh) \leq e^{-\delta^2 MNh/4}$ for $0 < \delta \leq 1/2$. Taking $\delta = 1/4$ and $C = 2(1 + \delta)M$, $\gamma = \delta^2 M/4$ completes the proof. \square

Let $u' = F(u)$ denote the MFE (5.1). As N becomes large, for small $h > 0$ we expect that with probability tending to 1, $u_{t+h} = u_t + hF(u_t) + o(h)$. Using Lemma 6.2 and re-using the estimate from Lemma 6.1 we obtain a quantitative bound on the remainder.

LEMMA 6.3. *Let $u_t = (y_t, i_t, si_t, ii_t)$. For each $\varepsilon > 0$ there are constants $C, \gamma > 0$ so that for small enough $h > 0$,*

$$\mathbb{P}(|u_{t+h} - u_t - hF(u)| \leq \varepsilon h) \geq 1 - Ce^{-\gamma Nh}.$$

PROOF. Let $Q_j(u)$, $j = 1, \dots, 10$, denote the transition rates of the ten transitions introduced in Section 5, as a function of u , and let $X_j(t, h)$ denote the number of type j transitions occurring in the time interval $[t, t + h]$. For each j , $Q_j(u) = Nq_j(u) + R_j(u)$ where $q_j(u)$ is a quadratic function of u and $R_j(u)$ is a remainder that satisfies $|R_j(u)| \leq M$ for some $M > 0$ and all $u \in [0, 1]^4$. It is easily verified that if $u_t = u$ and $X_j(t, h) = Nq_j(u)h$ for each j then $u_{t+h} = u + hF(u)$.

Since each transition changes u by at most $2/N$, it is therefore enough to show that there are constants $C, \gamma > 0$ so that for each j , small enough $h > 0$, and all u ,

$$\mathbb{P}(|X_j(t, h) - Nq_j(u)h| \leq \varepsilon Nh/20 | u_t = u) \geq 1 - Ce^{-\gamma Nh}.$$

Since the domain of $q_j(u)$ is a subset of $[0, 1]^4$, and thus bounded it follows that q_j is bounded and Lipschitz continuous, that is, for some $L > 0$ and all v, u in the domain of q_j , $q_j(u) \leq L$ and $|q_j(v) - q_j(u)| \leq L|v - u|$, and in particular, $|Q_j(v) - Q_j(u)| \leq NL|v - u| + 2M$; for what follows, take $L \geq \varepsilon$. Let $A(t, h)$ be the event

$$\left\{ \sup_{t \leq s \leq t+h} |u_s - u_t| \leq C_1 h \right\},$$

from Lemma 6.2, then on the event $\{u_t = u\} \cap A(t, h)$,

$$\begin{aligned} \sup_{t \leq s \leq t+h} |Q_j(u_s) - Nq_j(u)| &\leq \sup_{t \leq s \leq t+h} |Q_j(u_s) - Q_j(u)| + |Q_j(u) - Nq_j(u)| \\ &\leq N(LC_1 h + 3M/N). \end{aligned}$$

For ease of notation, let $q = q_j(u)$ and let $r = LC_1 h + 3M/N$, and note that $r \rightarrow 0$ as $\max(h, 1/N) \rightarrow 0$. Then, on $\{u_t = u\} \cap A(t, h)$, $X_j(t, h)$ is stochastically bounded above and below respectively by Poisson random variables with means $Nh(q + r)$ and $Nh(q - r)$, so from Lemma 6.1 it follows that for $0 < \delta \leq 1/2$,

$$\begin{aligned} (6.1) \quad &\mathbb{P}(\{|X_j(t, h) - Nhq| \leq Nh(q\delta + r(1 + \delta))\} \cap \{u_t = u\} \cap A(t, h)) \\ &\geq 1 - 2e^{-Nh(q-r)\delta^2/4}. \end{aligned}$$

Recalling that $q \leq L$, let $h, \delta, 1/N > 0$ be chosen small enough that $L\delta + r(1 + \delta) \leq \varepsilon/20$, then $Nh(q\delta + r(1 + \delta)) \leq \varepsilon Nh/20$. To bound the probability uniformly in q , we split into two cases according as $q \geq q\delta + r(1 + \delta)$ or not, that is, as $q \geq r(1 + \delta)/(1 - \delta)$ or not. If $q \geq r(1 + \delta)/(1 - \delta)$ then letting $\gamma_1 = r[(1 + \delta)/(1 - \delta) - 1]\delta^2/4$ which is > 0 it follows that $Nh(q - r)\delta^2/4 \geq \gamma_1 Nh$. If $q < q\delta + r(1 + \delta)$ the lower bound on $X_j(t, h) - Nhq$ is trivial and so in that case

$$\begin{aligned} &\mathbb{P}(\{|X_j(t, h) - Nhq| \leq Nh(q\delta + r(1 + \delta))\} \cap \{u_t = u\} \cap A(t, h)) \\ &\geq 1 - e^{-Nh(q+r)\delta^2/4}. \end{aligned}$$

Letting $\gamma_2 = r\delta^2/4$ which is > 0 it follows that $Nh(q + r)\delta^2/4 \geq \gamma_2 Nh$. Letting γ_3 be such that $\mathbb{P}(A(t, h)) \geq 1 - e^{-\gamma_3 Nh}$ and letting $\gamma = \min(\gamma_1, \gamma_2, \gamma_3)$ and $C = 3$ completes the proof. \square

Using the above estimate, we obtain finite-time control on the evolution of u_t , as N becomes large.

PROPOSITION 6.1. *Let $u_t = (y_t, i_t, si_t, ii_t)$. For each $\varepsilon, T > 0$ there are constants $\delta, C, \gamma > 0$ so that from any initial condition u_0 and any solution $u(t)$ to the MFE (5.1) satisfying $|u_0 - u(0)| \leq \delta$,*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |u_t - u(t)| \leq \varepsilon\right) \geq 1 - Ce^{-\gamma N}.$$

PROOF. The proof is analogous to the proof in numerical analysis that the Euler method is $O(h)$ accurate. Fix $h = T/M$ for integer M and define events A_1, \dots, A_M as follows: $A_1 = B_1 \cap D_1$ and given A_{j-1} , $A_j = A_{j-1} \cap B_j \cap D_j$ where

$$B_j = \left\{ \sup_{h(j-1) \leq t \leq hj} |u_t - u_{hj}| \leq C_1 h \right\}$$

is the event from Lemma 6.2 and

$$D_j = \{|u_{hj} - u_{h(j-1)} - hF(u_{h(j-1)})| \leq \mu h\}$$

is the event from Lemma 6.3, for $\mu > 0$ to be chosen. If $\mu, h > 0$ are fixed and h is small enough, then there are constants $C, \gamma > 0$ so that $\mathbb{P}(B_j \cap D_j) \geq 1 - (C/M)e^{-\gamma N}$, and since $A_M = \bigcap_{j=1}^M (B_j \cap D_j)$, $\mathbb{P}(A_M) \geq 1 - Ce^{-\gamma N}$. For $j = 1, \dots, M$ let

$$E_j = \sup_{\omega \in A_j} |u_{hj}(\omega) - u(hj)|,$$

where ω denotes an element of the probability space for the partner model. Letting $u' = F(u)$ denote (5.1), we have

