Estimation of the activity of jumps in time-changed Lévy models[*]

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Abstract: In this paper we consider a class of time-changed Lévy processes that can be represented in the form $Y_s = X_{\mathcal{T}(s)}$, where X is a Lévy process and $\mathcal{T}$ is a non-negative and non-decreasing stochastic process independent of X. The aim of this work is to infer on the Blumenthal-Getoor index of the process X from low-frequency observations of the time-changed Lévy process Y. We propose a consistent estimator for this index, derive the minimax rates of convergence and show that these rates can not be improved in general. The performance of the estimator is illustrated by numerical examples.

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1. Introduction

Nonparametric statistical inference for Lévy-type processes have been attracting the attention of researchers for many years starting from the works of Rubin and Tucker (1959) and Basawa and Brockwell (2007). The popularity of Lévy models is based on their simplicity on the one hand and the ability to reproduce many specific properties of the economic data on the other hand. In this article, we consider a class of processes known as the time-changed Lévy process. Let X be a Lévy process and $\mathcal{T}$ be a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$. Then the time-changed Lévy process is defined as $Y_s = X_{\mathcal{T}(s)}$. The change of time can be motivated by the fact that some economical effects (e.g., nervousness of the market which is indicated by volatility) can be better expressed in terms of "business" time which may run faster than the physical one in some periods (see Veraart and Winkel, 2010).

Theoretically it is known that even in the case of the Brownian motion X, the resulting class of time-changed processes is rather large and basically coincides with the class of all semimartingales (Monroe, 1978). Nevertheless, the practical application of this fact for financial modelling meets two major problems: first, the change of time $\mathcal{T}$ can be highly intricate - for instance, if Y has discontinuous trajectories (see Barndorff-Neilsen and Shiryaev, 2010); second, the dependence structure between X and $\mathcal{T}$ can be also quite sophisticated. In order to avoid the above difficulties we consider the whole class of Lévy processes for X and assume that X is independent of $\mathcal{T}$.

Suppose now that a time-changed process Y is observable on the equidistant time grid $0 < \Delta < \cdots < n\Delta$ with some $n \in \mathbb{N}$ and $\Delta > 0$. A natural question is which parameters of the underlying Lévy process X can be identified from the observations $Y_0, Y_\Delta, \ldots, Y_{n\Delta}$ as $n \to \infty$. This question has been recently addressed in the literature, and the answer turns out to crucially depend on the asymptotic behaviour of Δ and on the degree of our knowledge about $\mathcal{T}$. So in the case of high-frequency data with increasing time horizon, i.e., $\Delta_n \to 0$ with $n \cdot \Delta_n \to \infty$ (also known as rapidly increasing design), one basically can, under some regularity conditions, identify X completely, provided $\mathbb{E}[\mathcal{T}]$ is known (see Figueroa-López, 2009). If the time horizon remains fixed, only the diffusion part

of X and the behaviour of the Lévy measure of X at 0 can be identified (see Aït-Sahalia and Jacod, 2009 and 2012). The latter behaviour can be characterised in terms of the so-called Blumenthal-Getoor index or successive Blumenthal-Getoor indexes. The Blumenthal-Getoor index is a characteristic of the activity of small jumps and for a one-dimensional Lévy process $Z = (Z_t)_{t \geq 0}$ with a Lévy measure ν can be defined via

$$
\mathtt{BG}(Z) = \inf \left\{ r > 0 : \int_{|x| \leq 1} |x|^r \nu(dx) < \infty \right\}.
$$

The Blumenthal-Getoor index, its practical importance and its theoretical properties, have recently got much attention in the literature. For instance, Mijatovic and Tankov (2012) studied the impact of this index on the asymptotic behavior of the implied volatility. Rosenbaum and Tankov (2012) showed how the activity of small jumps influences the optimal discretization strategies for option pricing.

An important remark is that the statistical analysis of the time-changed Lévy models is much more difficult than the one of Lévy models. This lies in the fact that the increments of Y are not any longer independent and that neither the process X nor $\mathcal{T}$ is directly observable (see Belomestny, 2011).

This paper is devoted to the case of low-frequency data, i.e., the case when Δ is fixed and $n \to \infty$. For this case, Belomestny (2011) has proved that one can not in general identify the Lévy measure X from the low-frequency observations of Y. However, the question remains open whether the behaviour of ν at 0, expressed in terms of the BG index, can be recovered. It follows from the results of this work that a consistent estimation of the BG index of X is basically possible, provided the Lévy process X is independent of $\mathcal{T}$ and has a nonzero diffusion part.

The approach presented in this paper was already introduced by Belomestny and Panov (2013) in the context of affine stochastic volatility models. Nevertheless, the unique similarity between this paper and Belomestny and Panov (2013) is the main idea to use the asymptotic behaviour of the characteristic function of the increments of the price processes to infer on the Blumenthal-Getoor (BG) index. Indeed, the results are quite different: while the simultaneous estimation of the jump activities for Lévy process driving the volatility and the state process in ASV models is not possible, for the time-changed Lévy processes we are able to consistently estimate the BG indexes of the processes X and $\mathcal{T}$ provided X has a non-zero diffusion part. Furthermore, there are huge differences between Belomestny and Panov (2013) and the current paper on the methodological side. For example, the analysis in Belomestny and Panov (2013) is heavily based on the properties of affine processes that can not be used in the current framework.

The paper is organised as follows. In the next section, we present the main setup and give basic definitions and examples. Next, we introduce the main object of our study, the time-changed Lévy processes and formulate the main assumptions. Section 3 contains the so-called Abelian theorem describing the asymptotic behaviour of the characteristic function of the increments $Y_{t+\Delta} - Y_t$ for fixed $\Delta > 0$. Estimation algorithm for the Blumenthal-Getoor index of X

is presented in Section 4, where also some theoretical results showing the consistency of the proposed estimator and the corresponding rates of convergence, are presented. Some numerical examples can be found in Section 5. All proofs are collected in Section 6.

2. Setup

2.1. Lévy process X

Throughout this paper we assume that the process (X_t) is a one-dimensional Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. This particularly means that the characteristic function of X has the form:

$$\phi_X(u) := \mathbb{E}\left[\exp\{iuX_t\}\right] = \exp\{t\psi(u)\}, \quad t > 0,$$

where the function $\psi(u)$ is the so-called characteristic exponent of X. The Lévy-Khintchine formula yields

$$\psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \mathcal{V}(u),$$

$$\mathcal{V}(u) := \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux \cdot \mathbf{1}_{\{|x| \leq 1\}}\right) \nu(dx),$$

where $\mu \in \mathbb{R}$, σ is a non-negative number and ν is a Lévy measure on $\mathbb{R} \setminus \{0\}$, which satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty.$$

A triplet (μ, σ^2, ν) is called the characteristic triplet of the Lévy process X_t. We need the following assumptions.

(AL) σ is strictly positive and the function $\mathcal{V}(u)$ has the following representation:

$$\mathcal{V}(u) = -\lambda_1 |u|^\gamma \Psi_1(u), \tag{2.1}$$

where $\lambda_1 > 0$, $\gamma \in (0, 2)$, and for u large enough

$$|1 - \Psi_1(u)| \leq \vartheta_1 u^{-\chi_1} \tag{2.2}$$

with $\chi_1 \in (0, \gamma)$, $\vartheta_1 > 0$.

The assumption (AL) is, for example, fulfilled if there exist $\beta^{(0)} > 0$, $\beta^{(1)} \in \mathbb{R}$ such that

$$\int_{|x|>\varepsilon} \nu(dx) = \varepsilon^{-\gamma}(\beta^{(0)} + \beta^{(1)}\varepsilon^{\chi_1}(1 + O(\varepsilon))), \quad \varepsilon \to +0, \tag{2.3}$$

see Lemma C.1. Note that the parameter γ in (2.3) coincides with the Blumenthal-Getoor index, because this index can be equivalently defined as the smallest number $p \in [0, 2]$ such that

$$\lim_{\varepsilon \to 0+} \varepsilon^p \int_{|x| > \varepsilon} \nu(dx) \in (0, +\infty). \tag{2.4}$$

On the other hand, if the Assumption (AL) is fulfilled, then the parameter γ in (2.1) also coincides with the Blumenthal-Getoor index of the process X. In fact, Theorem 4.15 from Bismut (1983), known also as Abelian and Tauberian theorem for Lévy processes, states that the condition (2.4) is equivalent to

$$\operatorname{Re} \mathcal{V}(u) \asymp -Cu^\gamma, \qquad u \to +\infty \tag{2.5}$$

with some $C > 0$. Here and in the sequel the notation $f(u) \asymp g(u)$ as $u \to \infty$ (with some functions f and g such that $g(u) \neq 0$, $\forall u \in \mathbb{R}$) means that $\lim_{u \to \infty} f(u)/g(u) = 1$.

2.2. Time change

Let $\mathcal{T} = (\mathcal{T}(s))_{s \geq 0}$ be an increasing right-continuous process with left limits such that $\mathcal{T}(0) = 0$ and, for each fixed s, the random variable $\mathcal{T}(s)$ is a stopping time with respect to the filtration $\mathcal{F}$. In this paper, it is also assumed that

(AT1) the process $\mathcal{T}$ is independent of X;

(AT2) the sequence $T_k = \mathcal{T}(\Delta k) - \mathcal{T}(\Delta(k-1))$, $k \in \mathbb{N}$, is strictly stationary and α-mixing with the mixing coefficients $(\alpha_T(j))_{j \in \mathbb{N}}$ satisfying

$$\alpha_T(j) \leq \bar{\alpha}_0 \exp(-\bar{\alpha}_1 j), \quad j \in \mathbb{N},$$

for some positive constants $\bar{\alpha}_0$ and $\bar{\alpha}_1$ (the term "α-mixing sequence" means that $\alpha_T(j) \to 0$ as $j \to +\infty$);

(AT3) the Laplace transform of $\mathcal{T}(\Delta)$ has the following asymptotic behavior:

$$\mathcal{L}_\Delta(u) := \mathbb{E}[\exp(-u\mathcal{T}(\Delta))] \asymp A \exp\left\{-\lambda_2 u^\alpha \Psi_2(u)\right\}, \ u \to +\infty, \tag{2.6}$$

with $\lambda_2 > 0$, $A > 0$, $\alpha \in (0, 1]$, and $\Psi_2(u)$ such that for u large enough

$$|1 - \Psi_2(u)| \leq \vartheta_2 u^{-\chi_2}$$

with some $\chi_2, \vartheta_2 \geq 0$.

Note that in the case where $\mathcal{T}$ is an increasing Lévy process, i.e., a subordinator, the parameter α coincides with the Blumenthal-Getoor index of $\mathcal{T}$.

Remark 2.1. It is easy to see that the most restrictive assumption is (AT1). This assumption is made for the identifiability reasons. The corresponding class of processes remains rather large, but its full characterisation is still an open problem (see Barndorff-Nielsen and Shiryaev, 2010).

2.3. Examples

Tempered stable subordinator. The tempered stable distribution with parameters (a, b, α) can be defined via its Laplace transform:

$$\mathcal{L}^{TS}(u) := \exp\left\{ ab - a(b^{1/\alpha} + 2u)^\alpha \right\},$$

where $a > 0$, $b \geq 0$, $\alpha \in (0, 1)$, see Schoutens (2003). The tempered stable subordinator is a process Z_t, which has increments $Z_{t+s} - Z_t$ following a tempered stable law with parameters (sa, b, α). The Lévy measure of this process is of the form:

$$\rho(x) := \frac{c}{x^{\alpha+1}} \exp\{-\lambda x\} I\{x > 0\},$$

where $\lambda = b^{1/\alpha}/2$. Here the decay rate of big jumps, $c = -a * 2^\alpha / \Gamma(-\alpha)$ alters the intensity of all jumps simultaneously, and α is the Blumenthal-Getoor index of the process (see Cont and Tankov, 2004). Tempered stable subordinator satisfy (AT3) with $\chi_2 = 1$, $\lambda_2 = 2^\alpha a\Delta$, $\vartheta_2 = b^{1/\alpha}\alpha/2$ and $A = \exp\{ab\Delta\}$.

