

A quenched weak invariance principle

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Abstract. In this paper we study the almost sure conditional central limit theorem in its functional form for a class of random variables satisfying a projective criterion. Applications to strongly mixing processes and nonirreducible Markov chains are given. The proofs are based on the normal approximation of double indexed martingale-like sequences, an approach which has interest in itself.

Résumé. Dans cet article, nous étudions le théorème central limite conditionnel presque sûr, ainsi que sa forme fonctionnelle, pour des suites stationnaires de variables aléatoires réelles satisfaisant une condition de type projectif. Nous donnons des applications de ces résultats aux processus fortement mélangés ainsi qu'à des chaînes de Markov nonirréductibles. Les preuves sont essentiellement basées sur une approximation normale de suites doublement indexées de variables aléatoires de type martingale.

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1. Introduction

Let $(\xi_i)_{i \geq 0}$ be a Markov chain admitting an invariant probability π . Let f be a real-valued function such that $\pi(f^2) < \infty$ and $\pi(f) = 0$, and let $S_n = f(\xi_1) + \dots + f(\xi_n)$. If the central limit theorem (CLT) holds for $n^{-1/2}S_n$ starting from the initial distribution π , an interesting question is to know whether it remains true for another initial distribution ν . Maxwell and Woodroffe [24] have given a projective criterion under which S_n satisfies the so-called conditional CLT, which implies that the CLT holds for any initial distribution having a bounded density with respect to π . Necessary and sufficient conditions for the conditional CLT are given in Dedecker and Merlevède [10], and Wu and Woodroffe [36].

The question is more delicate if ν is a Dirac mass at point x . One says that the CLT is quenched if it holds for almost every starting point with respect to π . The quenched CLT implies the central limit theorem for the chain starting from an invariant probability measure π , referred as annealed CLT. The same terminologies are used for the functional central limit theorem (FCLT). For aperiodic Harris recurrent Markov chains, the quenched CLT question is solved by using Proposition 18.1.2 in Meyn and Tweedie [28]. More precisely, for an aperiodic Harris recurrent Markov chain, if the CLT holds for the initial distribution π , then it holds for any initial distribution, and hence for any starting point x (see Proposition 3.1 in Chen [4] and its proof). In the nonirreducible setting, the situation is not so clear. For instance,

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an example of a Markov chain with normal transition operator satisfying the annealed CLT but not the quenched is given at the end of Section 3 in Derriennic and Lin [12].

This question of the quenched CLT can be formulated in the more general context of stationary sequences: it means that, on a set of measure one, the central limit theorem holds when replacing the usual expectation by the conditional expectation with respect to the past σ -algebra. Some examples of stationary processes satisfying the CLT but not the quenched CLT can be found in Volný and Woodroffe [35].

The first general results on the quenched CLT and FCLT are given in Borodin and Ibragimov [1]: in the Markov chain setting, it says that the FCLT holds if there is a solution in $\mathbb{L}^2(\pi)$ to the Poisson equation (see Gordin and Lifschitz [20]); in a general setting it means that the FCLT is true under Gordin's condition [18]. This result has been improved by Derriennic and Lin [12,13], Zhao and Woodroffe [37], Cuny [5], Cuny and Peligrad [7], Cuny and Volný [8], Volný and Woodroffe [34] and Merlevède et al. [26]. In a recent paper, Cuny and Merlevède [6] have proved that the FCLT is quenched under the condition of Maxwell and Woodroffe [24].

All the papers cited above use a martingale approximation in \mathbb{L}^2 . Consequently, the projective condition obtained up to now are always expressed in terms of \mathbb{L}^2 norms of conditional expectations. In this paper, we prove the quenched FCLT under a projective condition involving \mathbb{L}^1 -norms, in the spirit of Gordin [19]. As a consequence, we obtain that the FCLT of Doukhan et al. [14] for strongly mixing sequences is quenched. Note that Doukhan et al. [14] have shown that their condition is optimal in some sense for the usual FCLT, so it is also sharp for the quenched FCLT. In Section 3.1, we study the example of the nonirreducible Markov chain associated to an intermittent map. Once again, we shall see through this example that our condition is essentially optimal.