$$u(hj) - u(h(j-1)) = \int_{h(j-1)}^{hj} F(u(s)) ds.$$

Since $F(u)$ is quadratic in u and its domain is bounded, it is bounded and Lipschitz continuous, that is, for some $L > 0$ and all u, v in the domain, $|F(u)| \leq L$ and $|F(v) - F(u)| \leq L|v - u|$. From the first inequality, it follows that $|u(s) - u(h(j-1))| \leq L(s - h(j-1))$ for $s \geq h(j-1)$ and from this and the second inequality it follows that

$$\begin{aligned} & |u(hj) - u(h(j-1)) - hF(u(h(j-1)))| \\ &= \left| \int_{h(j-1)}^{hj} (F(u(s)) - F(u(h(j-1)))) ds \right| \\ &\leq \int_{h(j-1)}^{hj} |F(u(s)) - F(u(h(j-1)))| ds \\ &\leq \int_{h(j-1)}^{hj} L|u(s) - u(h(j-1))| ds \\ &\leq \int_{h(j-1)}^{hj} L^2(s - hj) ds = L^2 \int_0^h s ds = L^2 h^2 / 2. \end{aligned}$$

Also,

$$\begin{aligned}
 |u_{hj} - u(hj)| &= |u_{hj} - u_{h(j-1)} - hF(u_{h(j-1)}) + u_{h(j-1)} - u(h(j-1)) \\
 &\quad + hF(u_{h(j-1)}) - hF(u(h(j-1))) + u(h(j-1)) \\
 &\quad + hF(u(h(j-1))) - u(hj)| \\
 &\leq |u_{hj} - u_{h(j-1)} - hF(u_{h(j-1)})| + |u_{h(j-1)} - u(h(j-1))| \\
 &\quad + |hF(u_{h(j-1)}) - hF(u(h(j-1)))| \\
 &\quad + |u(hj) - u(h(j-1)) - hF(u(h(j-1)))|
 \end{aligned}$$

so using the definition of A_j , letting $E_0 := |u_0 - u(0)| \leq \delta$ and using once more Lipschitz continuity of F it follows that for $j = 1, \dots, M$,

$$E_j \leq \mu h + E_{j-1} + hLE_{j-1} + L^2 h^2/2 = (1 + hL)E_{j-1} + h(\mu + hL^2/2).$$

Setting $q = (1 + hL)$ and $r = \mu + hL^2/2$ and iterating the inequality $E_j \leq qE_{j-1} + hr$, we find $E_M \leq q^M E_0 + [(q^M - 1)/(q - 1)]hr \leq q^M [E_0 + hr/(q - 1)] = (1 + hL)^M [E_0 + hr/(hL)] = (1 + LT/M)^M [E_0 + r/L] \leq e^{LT} [E_0 + r/L] \leq e^{LT} [\delta + r/L]$ and the same inequality holds for all E_j , $j = 1, \dots, M$. Since on A_j , $|u_s - u_{hj}| \leq C_1 h$ for $h(j-1) \leq s \leq hj$, on A_M we find for $j = 1, \dots, M$ and $h(j-1) \leq s \leq hj$ that

$$\begin{aligned}
 |u_s - u(s)| &\leq |u_s - u_{hj}| + |u_{hj} - u(hj)| + |u(hj) - u(s)| \\
 &\leq C_1 h + E_j + Lh \leq h(C_1 + L) + e^{LT} [\delta + r/L]
 \end{aligned}$$

and taking $h, \mu, \delta > 0$ small enough, this is $\leq \varepsilon$. \square

Our first application of Proposition 6.1 is to control y_t .

LEMMA 6.4. *For each $\varepsilon > 0$, there are constants $C, T, \gamma > 0$ so that from any value $y_0 \in [0, 1]$,*

$$\mathbb{P}\left(\sup_{T \leq t \leq e^{\gamma N}} |y_t - y^*| \leq \varepsilon\right) \geq 1 - Ce^{-\gamma N}.$$

Moreover, if $|y_0 - y^*| \leq 2\varepsilon/3$ we may take $T = 0$.

PROOF. Let $y' = f(y)$ denote the y' equation in (5.1) and let $\phi(t, y)$, $\phi : [0, 1] \times \mathbb{R}_+ \rightarrow [0, 1]$ denote the flow for this equation, that is, the unique function satisfying $\partial_t \phi(t, y) = f(\phi(t, y))$ and $\phi(0, y) = y$ for each (t, y) in its domain. Since $\phi(t, 0) \leq \phi(t, y) \leq \phi(t, 1)$ and $\lim_{t \rightarrow \infty} \phi(t, y) = y^*$ for each $y \in [0, 1]$, for each $\varepsilon > 0$ there is $T > 0$ so that $|\phi(T, y) - y^*| \leq \varepsilon/3$ for all $y \in [0, 1]$. Letting $y(t) = \phi(t, y_0)$ and using Proposition 6.1, there are constants $C_1, \gamma_1 > 0$ depending on ε but not on y_0 so that with probability $\geq 1 - C_1 e^{-\gamma_1 N}$,

$|y_T - y^*| \leq |y_T - y(T)| + |y(T) - y^*| \leq \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3$. Then, for $t \geq 0$ and $y \in [y^* - (2\varepsilon/3), y^* + (2\varepsilon/3)]$,

$$y^* - (2\varepsilon/3) \leq \phi(t, y^* - (2\varepsilon/3)) \leq \phi(t, y) \leq \phi(t, y^* + (2\varepsilon/3)) \leq y^* + (2\varepsilon/3)$$

and since all solutions approach y^* there is $h > 0$ so that $\phi(h, y^* - 2\varepsilon/3) \geq y^* - \varepsilon/3$ and $\phi(h, y^* + 2\varepsilon/3) \leq y^* + \varepsilon/3$. Thus, for the given value of h and any solution $y(t)$ of $y' = f(y)$, if $|y(T) - y^*| \leq 2\varepsilon/3$ then $|y(t) - y^*| \leq 2\varepsilon/3$ for $t \geq T$ and $|y(T+h) - y^*| \leq \varepsilon/3$. Given y_T such that $|y_T - y^*| \leq 2\varepsilon/3$ and setting $y(T) = y_T$, by Proposition 6.1 there are constants $C_2, \gamma_2 > 0$ so that $\sup_{T \leq t \leq T+h} |y_t - y(t)| \leq \varepsilon/3$ with probability $\geq 1 - C_2 e^{-2\gamma_2 N}$, in which case

$$\begin{aligned} \sup_{T \leq t \leq T+h} |y_t - y^*| &\leq \sup_{T \leq t \leq T+h} |y_t - y(t)| + \sup_{T \leq t \leq T+h} |y(t) - y^*| \\ &\leq \varepsilon/3 + 2\varepsilon/3 = \varepsilon \end{aligned}$$

and $|y_{T+h} - y^*| \leq |y_{T+h} - y(T+h)| + |y(T+h) - y^*| \leq \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3$ with the same probability. Iterating this for $e^{\gamma_2 N}/h$ time steps, we find that

$$\sup_{T \leq t \leq e^{\gamma_2 N}} |y_t - y^*| \leq \max_{i \in \{1, \dots, e^{\gamma_2 N}/h\}} \sup_{T+(i-1)h \leq t \leq T+ih} |y_t - y^*| \leq \varepsilon$$

with probability $\geq 1 - (C_2/h)e^{\gamma_2 N}e^{-2\gamma_2 N} = 1 - (C_2/h)e^{-\gamma_2 N}$, then choose $C = C_1 + C_2/h$ and $\gamma = \min(\gamma_1, \gamma_2)$. Note that if $|y_0 - y^*| \leq 2\varepsilon/3$, the iteration step is immediately applicable, in which case we may take $T = 0$. \square

7. Macroscopic behaviour. In this section, we prove the macroscopic side of Theorem 2.2 that is, when $|V_0| \geq \varepsilon N$. We begin with the analogue of Lemma 5.2 for the partner model, which we refer to later on as monotonicity. As for the MFE, define $ip_t := si_t + ii_t$.