Integrated CIR process. Another candidate for the time change process is given by the integrated Cox-Ingersoll-Ross (CIR) process. The CIR process is defined as a solution of the following SDE:

$$dZ_t = (a - bZ_t)dt + \zeta\sqrt{Z_t}\, dW_t,$$

where a, b and ζ are positive numbers, and W_t is a Wiener process. If Z_0 is sampled from the stationary invariant distribution π and $2a \geq \zeta^2$, then Z_t is strictly stationary and ergodic. The time change process $\mathcal{T}(s)$ is then defined as

$$\mathcal{T}(s) = \int_0^s Z_t\, dt.$$

The Laplace transform of $\mathcal{T}(\Delta)$ under π is given by

$$
\begin{aligned}
\mathcal{L}_\Delta^{iCIR}(u) &= \mathbb{E}_\pi\left[\mathbb{E}\left[e^{-u\mathcal{T}(\Delta)} | Z_0 \right] \right] \\
&= \frac{\exp\left\{ ab\Delta/\zeta^2 \right\}}{\left(\cosh(\Lambda\Delta/2) + \frac{b}{\Lambda}\sinh(\Lambda\Delta/2) \right)^{2a/\zeta^2}} \cdot \\
&\qquad\qquad \cdot \mathbb{E}_\pi\left[\exp\left\{ \frac{-2Z_0 u}{b + \Lambda\coth(\Lambda\Delta/2)} \right\} \right], \quad (2.7)
\end{aligned}
$$

where $\Lambda(u) = \sqrt{b^2 + 2\zeta^2 u}$, see Chapter 15.1.2 from Cont and Tankov (2004). Since the stationary distribution of the integrated CIR process is the Gamma distribution with parameters $2a/\zeta^2$ and $2b/\zeta^2$, the Laplace transform of Z_0 under π has a form

$$\mathbb{E}_\pi\left[e^{-hZ_0} \right] = \left(1 + h\frac{\zeta^2}{2b} \right)^{-2a/\zeta^2},$$

and therefore

$$\mathbb{E}_{\pi}\left[\exp\left\{\frac{-2Z_0 u}{b + \Lambda\coth(\Lambda\Delta/2)}\right\}\right] = \left(1 + B_1(\Lambda)\frac{\zeta^2}{b}\frac{u}{\Lambda}\right)^{-2a/\zeta^2}, \tag{2.8}$$

where $B_1(\Lambda) = (b/\Lambda + \coth(\Lambda\Delta/2))^{-1} \to 1$ as $\Lambda \to \infty$. From the definitions of the hyperbolic functions, it directly follows that

$$\cosh(\Lambda\Delta/2) + \frac{b}{\Lambda}\sinh(\Lambda\Delta/2) = \frac{1}{2}B_2(\Lambda)e^{\Lambda\Delta/2}. \tag{2.9}$$

where

$$B_2(\Lambda) := (1 + b/\Lambda)\left(1 + \frac{1 - b/\Lambda}{1 + b/\Lambda}e^{-\Lambda\Delta}\right) \to 1, \qquad \Lambda \to \infty.$$

Substituting (2.8) and (2.9) into (2.7), we get the following asymptotics for the function $\mathcal{L}_{\Delta}^{iCIR}(u)$ as $u \to \infty$:

$$\mathcal{L}_{\Delta}^{iCIR}(u) \asymp e^{ab\Delta/\zeta^2}2^{2a/\zeta^2}\exp\left\{-\frac{\Delta a}{\zeta^2}\Lambda B_3(u)\right\}, \tag{2.10}$$

with

$$B_3(u) := 1 + \frac{2}{\Delta\Lambda}\log\left(1 + B_1(\Lambda)\frac{\zeta^2}{b}\frac{u}{\Lambda}\right) \to 1, \qquad u \to \infty.$$

From (2.10) it follows that the assumption (AT3) is fulfilled with $\alpha = 1/2$ and

$$A = e^{ab\Delta/\zeta^2}2^{2a/\zeta^2}, \qquad \lambda_2 = \frac{\Delta a}{\zeta}\sqrt{2}, \qquad \Psi_2(u) = B_3(u)\sqrt{1 + \frac{b^2}{2\zeta^2 u}}.$$

Note that for u large enough,

$$|1 - \Psi_2(u)| = B_3(u)\sqrt{1 + b^2/(2\zeta^2 u)} - 1 \lesssim u^{-1/2}\log(u), \quad u \to \infty$$

and therefore $\chi_2 = 1/2 - \epsilon$ for arbitrary small $\epsilon < 1/2$.

3. The characteristic function of Y_Δ

Denote the characteristic function of Y_Δ by $\phi^\Delta(u)$, then

$$\phi^\Delta(u) \quad = \quad \mathbb{E}_{\mathcal{T}(\Delta)}\left[\mathbb{E}_X\left[\exp\left\{iuX_{\mathcal{T}(\Delta)}\right\}|\mathcal{T}(\Delta)\right]\right].$$

Since the inside (conditional) expectation is equal to $\exp\{\mathcal{T}(\Delta)\psi(u)\}$, we get

$$\phi^\Delta(u) = \mathbb{E}\exp\left\{\mathcal{T}(\Delta)\psi(u)\right\}. \tag{3.1}$$

The first objective of this paper is to infer on the asymptotic behavior of $\phi^\Delta(u)$.

Theorem 3.1. *Consider the process* $Y_s := X_{\mathcal{T}(s)}$*, where the processes* X *and* $\mathcal{T}$ *satisfy the conditions (AL), (AT1)–(AT3) with* $\chi_2 > 1 - \gamma/2$*. Then*

$$\left|\phi^\Delta(u)\right| = A \exp\left\{-\tau^{(1)}|u|^{2\alpha}\left(1 + \tau^{(2)}|u|^{\gamma-2} + r(u)\right)\right\} \tag{3.2}$$

with

$$\tau^{(1)} = \lambda_2\left(\sigma^2/2\right)^\alpha, \qquad \tau^{(2)} = 2\alpha\lambda_1/\sigma^2,$$

and

$$|r(u)| \le K \cdot \max\left\{\tau^{(2)}\vartheta_1\,|u|^{(\gamma-2)-\chi_1}, \vartheta_2\left(\sigma^2/2\right)^{-\chi_2}|u|^{-2\chi_2}\right\},$$

where $K > 0$ *doesn't depend on the parameters of the processes* X *and* $\mathcal{T}$*.*

Remark 3.2. The examples of Section 2.3 show that the condition $\chi_2 > 1 - \gamma/2$ is not restrictive. For example, if the tempered stable distribution is used as a time change, then this condition holds for any Lévy process X since $\chi_2 = 1$ in this case.

Remark 3.3. In the sequel we will use the notation

$$\tilde{\chi}_1 := \min\left\{\chi_1, 2\chi_2 + \gamma - 2\right\}$$

and

$$\begin{aligned}
\tau^{(3)} &:= K\tau^{(2)}\vartheta_1 I\left\{\tilde{\chi}_1 = \chi_1\right\} + K\left(\sigma^2/2\right)^{-\chi_2}\vartheta_2 I\left\{\tilde{\chi}_1 = 2\chi_2 + \gamma - 2\right\} \\
&\le K \max\left\{\tau^{(2)}\vartheta_1, \left(\sigma^2/2\right)^{-\chi_2}\vartheta_2\right\},
\end{aligned}$$

In these notations, $|r(u)| \le \tau^{(3)}|u|^{\gamma-2-\tilde{\chi}_1}$ for $|u|$ large enough.

Remark 3.4. In this remark, we draw attention of the special case $\alpha = 1$, which holds for instance if $\mathcal{T}$ is deterministic. In this case, the statement of Theorem 3.1 can be checked directly. Since $\lambda_2 = \mathcal{T}(\Delta)$, we get

$$\begin{aligned}
\left|\phi^\Delta(u)\right| = \left|e^{\psi(u)\mathcal{T}(\Delta)}\right| &= \exp\left\{-\left(\frac{1}{2}\sigma^2 u^2 + \lambda_1|u|^\gamma\Psi_1(u)\right)\mathcal{T}(\Delta)\right\} \\
&= \exp\left\{-\tau^{(1)}|u|^2\left(1 + \tau^{(2)}|u|^{\gamma-2} + r(u)\right)\right\}
\end{aligned}$$

with $\tau^{(1)} = \lambda_2\sigma^2/2$, $\tau^{(2)} = 2\lambda_1/\sigma^2$, and $|r(u)| \lesssim \tau^{(2)}\vartheta_1\,|u|^{(\gamma-2)-\chi_1}$, $u \to \infty$.

Remark 3.5. Theorem 3.1 can be viewed as the Abelian theorem for time-changed Lévy processes. For the detailed discussion of the term "Abelian theorem" and closely related term "Tauberian theorem", we refer to the book by Korevaar (2004). Perhaps the most famous result of this type is the Karamata Tauberian theorem (see Bingham et al., 1987). For the Lévy processes, the corresponding fact was proven by Bismut, 1983 and this yields the following corollary of Theorem 3.1.

Corollary 3.6. *If the process $\mathcal{T}$ is a subordinator with $\alpha < 1$, and the assumptions of the Theorem 3.1 are fulfilled, then the Blumenthal-Getoor index of the process Y is equal to 2α.*

4. Estimation of the Blumenthal-Getoor index

4.1. Main idea

Consider the processes X_t and $\mathcal{T}(s)$ satisfying the assumptions (AL), (AT1)–(AT3). First, assume that $A = 1$ in (AT3) and fix some $\theta > 2$. In this case, Theorem 3.1 yields

$$
\begin{aligned}
\mathcal{Y}_1(u) &:= \log\left\{-\log\left[|\phi^\Delta(u)|^{\theta^{2\alpha}}/|\phi^\Delta(\theta u)|\right]\right\} \\
&= \log(Q) + (2\alpha + \gamma - 2)\,\log|u| + \log(R_1(u)),
\end{aligned}
\tag{4.1}
$$

where

$$
Q := \tau^{(1)}\tau^{(2)}\theta^{2\alpha}(1 - \theta^{\gamma-2}) > 0 \quad \text{and} \quad R_1(u) := 1 + \frac{r(u) - r(\theta u)}{\tau^{(2)}|u|^{\gamma-2}(1 - \theta^{\gamma-2})}.
$$

Note that $R_1(u) \to 1$ as $u \to +\infty$ due to Remark 3.3. The representation (4.1) tells us that $\mathcal{Y}_1(u)$ is, up to a reminder term $\log(R_1(u))$, a linear function of $\log|u|$ with the slope $2\alpha + \gamma - 2$. If the parameter α is assumed to be known, then one can view the estimation of γ as a linear regression problem (at least for large u) and apply the (weighted) least-squares approach. Otherwise, if α is unknown, one should first estimate α. This can be also done by the method of (weighted) least-squares. Indeed, define

$$
\mathcal{Y}_2(u) := \log\left(-\log|\phi^\Delta(u)|\right) = \log(\tau^{(1)}) + 2\alpha\,\log|u| + \log(R_2(u)),
$$

where $R_2(u) \to 1$ as $u \to +\infty$. So, $\mathcal{Y}_2(u)$ is (at least for large u) a linear function of $\log|u|$ with the slope proportional to α. If $A \neq 1$, then one can first apply the transformation:

$$
\widetilde{\phi}^\Delta(u) := |\phi^\Delta(2u)|/|\phi^\Delta(u)| = \exp\left\{-\bar{\tau}^{(1)}|u|^{2\alpha}\left(1 + \bar{\tau}^{(2)}|u|^{\gamma-2} + \bar{r}(u)\right)\right\}
$$

with

$$
\bar{\tau}^{(1)} := \tau^{(1)}\left(2^{2\alpha} - 1\right), \quad \bar{\tau}^{(2)} := \tau^{(2)}\frac{2^{2\alpha+\gamma-2} - 1}{2^{2\alpha} - 1}, \quad \bar{r}(u) = \frac{2^{2\alpha}r(2u) - r(u)}{2^{2\alpha} - 1}
$$

and then work with $\widetilde{\phi}^\Delta(u)$ instead of $\phi^\Delta(u)$. The above discussion shows that one can consistently estimate the parameters α and γ, provided a consistent estimate for the c.f. of Y_Δ is available.

4.2. Estimation of the characteristic function

Suppose that the discrete observations $Y_0, Y_\Delta, \ldots, Y_{n\Delta}$ of the state process Y are available for some fixed $\Delta > 0$. We estimate $\phi^\Delta(u)$ by its empirical counterpart $\phi_n^\Delta(u)$ defined as

$$\phi_n^\Delta(u) := \frac{1}{n} \sum_{k=1}^{n} e^{iu(Y_{\Delta k} - Y_{\Delta(k-1)})}. \tag{4.2}$$

Note that due to the assumption (AT2), the process Y is ergodic and α-mixing with

$$\mathbb{E}\left[e^{iu(Y_{s+\Delta} - Y_s)} \right] = \phi^\Delta(u)$$

for any $s \geq 0$ (see Appendix D.1). Hence by virtue of the Birkhoff ergodic theorem (see Athreya and Lahiri, 2010),

$$\phi_n^\Delta(u) \longrightarrow \phi^\Delta(u), \quad n \to \infty,$$

almost surely and in $\mathcal{L}_1$.

4.3. The case of known α

Introduce a weighting function $w^{V_n}(u) = V_n^{-1} w^1(u/V_n)$, where V_n is a sequence of positive numbers tending to infinity, and w^1 is an almost everywhere smooth function supported on $[\varepsilon, 1]$ for some $\varepsilon > 0$ that satisfies

$$\int_\varepsilon^1 w^1(u)\, du = 0, \quad \int_\varepsilon^1 w^1(u) \log u\, du = 1. \tag{4.3}$$

Some examples of such weighting functions can be found in Panov (2012). If $2 - \gamma < 2\alpha$, we define an estimator of γ by

$$\hat{\gamma}_n(\alpha) := 2(1 - \alpha) + \int_0^\infty w^{V_n}(u) \log\left(-\log \frac{|\phi_n^\Delta(u)|^{\theta^{2\alpha}}}{|\phi_n^\Delta(\theta u)|} \right) du. \tag{4.4}$$

For $2 - \gamma > 2\alpha$ consider the estimate

$$\hat{\gamma}_n^*(\alpha) := 2(1 - \alpha) + \int_0^\infty w^{V_n}(u) \log\left(1 - \frac{|\phi_n^\Delta(u)|^{\theta^{2\alpha}}}{|\phi_n^\Delta(\theta u)|} \right) du. \tag{4.5}$$

In our theoretical study we mainly focus on the first case (a couple of remarks about the second case can be found in Section A). The estimate $\hat{\gamma}_n(\alpha)$ can be represented as $\hat{\gamma}_n(\alpha) = 2(1 - \alpha) + m_{n,2}$, where $m_{n,2}$ is a solution of the following optimization problem:

$$(m_{n,1}, m_{n,2}) := \arg\min_{\beta_1, \beta_2} \int_0^\infty \widetilde{w}^{V_n}(u) \Big\{ \log\left(-\log \frac{|\phi_n^\Delta(u)|^{\theta^{2\alpha}}}{|\phi_n^\Delta(\theta u)|} \right)$$

$$- \beta_2 \log(u) - \beta_1 \Big\}^2 du, \tag{4.6}$$

where $\widetilde{w}^{V_n}(u)$ is an almost everywhere smooth positive function on $\mathbb{R}$ that can represented in the form $\widetilde{w}^{V_n}(u) = \widetilde{w}^1(u/V_n)/V_n$ with some function $\widetilde{w}^1$ supported on the interval $[\varepsilon, 1]$. The proof of the fact that (4.4) and (4.6) are equivalent follows the same lines as the proof of Lemma A.4 in Belomestny and Panov (2013). In particular, the weighting functions w^1 and $\widetilde{w}^1$ are related as follows:

$$w^1(u) := \widetilde{w}^1(u) \frac{\int_\varepsilon^1 \widetilde{w}^1(s) \log s \, ds - \left(\int_\varepsilon^1 \widetilde{w}^1(s) \, ds \right) \log u}{\left(\int_\varepsilon^1 \widetilde{w}^1(s) \log s \, ds \right)^2 - \int_\varepsilon^1 \widetilde{w}^1(s) \log^2 s \, ds \cdot \int_\varepsilon^1 \widetilde{w}^1(s) \, ds}.$$

Next, we introduce the deterministic quantity

$$\bar{\gamma}_n(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log \left(-\log \frac{|\phi^\Delta(u)|^{\theta^{2\alpha}}}{|\phi^\Delta(\theta u)|} \right) du. \tag{4.7}$$

The next lemma shows that $\bar{\gamma}_n(\alpha)$ is close to γ.