Our main result, Theorem 2.1 below, is a consequence of the more general Proposition 4.1, where the conditions are expressed in terms of conditional expectations of partial sums. The proof of this proposition is done via a blocking argument followed by a two step martingale decomposition. We start with a finite number of consecutive blocks of random variables. The sum in blocks are approximated by martingales. This decomposition introduces the need of studying the normal approximation for a family of double indexed martingales. This approximation has interest in itself and is presented in Section 6.

2. Results

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T: \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability \mathbb{P} . An element A is said to be invariant if $T(A) = A$. We denote by \mathcal{I} the σ -algebra of all invariant sets. The probability \mathbb{P} is ergodic if each element of \mathcal{I} has measure 0 or 1.

Let \mathcal{F}_0 be a σ -algebra of \mathcal{A} satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$ and define the nondecreasing filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. We assume that there exists a regular version $P_{T|\mathcal{F}_0}$ of T given \mathcal{F}_0 , and for any integrable random variable f from Ω to \mathbb{R} we write $K(f) = P_{T|\mathcal{F}_0}(f)$. Since \mathbb{P} is invariant by T , for any integer k , a regular version $P_{T|\mathcal{F}_k}$ of T given \mathcal{F}_k is then obtained via $P_{T|\mathcal{F}_k}(f) = K(f \circ T^{-k}) \circ T^k$. In the sequel, all the conditional expectations with respect to \mathcal{F}_k are obtained through these conditional probabilities. More precisely, we shall use the following notations:

$$\mathbb{E}_0(X) := \mathbb{E}(X|\mathcal{F}_0) = K(X \circ T^{-1}) \quad \text{and} \quad \mathbb{E}_k(X) := \mathbb{E}(X|\mathcal{F}_k) = K(X \circ T^{-k-1}) \circ T^k.$$

With these notations, $\mathbb{E}(f \circ T^2|\mathcal{F}_0) = \mathbb{E}(K(f) \circ T|\mathcal{F}_0) = K^2(f)$, and more generally, for any positive integer ℓ , $\mathbb{E}(f \circ T^\ell|\mathcal{F}_0) = K^\ell(f)$.

Let X_0 be an \mathcal{F}_0 -measurable, square integrable and centered random variable. Define the sequence $\mathbf{X} = (X_i)_{i \in \mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Let $S_n = X_1 + \dots + X_n$, and define the Donsker process W_n by $W_n(t) = n^{-1/2}(S_{[nt]} + (nt - [nt])X_{[nt]+1})$.

Let \mathcal{H}^* the space of continuous functions φ from $(C([0, 1]), \|\cdot\|_\infty)$ to \mathbb{R} such that $x \mapsto |(1 + \|x\|_\infty^2)^{-1}\varphi(x)|$ is bounded. Our main result is the following:

Theorem 2.1. *Assume that*

$$\sum_{k \geq 0} \|X_0 \mathbb{E}_0(X_k)\|_1 < \infty, \tag{2.1}$$

then the series

$$\eta = \mathbb{E}(X_0^2|\mathcal{I}) + 2 \sum_{k>0} \mathbb{E}(X_0 X_k|\mathcal{I}) \tag{2.2}$$

converges almost surely and in \mathbb{L}^1 . Moreover, on a set of probability one, for any φ in \mathcal{H}^* ,

$$\lim_{n \rightarrow \infty} \mathbb{E}_0(\varphi(W_n)) = \int \varphi(z\sqrt{\eta}) W(dz), \tag{2.3}$$

where W is the distribution of a standard Wiener process. The convergence in (2.3) also holds in \mathbb{L}^1 .

Note that the \mathbb{L}^1 -convergence in (2.3) has been proved in Dedecker and Merlevède [10]. In this paper, we shall prove the almost sure convergence. Various classes of examples satisfying (2.1) can be found in Dedecker and Rio [11].