LEMMA 7.1. *Let \leq denote the partial order on \mathbb{R}^3 given by $u \leq v \Leftrightarrow u_j \leq v_j, \forall j \in \{1, 2, 3\}$, and let $u_t = (i_t, ip_t, ii_t)$. If $(V_t^{(1)}, E_t^{(1)})$ and $(V_t^{(2)}, E_t^{(2)})$ are two copies of the partner model with $E_0^{(1)} = E_0^{(2)}$ and $V_0^{(1)} \subseteq V_0^{(2)}$ then with respect to the coupling given by the graphical construction, $E_t^{(1)} = E_t^{(2)}$ and $V_t^{(1)} \subseteq V_t^{(2)}$ for $t > 0$ and correspondingly $y_t^{(1)} = y_t^{(2)}$ and $u_t^{(1)} \leq u_t^{(2)}$.*

PROOF. If $E_0^{(1)} = E_0^{(2)}$ then $E_t^{(1)} = E_t^{(2)} =: E_t$ for $t > 0$. Given $\{E_t : t \geq 0\}$, the only transitions affecting $V_t^{(1)}$ and $V_t^{(2)}$ are recovery of infectious sites and transmission from infectious to healthy sites along open edges, both of which preserve the order $V_t^{(1)} \subseteq V_t^{(2)}$. The equality $y_t^{(1)} = y_t^{(2)}$ follows directly from $|E_t^{(1)}| = |E_t^{(2)}|$ and the inequality $u_t^{(1)} \leq u_t^{(2)}$ follows directly from $V_t^{(1)} \subseteq V_t^{(2)}$. \square

Using Proposition 6.1 and Lemma 7.1, we can prove the macroscopic part of Theorem 2.2 when $R_0 \leq 1$. In this section, u_t will generally refer to (i_t, si_t, ii_t) or (i_t, ip_t, ii_t) , with y_t written separately.

PROPOSITION 7.1. *If $R_0 \leq 1$, for each $\varepsilon > 0$ there are constants $C, T, \gamma > 0$ so that, from any initial configuration, with probability $\geq 1 - Ce^{-\gamma N}$, $|V_T| \leq \varepsilon N$.*

PROOF. By Lemma 7.1, it is enough to show the result holds when $V_0 = V$ that is, everyone is initially infectious; in this case $y_0 = 1 - 2E_0/N$, $i_0 = y_0$ and $ip_0 = ii_0 = (1 - y_0)/2$. Let $u_t = (i_t, ip_t, ii_t)$ and let $(y(t), u(t))$ be the solution to the MFE with $y(0) = y_0$ and $u(0) = u_0$. By Lemma 6.4 and Proposition 6.1, for each $\delta > 0$ there are constants $C_1, T_1, \gamma_1 > 0$ so that with probability $\geq 1 - C_1 e^{-\gamma_1 N}$, $|y_{T_1} - y^*| \leq \delta$ and $|u_{T_1} - u(T_1)| \leq \delta$, so with the same probability $|(y_{T_1}, u_{T_1}) - (y^*, u(T_1))| \leq \delta$.

Recall the set Λ^* and let $(y^*, \bar{u}(t))$ be the solution to the MFE with $\bar{u}(0) = (y^*, (1 - y^*)/2, (1 - y^*)/2)$. As shown in the proof of Lemma 5.3, $\bar{u}(t)$ decreases to an equilibrium. Since $R_0 \leq 1$, $(0, 0, 0)$ is the only equilibrium, so $\bar{u}(t) \rightarrow (0, 0, 0)$ as $t \rightarrow \infty$. Moreover, $\bar{u}(0) \geq v$ for each $v \in \Lambda^*$ so for any solution $(y^*, u(t))$, $\bar{u}(0) \geq u(0)$. By Lemma 5.2, $\bar{u}(t) \geq u(t)$ for $t \geq 0$, so there is T_2 not depending on $u(0)$ so that $|u(T_2)| \leq \varepsilon/2$. Using Proposition 6.1, there are constants $C_2, \gamma_2, \delta > 0$ not depending on $u(0)$ so that with probability $\geq 1 - C_2 e^{-\gamma_2 N}$, if $|(y_0, u_0) - (y^*, u(0))| \leq \delta$ then $|u_{T_2}| \leq |u(T_2)| + |u_{T_2} - u(T_2)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Letting $T = T_1 + T_2$, $C = C_1 + C_2$ and $\gamma = \min(\gamma_1, \gamma_2)$ and combining the two steps completes the proof. \square

Using similar ideas, we can prove the macroscopic part of Theorem 2.2 when $R_0 > 1$. Before showing the approach to equilibrium, we first have to show long time survival of the infection, and to do that we need the following result concerning the MFEs.

LEMMA 7.2. *Suppose $R_0 > 1$ and let $v \in \mathbb{R}^3$ with $|v| = 1$ be an unstable eigenvector of the MFEs on Λ^* as given in the proof of Theorem 5.1, written in (i, ip, ii) coordinates. For $0 < \delta' \leq \delta$ let $(y(t), u(t))$ be a solution to the MFE with $|y(0) - y^*| \leq \delta$ and $u(0) := (i(0), ip(0), ii(0)) = \delta'v$. If $\delta > 0$ is small enough, then there is $T > 0$ so that $\min_j u_j(T) \geq 2\delta'$ for all $0 < \delta' \leq \delta$.*

PROOF. First, write the MFE (5.2), without the y equation, in matrix form as follows:

$$(7.1) \quad \begin{pmatrix} i' \\ ip' \\ ii' \end{pmatrix} = \begin{pmatrix} -(1+r_+y) & r_- & r_- \\ r_+(y^* - i/2) & -(1+r_-) & 1 \\ r_+i/2 & \lambda & -(2+r_- + \lambda) \end{pmatrix} \begin{pmatrix} i \\ ip \\ ii \end{pmatrix}.$$