Lemma 4.1. *In the setup of Theorem 3.1, it holds for n large enough,*

$$|\gamma - \bar{\gamma}_n(\alpha)| \le C V_n^{-\tilde{\chi}_1}, \tag{4.8}$$

where $\tilde{\chi}_1$ was introduced in Remark 3.3, and C is some positive constant. More precisely, for n large enough,

$$|\gamma - \bar{\gamma}_n(\alpha)| \le C^{(1)} \frac{\tau^{(3)}}{\tau^{(2)}} \frac{1 + \theta^{\gamma-2-\tilde{\chi}_1}}{1 - \theta^{\gamma-2}} (\varepsilon V_n)^{-\tilde{\chi}_1}, \tag{4.9}$$

where $C^{(1)} > 0$ does not depend on the parameters of Y.

In the sequel we shall use the notation

$$|\gamma - \bar{\gamma}_n(\alpha)| \lesssim V_n^{-\tilde{\chi}_1}, \qquad n \to \infty,$$

which means the existence of some $C > 0$ such that (4.8) is fulfilled. The next lemma shows that $\hat{\gamma}_n(\alpha)$ converges to $\bar{\gamma}_n(\alpha)$ in probability.

Lemma 4.2. *Let the sequence V_n be such that*

$$\varepsilon_n := \frac{\log n}{\sqrt{n}} \exp \left\{ \tau^{(4)} (\theta V_n)^{2\alpha} \right\} = o(1), \quad n \to \infty, \tag{4.10}$$

where $\tau^{(4)} := \tau^{(1)}(1 + \tau^{(2)} + \tau^{(3)})$. Assume that the conditions of Theorem 3.1 are fulfilled and moreover $2 - \gamma < 2\alpha$. Then there exist positive constants $C^{(2)}$, κ and δ such that

$$\mathbb{P} \left\{ |\bar{\gamma}_n(\alpha) - \hat{\gamma}_n(\alpha)| \le C^{(2)} \varepsilon_n V_n^{(2-\gamma)-2\alpha} \right\} > 1 - \kappa \, n^{-1-\delta}. \tag{4.11}$$

The last two lemmas can be combined into the following theorem, which presents the minimax upper bound for $\hat{\gamma}_n(\alpha)$.

Theorem 4.3 (minimax upper bound for $\hat{\gamma}_n(\alpha)$). *Fix some set of positive numbers*

$$\mathscr{P} = (\alpha_\circ, \alpha^\circ, \chi_{1\circ}, \lambda_{1\circ}, \lambda_1^\circ, \vartheta_1^\circ, \chi_{2\circ}, \lambda_{2\circ}, \lambda_2^\circ, \vartheta_2^\circ)$$

and consider a class of time-changed Lévy models $\mathscr{A} = \mathscr{A}(\mathscr{P})$ such that

- *Assumptions (AT1), (AT2) hold;*
- *Assumption (AT3) is fulfilled with $\alpha \in [\alpha_\circ, \alpha^\circ] \subset (0,1)$, $\chi_2 \geq \chi_{2\circ} > 0$, $\lambda_2 \in [\lambda_{2\circ}, \lambda_2^\circ]$ and $\vartheta_2 \in (0, \vartheta_2^\circ)$;*
- *Assumption (AL) is fulfilled with $\gamma \in (\gamma_\circ, 2)$, where $\gamma_\circ := \max\{\chi_{1\circ}, 2(1 - \alpha)\}$, $\chi_1 \in [\chi_{1\circ}, \gamma)$, $\lambda_1 \in [\lambda_{1\circ}, \lambda_1^\circ]$, and $\vartheta_1 \in (0, \vartheta_1^\circ)$.*

Take $V_n = (q \log n)^{1/(2\alpha^\circ)}$ with $q < (2\theta^{2\alpha^\circ} \min_{\mathscr{A}} \tau^{(4)}(\mathscr{A}))^{-1}$, where $\tau^{(4)} = \tau^{(4)}(\mathscr{A})$ is defined in Lemma 4.2. Then there exists $\Xi > 0$ such that for any $\Xi_1 > \Xi$ it holds

$$\sup_{\mathscr{A}} \mathbb{P}\left\{ |\hat{\gamma}_n(\alpha) - \gamma| \geq \Xi_1 (\log n)^{-\tilde{\chi}_{1\circ}/(2\alpha^\circ)} \right\} < \kappa \, n^{-1-\delta}, \qquad (4.12)$$

where $\tilde{\chi}_{1\circ} := \min\{\chi_{1\circ}, 2\chi_{2\circ} + \gamma_\circ - 2\}$, the supremum is taken over the set of all models from $\mathscr{A}(\mathscr{P})$, constants κ and δ do not depend on $\mathscr{P}$.

The next theorem states that the rates obtained in Theorem 4.3 are optimal.

Theorem 4.4 (lower bound). *For any $\Xi_2 < \Xi$, it holds*

$$\varliminf_{n \to \infty} \inf_{\hat{\gamma}_n^*} \sup_{\mathscr{A}} \mathbb{P}\left\{ |\hat{\gamma}_n^* - \gamma| \geq \Xi_2 (\log n)^{-\tilde{\chi}_{1\circ}/(2\alpha^\circ)} \right\} > 0 \qquad (4.13)$$

where the infimum is taken over all possible estimates of the parameter γ, the supremum - over the set of all models from $\mathscr{A}$.

4.4. The case of unknown α

Estimation of α. Define an estimate of the parameter α via

$$\hat{\alpha}_n := \frac{1}{2} \int_0^\infty w_\alpha^{U_n}(u) \log\left(-\log |\phi_n^\Delta(u)| \right) \, du, \qquad (4.14)$$

where U_n is a sequence of positive numbers tending to infinity, and a weighting function $w_\alpha^{U_n}$ satisfies the same properties as the function w^{V_n}, see (4.3).

This estimate can be alternatively defined as a solution $\hat{\alpha}_n = l_{n,1}$ of the following optimization problem:

$$(l_{n,1}, l_{n,2}) := \operatorname*{arg\,min}_{\beta_1, \beta_2} \int_0^\infty \tilde{w}_\alpha^{U_n}(u) \Big(\frac{1}{2} \log\left(-\log |\phi_n^\Delta(u)| \right)$$

$$- \beta_2 \log(u) - \beta_1 \Big)^2 \, du, \quad (4.15)$$

where $\widetilde{w}_\alpha^{U_n}(u)$ is an almost everywhere smooth positive function on $\mathbb{R}$ having the representation:

$$\widetilde{w}_\alpha^{U_n}(u) = \frac{1}{U_n}\,\widetilde{w}_\alpha^1\left(\frac{u}{U_n}\right)$$

with some function $\widetilde{w}_\alpha^1$ supported on the interval $[\varepsilon, 1]$. The upper bound for the estimate $\hat{\alpha}_n$ is given in the next theorem.

Theorem 4.5 (upper bound for $\hat{\alpha}_n$). *Take the sequence*

$$U_n = (q \log n)^{1/(2\alpha^\circ)} \quad \textit{with } q < q^\circ := \left(2^{1-\alpha_\circ}\lambda_2^\circ \max\left\{\sigma^{2\alpha_\circ}, \sigma^{2\alpha^\circ}\right\}\right)^{-1}.$$

Then there exists a positive constant Ξ_3 such that

$$\sup_{\mathscr{A}} \mathbb{P}\left\{|\hat{\alpha}_n - \alpha| \geq \Xi_3(\log n)^{(\gamma-2)/\alpha^\circ}\right\} < \kappa\, n^{-1-\delta}, \qquad (4.16)$$

where the supremum is taken over the set of all models from $\mathscr{A}$, and the constants κ and δ are defined in Theorem 4.3.

Estimation of γ in the case of unknown α. After estimating α using the sequence U_n, one can define an estimate of γ via

$$\hat{\gamma}_n(\hat{\alpha}_n) := 2(1 - \hat{\alpha}_n) + \int_0^\infty w^{V_n}(u) \log\left(-\log\frac{|\phi_n^\Delta(u)|^{\theta^{2\hat{\alpha}_n}}}{|\phi_n^\Delta(\theta u)|}\right) du. \qquad (4.17)$$

The next theorem shows that the upper bound for the estimate $\hat{\gamma}_n(\hat{\alpha}_n)$ is the same as in the case of the known α as long as $\tilde{\chi}_{1\circ} \leq 2(2 - \gamma)$. The latter inequality holds true if, e.g., $\gamma \leq 4/3$.

Theorem 4.6 (upper bound for $\hat{\gamma}_n(\hat{\alpha}_n)$). *Take the sequences*

$$U_n = (q \log n)^{1/(2\alpha^\circ)} \quad \textit{with } q < q^\circ := \left(2^{1-\alpha_\circ}\lambda_2^\circ \max\left\{\sigma^{2\alpha_\circ}, \sigma^{2\alpha^\circ}\right\}\right)^{-1},$$

$$V_n = (p \log n)^{1/(2\alpha^\circ)} \quad \textit{with } p < p^\circ := q^\circ / \left(1 + \varepsilon^{2\alpha^\circ}\right).$$

Then

$$\sup_{\mathscr{A}} \mathbb{P}\left\{|\hat{\gamma}_n(\hat{\alpha}_n) - \gamma| \geq \Xi_4(\log n)^{\max\{-\tilde{\chi}_{1\circ}, 2(\gamma-2)\}/(2\alpha^\circ)}\right\} < \kappa\, n^{-1-\delta}, \quad (4.18)$$

where the supremum is taken over the set of all models from $\mathscr{A}$, constants κ and δ do not depend on the parameters of Y, and Ξ_4 depends on $\mathscr{P}$ only. In particular, if we additionally assume that $\tilde{\chi}_{1\circ} \leq 2(2 - \gamma)$, then

$$\sup_{\mathscr{A}} \mathbb{P}\left\{|\hat{\gamma}_n(\hat{\alpha}_n) - \gamma| \geq \Xi_4(\log n)^{-\tilde{\chi}_{1\circ}/(2\alpha^\circ)}\right\} < \kappa\, n^{-1-\delta}. \qquad (4.19)$$

5. Numerical examples

In our numerical study we consider the following time-changed Lévy model. Let Y_s be in the form

$$Y_s := X_{\mathcal{T}(s)}, \qquad X_t := \sigma W_t + G_t, \tag{5.1}$$

where

- W_t is a Brownian motion and parameter $\sigma = 0.25/\sqrt{252} \approx 0.016$, which corresponds to the annual volatility equal to 0.25,
- G_t is a Lévy process which will be specified later,
- $\mathcal{T}(s)$ is the integrated CIR process with parameters $a = 1.763$, $b = 1.763$, $\zeta = 0.563$, see Section 2.3 for notation; note that the assumptions (AT2) and (AT3) hold for this $\mathcal{T}(s)$ with $\alpha = 1/2$, if Z_0 has the invariant distribution.

Note that the values of parameters for $\mathcal{T}(s)$ are taken from Figueroa-López (2012). As for the process G_t, we focus on the following two cases:

1. *G_t is a stable process with index $\gamma = 1.2$* . In this case, the characteristic exponent of X_t is equal to

$$\psi(u) = -\sigma^2 u^2/2 - \delta_1 |u|^\gamma \Big(1 - \imath \delta_2 \operatorname{sign}(u) \tan(\pi\gamma/2) \Big), \tag{5.2}$$

   where we take $\delta_1 = 0.25$ and $\delta_2 = 0.3$. Note that the assumption (AL) is fulfilled with $\lambda_1 = \delta_1$.
2. *G_t is a Normal inverse Gaussian (NIG) process (i.e., $\gamma = 1$).* This process is defined by

$$G_t = \sigma_G \tilde{W}_{U_t} + \theta_G U_t,$$

   where $\tilde{W}$ is a Brownian motion independent of W, U is an inverse Gaussian subordinator independent of $\tilde{W}$ such that $\mathbb{E}[U_t] = 0.21\, t$, and $\sigma_G = 0.5$, $\theta_G = -0.8$. Note that these parameter values are used in Figueroa-López (2012). In this case, the assumption (AL) is guaranteed by the fact that the characteristic exponent of X can be represented in the form

$$\psi(u) = -\sigma^2 u^2/2 + \chi_1 \left(\sqrt{\chi_2^2 - \chi_3^2} - \sqrt{\chi_2^2 - (\chi_3 + \imath u)^2} \right) + \imath \chi_4 u, \tag{5.3}$$

   with some $\chi_1 \in \mathbb{R}_+$, $\chi_2 \in \mathbb{R}_+$, $\chi_3 \in \mathbb{R}$, $\chi_4 \in \mathbb{R}$, see Belomestny (2011).