This result has an interesting interpretation in the terminology of additive functionals of Markov chains. Let $(\xi_n)_{n \geq 0}$ be a Markov chain with values in a Polish space S , so that there exists a regular transition probability $P_{\xi_1|\xi_0=x}$. Let P be the transition kernel defined by $P(f)(x) = P_{\xi_1|\xi_0=x}(f)$ for any bounded measurable function f from S to \mathbb{R} , and assume that there exists an invariant probability π for this transition kernel, that is a probability measure on S such that $\pi(f) = \pi(P(f))$ for any bounded measurable function f from S to \mathbb{R} . Let then $\mathbb{L}_0^2(\pi)$ be the set of functions from S to \mathbb{R} such that $\pi(f^2) < \infty$ and $\pi(f) = 0$. For $f \in \mathbb{L}_0^2(\pi)$ define $X_i = f(\xi_i)$. Notice that any stationary sequence $(Y_k)_{k \in \mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (Y_i; i \leq k)$, for the function $g(\xi_k) = Y_k$.

In this setting the condition (2.1) is $\sum_{k \geq 0} \pi(|f P^k(f)|) < \infty$. Also, the random variable η defined in Theorem 2.1 is the limit almost surely and in \mathbb{L}^1 of $n^{-1} \mathbb{E}(S_n^2|\xi_0)$, in such a way that $\eta = \bar{\eta}(\xi_0)$. By stationarity, it is also the limit in \mathbb{L}^1 of the sequence $n^{-1} \mathbb{E}((X_2 + \dots + X_{n+1})^2|\xi_1)$, so that $\bar{\eta}(\xi_0) = \bar{\eta}(\xi_1)$ almost surely. Consequently $\bar{\eta}$ is an harmonic function for P in the sense that π -almost surely $P(\bar{\eta}) = \bar{\eta}$.

In the context of Markov chain the conclusion of Theorem 2.1 is also known under the terminology of FCLT started at a point. To rephrase it, let \mathbb{P}^x be the probability associated to the Markov chain started from x and let \mathbb{E}^x be the corresponding expectation. Then, for π -almost every $x \in S$, for any φ in \mathcal{H}^* ,

$$\lim_{n \rightarrow \infty} \mathbb{E}^x(\varphi(W_n)) = \int \varphi(z\sqrt{\bar{\eta}(x)}) W(dz).$$

Moreover,

$$\lim_{n \rightarrow \infty} \int \left| \mathbb{E}^x(\varphi(W_n)) - \int \varphi(z\sqrt{\bar{\eta}(x)}) W(dz) \right| \pi(dx) = 0.$$

We mention that in Theorem 2.1 no assumption of irreducibility nor of aperiodicity is imposed. Under the additional assumptions that the Markov chain is irreducible, aperiodic and positively recurrent, Chen [4] showed that the CLT holds for the stationary Markov chain under the condition $\sum_{k \geq 0} \pi(f P^k(f))$ is convergent, and the quenched CLT holds under the same condition by applying his Proposition 3.1.

Remark 2.2. Let us present an alternative condition to the criterion (2.1) in case where T is ergodic. We do not require here X_0 to be in \mathbb{L}^2 but only in \mathbb{L}^1 . The so-called Gordin criterion in \mathbb{L}^1 is:

$$\sup_{n \in \mathbb{N}} \|\mathbb{E}_0(S_n)\|_1 < \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\mathbb{E}|S_n|}{\sqrt{n}} < \infty. \tag{2.4}$$

By Esseen and Janson [17], it is known that (2.4) is equivalent to the following \mathbb{L}^1 -coboundary decomposition:

$$X_0 = m_0 + z_0 - z_0 \circ T, \tag{2.5}$$

where $z_0 \in \mathbb{L}^1$ and m_0 is a \mathcal{F}_0 -measurable random variable in \mathbb{L}^2 such that $\mathbb{E}_{-1}(m_0) = 0$ almost surely. Therefore, the criterion (2.4) leads to the annealed CLT. Note that one can easily prove that the condition (3.2) of the next section also implies (2.4). However, the condition (2.4) is not sufficient to get the annealed FCLT (see Volný and Samek [33]). In addition, from Corollary 2 in Volný and Woodroffe [35], it follows that (2.4) is not sufficient to get the quenched

CLT either. In Proposition 5.4 of Section 5.2, we shall provide an example of stationary process for which (2.1) holds but (2.4) fails.

3. Applications

As a consequence of Theorem 2.1, we obtain the following corollary for a class of weakly dependent sequences. We first need some definitions.