The y dynamics proceeds as in (5.1), and note $|y(t) - y^*| \leq |y(0) - y^*|$ for $t > 0$. Write (7.1) as $u' = A(i, y)u$ with $u = (i, si, ii)^\top$ to emphasize the dependence on

i, y . As noted in the proof of Theorem 5.1, if $R_0 > 1$ then $A := A(0, y^*)$ has a positive eigenvalue $\mu > 0$ with positive eigenvector v such that $|v| = 1$, so the system $v' = Av$ has solutions $v(t) = cve^{\mu t}$ for any $c > 0$. Let $|\cdot|$ denote the operator norm and let

$$L = \sup_{(i,y) \in [0,1]^2} |A(i, y)|$$

then any solution $u(t)$ to (7.1) has $|u(t)| \leq |u(0)|e^{Lt}$ for $t > 0$. Fix $T > 0$, then for each $\varepsilon > 0$, by continuity there is $\delta > 0$ so that if $\max(y - y^*, i) \leq e^{LT}\delta$ then $|A(i, y) - A| \leq \varepsilon$. Let $|y(0) - y^*| \leq \delta$ and for $0 < \delta' \leq \delta$ let $u(t)$ be the solution to (7.1) with $u(0) = \delta'v$, then for $0 \leq t \leq T$,

$$\begin{aligned} |(u - v)'| &= |A(i, y)u - Av| \leq |(A(i, y) - A)u| + |A(u - v)| \\ &\leq |A(i, y) - A||u| + |A||u - v| \\ &\leq \varepsilon|u| + L|u - v| \\ &\leq \varepsilon e^{Lt}\delta' + L|u - v|. \end{aligned}$$

Letting $v(0) = u(0)$, defining $E(t) := |u(t) - v(t)|$, noting that $E(0) = 0$ and integrating,

$$E(T) \leq e^{LT} \varepsilon \delta' T.$$

Since $v(T) = \delta v e^{\mu T}$,

$$\begin{aligned} \min_j u_j(T) &\geq \min_j v_j(T) - \max_j |v_j(T) - u_j(T)| \\ &\geq \delta' e^{\mu T} \min_j v_j - \varepsilon e^{LT} \delta' T \\ &= \delta' e^{\mu T} \left(\min_j v_j - \varepsilon e^{(L-\mu)T} T \right). \end{aligned}$$

Choose $T > 0$ so that $e^{\mu T} \min_j v_j / 2 \geq 2$, then choose $\varepsilon > 0$ so that $\varepsilon e^{(L-\mu)T} T \leq \min_j v_j / 2$, then it follows that $\min_j u_j(T) \geq 2\delta'$. \square

Now we can show long-time survival of the infection when $R_0 > 1$ and $|V_0| \geq \varepsilon N$.

LEMMA 7.3. *Suppose $R_0 > 1$. For each $\varepsilon > 0$, there are constants $\delta, C, \gamma > 0$ so that if $|V_0| \geq \varepsilon N$ then*

$$\mathbb{P}\left(\inf_{0 \leq t \leq e^{\gamma N}} |V_t| \geq \delta N\right) \geq 1 - Ce^{-\gamma N}.$$

PROOF. Recall that an event holds with high probability or w.h.p. in N if for N large enough it occurs with probability $\geq 1 - Ce^{-\gamma N}$ for some $C, \gamma > 0$. If $|V_0| \geq \varepsilon N$ then $\max(i_0, ip_0, ii_0) \geq \varepsilon/3$, so in view of Lemma 7.1 it is enough to prove the result starting from $u_0 := (i_0, ip_0, ii_0) \in \mathcal{E} := \{(\varepsilon/3, 0, 0), (0, \varepsilon/3, 0), (0, 0, \varepsilon/3)\}$. For $\delta_1 > 0$, by Proposition 6.4 there are $T, \gamma_1 > 0$ so that w.h.p. $|y_t - y^*| \leq \delta_1$ for $T \leq t \leq e^{\gamma_1 N}$. If $u(0) \neq (0, 0, 0)$ then for $t > 0$, $\min_j u_j(t) > 0$; this is shown for $u(0) \in \Lambda^*$ in the proof of Lemma 5.3, but the same proof applies if $y \neq y^*$. Also, since $(0, 0, 0)$ is an equilibrium solution, by uniqueness of solutions $u(t) \neq (0, 0, 0)$ for $0 \leq t \leq T$, so by continuity of solutions $\inf\{|u(t)| : 0 \leq t \leq T\} > 0$. Therefore, there exists $0 < \delta_2 \leq \delta_1$ so that $\min_j u_j(T) \geq \delta_2$ and $\inf\{\max_j u_j(t) : 0 \leq t \leq T\} \geq \delta_2$ for all $u(0) \in \mathcal{E}$. For $u_0 = u(0) \in \mathcal{E}$ with $y_0 = y(0) \in [0, 1]$, by Proposition 6.1, w.h.p. $|u_t - u(t)| \leq \delta_2/2$ for $0 \leq t \leq T$ in which case $\min(i_T, ip_T, ii_T) \geq \delta_2/2$ and $\inf\{\max(i_t, ip_t, ii_t) : 0 \leq t \leq T\} \geq \delta_2/2$, which means that for the eigenvector v with $|v| = 1$ mentioned in the proof of Lemma 7.2, $(i_T, si_T, ii_T) \geq (\delta_2/2)v$, and also $|V_t| \geq (\delta_2/2)N$ for $0 \leq t \leq T$.

Taking $y(t) = y_t$ and $u(T) = (\delta_2/2)v$, if $|y_t - y^*| \leq \delta_1$ then by Lemma 7.2 there is $h > 0$ so that $\min_j u_j(T + h) \geq \delta_2$, and as before there is $\delta_3 > 0$ so that $\inf\{\max_j u_j(t) : T \leq t \leq T + h\} \geq \delta_3$. By Lemma 7.1 and the last paragraph, it is enough to consider the case $u_T = u(T) = (\delta_2/2)v$. Letting $\delta = \min(\delta_2/2, \delta_3/2)$ and using Proposition 6.1, with probability $\geq 1 - Ce^{\gamma_2 N}$, $|u_t - u(t)| \leq \delta$ for $T \leq t \leq T + h$, in which case $u_{T+h} \geq (\delta_2/2)v$ and $|V_t| \geq N \min(i_t, ip_t, ii_t) \geq (\delta_3/2)N$ for $T \leq t \leq T + h$. Letting $\gamma = \min(\gamma_1/2, \gamma_2/2)$ and iterating for $e^{\gamma N}/h$ time steps as in the proof of Lemma 6.4, w.h.p. $|V_t| \geq N \min(i_t, ip_t, ii_t) \geq (\delta_3/2)N$ for $T \leq t \leq e^{\gamma N}$. Combining with the previous estimate, w.h.p. $|V_t| \geq \delta N$ for $0 \leq t \leq e^{\gamma N}$ as we wanted to show. \square

We now wrap up the macroscopic side of Theorem 2.2.