For both choices of the process G_t, we estimate the parameters α and γ by the following procedure. First, we generate a trajectory $Y_0, Y_\Delta, \ldots, Y_{n\Delta}$ with $\Delta = 1$. Next, we estimate the characteristic function $\phi_n(u)$ by $\phi_n^\Delta(u)$, see (4.2), and consider the optimization problem (4.15):

$$(l_{n,1}, l_{n,2}) := \operatorname*{arg\,min}_{\beta_1, \beta_2} \int_{U_{low}}^{U_{up}} \Big(\frac{1}{2} \log \big(-\log |\phi_n^\Delta(uU_n)| \big)$$

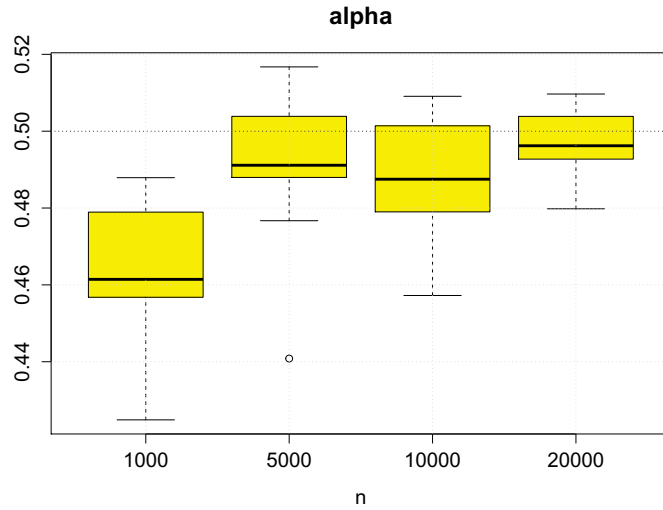$$- \beta_2 \log(uU_n) - \beta_1 \Big)^2 du. \tag{5.4}$$

FIG 1. *Boxplots of the estimate $\hat{\alpha}_n$ for different values of n based on 100 simulation runs for the model with stable process G_t.*
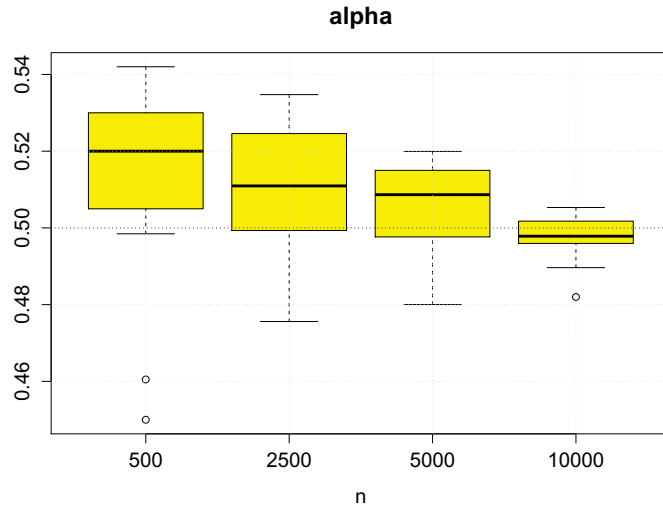

FIG 2. *Boxplots of the estimate $\hat{\alpha}_n$ for different values of n based on 100 simulation runs for the model with Normal inverse Gaussian process G_t.*

where U_{low} and U_{up} are the truncation levels. The solution $l_{n,1}$ of this problem gives an estimate of α, which we denote by $\hat{\alpha}_n$. Figures 1 and 2 show the boxplots of $\hat{\alpha}_n$ as a function of n based on 100 simulation runs for the first and the second choice of the process G_t, respectively.
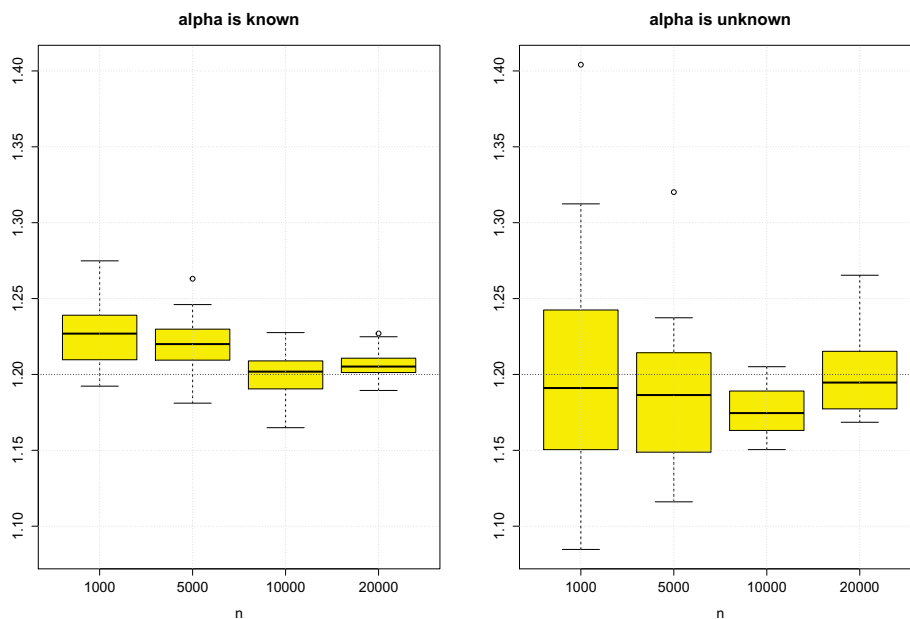
FIG 3. *Boxplots of the estimates $\hat{\gamma}_n(\alpha)$ and $\hat{\gamma}_n(\hat{\alpha}_n)$ for different values of n based on 100 simulation runs for the model with stable process G_t.*

Next, we proceed to the estimation of γ by considering the optimization problem (4.6) with $\alpha = \hat{\alpha}_n$:

$$(m_{n,1}, m_{n,2}) := \underset{\beta_1, \beta_2}{\arg\min} \int_{V_{low}}^{V_{up}} \left\{ \log\left(-\log \frac{|\phi_n^\Delta(uV_n)|^{\theta^{2\hat{\alpha}_n}}}{|\phi_n^\Delta(\theta u V_n)|} \right) \right.$$
$$\left. - \beta_2 \log(u) - \beta_1 \right\}^2 du, \quad (5.5)$$

where V_{low} and V_{up} are the truncation levels, and $\theta = 2$. The estimate of γ is then defined as $\hat{\gamma}_n(\hat{\alpha}_n) = 2(1 - \hat{\alpha}_n) + m_{n,1}$.

To illustrate Theorem 4.6, we compute also the estimates $\hat{\gamma}_n(\alpha)$, which are the solutions of the optimization problem (5.5) with the true value $\alpha = 1/2$ instead of its estimate $\hat{\alpha}_n$. The boxplots given on Figure 3 and 4 indicate that the quality of the estimates $\hat{\gamma}_n(\alpha)$ and $\hat{\gamma}_n(\hat{\alpha}_n)$ is quite similar. Note that the condition $2 - \gamma > 2\alpha$ is not fulfilled for the model with NIG process (and therefore the theory presented in this paper formally cannot be applied to this situation), but the use of estimator (5.5) still makes sense.

6. Proofs

In the sequel we use the simplified notation: $\phi(u) := \phi^\Delta(u)$ and $\phi_n(u) := \phi_n^\Delta(u)$.
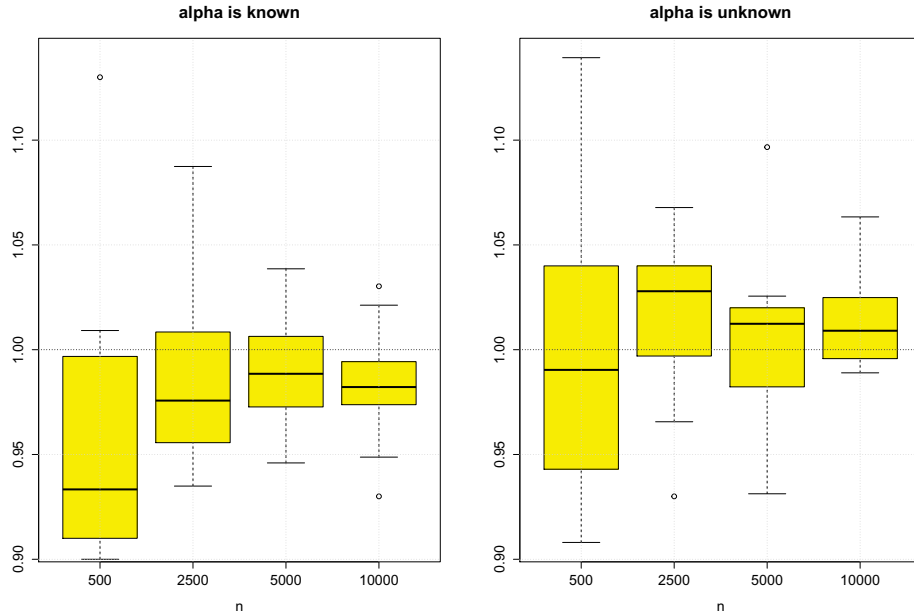
FIG 4. *Boxplots of the estimates $\hat{\gamma}_n(\alpha)$ and $\hat{\gamma}_n(\hat{\alpha}_n)$ for different values of n based on 100 simulation runs for the model with Normal inverse Gaussian process G_t.*

6.1. Proof of Theorem 3.1

First note that by (3.1),

$$|\phi(u)| \asymp \mathbb{E} \exp\left\{\mathcal{T}(\Delta) \operatorname{Re}(\psi(u))\right\} = \mathcal{L}_\Delta\left(-\operatorname{Re}(\psi(u))\right), \qquad u \to \infty, \qquad (6.1)$$

i.e., $|\phi(u)|$ is asymptotically equivalent to the Laplace transform of $\mathcal{T}(\Delta)$ computed at the point $-\operatorname{Re}(\psi(u))$. Substitung (2.6) into (6.1), we get

$$|\phi(u)| \asymp A \exp\left\{-\lambda_2\left(-\operatorname{Re}\psi(u)\right)^\alpha \Psi_2\left(-\operatorname{Re}\psi(u)\right)\right\}.$$

Using now Assumption (AT3), we conclude that

$$|\phi(u)| \asymp A \exp\left\{-\lambda_2\left(-\operatorname{Re}\psi(u)\right)^\alpha \left(1 + S(-\operatorname{Re}\psi(u))\right)\right\}, \qquad (6.2)$$

where $|S(u)| \le \vartheta_2 u^{-\chi_2}$. Recall that by Assumption (AL),

$$-\operatorname{Re}\psi(u) = \frac{1}{2}\sigma^2 u^2 + \lambda_1 |u|^\gamma \operatorname{Re}(\Psi_1(u)). \qquad (6.3)$$

Therefore for u large enough,

$$S(-\operatorname{Re}\psi(u)) \le K_1 \vartheta_2 \left(\sigma^2/2\right)^{-\chi_2} |u|^{-2\chi_2}, \qquad (6.4)$$

where $K_1 > 0$ doesn't depend on the parameters of the processes X and $\mathcal{T}$. Substituting (6.3) and (6.4) into (6.2), we get

$$|\phi(u)| \;\asymp\; A \exp\left\{-\lambda_2 \left(\frac{1}{2}\sigma^2\right)^\alpha |u|^{2\alpha}\left(1 + \frac{2\alpha\lambda_1}{\sigma^2}|u|^{\gamma-2}\operatorname{Re}(\Psi_1(u)) + R(u)\right)\right\}$$

where

$$|R(u)| = S(-\operatorname{Re}\psi(u))\left(1 + 2\alpha\lambda_1|u|^{\gamma-2}\operatorname{Re}(\Psi_1(u))/\sigma^2\right)$$
$$\leq K_2\vartheta_2\left(\sigma^2/2\right)^{-\chi_2}|u|^{-2\chi_2}, \quad (6.5)$$

with $K_2 = 2K_1$, since $|u|^{\gamma-2}\operatorname{Re}(\Psi_1(u)) \to 0$ as $u \to \infty$. The inequality

$$|1 - \operatorname{Re}(\Psi_1(u))| \leq |1 - \Psi_1(u)| \leq \vartheta_1|u|^{-\chi_1}, \qquad u \to \infty$$

yields the required upper bound for $r(u)$ and completes the proof.

6.2. Estimation of γ when α is known

6.2.1. Upper bounds

Proof of Lemma 4.1. First note that

$$|\gamma - \bar{\gamma}_n(\alpha)| \;=\; \left|\int_{\varepsilon V_n}^{V_n} w^{V_n}(u)\log\left(1 + \frac{r(u) - r(\theta u)}{u^{\gamma-2}\tau^{(2)}\left(1 - \theta^{\gamma-2}\right)}\right) du\right|. \quad (6.6)$$

Since $|r(u) - r(\theta u)|/u^{\gamma-2} \leq \tau^{(3)}\left(1 + \theta^{\gamma-2-\tilde{\chi}_1}\right)u^{-\tilde{\chi}_1}$ as $u \to \infty$, and $|\log(1 + x)| \leq 2|x|$ for any $|x| \leq 1/2$, it follows that for u large enough,

$$|\gamma - \bar{\gamma}_n(\alpha)| \leq (\varepsilon V_n)^{-\tilde{\chi}_1}\, 2\frac{\tau^{(3)}(1 + \theta^{\gamma-2-\tilde{\chi}_1})}{\tau^{(2)}\left(1 - \theta^{\gamma-2}\right)}\int_\varepsilon^1 |w^1(u)| du$$

with $C > 0$. The statement of the lemma follows with $C^{(1)} := 2\int_\varepsilon^1 |w^1(u)| du$.