Definition 3.1. For a sequence $\mathbf{Y} = (Y_i)_{i \in \mathbb{Z}}$, where $Y_i = Y_0 \circ T^i$ and Y_0 is an \mathcal{F}_0 -measurable and real-valued random variable, let for any $k \in \mathbb{N}$,

$$\alpha_{\mathbf{Y}}(k) = \sup_{t \in \mathbb{R}} \|\mathbb{E}(\mathbf{1}_{Y_k \leq t} | \mathcal{F}_0) - \mathbb{E}(\mathbf{1}_{Y_k \leq t})\|_1.$$

Definition 3.2. Recall that the strong mixing coefficient of Rosenblatt [31] between two σ -algebras \mathcal{F} and \mathcal{G} is defined by $\alpha(\mathcal{F}, \mathcal{G}) = \sup_{A \in \mathcal{F}, B \in \mathcal{G}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|$. For a strictly stationary sequence $(Y_i)_{i \in \mathbb{Z}}$ of real valued random variables, and the σ -algebra $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$, define then

$$\alpha(0) = 1 \quad \text{and} \quad \alpha(k) = 2\alpha(\mathcal{F}_0, \sigma(Y_k)) \quad \text{for } k > 0. \tag{3.1}$$

Between the two above coefficients, the following relation holds: for any positive k , $\alpha_{\mathbf{Y}}(k) \leq \alpha(k)$. In addition, the α -dependent coefficient as defined in Definition 3.1 may be computed for instance for many Markov chains associated to dynamical systems that fail to be strongly mixing in the sense of Rosenblatt (see Section 3.1).

Definition 3.3. A quantile function Q is a function from $]0, 1[$ to \mathbb{R}_+ , which is left-continuous and nonincreasing. For any nonnegative random variable Z , we define the quantile function Q_Z of Z by $Q_Z(u) = \inf\{t \geq 0: \mathbb{P}(|Z| > t) \leq u\}$.

Definition 3.4. Let μ be the probability distribution of a random variable X . If Q is an integrable quantile function, let $\text{Mon}(Q, \mu)$ be the set of functions g which are monotonic on some open interval of \mathbb{R} and null elsewhere and such that $Q_{|g(X)|} \leq Q$. Let $\mathcal{F}(Q, \mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \leq 1$ and f_ℓ belongs to $\text{Mon}(Q, \mu)$.

Corollary 3.5. Let Y_0 be a real-valued random variable with law P_{Y_0} , and $Y_i = Y_0 \circ T^i$. Let Q be a quantile function such that

$$\sum_{k \geq 0} \int_0^{\alpha_{\mathbf{Y}}(k)} Q^2(u) \, du < \infty. \tag{3.2}$$

Let $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$, where f belongs to $\mathcal{F}(Q, P_{Y_0})$. Then (2.1) is satisfied and consequently, the conclusion of Theorem 2.1 holds.

To prove that (3.2) implies (2.1), it suffices to apply Proposition A.3 with $m = q = 1$ of Merlevède and Rio [27].

Notice that if $(\alpha(k))_{k \geq 0}$ is the usual sequence of strong mixing coefficients of the stationary sequence $(X_i)_{i \in \mathbb{Z}}$ as defined in (3.1), then it follows from Corollary 3.5 that if

$$\sum_{k \geq 0} \int_0^{\alpha(k)} Q_{|X_0|}^2(u) \, du < \infty, \tag{3.3}$$

then the conclusion of Theorem 2.1 holds. Hence the weak invariance principle of Doukhan et al. [14] is also quenched. We refer to Theorem 2 in Doukhan et al. [14] and to Bradley [2] for a discussion on the optimality of the condition (3.3).

3.1. Application to functions of Markov chains associated to intermittent maps

For γ in $]0, 1[$, we consider the intermittent map T_γ from $[0, 1]$ to $[0, 1]$, which is a modification of the Pomeau–Manneville map [30]:

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[, \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

Recall that T_γ is ergodic (and even mixing in the ergodic theoretic sense) and that there exists a unique T_γ -invariant probability measure ν_γ on $[0, 1]$, which is absolutely continuous with respect to the Lebesgue measure. We denote by L_γ the Perron–Frobenius operator of T_γ with respect to ν_γ . Recall that for any bounded measurable functions f and g , $\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(L_\gamma(f)g)$. Let $(Y_i)_{i \geq 0}$ be a Markov chain with transition Kernel L_γ and invariant measure ν_γ .