PROPOSITION 7.2. *Suppose $R_0 > 1$ and let (y^*, i^*, ip^*, ii^*) with $i^* > 0$ be the nontrivial equilibrium solution to the MFE (5.2). Let $u_t = (i_t, ip_t, ii_t)$ and let $u^* = (i^*, ip^*, ii^*)$. For each $\varepsilon > 0$, there are constants $C, T, \gamma > 0$ so that if $|V_0| \geq \varepsilon N$ then*

$$\mathbb{P}\left(\sup_{T \leq t \leq e^{\gamma N}} |(y_t, u_t) - (y^*, u^*)| \leq \varepsilon\right) \geq 1 - Ce^{-\gamma N}.$$

PROOF. We begin with the lower bound. As shown in the proof of Lemma 7.3 there are $T_1, h_1, \delta_1, \gamma_1 > 0$ so that w.h.p. $\min(i_t, ip_t, ii_t) \geq \delta_1$, and thus $u_t \geq \delta_1 v$, for $t = T_1 + kh_1, k = 1, \dots, (e^{\gamma_1 N} - T_1)/h_1$, where v with $|v| = 1$ is the eigenvector from Lemma 7.2. Let $y(0) = y^*$ and $u(0) := (i(0), ip(0), ii(0)) = \delta_1 v$. If $\delta_1 > 0$ is small enough, then $u'_j(0) > 0$ in each coordinate and since $u^* \neq (0, 0, 0)$ is unique, as shown in the proof of Lemma 5.3 $u(t)$ is increasing with respect to (i, ip, ii) coordinates and $\lim_{t \rightarrow \infty} u(t) = u^*$, and in particular $u(t) \leq u^*$ for $t \geq 0$.

We will need the stronger fact $u_j(t) < u_j^*$ for $j = 1, 2, 3$. Looking to the equations for i', ip', ii' in (5.2), the derivative of each variable increases with the other two variables, and of course is equal to 0 at u^* . If we had $i(t) = i^*$, then since $ip(t) \leq ip^*$ and $ii(t) \leq ii^*$ we would have $i' < 0$ which contradicts the fact that $u(t)$ is increasing, and the same applies to $ip(t)$ and $ii(t)$.

Using the above facts, there is T_2 so that $u(T_2) \geq u^* - \varepsilon/2$, and since $0 < \min_j(u_j^* - u_j(T_2)) =: \varepsilon' \leq \varepsilon$, there is h_2 so that $u(T_2 + h_2) \geq u^* - \varepsilon'/2$. By Proposition 6.1, there is $\delta_2 > 0$ so that if $u_0 = u(0)$ and $|y_0 - y^*| \leq \delta_2$ then w.h.p. $|u_t - u(t)| \leq \varepsilon'/2$ for $T_2 \leq t \leq T_2 + h_2$ in which case $u_t \geq u^* - \varepsilon$ for $T_2 \leq t \leq T_2 + h_2$ and $u_{T_2+h_2} \geq u^* - \varepsilon'$, which means that $u_{T_2+h_2} \geq u(T_2)$. By Lemma 6.4, there are T_3, γ_2 so that w.h.p. $|y_t - y^*| \leq \min(\delta_2, \varepsilon)$ for $T_3 \leq t \leq e^{\gamma_2 N}$. Let k be such that $T_1 + kh_1 \geq T_3$ and let $T_4 = T_1 + kh_1$, then setting $u(T_4) = \delta_1 v$, w.h.p. $u_{T_4} \geq u(T_4)$ so it is enough to consider the case where $u_{T_4} = u(T_4)$. Letting $T = T_4 + T_2$, then for some $\gamma_3 > 0$, with probability $\geq 1 - Ce^{-\gamma_3 N}$, $u_t \geq u^* - \varepsilon$ for $T \leq t \leq T + h_2$ and $u_{T+h_2} \geq u(T)$. Letting $\gamma = \min(\gamma_2/2, \gamma_3/2)$ and iterating for $(e^{\gamma N} - T)/h_2$ time steps (subtracting T to make sure y_t stays in bounds) as in the proof of Lemma 6.4 it follows that $u_t \geq u^* - \varepsilon$ for $T \leq t \leq e^{\gamma N}$.

To prove the upper bound, it is enough to consider any value of y_0 and let $u_0 = (y_0, (1/2)(1 - y_0), (1/2)(1 - y_0))$. Setting $y(0) = y^*$ and $u(0) = (y^*, (1/2)(1 - y^*), (1/2)(1 - y^*))$, then as shown in the proof of Lemma 5.3, $u(t)$ decreases to u^* . Moreover, $u_j(t) - u_j^* > 0$ for the same reason as above, so there is T_1 so that $u(T_1) \leq u^* + \varepsilon/2$, and since $0 < \min_j(u_j(T_1) - u_j^*) =: \varepsilon' \leq \varepsilon$, there is h so that $u(T_1 + h) \geq u^* - \varepsilon'/2$. By Proposition 6.1, there is $\delta > 0$ so that if $\max(|u_0 - u(0)|, |y_0 - y(0)|) \leq \delta$ then w.h.p. $|u_t - u(t)| \leq \varepsilon'/2$ for $T_1 \leq t \leq T_1 + h$ in which case $u_t \leq u^* + \varepsilon$ for $T_1 \leq t \leq T_1 + h$ and $u_{T_1+h} \leq u^* + \varepsilon'$ which means that $u_{T_1+h} \leq u(T_1)$. By Lemma 6.4, there are T_2, γ_1 so that w.h.p. $|y_t - y^*| \leq \delta$ for $T_2 \leq t \leq e^{\gamma_1 N}$. Letting $T = T_1 + T_2$ and setting $u(T_2) = (y^*, (1/2)(1 - y^*), (1/2)(1 - y^*))$ and $u_{T_2} = (y_{T_2}, (1/2)(1 - y_{T_2}), (1/2)(1 - y_{T_2}))$, then for some $\gamma_2 > 0$, with probability $\geq 1 - Ce^{-\gamma_2 N}$, $u_t \leq u^* + \varepsilon$ for $T \leq t \leq T + h$ and $u_{T+h} \leq u(T)$. Letting $\gamma = \min(\gamma_1/2, \gamma_2/2)$ and iterating for $(e^{\gamma N} - T)/h$ time steps it follows as before that $u_t \leq u^* + \varepsilon$ for $T \leq t \leq e^{\gamma N}$. \square

In the next section, we use a comparison to prove that if $R_0 < 1$ the infection disappears quickly from the population. To make this work, we will need a complementary result to Lemma 7.3.

LEMMA 7.4. *If $R_0 \leq 1$, then for each $\varepsilon > 0$ there are $C, T, \gamma > 0$ so that*

$$\mathbb{P}\left(\sup_{T \leq t \leq e^{\gamma N}} |V_t| \leq \varepsilon N\right) \geq 1 - Ce^{-\gamma N}.$$