Proof of Lemma 4.2. The proof of this theorem follows the same lines as the proof of its analogue for the case of affine stochastic volatility models (Belomestny and Panov, 2013).

We begin the proof with the following lemma.

Lemma 6.1. *Suppose that*

$$\widetilde{\varepsilon}_n := \left[\inf_{u\in[\varepsilon V_n, V_n]}|\phi(u)|\right]^{-\theta^{2\alpha}}\frac{\log n}{\sqrt{n}} = o(1), \quad n \to \infty. \quad (6.7)$$

Then there exist positive constants B_1, κ and δ such that for any $n > 1$

$$\mathbb{P}\left\{|\bar{\gamma}_n(\alpha) - \hat{\gamma}_n(\alpha)| \le B_1\widetilde{\varepsilon}_n \int_{\varepsilon V_n}^{V_n} \left|w^{V_n}(u)\right| \left|\log^{-1}(\mathcal{G}(u))\right| du\right\}$$
$$> 1 - \kappa n^{-1-\delta}, \quad (6.8)$$

where $\mathcal{G}(u) = |\phi(u)|^{\theta^{2\alpha}} / |\phi(u\theta)|$.

Proof. We divide the proof of the lemma into several steps.
1. Denote $\mathcal{G}_n(u) = |\phi_n(u)|^{\theta^{2\alpha}} / |\phi_n(\theta u)|$. It holds

$$\begin{aligned}
\mathcal{G}_n(u) - \mathcal{G}(u) &= \frac{|\phi_n(u)|^{\theta^{2\alpha}} - |\phi(u)|^{\theta^{2\alpha}}}{|\phi_n(u\theta)|} + \frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(u\theta)|}\frac{|\phi(u\theta)| - |\phi_n(u\theta)|}{|\phi_n(u\theta)|} \\
&= \mathcal{G}(u)\left[\frac{\xi_{1,n}(u) + \xi_{2,n}(u)}{1 - \xi_{2,n}(u)}\right] =: \mathcal{G}(u)\Lambda_n(u)
\end{aligned} \quad (6.9)$$

with

$$\xi_{1,n}(u) = \frac{|\phi_n(u)|^{\theta^{2\alpha}} - |\phi(u)|^{\theta^{2\alpha}}}{|\phi(u)|^{\theta^{2\alpha}}} \quad \text{and} \quad \xi_{2,n}(u) = \frac{|\phi(u\theta)| - |\phi_n(u\theta)|}{|\phi(u\theta)|}.$$

2. Lemma 7.5 from Belomestny and Panov (2013) shows that the event

$$\mathcal{W}_n = \left\{\sup_{u \in [\varepsilon V_n, V_n]} |\xi_{k,n}(u)| \le B_2\,\widetilde{\varepsilon}_n, \; k = 1, 2\right\}$$

has the probability tending to 1 as n tends to infinity. More precisely, it holds

$$\mathbb{P}(\mathcal{W}_n) \ge \mathbb{P}\left\{\sup_{u \in [0, V_n]} |\xi_{k,n}(u)| \le B_2\widetilde{\varepsilon}_n\right\} \ge 1 - \kappa n^{-1-\delta}, \quad k = 1, 2 \quad (6.10)$$

for some positive constants B_2, κ and δ.
3. For any $u \in [\varepsilon V_n, V_n]$, the Taylor expansion for the function $f(x) = \log(-\log(x))$ in the vicinity of the point $x = \mathcal{G}(u)$ yields

$$\mathcal{Y}_n(u) - \mathcal{Y}(u) = K_1(u)(\mathcal{G}_n(u) - \mathcal{G}(u)) + K_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2 \quad (6.11)$$

with

$$\begin{aligned}
K_1(u) &= \mathcal{G}^{-1}(u)\log^{-1}(\mathcal{G}(u)), \\
|K_2(u)| &\le 2^{-1}\max_{z \in I_n(u)}\left[\frac{1 + |\log(z)|}{z^2\log^2(z)}\right],
\end{aligned} \quad (6.12)$$

where $I_n(u)$ stands for the interval between $\mathcal{G}(u)$ and $\mathcal{G}_n(u)$. Due to Theorem 3.1,

$$\mathcal{G}(u) = \exp\left\{-\tau^{(1)}\theta^{2\alpha}u^{2\alpha}\left[1 + \tau^{(2)}|u|^{\gamma-2} + r(u)\right]\right.$$

$$+ \tau^{(1)}(\theta u)^{2\alpha}\Big[1 + \tau^{(2)}|\theta u|^{\gamma-2} + r(\theta u)\Big]\Big\} \qquad (6.13)$$

$$= \exp\Big\{-A_1|u|^{2\alpha+(\gamma-2)} + R(u)\Big\},$$

where $A_1 = \tau^{(1)}\tau^{(2)}\theta^{2\alpha}(1-\theta^{\gamma-2}) > 0$ and $|R(u)| \leq A_2|u|^{2\alpha+(\gamma-2)-\tilde{\chi}_1}$ for u large enough with $A_2 = \tau^{(1)}\tau^{(3)}\theta^{2\alpha}[1+\theta^{(\gamma-2)-\tilde{\chi}_1}] > 0$. The condition $2\alpha+(\gamma-2) > 0$ guarantees that $\mathcal{G}(u) \to 0$ as $u \to +\infty$. The length of the interval $I_n(u)$ is equal to $\mathcal{G}(u)|\Lambda_n(u)|$; therefore, the length of $I_n(u)$ tends to 0 on the event $\mathcal{W}_n$, uniformly in $u \in [\varepsilon V_n, V_n]$. Thus, $I_n(u) \subset (0,1)$ on $\mathcal{W}_n$ for n large enough and the maximum on the right hand side of the inequality in (6.12) is attained at one of the endpoints of interval $I_n(u)$.

4. Denote $Q(u) = K_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2$. Lemma 7.6 from Belomestny and Panov (2013) shows that there exists a positive constant B_3 such that for any $u \in [\varepsilon V_n, V_n]$ and for n large enough

$$\mathcal{W}_n \subset \big\{|Q(u)| \leq B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u))\big|\log^{-1}(\mathcal{G}(u))\big|\big\}. \qquad (6.14)$$

5. The Taylor expansion (6.11) and the previous discussion yield on the set $\mathcal{W}_n$,

$$
\begin{aligned}
|\bar{\gamma}_n(\alpha) - \hat{\gamma}_n(\alpha)| &= \left|\int_{\varepsilon V_n}^{V_n} w^{V_n}(u)(\mathcal{Y}_n(u) - \mathcal{Y}(u))\,du\right| \\
&\leq \int_{\varepsilon V_n}^{V_n} |w^{V_n}(u)|\left(\frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|}\left|\log^{-1}(\mathcal{G}(u))\right| \right.\\
&\hspace{6cm}\left.+ |Q(u)|\right)du \\
&\leq \int_{\varepsilon V_n}^{V_n} |w^{V_n}(u)|\log^{-1}(\mathcal{G}^{-1}(u))\left(\frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|}\right.\\
&\hspace{4cm}\left.+ B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u))\right)du.
\end{aligned}
$$

By (6.9), the expression in the brackets is equal to

$$
\begin{aligned}
P &:= \frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \\
&= \frac{|\xi_{1,n}(u) + \xi_{2,n}(u)|}{|1 - \xi_{2,n}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)).
\end{aligned}
$$

Taking into account that $\xi_{2,n} < 1$ on the set $\mathcal{W}_n$ by (6.10), we conclude that P can be upper bounded on $\mathcal{W}_n$ as follows (all supremums are taken over $[\varepsilon V_n, V_n]$):

$$
\begin{aligned}
P &\leq \frac{\sup|\xi_{1,n}(u)| + \sup|\xi_{2,n}(u)|}{1 - \sup|\xi_{2,n}(u)|} + B_3\left((\sup|\xi_{1,n}(u)|)^2 + (\sup|\xi_{2,n}(u)|)^2\right) \\
&\leq \frac{2B_2\widetilde{\varepsilon}_n}{1 - B_2\widetilde{\varepsilon}_n} + 2B_3B_2^2\widetilde{\varepsilon}_n^2 \leq B_1\widetilde{\varepsilon}_n
\end{aligned}
$$

with $B_1 > 0$. This completes the proof of the lemma. $\qquad\square$

Next, we proceed with the proof of Lemma 4.2. First, we get a lower bound for the infimum of the function $|\phi(u)|$ over $[\varepsilon V_n, V_n]$:

$$
\begin{aligned}
\inf_{u \in [\varepsilon V_n, V_n]} |\phi(u)| &\geq \exp\left\{-\tau^{(1)} V_n^{2\alpha}\left(1 + \tau^{(2)}|V_n|^{\gamma - 2} + \tau^{(2)}\vartheta_1 V_n^{(\gamma - 2) - \tilde{\chi}_1}\right)\right\} \\
&\geq \exp\left\{-\tau^{(1)}\left(1 + \tau^{(2)} + \tau^{(3)}\right) V_n^{2\alpha}\right\}.
\end{aligned}
$$

Applying Lemma 6.1 and taking into account that by (6.13), $|\log(\mathcal{G}(u))| \gtrsim u^{2\alpha + (\gamma - 2)}$, we arrive at the desired result.

Proof of Theorem 4.3. Now we combine Lemma 4.1 with Lemma 4.2. We choose the sequence V_n in the form $V_n^{2\alpha^{\circ}} = q \log n$. The assumption

$$
q < \min_{\mathscr{A}}\left(2\tau^{(4)}\theta^{2\alpha^{\circ}}\right)^{-1}
$$

guarantees (4.10) for any model from $\mathscr{A}$. With the above V_n we have $\varepsilon_n \leq (\log n)/n^{\kappa_1}$ where $\kappa_1 > 0$. Therefore, on a set of probability $1 - \kappa n^{-\delta - 1}$, for any models from $\mathscr{A}$, it holds

$$
|\hat{\gamma}_n(\alpha) - \gamma| \leq C_1 (\log n)^{-\tilde{\chi}_1/(2\alpha^{\circ})} + C_2 \frac{(\log n)^{\kappa_2}}{n^{\kappa_1}}
$$

with $\tilde{\chi}_1 := \min\{\chi_1, 2\chi_2 + \gamma - 2\}$, some κ_2, positive C_1, C_2, n large enough. From here it follows that the estimate $\hat{\gamma}_n(\alpha)$ is of logarithmic order, i.e.,

$$
\mathbb{P}\left\{|\hat{\gamma}_n(\alpha) - \gamma| \leq C(\log n)^{-\tilde{\chi}_1/(2\alpha^{\circ})}\right\} \geq 1 - \kappa n^{-\delta - 1}, \tag{6.15}
$$

where

$$
C = C^{(1)} \tau^{(3)} \frac{1 + \theta^{\gamma - 2 - \tilde{\chi}_1}}{1 - \theta^{\gamma - 2}} \varepsilon^{-\tilde{\chi}_1} q^{-\tilde{\chi}_1/(2\alpha^{\circ})}. \tag{6.16}
$$

Note that the constant C can be uniformly bounded on the set of models from $\mathscr{A}$. This completes the proof.

6.2.2. Lower bounds

Proof of Theorem 4.4. The aim of this proof is to show that

$$
\varliminf_{n \to \infty} \inf_{\hat{\gamma}_n^*} \sup_{\mathscr{A}'} \mathbb{P}\left\{|\hat{\gamma}_n^* - \gamma| \geq \psi_n\right\} > 0, \qquad \psi_n := \Xi_2 (\log n)^{-\tilde{\chi}_{1\circ}/(2\alpha^{\circ})}, \tag{6.17}
$$

where Ξ_2 is any positive constant smaller than Ξ, infimum is taken over all possible estimates of the parameter γ, supremum - over all models from the class $\mathscr{A}'$, which includes the models from $\mathscr{A}$ constructed with *subordinators* $\mathcal{T}$.

The main ingredient of the proof is the following lemma, which follows from Tsybakov (2009), Section 2.2 and Theorem 2.2.

Lemma 6.2. *Let $\mathcal{P} = \{\mathbb{P}_\gamma\}$ be a (nonparametric) family of models in $\mathbb{R}^m$. Assume that there exist two values of parameter γ, say γ_1 and γ_2, such that $|\gamma_1 - \gamma_2| > 2\psi_n$ and moreover the corresponding measures $\mathbb{P}_1 := \mathbb{P}_{\gamma_1}$ and $\mathbb{P}_2 := \mathbb{P}_{\gamma_2}$ satisfy the following properties:*

1. *there exists a measure μ such that $\mathbb{P}_1 \ll \mu$ and $\mathbb{P}_2 \ll \mu$;*
2. *the χ^2 - distance between $\mathbb{P}_1^{\otimes n}$ and $\mathbb{P}_2^{\otimes n}$ is bounded by some constant $\eta \in \mathbb{R}_+$, where the χ^2 - distance is defined for any two measures $\mathbb{P}$ and $\mathbb{Q}$ as*

$$\chi^2\left(\mathbb{P}, \mathbb{Q}\right) := \begin{cases} \int \left(\frac{d\mathbb{P}}{d\mathbb{Q}} - 1\right)^2 d\mathbb{Q}, & \text{if } \mathbb{P} \ll \mathbb{Q}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then (6.17) *holds.*

It holds

$$\chi^2\left(\mathbb{P}_1^{\otimes n}, \mathbb{P}_2^{\otimes n}\right) = \left(1 + \chi^2\left(\mathbb{P}_1, \mathbb{P}_2\right)\right)^n - 1,$$

see Tsybakov (2009). Therefore the boundedness of the χ^2 divergence is equivalent to the condition:

$$\chi^2\left(\mathbb{P}_1, \mathbb{P}_2\right) \leq q^{1/n} - 1 \asymp \frac{\log q}{n}, \tag{6.18}$$

where $q := 1 + \eta > 1$. Further properties of the χ_1^2 -divergence are discussed in Liese and Vajda (1987).