Definition 3.6. A function H from \mathbb{R}_+ to $[0, 1]$ is a tail function if it is nonincreasing, right continuous, converges to zero at infinity, and $x \rightarrow xH(x)$ is integrable. If μ is a probability measure on \mathbb{R} and H is a tail function, let $\text{Mon}^*(H, \mu)$ denote the set of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ which are monotonic on some open interval and null elsewhere and such that $\mu(|f| > t) \leq H(t)$. Let $\mathcal{F}^*(H, \mu)$ be the closure in $\mathbf{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \leq 1$ and $f_\ell \in \text{Mon}^*(H, \mu)$.

Corollary 3.7. Let $\gamma \in (0, 1/2)$ and $(Y_i)_{i \geq 1}$ be a stationary Markov chain with transition kernel L_γ and invariant measure ν_γ . Let H be a tail function such that

$$\int_0^\infty x(H(x))^{(1-2\gamma)/(1-\gamma)} dx < \infty. \quad (3.4)$$

Let $X_i = f(Y_i) - \nu_\gamma(f)$ where f belongs to $\mathcal{F}^*(H, \nu_\gamma)$. Then (2.1) is satisfied and the conclusion of Theorem 2.1 holds with

$$\eta = \nu_\gamma((f - \nu_\gamma(f))^2) + 2 \sum_{k>0} \nu_\gamma((f - \nu_\gamma(f))f \circ T_\gamma^k). \quad (3.5)$$

Proof. To prove this corollary, it suffices to see that (3.4) implies (3.2). For this purpose, we use Proposition 1.17 in Dedecker et al. [9] stating that there exist two positive constant B, C such that, for any $n > 0$, $Bn^{(\gamma-1)/\gamma} \leq \alpha_Y(n) \leq Cn^{(\gamma-1)/\gamma}$, together with their computations page 817. \square

In particular, if f is BV and $\gamma < 1/2$, we infer from Corollary 3.7 that the conclusion of Theorem 2.1 holds with η defined by (3.5). Note also that (3.4) is satisfied if H is such that $H(x) \leq Cx^{-2(1-\gamma)/(1-2\gamma)}(\ln(x))^{-b}$ for x large enough and $b > (1-\gamma)/(1-2\gamma)$. Therefore, since the density h_{ν_γ} of ν_γ is such that $h_{\nu_\gamma}(x) \leq Cx^{-\gamma}$ on $(0, 1]$, one can easily prove that if f is positive and nonincreasing on $(0, 1)$, with

$$f(x) \leq \frac{C}{x^{(1-2\gamma)/2} |\ln(x)|^d} \quad \text{near } 0 \text{ for some } d > 1/2,$$

then (3.4) and the quenched FCLT hold. Notice that when f is exactly of the form $f(x) = x^{-(1-2\gamma)/2}$, Gouëzel [21] proved that the central limit theorem holds for $\sum_{i=1}^n (f(Y_i) - \nu_\gamma(f))$ but with the normalization $\sqrt{n \ln(n)}$. This shows that the condition (3.4) is essentially optimal for the quenched CLT with the normalization \sqrt{n} .

4. Some general results

In this section we develop sufficient conditions imposed to conditional expectations of partial sums for the validity of the quenched CLT and FCLT.

For any positive integers i and p , define $S_p^{(i)} = S_{pi} - S_{p(i-1)}$.

4.1. A quenched CLT

Let us introduce the following three conditions under which the quenched central limit theorem holds:

$$C_1 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{\sqrt{mp}} \sum_{i=2}^{m+1} \mathbb{E}_0 |\mathbb{E}_{(i-2)p} (S_p^{(i)})| = 0 \quad \text{a.s.}$$