PROOF. The proof is similar to that of Lemma 7.3. Letting $\bar{u} = (y^*, (1 - y^*)/2, (1 - y^*)/2)$ as in Lemma 5.3 and letting $(y^*, \bar{u}(t))$ be the solution to

the MFE with $\bar{u}(0) = \bar{u}$, since $\bar{u}(t)$ decreases to $(0, 0, 0)$ and $\bar{u} \geq v$ for all $v \in \Lambda^*$, there is T_1 so that for any solution $(y^*, u(t))$, $|u(T_1)| \leq \varepsilon/6$, and since $\varepsilon' := \min_j u_j(T_1) > 0$, there is h so that $|u(T_1 + h)| \leq \varepsilon'/2$. There is $\delta > 0$ so that if $\max(|u_0 - u(0)|, |y_0 - y^*|) \leq \delta$ then w.h.p. $|u_t - u(t)| \leq \min(\varepsilon'/2, \varepsilon/6)$ for $T_1 \leq t \leq T_1 + h$ in which case $|u_t| \leq \varepsilon/3$ for $T_1 \leq t \leq T_1 + h$ and $|u_{T_1+h}| \leq \varepsilon'$ which means $u_{T_1+h} \leq u(T_1)$. There are $\gamma_1, T_2 > 0$ so that w.h.p. $|y_t - y^*| \leq \delta$ for $T_2 \leq t \leq e^{\gamma_1 N}$. By monotonicity, it is enough to consider $u_{T_2} = \bar{u}$. Letting $u(T_2) = u_{T_2}$ and $T = T_1 + T_2$, there are C_1, γ_2 so that with probability $\geq 1 - C_1 e^{\gamma_2 N}$, $|u_t| \leq \varepsilon/3$ for $T \leq t \leq T + h$ and $u_T \leq u(T)$. Letting $\gamma = \min(\gamma_1, \gamma_2)$ and iterating for $(e^{\gamma N} - T)/h$ time steps, w.h.p. $|u_t| \leq \varepsilon/3$ and thus $|V_T| \leq \varepsilon N$ for $T \leq t \leq e^{\gamma N}$. \square

8. Microscopic behaviour. In this section, we compare the partner model in the regime $|V| \leq \varepsilon N$ for small $\varepsilon > 0$ to a branching process to get decisive information when $R_0 \neq 1$.

8.1. Subcritical case: $R_0 < 1$. First, we introduce the comparison process to use when $R_0 < 1$.

DEFINITION 8.1. Define the *upperbound process* (UBP) $B_t = (\mathcal{I}_t, \mathcal{SI}_t, \mathcal{II}_t)$ on state space $\{0, 1, 2, \dots\}^3$ with parameter $0 \leq \delta \leq y^*$ by the following transitions:

- $\mathcal{I} \rightarrow \mathcal{I} - 1$ at rate \mathcal{I} ,
- $\mathcal{I} \rightarrow \mathcal{I} - 1$ and $\mathcal{SI} \rightarrow \mathcal{SI} + 1$ at rate $r_+(y^* - \delta)\mathcal{I}$,
- $\mathcal{SI} \rightarrow \mathcal{SI} + 1$ at rate $2r_+\delta\mathcal{I}$,
- $\mathcal{II} \rightarrow \mathcal{II} + 1$ at rate $r_+\delta\mathcal{I}$,
- $\mathcal{SI} \rightarrow \mathcal{SI} - 1$ at rate \mathcal{SI} ,
- $\mathcal{SI} \rightarrow \mathcal{SI} - 1$ and $\mathcal{I} \rightarrow \mathcal{I} + 1$ at rate $r_-\mathcal{SI}$,
- $\mathcal{SI} \rightarrow \mathcal{SI} - 1$ and $\mathcal{II} \rightarrow \mathcal{II} + 1$ at rate $\lambda\mathcal{SI}$,
- $\mathcal{II} \rightarrow \mathcal{II} - 1$ and $\mathcal{SI} \rightarrow \mathcal{SI} + 1$ at rate $2\mathcal{II}$,
- $\mathcal{II} \rightarrow \mathcal{II} - 1$ and $\mathcal{I} \rightarrow \mathcal{I} + 2$ at rate $r_-\mathcal{II}$.

Note the UBP describes the evolution of the total number of particles of each of the three types $\mathcal{I}, \mathcal{SI}, \mathcal{II}$ in a multi-type continuous-time branching process; for an introduction to branching processes, see [7]. We now show that for fixed $R_0 < 1$, if $\delta > 0$ is small enough the UBP quickly dies out.

LEMMA 8.1. For fixed λ, r_+, r_- , let B_t denote the UBP with parameter δ' and let R_0 be as defined in (4.1). If $R_0 < 1$, there are $C, \delta > 0$ so that if $|B_0| \leq N$ and $\delta' \leq \delta$ then

$$\mathbb{P}(|B_{C \log N}| = 0) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

PROOF. For a multi-type continuous time branching process $B_t = (b_1(t), \dots, b_n(t))$, with $b_j(t)$ denoting the number of type j particles alive at time t , we can extract some useful information from the *mean matrix* M_t defined by $m_{ij}(t) = \mathbb{E}(b_j(t) | b_k(0) = \delta_{ik})$. Since particles evolve independently, $\mathbb{E}(B_t) = B_0 M_t$ and it is not hard to show that M_t satisfies the equation

$$\frac{d}{dt} M_t = A M_t$$

and, therefore, $M_t = \exp(At)$, where $A = (r_{ij})$ is the matrix whose entries r_{ij} give the rate at which a particle of type i produces particles of type j . If $\operatorname{Re}(\lambda) < 0$ for each eigenvalue λ of A , then letting $\gamma_0 = \min\{|\operatorname{Re}(\lambda)| : \lambda \in \sigma(A)\}$ where $\sigma(\cdot)$ denotes the spectrum, from standard matrix theory it follows that for any $\gamma_1 < \gamma_0$, there is $C_1 > 0$ so that $m_{ij} \leq C_1 e^{-\gamma_1 t}$ for each pair ij . Since each $b_i(t)$ is valued on nonnegative integers,

$$\begin{aligned} \mathbb{P}(B_t \neq (0, \dots, 0)) &\leq \sum_i \mathbb{P}(b_i(t) \neq 0) \leq \sum_i \mathbb{E} b_i(t) \\ &= \sum_{ij} b_i(0) m_{ij}(t) \leq |B(0)| n^2 C_1 e^{-\gamma_1 t}. \end{aligned}$$

If $|B(0)| \leq N$, then letting $t = C \log N$ for $C > 1/\gamma_1$ and setting $\gamma = C\gamma_1 - 1$ and $C_2 = n^2 C_1$ we find

$$\begin{aligned} \mathbb{P}(B_{C \log N} \neq (0, \dots, 0)) &\leq N C_2 e^{-\gamma_1 C \log N} = N C_2 N^{-\gamma_1 C} \\ &= C_2 N^{1-\gamma_1 C} = C_2 N^{-\gamma} \end{aligned}$$

which tends to 0 as $N \rightarrow \infty$. In our case,

$$A = A(\delta) = \begin{pmatrix} -(1 + r_+(y^* - \delta)) & r_+(y^* + \delta) & r_+\delta \\ r_- & -(1 + r_- + \lambda) & \lambda \\ 2r_- & 2 & -(2 + r_-) \end{pmatrix}.$$

Letting $\sigma(A)$ denote the spectrum and defining the *spectral abscissa* $\mu(A) := \max\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}$, if $\mu(A(\delta)) < 0$, then the real part of each eigenvalue of A is negative, and the above argument applies. By continuity of eigenvalues in the entries of a matrix, it is enough to show $\mu(A(0)) < 0$, since then there is $\delta > 0$ so that if $\delta' \leq \delta$ then $\mu(A(\delta')) \leq \mu(A(0))/2 < 0$. Setting $\delta = 0$,

$$A(0) = \begin{pmatrix} -(1 + r_+ y^*) & r_+ y^* & 0 \\ r_- & -(1 + r_- + \lambda) & \lambda \\ 2r_- & 2 & -(2 + r_-) \end{pmatrix}$$

and looking to Section 5 we see that $A(0, 0)$ is the (transpose of the) linearized matrix at $(0, 0, 0)$ for the MFE on Λ^* , which we denote A . As noted in Remark 5.1, $(0, 0, 0)$ is locally asymptotically stable when $R_0 < 1$, and in the proof of Theorem 2 in [14] this is done by showing that $\mu(A) < 0$. \square

We now complete the proof of the case $R_0 < 1$ in Theorem 2.2.