Lemma 6.3. *There exist two parameters γ_1 and γ_2 such that $|\gamma_1 - \gamma_2| > 2\psi_n$, and the measures $\mathbb{P}_1 = \mathbb{P}_{\gamma_1}$ and $\mathbb{P}_2 = \mathbb{P}_{\gamma_2}$ belong to $\mathscr{A}'$ and satisfy the condition* (6.18).

Proof. **1.** *Presentation of the models.* Let us fix some set of parameters $\mathscr{P}$. Our aim is to construct two time-changed models from $\mathscr{A}' = \mathscr{A}'(\mathscr{P})$. Let $\mathcal{T}(s)$ follow the tempered stable process with $b = 0$ in both models. The choice of parameters $\alpha \in (\alpha_\circ, \alpha^\circ)$ and $a \in [2^{-\alpha}\lambda_{2\circ}, 2^{-\alpha}\lambda_2^\circ]$ guarantees that the assumption (AT3) holds with $A = 1$, $\lambda_2 \in [\lambda_{2\circ}, \lambda_2^\circ]$ and any ϑ_2, γ_2. For the Lévy process X in the first model, we take the sum of two independent processes: the Brownian motion W and γ-stable process $\tilde{X}$ with $\gamma \in (\gamma_\circ, 2)$ and $\lambda_1 \in [\lambda_{1\circ}, \lambda_1^\circ]$, so that the characteristic exponent of X is equal to $\psi(u) = -u^2/2 - \lambda_1|u|^\gamma$. Note that the condition (AL) holds for any values of ϑ_1 and χ_1. According to (6.1), the characteristic function of the increments in the first model has the following asymptotics:

$$\phi(u) \asymp \exp\left\{-\lambda_2 \left(\frac{1}{2}u^2 + \lambda_1|u|^\gamma\right)^\alpha\right\}, \qquad u \to \infty.$$

We define the Lévy process X for the second model by its characteristic exponent

$$\breve{\psi}(u) = -\frac{1}{2}u^2 - \lambda_1|u|^\gamma I\{|u| \leq M\} - \lambda_1 b|u|^{\gamma - 2\psi_n}\left(1 + c|u|^{-\breve{\chi}_1}\right) I\{|u| \geq M\},$$

where $M, c > 0$, $\check{\chi}_1 \in [\chi_{1\circ}, \gamma)$ and $b = M^{2\psi_n}/(1 + c|M|^{-\check{\chi}_1})$. As it is explained in Belomestny (2010), Appendix A.4, this function determines some Lévy process with the BG index equal to $\check{\gamma} = \gamma - 2\psi_n$ for M and c large enough. Moreover, for $b = 1$, any fixed M and c, this process satisfies the assumption (AL), since

$$\Psi_1(u) = |u|^{2\psi_n} I\{|u| \leq M\} + \left(1 + c|u|^{-\check{\chi}_1}\right) I\{|u| \geq M\}$$

lies between $1 - c|u|^{-\check{\chi}_1}$ and $1 + c|u|^{-\check{\chi}_1}$ for $|u|$ large enough. Assumption $b = 1$ yields the following relation between M and n:

$$\psi_n = \frac{\log\left(1 + M^{-\check{\chi}_1}\right)}{2\log M} \asymp 1/\left(M^{\check{\chi}_1}\log M\right), \qquad M \to \infty. \tag{6.19}$$

The characteristic function of the second time-changed Lévy process is given by

$$\check{\phi}^{\Delta}(u) = \exp\left\{\lambda_2 \left(-\check{\psi}(u)\right)^{\alpha}\right\}.$$

Note that both time-changed Lévy processes have absolute continuous distributions. Denoting the corresponding densities in the time moment Δ by $p_\Delta(x)$ and $\check{p}_\Delta(x)$, we can express the χ^2 - divergence between $p_\Delta(x)$ and $\check{p}_\Delta(x)$ in the following way:

$$\chi^2(\mathbb{P}_1, \mathbb{P}_2) = \int_{\mathbb{R}} \frac{(p_\Delta(x) - \check{p}_\Delta(x))^2}{p_\Delta(x)} dx, \tag{6.20}$$

since $p_\Delta(x)\check{p}_\Delta(x) > 0$, $\forall x \in \mathbb{R}$.

2. *Lower bound for $p_\Delta(x)$.* The density function $p_\Delta(x)$ can be written as

$$p_\Delta(x) = \int_{\mathbb{R}+} q_t(x)\pi_\Delta(t)dt,$$

where $\pi_\Delta(t)$ is the density function of the tempered stable process at the time moment t, and $q_t(x)$ is the density function of the sum of $\tilde{X}_t$ and W_t. Since $q_t(x)$ is a convolution of two density functions, and the (strictly) γ-stable process $\tilde{X}_t$ posseses the property $\tilde{X}_t \stackrel{d}{=} t^{1/\gamma}\tilde{X}_1$ (see Cont and Tankov, 2004), we conclude that

$$q_t(x) = \frac{1}{(2\pi)^{1/2}t^{1/2+1/\gamma}} \int_{\mathbb{R}} \exp\left\{-\frac{(x-v)^2}{2t}\right\} p_{st}\left(\frac{v}{t^{1/\gamma}}\right) dv$$
$$\gtrsim |x|^{-(\gamma+1)}, \qquad x \to \infty,$$

where p_{st} is the density of $\tilde{X}_1$; the last inequality follows from Zolotarev (1986). Fixing some $0 < d_1 < d_2 < 1/2$, we arrive at

$$p_\Delta(x) \geq \int_{d_1}^{d_2} q_t(x)\pi_\Delta(t)dt \gtrsim |x|^{-(\gamma+1)}.$$

Returning now to (6.20). Taking into account that $p_\Delta(x)$ is bounded on any set of the form $\{|x| \le \mathcal{C}\}$ by some constant $\mathcal{D}$, we get with $\mathcal{C}$ large enough,

$$\chi^2(\mathbb{P}_1, \mathbb{P}_2) \le \mathcal{D} \int_{|x| \le \mathcal{C}} (p_\Delta(x) - \breve{p}_\Delta(x))^2 \, dx$$

$$+ \int_{|x| > \mathcal{C}} |x|^{1+\gamma} (p_\Delta(x) - \breve{p}_\Delta(x))^2 \, dx =: I_1 + I_2$$

3. *Upper bound for I_1.* By the Parseval-Plancherel theorem (see, e.g., Ushakov, 1999),

$$I_1 \le \mathcal{D} \int_{\mathbb{R}} (p_\Delta(x) - \breve{p}_\Delta(x))^2 \, dx = \frac{\mathcal{D}}{2\pi} \int_{|x| > M} \left| \phi(x) - \breve{\phi}(x) \right|^2 \, dx,$$

as $\phi(x)$ coincides with $\breve{\phi}(x)$ for $|x| \le M$. Next, note that

$$\int_{|x| > M} \left| \phi(x) - \breve{\phi}(x) \right|^2 \, dx = \int_{|x| > M} e^{-2\lambda_2 \left(x^2/2 + \lambda_1 |x|^{\gamma - \psi_n} \right)^\alpha} \left(e^{\kappa_1} - e^{\kappa_2} \right)^2 \, du,$$

where

$$\kappa_1 = -\lambda_2 \left(x^2/2 + \lambda_1 |x|^\gamma \right)^\alpha + \lambda_2 \left(x^2/2 + \lambda_1 |x|^{\gamma - \psi_n} \right)^\alpha$$
$$\asymp -2\alpha\lambda_1\lambda_2 \left(x^2/2 \right)^\alpha |x|^{\gamma - 2} < 0;$$
$$\kappa_2 = -\lambda_2 \left(x^2/2 + \lambda_1 |x|^{\gamma - 2\psi_n} \left(1 + c|x|^{-\breve{\chi}_1} \right) \right)^\alpha + \lambda_2 \left(x^2/2 + \lambda_1 |x|^{\gamma - \psi_n} \right)^\alpha$$
$$\asymp -2\alpha\lambda_1\lambda_2 c \left(x^2/2 \right)^\alpha |x|^{\gamma - \psi_n - 2} < 0.$$

Therefore,

$$I_1 \lesssim \int_{|x| > M} e^{-2\lambda_2 \left(x^2/2 + \lambda_1 |x|^{\gamma - \psi_n} \right)^\alpha} \, dx$$

$$\lesssim e^{-2\lambda_2 \left(M^2/2 \right)^\alpha} \int_{|x| > M} e^{-4\lambda_1\lambda_2\alpha |x|^{\gamma - \psi_n - 2}} \, du.$$

The asymptotic bound for the last integral can be obtained using the change of variables and integration by parts. Denote $\xi_1 = 4\lambda_1\lambda_2\alpha$ and $\xi_2 = \gamma - \psi_n - 2$, then

$$\frac{1}{2} \int_{|x| > M} e^{-\xi_1 |x|^{\xi_2}} \, dx \le \int_{v > M^{\xi_2}} e^{-\xi_1 v} d(v^{1/\xi_2})$$

$$\le -e^{-\xi_1 M^{\xi_2}} M + \xi_1 \int_{v > M^{\xi_2}} e^{-\xi_1 v} v^{1/\xi_2} \, dv \qquad (6.21)$$

$$\le -e^{-\xi_1 M^{\xi_2}} M + \xi_1 e^{-\xi_1 M^{\xi_2}} \int_{v > M^{\xi_2}} v^{1/\xi_2} \, dv$$

$$\lesssim e^{-\xi_1 M^{\xi_2}} M^{1+\xi_2}.$$

This in turn leads to the following upper bound for the integral I_1:

$$I_1 \lesssim e^{-2\lambda_2\left(M^2/2\right)^\alpha} e^{-4\lambda_1\lambda_2\alpha M^{\gamma-\psi_n-2}} M^{\gamma-\psi_n-1} \le e^{-2\lambda_2\left(M^2/2\right)^\alpha} M^{\gamma-\psi_n-1}.$$

4. *Upper bound for I_2.* Note that

$$I_2 \le \int_{|x|>M} \left[x^2\left(p_\Delta(x) - \breve{p}_\Delta(x)\right)\right]^2 dx \le \frac{1}{2\pi}\int_{|x|>M} \left|\widehat{x^2 p_\Delta(x)} - \widehat{x^2 \breve{p}_\Delta(x)}\right|^2 dx,$$

where $\widehat{g(x)}$ stands for the Fourier transform of a function $g(x)$. Making use of the property $\widehat{x^2 g(x)} = \partial^2\widehat{g(x)}/\partial x^2$, we conclude

$$I_2 \lesssim \int_{|x|>M} x^4 \left|\phi(x) - \breve{\phi}^\Delta(x)\right|^2 dx \lesssim e^{-2\lambda_2\left(M^2/2\right)^\alpha} M^{\gamma-\psi_n+3},$$

by the arguments similar to (6.21),

$$\frac{1}{2}\int_{|x|>M} |x|^n e^{-\xi_1|x|^{\xi_2}} dx \lesssim e^{-\xi_1 M^{\xi_2}} M^{n+1+\xi_2}$$

for any $n > 0$.

5. *Choice of M.* Finally, we get

$$\chi^2(\mathbb{P}_1, \mathbb{P}_2) \lesssim e^{-\mu_1 M^{2\alpha}} M^{\mu_2}, \qquad M \to +\infty,$$

where $\mu_1 = \lambda_2\, 2^{1-\alpha} > 0$, $\mu_2 = \gamma - \psi_n + 3 > 0$. The aim now is to choose the parameter M in such way that the conditions (6.18) and (6.19) are fulfilled simultaneously. Take $M = (A_n/\mu_1)^{1/(2\alpha)}$. Substituting this M into (6.18) results in the following condition on A_n:

$$-A_n + \frac{\mu_2}{2\alpha}\log\left(\frac{A_n}{\mu_1}\right) \lesssim -\log n,$$

suggesting the choice $A_n = \log(n\log^\beta n)$ with $\beta > \mu_2/(2\alpha)$. Hence

$$M = \left(\frac{\log\left(n\log^\beta n\right)}{\lambda_2\, 2^{1-\alpha}}\right)^{1/(2\alpha)}, \qquad \beta > (\gamma - \psi_n + 3)/(2\alpha),$$

satisfies (6.18) and (6.19). This completes the proof. $\qquad\square$

6.3. Estimation of α

The empirical counterpart of the estimate $\hat{\alpha}_n$ is given by

$$\bar{\alpha}_n := \frac{1}{2}\int_0^\infty w_\alpha^{U_n}(u)\log\left(-\log|\phi(u)|\right)\, du.$$

The closeness of $\bar{\alpha}_n$ and α is proved in the next lemma.

Lemma 6.4.

$$|\alpha - \bar{\alpha}_n| \lesssim U_n^{\gamma-2}, \qquad n \to \infty. \tag{6.22}$$

Proof. The proof follows the same lines as one of Lemma 4.1. The basic observation is that

$$
\begin{aligned}
|\alpha - \bar{\alpha}_n| &= \frac{1}{2} \int_{\varepsilon U_n}^{U_n} w_\alpha^{U_n}(u) \log\left(1 + \tau^{(2)} u^{\gamma-2} + r(u)\right) du \\
&\leq \int_{\varepsilon U_n}^{U_n} w_\alpha^{U_n}(u) \left(\tau^{(2)} u^{\gamma-2} + r(u)\right) du \\
&\leq U_n^{\gamma-2} \left(\tau^{(2)} + \tau^{(3)} U_n^{-\tilde{\chi}_1}\right) \int_\varepsilon^1 w_\alpha^1(u) du.
\end{aligned}
$$

This completes the proof. □

The next lemma is an analogue of Lemma 4.2 for the estimate $\hat{\alpha}_n$.