C_2 there exists a T -invariant r.v. η that is \mathcal{F}_0 -measurable and such that

$$\lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \mathbb{E}_0 \left| \sum_{i=1}^m \frac{1}{mp} \mathbb{E}_{(i-1)p} ((S_p^{(i+1)})^2) - \eta \right| = 0 \quad \text{a.s.},$$

$$\lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \mathbb{E}_0 \left| \sum_{i=1}^m \frac{1}{mp} \mathbb{E}_{(i-1)p} ((S_p^{(i)} + S_p^{(i+1)})^2) - 2\eta \right| = 0 \quad \text{a.s.}$$

$$C_3 \quad \text{for each } \varepsilon > 0 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \frac{1}{p} \mathbb{E}_0 ((S_p^{(i)})^2 \mathbf{1}_{|S_p^{(i)}|/\sqrt{p} > \varepsilon \sqrt{m}}) = 0 \quad \text{a.s.}$$

Proposition 4.1. Assume that C_1 , C_2 and C_3 hold. Then, on a set of probability one, for any continuous and bounded function f ,

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 (f(n^{-1/2} S_n)) = \int f(x\sqrt{\eta})g(x) dx,$$

where g is the density of a standard normal.

This proposition is designed especially for the proof of Theorem 2.1. Notice that in the expression $\mathbb{E}_{(i-2)p} (S_p^{(i)})$ of condition C_1 there is a gap of p variables between $S_p^{(i)}$ and the variables used for conditioning. This gap is important for weakening the dependence and is essentially used in the proof of Theorem 2.1.

Proof of Proposition 4.1. The result will follow from Proposition 4.2 below, for double indexed arrays of random variables:

Proposition 4.2. Assume that $(Y_{n,m,i})_{i \geq 1}$ is an array of random variables in \mathbb{L}^2 adapted to an array $(\mathcal{G}_{n,m,i})_{i \geq 1}$ of nested sigma fields. Let $\mathbb{E}_{n,m,i}$ denote the conditional expectation with respect to $\mathcal{G}_{n,m,i}$. Suppose that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=2}^{m+1} \mathbb{E} |\mathbb{E}_{n,m,i-2} (Y_{n,m,i})| = 0, \tag{4.1}$$

and that there exists $\sigma^2 \geq 0$ such that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1} (Y_{n,m,i+1}^2) - \sigma^2 \right| = 0 \tag{4.2}$$

and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1} ((Y_{n,m,i} + Y_{n,m,i+1})^2) - 2\sigma^2 \right| = 0. \tag{4.3}$$

Assume in addition that for each $\varepsilon > 0$

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^{m+1} \mathbb{E} (Y_{n,m,i}^2 \mathbf{1}_{|Y_{n,m,i}| > \varepsilon}) = 0. \tag{4.4}$$

Then for any continuous and bounded function f ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left(f \left(\sum_{i=1}^m Y_{n,m,i} \right) \right) - \mathbb{E}(f(\sigma N)) \right| = 0,$$

where N is a standard Gaussian random variable.

Before proving Proposition 4.2, let us show how it leads to Proposition 4.1. Let m be a fixed positive integer less than n . Set $p = \lfloor n/m \rfloor$. We apply Proposition 4.2 to the sequence $Y_{n,m,i} = S_p^{(i)} / \sqrt{mp}$ and the filtration $\mathcal{G}_{n,m,i} = \mathcal{F}_{ip}$. We also replace the expectation \mathbb{E} by the conditional expectation \mathbb{E}_0 (recall that all the conditional expectations of functions of T with respect to \mathcal{F}_0 are obtained through the regular conditional probability $P_{T|\mathcal{F}_0}$), and σ^2 by the nonnegative \mathcal{F}_0 -measurable random variable η . With these notations, the conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_3 imply that (4.1), (4.2), (4.3) and (4.4) hold almost surely. It follows from Proposition 4.2 that, on a set of probability one, for any continuous and bounded function f ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E}_0 \left(f \left(n^{-1/2} \sum_{i=1}^{m\lfloor n/m \rfloor} X_i \right) \right) - \int f(x\sqrt{\eta})g(x) dx \right| = 0,$$

where g is the density of a standard normal. Proposition 4.1 will then follow if we can prove that for any $\varepsilon > 0$,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_0 \left(\left| \sum_{i=1}^n X_i - \sum_{i=1}^{m\lfloor n/m \rfloor} X_i \right| \geq \varepsilon\sqrt{n} \right) = 0 \quad \text{a.s.} \tag{4.5}$$

With this aim, we notice that

$$\mathbb{P}_0 \left(\left| \sum_{i=1}^n X_i - \sum_{i=1}^{m\lfloor n/m \rfloor} X_i \right| \geq \varepsilon\sqrt{n} \right) \leq \mathbb{P}_0 \left(m^2 \max_{1 \leq i \leq n} X_i^2 \geq \varepsilon^2 n \right)$$

and therefore (4.5) holds by relation (7.2) in Lemma 7.1 applied to $Z_i = X_i^2$. It remains to prove Proposition 4.2.