PROPOSITION 8.1. *If $R_0 < 1$ there are constants $C, T, \gamma > 0$ so that, from any initial configuration,*

$$\mathbb{P}(|V_{T+C \log N}| = 0) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

PROOF. Let $U_t := (I_t, SI_t, II_t)$ denote variables in the partner model and for $\delta > 0$ such that $y^* - \delta \geq 0$ and $y^* + \delta \leq 1$, let B_t denote the UBP with parameter δ . We first describe a coupling with the property that $U_0 \leq B_0 \Rightarrow U_t \leq B_t$ for $t > 0$, with respect to the usual partial order $U \leq V \Leftrightarrow U_j \leq V_j, j = 1, 2, 3$. For $j = 1, \dots, 10$, define a countable number of independent Poisson point processes (p.p.p.'s) $\{e_j(n) : n = 1, 2, \dots\}$ with respective rates $1, r_+, r_+, 1, r_-, \lambda, 2, r_-, r_+, r_-$, together with independent uniform $[0, 1]$ random variables attached to each event in $e_2(n), e_3(n), e_9(n), n = 1, 2, \dots$. These correspond to the nine transitions listed in the definition of the UBP, except that the second and third transition in the UBP are lumped into e_2 , plus an additional transition for $S + S \rightarrow SS$ and one for $SS \rightarrow S + S$. Note that the rates of e_2, e_3, e_9 appear too large at the moment and are corrected in the next paragraph.

Construct the UBP one transition at a time as follows, letting $(\mathcal{I}, S\mathcal{I}, \mathcal{II})$ denote the present state. Each event in $e_1(1), \dots, e_1(\mathcal{I})$ reduces \mathcal{I} by 1. For an event in e_2, e_3 let p denote the corresponding uniform $[0, 1]$ random variable. If an event in $e_2(1), \dots, e_2(\mathcal{I})$ occurs and $p \leq (y^* - \delta)$, reduce \mathcal{I} by 1 and increase $S\mathcal{I}$ by 1, while if $y^* - \delta < p \leq y^* + \delta$ simply increase $S\mathcal{I}$ by 1. If an event in $e_3(1), \dots, e_3(\mathcal{I})$ occurs and $p \leq \delta$, increase \mathcal{II} by 1. Each event in $e_4(1), \dots, e_4(S\mathcal{I})$ reduces $S\mathcal{I}$ by 1, each event in $e_5(1), \dots, e_5(S\mathcal{I})$ reduces $S\mathcal{I}$ by 1 and increases \mathcal{I} by 1, each event in $e_6(1), \dots, e_6(S\mathcal{I})$ reduces $S\mathcal{I}$ by 1 and increases \mathcal{II} by 1, each event in $e_7(1), \dots, e_7(\mathcal{II})$ reduces \mathcal{II} by 1 and increases $S\mathcal{I}$ by 1, and each event in $e_8(1), \dots, e_8(\mathcal{II})$ reduces \mathcal{II} by 1 and increases \mathcal{I} by 2. It can be checked that the transition rates are correct.

Similarly, construct the Markov chain $(S_t, I_t, SS_t, SI_t, II_t)$ for the partner model as follows, letting (S, I, SS, SI, II) denote the present state. Define $\alpha_t = y_t - y^* - i_t$ and $\beta_t = i_t/2 - 1/(2N)$ and note that α_t and β_t are piecewise constant in time. Each event in $e_1(1), \dots, e_1(I)$ reduces I by 1 and increases S by 1. If an event in $e_2(1), \dots, e_2(I)$ occurs and $p \leq y^* + \alpha_t$ reduce S and I by 1 and increase SI by 1. If an event in $e_3(1), \dots, e_3(I)$ occurs and $p \leq \beta_t$ reduce I by 2 and increase II by 1. Each event in $e_4(1), \dots, e_4(SI)$ reduces SI by 1 and increases SS by 1, each event in $e_5(1), \dots, e_5(SI)$ reduces SI by 1 and increases S and I by 1, and events in e_6, e_7, e_8 have the same effect as before. If an event in $e_9(1), \dots, e_9(S)$ occurs and $p \leq s_t/2 - 1/(2N)$ reduce S by 2 and increase SS by 1, and each event in $e_{10}(1), \dots, e_{10}(SS)$ reduces SS by 1 and increases S by 2. Recalling that $U_t := (I_t, SI_t, II_t)$, if $U_0 \leq B_0$ and $\sup_{s \leq t} \max(|\alpha_s|, \beta_s) \leq \delta$ then $U_t \leq B_t$ since (as can be easily checked) the order is preserved at each transition.

By Lemma 6.4, there are $T_1, \gamma_1 > 0$ so that w.h.p. $|y_t - y^*| \leq \delta/2$ for $T_1 \leq t \leq e^{\gamma_1 N}$ and since $R_0 < 1$, by Lemma 7.4 there are T_2, γ_2 so that $|V_t| \leq (\delta/2)N$, and

thus $i_t \leq \delta/2$ for $T_2 \leq t \leq e^{\gamma_2 N}$. Letting $T = \max(T_1, T_2)$ and $\gamma = \min(\gamma_1, \gamma_2)$, w.h.p. $\max(|\alpha_t|, \beta_t) \leq \delta$ for $T \leq t \leq e^{\gamma N}$. Setting $B_T = U_T$ and using Lemma 8.1 completes the proof. \square

8.2. Supercritical case: $R_0 > 1$. We introduce the comparison process for $R_0 > 1$, which is similar to the UBP, but different.

DEFINITION 8.2. Define the *lowerbound process* (LBP) $B_t = (\mathcal{I}_t, \mathcal{SI}_t, \mathcal{II}_t)$ on state space $\{0, 1, 2, \dots\}^3$ with parameters $\delta \geq 0$ such that $y^* - \delta \geq 0$ by the following transitions:

- $\mathcal{I} \rightarrow \mathcal{I} - 1$ at rate $(1 + 2r_+ \delta)\mathcal{I}$,
- $\mathcal{I} \rightarrow \mathcal{I} - 1$ and $\mathcal{SI} \rightarrow \mathcal{SI} + 1$ at rate $r_+(y^* - \delta)\mathcal{I}$,
- $\mathcal{I} \rightarrow \mathcal{I} - 2$ at rate $r_+ \delta \mathcal{I}$,
- $\mathcal{SI} \rightarrow \mathcal{SI} - 1$ at rate \mathcal{SI} ,
- $\mathcal{SI} \rightarrow \mathcal{SI} - 1$ and $\mathcal{I} \rightarrow \mathcal{I} + 1$ at rate $r_- \mathcal{SI}$,
- $\mathcal{SI} \rightarrow \mathcal{SI} - 1$ and $\mathcal{II} \rightarrow \mathcal{II} + 1$ at rate $\lambda \mathcal{SI}$,
- $\mathcal{II} \rightarrow \mathcal{II} - 1$ and $\mathcal{SI} \rightarrow \mathcal{SI} + 1$ at rate $2\mathcal{II}$,
- $\mathcal{II} \rightarrow \mathcal{II} - 1$ and $\mathcal{I} \rightarrow \mathcal{I} + 2$ at rate $r_- \mathcal{II}$.