Lemma 6.5. *There exist positive constants D, κ and δ such that*

$$
\mathbb{P}\left\{|\bar{\alpha}_n - \hat{\alpha}_n| \leq D\, \frac{\log n}{\sqrt{n}}\, \frac{\exp\{\tau^{(1)} U_n^{2\alpha}\}}{U_n^{2\alpha}}\right\} \geq 1 - \kappa n^{-1-\delta}.
$$

Proof. The main ingredient of the proof is the observations that the difference between $\hat{\alpha}_n$ and $\bar{\alpha}_n$ has the representation:

$$
\begin{aligned}
|\hat{\alpha}_n - \bar{\alpha}_n| &= \frac{1}{2} \left| \int_{\varepsilon U_n}^{U_n} w_\alpha^{U_n}(u) \Big(\log\left(-\log|\phi(u)|\right) - \log\left(-\log|\phi_n(u)|\right)\Big) du \right| \\
&\leq \frac{1}{2} \int_{\varepsilon U_n}^{U_n} \left|w_\alpha^{U_n}(u)\right| \cdot \left| \max_{\xi \in I_n(u)} \frac{1}{\xi \log \xi} \right| \cdot \left| |\phi(u)| - |\phi_n(u)| \right| du,
\end{aligned}
\tag{6.23}
$$

where $I_n(u)$ is the interval between $\phi(u)$ and $\phi_n(u)$. Since $\phi_n(u)$ converges uniformly to $\phi(u)$ (see Section 4.2), we conclude that

$$
\begin{aligned}
\left| \max_{\xi \in I_n(u)} \frac{1}{\xi \log \xi} \right| &\leq \max_{a \in (1/2, 3/2)} \frac{1}{a|\phi(u)| \cdot |\log\left(a\,|\phi(u)|\right)|} \\
&= \frac{2}{|\phi(u)| \cdot (\log 2 + |\log|\phi(u)||)}. \tag{6.24}
\end{aligned}
$$

Theorem 3.1 yields

$$
\begin{aligned}
\min_{u \in [\varepsilon U_n, U_n]} |\phi(u)| &= \left(\max_{u \in [\varepsilon U_n, U_n]} \exp\left\{\tau^{(1)}|u|^{2\alpha}\left(1 + \tau^{(2)}|u|^{\gamma-2} + r(u)\right)\right\} \right)^{-1} \\
&= \exp\left\{-\tau^{(1)} U_n^{2\alpha}\left(1 + \tau^{(2)} U_n^{\gamma-2} + r(U_n)\right)\right\} \\
&\asymp \exp\left\{-\tau^{(1)} U_n^{2\alpha}\right\}.
\end{aligned}
\tag{6.25}
$$

Similar to (6.10), there exist positive constants B, κ and δ such that

$$\mathbb{P}\left\{\sup_{u\in[\varepsilon U_n,U_n]}\Big||\phi(u)|-|\phi_n(u)|\Big|\leq B\frac{\log n}{\sqrt{n}}\right\}\geq 1-\kappa n^{-1-\delta}. \qquad (6.26)$$

Combining (6.24), (6.25), (6.26) with (6.23), we arrive at the desired result. $\square$

Proof of Theorem 4.5. The choice $U_n^{2\alpha^\circ}=q\log n$ yields that on a set $\mathcal{W}_n$ of the probability larger than $1-\kappa n^{-1-\delta}$, it holds for any model from $\mathscr{A}$

$$|\bar{\alpha}_n-\hat{\alpha}_n|\quad\leq\quad D\,\frac{\log n}{\sqrt{n}}\,\frac{\exp\{\tau^{(1)}U_n^{2\alpha^\circ}\}}{U_n^{2\alpha_\circ}}=D\,\frac{\log n}{\sqrt{n}}\,\frac{n^{q\tau^{(1)}}}{(q\log n)^{\alpha_\circ/\alpha^\circ}}\lesssim\frac{(\log n)^{\kappa_2}}{n^{\kappa_1}},$$

with $\kappa_1=1/2-q\tau^{(1)}$ and some κ_2. Therefore, choosing $q<q^\circ=\min_{\mathscr{A}}\left\{1/(2\tau^{(1)})\right\}$ we get on $\mathcal{W}_n$

$$|\hat{\alpha}_n-\alpha|\leq|\bar{\alpha}_n-\hat{\alpha}_n|+|\hat{\alpha}_n-\alpha|\lesssim\left(\log n\right)^{(\gamma-2)/\alpha^\circ}.$$

This completes the proof.

6.4. Estimation of γ when α is unknown

Proof of Theorem 4.6.
1. _Preliminary remarks._ Note that

$$
\begin{aligned}
|\gamma_n(\hat{\alpha}_n)-\gamma|\quad&\leq\quad|\gamma_n(\hat{\alpha}_n)-\bar{\gamma}_n(\alpha)|+|\bar{\gamma}_n(\alpha)-\gamma|\\
&\leq\quad 2\,|\hat{\alpha}_n-\alpha|+|\bar{\gamma}_n(\alpha)-\gamma|\\
&\quad+\left|\int_0^\infty w^{V_n}(u)\log\left(\frac{-\theta^{2\hat{\alpha}_n}\log|\phi_n(u)|+\log|\phi_n(\theta u)|}{-\theta^{2\alpha}\log|\phi(u)|+\log|\phi(\theta u)|}\right)\right|du.
\end{aligned}
$$

The upper bound for the first two summands are given in Theorem 4.5 and Lemma 4.1, respectively. So, the aim now is to find the upper bound for the last summand, which we denote by I.

$$
\begin{aligned}
I\quad&\leq\quad\left|\int_0^\infty w^{V_n}(u)\log\left(\frac{-\theta^{2\hat{\alpha}_n}\log|\phi_n(u)|+\log|\phi_n(\theta u)|}{-\theta^{2\alpha}\log|\phi(u)|+\log|\phi(\theta u)|}\right)\right|du\\
&\leq\quad 2\left|\int_0^\infty w^{V_n}(u)\frac{-\theta^{2\alpha}\Upsilon_1+\Upsilon_2}{-\theta^{2\alpha}\log|\phi(u)|+\log|\phi(\theta u)|}\right|,
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon_1\quad&:=\quad\theta^{2(\hat{\alpha}_n-\alpha)}\log|\phi_n(u)|-\log|\phi(u)|,\\
\Upsilon_2\quad&:=\quad\log|\phi_n(\theta u)|-\log|\phi(\theta u)|.
\end{aligned}
$$

2. _Upper bounds for Υ_1 and Υ_2._ Note that for any $u>1$ and any $\beta\in\mathbb{R}$,

$$\left|u^\beta-1\right|\leq u^{|\beta|}-1.$$

Moreover, for β tending to zero,

$$u^{|\beta|} - 1 \leq C|\beta|\log(u),$$

where $C > 0$. This in particular implies that for any small $\upsilon > 0$ and n large enough,

$$\left|\theta^{2(\hat{\alpha}_n - \alpha)} - 1\right| \leq 2C\log(\theta)|\hat{\alpha}_n - \alpha| \lesssim \left(\log n\right)^{(\gamma - 2)/\alpha^\circ} := \upsilon_n,$$

where the last asymptotic inequality follows from Theorem 4.5 (here we choose the sequence U_n as it is described in Theorem 4.5). Therefore, for n large enough,

$$|\Upsilon_1| \leq \left|\log\left(\frac{|\phi_n(u)|}{|\phi(u)|}\right)\right| + \upsilon_n |\log|\phi_n(u)||$$

$$\leq 2\frac{||\phi_n(u)| - |\phi(u)||}{|\phi(u)|} + \upsilon_n |\log|\phi_n(u)||. \quad (6.27)$$

On the other hand, (6.26) together with Theorem 3.1 yield that for n large enough,

$$|\log|\phi_n(u)|| = \left|\log|\phi(u)| + \log\left(\frac{|\phi_n(u)| - |\phi(u)|}{|\phi(u)|} + 1\right)\right|$$

$$\lesssim |\log|\phi(u)|| + 2\frac{||\phi_n(u)| - |\phi(u)||}{|\phi(u)|}.$$

Substituting this bound into (6.27), we conclude that

$$|\Upsilon_1| \leq 2(1 + \upsilon_n)\frac{||\phi_n(u)| - |\phi(u)||}{|\phi(u)|} + \upsilon_n |\log|\phi(u)||. \quad (6.28)$$

Next, the arguments similar to given in the proof of Theorem 6.1, we get that for $u \in [\varepsilon V_n, V_n]$,

$$\frac{||\phi_n(u)| - |\phi(u)||}{|\phi(u)|} \leq \frac{\log n}{\sqrt{n}}\left(\inf_{u \in [\varepsilon V_n, V_n]}|\phi(u)|\right)^{-1} \lesssim \frac{\log n}{\sqrt{n}}\exp\left\{\tau^{(1)}V_n^{2\alpha}\right\},$$

$$|\log|\phi(u)|| \lesssim \exp\left\{-\tau^{(1)}(\varepsilon V_n)^{2\alpha}\right\}.$$

Combining the last inequalities and the definition of υ_n, we conlcude that

$$|\Upsilon_1| \lesssim 3\frac{\log n}{\sqrt{n}}\exp\left\{\tau^{(1)}V_n^{2\alpha}\right\} + \left(\log n\right)^{(\gamma - 2)/\alpha^\circ}\exp\left\{-\tau^{(1)}(\varepsilon V_n)^{2\alpha}\right\}.$$

As for Υ_2, it is bounded for large n, up to a constant, by the absolute value of $\xi_{2,n}$ from Lemma 6.1:

$$|\Upsilon_2| = \left|\log\left(\frac{|\phi_n(\theta u)|}{|\phi(\theta u)|}\right)\right| \leq 2\frac{||\phi_n(\theta u)| - |\phi(\theta u)||}{|\phi(\theta u)|}$$

$$:= 2|\xi_{2,n}(u)| \lesssim \frac{\log n}{\sqrt{n}}\exp\left\{\tau^{(1)}V_n^{2\alpha}\right\}.$$

3. *Upper bound for I.* Taking into account

$$-\theta^{2\alpha} \log|\phi(u)| + \log|\phi(\theta u)| \asymp -\tau^{(1)}\tau^{(2)}\theta^{2\alpha}\left(1 - \theta^{\gamma-2}\right)|u|^{2\alpha+(\gamma-2)}$$

(see (6.13) for details), we arrive at

$$I \lesssim V_n^{-(2\alpha+(\gamma-2))}\left((3\,\theta^{2\alpha} + 1)\frac{\log n}{\sqrt{n}}\exp\left\{\tau^{(1)}V_n^{2\alpha}\right\}\right.$$
$$\left. + \left(\log n\right)^{(\gamma-2)/\alpha^\circ}\exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}\right). \quad (6.29)$$

Denote

$$G_1(\alpha) \; := \; (3\,\theta^{2\alpha} + 1)\frac{\log n}{\sqrt{n}}\exp\left\{\tau^{(1)}V_n^{2\alpha}\right\},$$

$$G_2(\alpha) \; := \; \left(\log n\right)^{(\gamma-2)/\alpha^\circ}\exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}.$$

It is a worth mentioning that for any model from $\mathscr{A}$ and n large enough,

$$G_1(\alpha) \leq G_1(\alpha^\circ) \lesssim G_2(\alpha^\circ) \leq G_2(\alpha)$$

provided that $V_n^{2\alpha^\circ} = p\log n$ with $p \leq p^\circ$. This means that the asymptotic bound for I is given by the second summand in (6.29), i.e.,

$$I \lesssim V_n^{-(2\alpha+(\gamma-2))}\left(\log n\right)^{(\gamma-2)/\alpha^\circ}\exp\left\{-\tau^{(1)}\left(\varepsilon V_n\right)^{2\alpha}\right\}$$
$$\lesssim \left(\log n\right)^{(-2\alpha+(\gamma-2))/(2\alpha^\circ)}. \quad (6.30)$$

4. *Upper bound for $|\hat{\gamma}_n(\hat{\alpha}_n) - \gamma|$.* To conclude the proof, we combine (6.30), Theorem 4.5 and Lemma 4.1:

$$I \; \lesssim \; \left(\log n\right)^{(-2\alpha+(\gamma-2))/(2\alpha^\circ)},$$

$$|\hat{\alpha}_n - \alpha| \; \lesssim \; \left(\log n\right)^{(\gamma-2)/\alpha^\circ},$$

$$|\bar{\gamma}_n(\alpha) - \gamma| \; \lesssim \; \left(\log n\right)^{-\tilde{\chi}_1/(2\alpha^\circ)}.$$

Since $2 - \gamma < 2\alpha$ for class $\mathscr{A}$,

$$(\gamma - 2)/\alpha^\circ > (-2\alpha + (\gamma - 2))/(2\alpha^\circ),$$

and we arrive at (4.18).