Proof of Proposition 4.2. For any positive integer i , let

$$U_{n,m,i} = Y_{n,m,i} + \mathbb{E}_{n,m,i}(Y_{n,m,i+1}) - \mathbb{E}_{n,m,i-1}(Y_{n,m,i}). \tag{4.6}$$

To ease the notation, we shall drop the first two indexes (the pair n, m) when no confusion is possible. With this notation,

$$Y_i = U_i - \mathbb{E}_i(Y_{i+1}) + \mathbb{E}_{i-1}(Y_i),$$

and since we have telescoping sum,

$$\sum_{i=1}^m Y_i = \sum_{i=1}^m U_i + \mathbb{E}_0(Y_1) - \mathbb{E}_m(Y_{m+1}).$$

Notice that for any $i \in \{1, m + 1\}$ and any $\varepsilon > 0$,

$$\mathbb{E}(|\mathbb{E}_{i-1}(Y_i)|^2) \leq \varepsilon^2 + \mathbb{E}(Y_i^2 \mathbf{1}_{|Y_i| > \varepsilon}). \tag{4.7}$$

Therefore by condition (4.4),

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left((\mathbb{E}_{n,m,m}(Y_{n,m,m+1}))^2 + (\mathbb{E}_{n,m,0}(Y_{n,m,1}))^2 \right) = 0. \tag{4.8}$$

The theorem will be proven if we can show that the sequence $(U_{n,m,i})_{i \geq 1}$ defined by (4.6) satisfies the conditions of Theorem 6.1. We first notice that $\mathbb{E}_{i-1}(U_i) = \mathbb{E}_{i-1}(Y_{i+1})$. Hence condition (6.1) is clearly satisfied under (4.1). On an other hand,

$$\begin{aligned} \text{Var}(U_i | \mathcal{G}_{i-1}) &= \mathbb{E}_{i-1}(Y_i^2 + 2Y_i \mathbb{E}_i(Y_{i+1})) + \mathbb{E}_{i-1}((\mathbb{E}_i(Y_{i+1}))^2) - (\mathbb{E}_{i-1}(Y_i))^2 \\ &\quad - 2(\mathbb{E}_{i-1}(Y_i))(\mathbb{E}_{i-1}(Y_{i+1})) - (\mathbb{E}_{i-1}(Y_{i+1}))^2. \end{aligned} \tag{4.9}$$

Notice that for any $\varepsilon > 0$

$$\begin{aligned} \sum_{i=1}^m \mathbb{E}((\mathbb{E}_{i-1}(Y_{i+1}))^2) &\leq \varepsilon \sum_{i=1}^m \mathbb{E}|\mathbb{E}_{i-1}(Y_{i+1})| + \varepsilon \sum_{i=1}^m \mathbb{E}(|Y_{i+1}| \mathbf{1}_{|Y_{i+1}| > \varepsilon}) + \sum_{i=1}^m \mathbb{E}(Y_{i+1}^2 \mathbf{1}_{|Y_{i+1}| > \varepsilon}) \\ &\leq \varepsilon \sum_{i=1}^m \mathbb{E}|\mathbb{E}_{i-1}(Y_{i+1})| + 2 \sum_{i=2}^{m+1} \mathbb{E}(Y_i^2 \mathbf{1}_{|Y_i| > \varepsilon}). \end{aligned} \tag{4.10}$$

Similarly, for any $\varepsilon > 0$,

$$\sum_{i=1}^m \mathbb{E}|(\mathbb{E}_{i-1}(Y_i))(\mathbb{E}_{i-1}(Y_{i+1}))| \leq \varepsilon \sum_{i=1}^m \mathbb{E}|\mathbb{E}_{i-1}(Y_{i+1})| + 2 \sum_{i=1}^{m+1} \mathbb{E}(Y_i^2 \mathbf{1}_{|Y_i| > \varepsilon}).$$