As before, the LBP describes the evolution of the total number of particles of each of the three types $\mathcal{I}, \mathcal{SI}, \mathcal{II}$ in a multi-type continuous-time branching process. We now show that for fixed $R_0 > 1$, if $\delta > 0$ is small enough then the LBP survives.

LEMMA 8.2. *Let B_t denote the LBP with parameter δ' . If λ, r_+, r_- are such that $R_0 > 1$ then there are $C, \delta > 0$ so that if $\delta' \leq \delta$ then $\liminf_{N \rightarrow \infty} \mathbb{P}(B_{C \log N} \neq (0, 0, 0)) > 0$ and*

$$\mathbb{P}(|B_{C \log N}| \geq \delta N | B_{C \log N} \neq (0, 0, 0)) \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

PROOF. As in the proof of Lemma 8.1, define the mean matrix $M(t) = \exp(At)$ and the spectral abscissa $\mu(A)$. If $\delta' = 0$ for both the UBP and the LBP they coincide, in which case A is the transpose of the linearized matrix at $(0, 0, 0)$ of the MFE on Λ^* . As shown in the proof of Theorem 5.1, if $R_0 > 1$ then $\mu(A) > 0$. By continuity of eigenvalues in the entries of a matrix, there is $\delta > 0$ so that if $\delta' \leq \delta$ then $\mu(A(\delta')) \geq \mu(A)/2 > 0$. As shown in V.7 of [7], if $M(t)$ is such that for some $t_0 > 0$ and each entry $m_{ij}(t)$ of $M(t)$ one has $m_{ij}(t_0) > 0$ (which is the case here), then $\mu(A) =: \lambda_1$ is an eigenvalue of A , and if $\lambda_1 > 0$ the process is said to be *supercritical*. In this case, $B_t e^{-\lambda_1 t} \rightarrow Wv$ where v is a left eigenvector of A with eigenvalue λ_1 and W is a real-valued random variable. Setting $t = C \log N$ with $C > 1/\lambda_1$ and letting $\gamma = C\lambda_1 > 1$, $B_{C \log N} N^{-\gamma} \rightarrow Wv$, so for each $\varepsilon > 0$,

$$\liminf_{N \rightarrow \infty} \mathbb{P}(|B_{C \log N}| \geq \delta N) \geq \lim_{N \rightarrow \infty} \mathbb{P}(|B_{C \log N}| \geq \varepsilon N^\gamma) = \mathbb{P}(W|v| \geq \varepsilon)$$

and letting $\varepsilon \rightarrow 0^+$ and using continuity of measure,

$$\liminf_{N \rightarrow \infty} \mathbb{P}(|B_C \log N| \geq \delta N) \geq \mathbb{P}(W > 0).$$

Under a mild regularity assumption on the offspring distribution that holds trivially in this case, $\mathbb{P}(W > 0) = \lim_{t \rightarrow \infty} \mathbb{P}(B_t \neq (0, 0, 0)) > 0$. Since $|B_t| \geq \delta N$ implies $B_t \neq (0, 0, 0)$, this means $\limsup_{N \rightarrow \infty} \mathbb{P}(|B_C \log N| \geq \delta N) \leq \lim_{t \rightarrow \infty} \mathbb{P}(B_t \neq (0, 0, 0)) = \mathbb{P}(W > 0)$, so $\lim_{N \rightarrow \infty} \mathbb{P}(|B_C \log N| \geq \delta N)$ exists and is equal to $\mathbb{P}(W > 0)$. The result then follows by observing that for $t, x > 0$, $\mathbb{P}(|B_t| \geq x | B_t \neq (0, 0, 0)) = \mathbb{P}(|B_t| \geq x) / \mathbb{P}(B_t \neq (0, 0, 0))$. \square

We now complete the proof of Theorem 2.2.

PROPOSITION 8.2. *If $R_0 > 1$, there are constants $\delta, p, C, T > 0$ so that if $|V_0| > 0$ then $\mathbb{P}(|V_{T+C \log N}| \geq \delta N) \geq p$.*

PROOF. We use the same approach as in the proof of Proposition 8.1. Let $U_t := (I_t, SI_t, II_t)$ denote variables in the partner model and for $\delta_1 > 0$ such that $\delta_1 \leq 1$, $y^* - \delta_1 \geq 0$ and $y^* + \delta_1 \leq 1$, let B_t denote the LBP with parameter δ_1 . Let e_1, \dots, e_{10} be as in the proof of Proposition 8.1.

Construct the LBP one transition at a time as follows, letting $(\mathcal{I}, S\mathcal{I}, \mathcal{II})$ denote the present state. Each event in $e_1(1), \dots, e_1(\mathcal{I})$ reduces \mathcal{I} by 1. For an event in e_2, e_3 let p denote the corresponding uniform $[0, 1]$ random variable. If an event in $e_2(1), \dots, e_2(\mathcal{I})$ occurs and $p \leq (y^* - \delta_1)$, reduce \mathcal{I} by 1 and increase $S\mathcal{I}$ by 1, while if $y^* - \delta_1 < p \leq y^* + \delta_1$ simply reduce \mathcal{I} by 1. If an event in $e_3(1), \dots, e_3(\mathcal{I})$ occurs and $p \leq \delta_1$, reduce \mathcal{I} by 2. Events in e_4, e_5, e_6, e_7, e_8 have the same effect as in the dynamics of the UBP. The Markov chain $(S_t, I_t, SS_t, SI_t, II_t)$ for the partner model is constructed in the same way as in the proof of Proposition 8.1, with α_t, β_t defined in the same way, and it is easy to check in this case that if $U_0 \geq B_0$ and $\sup_{s \leq t} \max(|\alpha_s|, \beta_s) \leq \delta_1$ then $U_t \geq B_t$.

Define the stopping time $\tau = \inf\{t : |U_t| \geq \delta_1 N/2\}$ and note that $|V_\tau| \geq (\delta_1/2)N$. By Lemma 7.3 and using the strong Markov property, there are $\delta, \gamma > 0$ so that w.h.p. $|V_t| \geq \delta N$ for $\tau \leq t \leq \tau + e^{\gamma N}$. There are $T, \gamma > 0$ so that w.h.p. $|y_t - y^*| \leq \delta_1/2$ for $T \leq t \leq e^{\gamma N}$. If $\tau \leq T$, then since T is fixed, we are done. If $t < \tau$ then $i_t \leq \delta_1/2$, so letting $B_T = U_T$, if $T \leq t < \tau$ then $\max(|\alpha_t|, \beta_t) \leq \delta_1$, so $U_t \geq B_t$ for $T \leq t < \tau$. The result follows from this inequality and from Lemma 8.2. \square

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