Appendix A: The case $2 - \gamma > 2\alpha$

Assume $A = 1$ and introduce the estimate:

$$\hat{\gamma}_n^*(\alpha) := 2(1 - \alpha) + \int_0^\infty w^{V_n}(u)\log\left(1 - \frac{|\phi_n(u)|^{\theta^{2\alpha}}}{|\phi_n(\theta u)|}\right)\,du. \quad (A.1)$$

The main idea behind (A.1) comes from

$$\frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(\theta u)|} = \exp\left\{-A_1|u|^{2\alpha+(\gamma-2)}R(u)\right\}, \quad |R(u)| \lesssim 1 + A_2|u|^{-\tilde{\chi}_1}$$

with $A_1 = \tau^{(1)}\tau^{(2)}\theta^{2\alpha}(1-\theta^{\gamma-2}) > 0$ and $A_1 A_2 = \tau^{(1)}\tau^{(3)}\theta^{2\alpha}(1+\theta^{(\gamma-2)-\chi_1})$ meaning that

$$\mathcal{Y}_3(u) := 1 - \frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(\theta u)|} \asymp A_1|u|^{2\alpha+(\gamma-2)}R(u)$$

Taking logarithms of both parts, we conclude that $\mathcal{Y}_3(u)$ is linear in $\log|u|$ (at least for large $|u|$) with slope $2\alpha + \gamma - 2$. Therefore, the weighted least squares can be used here to estimate α as well. The analysis of $\hat{\gamma}_n^*(\alpha)$ is quite similar to the analysis of $\hat{\gamma}_n(\alpha)$.

Theorem A.1. *Introduce the deterministic quantity:*

$$\bar{\gamma}_n^*(\alpha) := 2(1-\alpha) + \int_0^\infty w^{V_n}(u) \log\left(1 - \frac{|\phi(u)|^{\theta^{2\alpha}}}{|\phi(\theta u)|}\right) du.$$

(i) In the setup of Theorem 3.1, it holds for n large enough,

$$|\gamma - \bar{\gamma}_n^*(\alpha)| \lesssim V_n^{-\tilde{\chi}_1}, \qquad n \to \infty.$$

(ii) Let the sequence V_n be such that

$$\varepsilon_n := \frac{\log n}{\sqrt{n}} \exp\left\{\tau^{(4)} (\theta V_n)^{2\alpha}\right\} = o(1), \quad n \to \infty.$$

Then there exist positive constants $C^{(3)}$, κ and δ such that

$$\mathbb{P}\left\{|\bar{\gamma}_n^*(\alpha) - \hat{\gamma}_n^*(\alpha)| \geq C^{(3)}\varepsilon_n V_n^{(2-\gamma)-2\alpha}\right\} < \kappa\, n^{-1-\delta}. \tag{A.2}$$

(iii) Take $V_n = (q \log n)^{1/(2\alpha^\circ)}$ with $q < \left(2\theta^{2\alpha^\circ} \min_{\mathscr{A}} \tau^{(4)}\right)^{-1}$. Then

$$\sup_{\mathscr{A}} \mathbb{P}\left\{|\hat{\gamma}_n(\alpha) - \gamma| \geq \Xi_5(\log n)^{-\tilde{\chi}_1/(2\alpha^\circ)}\right\} < \kappa\, n^{-1-\delta}, \tag{A.3}$$

where $\tilde{\chi}_1 := \min\{\chi_1, 2\chi_2 + \gamma - 2\}$, the supremum is taken over the set of all models from $\mathscr{A}$, constants κ and δ do not depend on the parameters of the underlined models, and Ξ_5 depends on $\mathscr{P}$ only.

Appendix B: The case $\sigma = 0$

This paper is mainly devoted to the case $\sigma > 0$. Nevertheless, in the case $\sigma = 0$ it is still possible to identify γ when α is known. The analogue of Theorem 3.1 can be formulated for models with $\sigma = 0$ as follows.

Theorem B.1. *Consider the process $Y_s := X_{\mathcal{T}(s)}$, where the processes X_t and $\mathcal{T}(s)$ satisfy the conditions (AT1)–(AT3) with $\chi_2 > 1 - \gamma/2$, and moreover it holds*

(AL') $\sigma = 0$ *and the function $\mathcal{V}(u)$ has the following representation:*

$$\mathcal{V}(u) = -\lambda_1 \, |u|^\gamma \, \Psi_1(u),$$

where $\lambda_1 > 0$, $\gamma \in (0, 2)$, and for u large enough

$$|1 - \Psi_1(u)| \le \vartheta_1 u^{-\chi_1},$$

with $\chi_1 \in (0, \gamma)$, $\vartheta_1 > 0$.

Then

$$\left|\phi^\Delta(u)\right| = A \exp\left\{-\breve{\tau}^{(1)}|u|^{\alpha\gamma}\left(1 + \breve{r}(u)\right)\right\}, \tag{B.1}$$

where $\breve{\tau}^{(1)} = \lambda_1^\alpha \lambda_2$ and for u large enough

$$|\breve{r}(u)| \le K \max\left\{\alpha\vartheta_1|u|^{-\chi_1}, \vartheta_2(\lambda_1)^{-\chi_2}|u|^{-\gamma\chi_2}\right\} \tag{B.2}$$

with some constant $K > 0$ not depending on the parameters of the processes X and $\mathcal{T}$.

Proof. The proof follows the same lines as the proof of Theorem 3.1, see Section 6.1. Substituting (6.3) into (6.2), and taking into account that $\sigma = 0$, we arrive at

$$|\phi(u)| \;\asymp\; A \exp\left\{-\lambda_2\Big(\lambda_1|u|^\gamma \operatorname{Re}\Psi_1(u)\Big)^\alpha\Big(1 + S\left(\lambda_1|u|^\gamma \operatorname{Re}\Psi_1(u)\right)\Big)\right\}.$$

The upper bound (B.2) for the remainder $\breve{r}(u)$ follows from the facts that for u large enough

$$
\begin{aligned}
\operatorname{Re}\Psi_1(u) &\le 1 + \vartheta_1|u|^{-\chi_1}, \\
|S\left(\lambda_1|u|^\gamma \operatorname{Re}\Psi_1(u)\right)| &\le K_1\vartheta_2\left(\lambda_1\right)^{-\chi_2}|u|^{-\gamma\chi_2}
\end{aligned}
$$

with some constant $K_1 > 0$ not depending on the parameters of the considered processes. $\qquad\square$

Similar to Corollary 3.6, we conclude from Theorem B.1 that the BG index of the compound process Y is equal to $\delta := \alpha\gamma$ in this case. Motivated by (B.1) and assuming for simplicity that $A = 1$, we define the estimate of δ by

$$\hat{\delta}_n = \int_0^\infty w^{V_n}(u) \log\left(-\log\left(\phi_n^\Delta(u)\right)\right) du.$$

This kind of estimators has been considered in details in Section 4.4. If the parameter α is assumed to be known, we are able to estimate the parameter γ by $\hat{\gamma}_n := \hat{\delta}_n/\alpha$.

Appendix C: Asymptotic behavior of the characteristic exponent

Lemma C.1. *Consider a Lévy measure ν on $\mathbb{R} \setminus \{0\}$ that fulfilles*

$$G(\varepsilon) := \int_{|x|>\varepsilon} \nu(dx) = \varepsilon^{-\gamma}(\beta^{(0)} + \beta^{(1)}\varepsilon^{\chi_1}(1 + O(\varepsilon))), \quad \varepsilon \to +0 \qquad \text{(C.1)}$$

with $0 < \chi_1 < \gamma < 2$, and $\beta^{(0)} > 0$. Denote

$$\mathcal{V}(u) = \operatorname{Re}(\psi(u)) + \frac{1}{2}\sigma^2 u^2 = \int_{\mathbb{R}} \big(\cos(ux) - 1\big) d\nu(x).$$

Then as $u \to +\infty$,

$$\mathcal{V}(u) = -u^\gamma \Big(\beta^{(0)}d_\gamma + \beta^{(1)}d_{\gamma-\chi_1}u^{-\chi_1}\Big) + O(1).$$

where $d_\gamma = \Gamma(1-\gamma)\sin((1-\gamma)\pi/2)$.

Appendix D: Mixing properties of Y

Lemma D.1. *Let X_t be a one-dimensional Lévy process with the Lévy measure ν and let $\mathcal{T}(t)$ be a time change independent of X_t. Fix some $\Delta > 0$ and consider two sequences $T_k = \mathcal{T}(\Delta k) - \mathcal{T}(\Delta(k-1))$ and $Z_k = Y_{\Delta k} - Y_{\Delta(k-1)}$, $k = 1, \ldots, n$, where $Y_t = X_{\mathcal{T}(t)}$. If the sequence $(T_k)_{k\in\mathbb{N}}$ is strictly stationary and α-mixing with the mixing coefficients $(\alpha_T(j))_{j\in\mathbb{N}}$, then the sequence $(Z_k)_{k\in\mathbb{N}}$ is also strictly stationary and α-mixing with the mixing coefficients $(\alpha_Z(j))_{j\in\mathbb{N}}$, satisfying*

$$\alpha_Z(j) \leq \alpha_T(j), \quad j \in \mathbb{N}. \qquad \text{(D.1)}$$

Proof. Fix some natural k, l with $k + l < n$. Using the independence of increments of the Lévy process X_t and the fact that $\mathcal{T}$ is a non-decreasing process, we get $\mathbb{E}[\phi(Z_1, \ldots, Z_k)] = \mathbb{E}[\widetilde{\phi}(T_1, \ldots, T_k)]$ and

$$\mathbb{E}[\phi(Z_1, \ldots, Z_k)\psi(Z_{k+l}, \ldots, Z_n)] = \mathbb{E}[\widetilde{\phi}(T_1, \ldots, T_k)\widetilde{\psi}(T_{k+l}, \ldots, T_n)], \quad k, l \in \mathbb{N}$$

for any two functions $\phi : \mathbb{R}^k \to [0, 1]$ and $\psi : \mathbb{R}^{n-l-k} \to [0, 1]$, where $\widetilde{\phi}(t_1, \ldots, t_k) = \mathbb{E}[\phi(X_{t_1}, \ldots, X_{t_k})]$ and $\widetilde{\psi}(t_1, \ldots, t_k) = \mathbb{E}[\psi(X_{t_1}, \ldots, X_{t_k})]$. This implies that the sequence Z_k is strictly stationary and α-mixing with the mixing coefficients satisfying (D.1). $\qquad \square$

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References

AÏT-SAHALIA, Y. AND JACOD, J. (2012). Identifying the succesive Blumenthal-Getoor indices of a discretly observed process. *The Annals of Statistics* **40** 1430–1464. MR3015031

AÏT-SAHALIA, Y. AND JACOD, J. (2009). Estimating the degree of activity of jumps in high frequency financial data. *The Annals of Statistics* **37** 2202–2244. MR2543690

ATHREYA, K. AND LAHIRI, S. (2010). *Measure Theory and Probability Theory.*

BARNDORFF-NIELSEN, OLE E. AND SHIRYAEV, A. N. (2010). *Change of Time and Change of Measure.* World Scientific. MR2779876

BELOMESTNY, D. (2010). Spectral estimation of the fractional order of a Lévy process. *The Annals of Statistics* **38** 317–351. MR2589324

BELOMESTNY, D. (2011). Statistical inference for time-changed Lévy processes via composite characteristic function estimation. *The Annals of Statistics* **39** 2205–2242. MR2893866

BELOMESTNY, D., PANOV, V. (2013). Abelian theorems for stochastic volatility models with application to the estimation of jump activity. *Stochastic Processes and Their Applications* **123** 15–44. MR2988108

BINGHAM, N. H., GOLDIE, C. M. AND TEUGELS, J. L. (1987). *Regular Variation.* MR0898871

BISMUT, J.-M. (1983). Calcul des variations stochastique et processus de sauts. *Zeitschrift für Wahrscheinlichkeitstheorie und Verw. Gebiete* **63** 147–235. MR0701527

BOSQ, D. AND BLANKE, D. (2007). Inference and prediction in large dimensions. In *Wiley Series in Probability And Statistics.* John Wiley. MR2364006

CONT, R. AND TANKOV, P. (2004). *Financial Modelling with Jump Process.* MR2042661

FIGUEROA-LÓPEZ, J. E. (2009). Nonparametric estimation of time-changed Lévy models under high-frequency data. *Advances in Applied Probability* **41** 1161–1188. MR2663241

FIGUEROA-LÓPEZ, J. E. (2012). Statistical estimation of Lévy-type stochastic volatility models. *Annals of Finance* **8** 309–335. MR2922800

KOREVAAR, J. (2004). *Tauberian Theory: A Century of Developments.* Springer. MR2073637

LIESE, F. AND VAJDA, I. (1987). *Convex Statistical Distance.* BSB Teubner. MR0926905

MIJATOVIC, A. AND TANKOV, P. (2012). A new look at short-term implied volatility in asset price models with jumps. *Preprint.*

MONROE, I. (1978). Processes that can be embedded in Brownian motion. *The Annals of Probability* **6** 42–56. MR0455113

PANOV, V. (2012). Abelian theorems for stochastic volatility models and semi-parametric estimation of the signal space. PhD thesis, Humboldt University.

ROSENBAUM, M. AND TANKOV, P. (2012). Asymptotically optimal discretization of hedging strategies with jumps. arXiv:1108.5940v2.

Rubin, H. and Tucker, H. (1959). Estimating the parameters of a differential process. *The Annals of Mathematical Statistics* **30** 641–658. MR0110174

Schoutens, W. (2003). *Lévy Processes in Finance*. John Wiley and Sons.

Tsybakov, A. (2009). *Introduction to Nonparametric Estimation*. Springer, New York. MR2724359

Ushakov, N. G. (1999). *Selected Topics in Characteristic Functions*. VSP, Utrecht. MR1745554

Veraart, A. and Winkel, M. (2010). Time change. In *Encyclopedia of Quantitative Finance* (R. Cont, ed.) Wiley.

Zolotarev, V. M. (1986). *One-dimensional Stable Distributions*. Providence, RI : American Mathematical Soc. MR0854867