In addition since $\mathbb{E}_{i-1}(Y_i^2 + 2Y_i \mathbb{E}_i(Y_{i+1})) = \mathbb{E}_{i-1}((Y_i + Y_{i+1})^2) - \mathbb{E}_{i-1}(Y_{i+1}^2)$, the conditions (4.2) and (4.3) imply that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^{m+1} \mathbb{E}_{n,m,i-1}(Y_{n,m,i}^2 + 2Y_{n,m,i} \mathbb{E}_{n,m,i}(Y_{n,m,i+1})) - \sigma^2 \right| = 0. \tag{4.11}$$

Starting from (4.9) and considering (4.10), (4.1) and (4.11), it follows that condition (6.2) will be satisfied provided that (4.1) and (4.4) hold and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m (\mathbb{E}_{n,m,i-1}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2) - (\mathbb{E}_{n,m,i-1}(Y_{n,m,i}))^2) \right| = 0. \tag{4.12}$$

To prove (4.12), we first write that

$$\begin{aligned} \sum_{i=1}^m (\mathbb{E}_{i-1}((\mathbb{E}_i(Y_{i+1}))^2) - (\mathbb{E}_{i-1}(Y_i))^2) &= \mathbb{E}_m(Y_{m+1})^2 - (\mathbb{E}_0(Y_1))^2 \\ &\quad - \sum_{i=1}^m ((\mathbb{E}_i(Y_{i+1}))^2 - \mathbb{E}_{i-1}((\mathbb{E}_i(Y_{i+1}))^2)). \end{aligned}$$

By (4.8), it follows that (4.12) will hold if we can show that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m ((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 - \mathbb{E}_{n,m,i-1}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2)) \right| = 0. \tag{4.13}$$

This follows from an application of Lemma 6.2 with

$$d_{n,m,i} = (\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 - \mathbb{E}_{n,m,i-1}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2).$$

Indeed

$$\sum_{i=1}^m \mathbb{E}(|d_{n,m,i}|) \leq 2 \sum_{i=1}^{m+1} \mathbb{E}(Y_{n,m,i}^2),$$

and by Lemma 6.3, for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}| > 8\varepsilon^2}) &\leq 2 \sum_{i=1}^m \mathbb{E}((\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 \mathbf{1}_{(\mathbb{E}_{n,m,i}(Y_{n,m,i+1}))^2 > 4\varepsilon^2}) \\ &\leq 2 \sum_{i=1}^m \mathbb{E}(Y_{n,m,i+1}^2 \mathbf{1}_{|\mathbb{E}_{n,m,i}(Y_{n,m,i+1})| > 2\varepsilon}) \leq 4 \sum_{i=1}^{m+1} \mathbb{E}(Y_{n,m,i}^2 \mathbf{1}_{|Y_{n,m,i}| > \varepsilon}). \end{aligned}$$

So condition (6.15) holds by using (4.2) and (4.4).

It remains to prove that (6.3) holds. Clearly this can be achieved by using (4.4) combined with Lemma 6.3. □

4.2. *Finite dimensional convergence*

For $0 < t_1 < \dots < t_d \leq 1$, define the function π_{t_1, \dots, t_d} from $C([0, 1])$ to \mathbb{R}^d by $\pi_{t_1, \dots, t_d}(x) = (x(t_1), x(t_2) - x(t_1), \dots, x(t_d) - x(t_{d-1}))$. For any a in \mathbb{R}^d define the function f_a from \mathbb{R}^d to \mathbb{R} by $f_a(x) = \langle a, x \rangle = \sum_{i=1}^d a_i x_i$.

Proposition 4.3. *Assume that C_1, C_2 and C_3 hold. Then, on a set of probability one, for any continuous and bounded function h , for any $a \in \mathbb{Q}^d$ and any t_1, t_2, \dots, t_d rational numbers such that $0 < t_1 < \dots < t_d \leq 1$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_0(h \circ f_a \circ \pi_{t_1, \dots, t_d}(W_n)) = \int h \circ f_a \circ \pi_{t_1, \dots, t_d}(z\sqrt{\eta})W(dz), \tag{4.14}$$

where W is the distribution of a standard Wiener process.

Proof. Since $\bigcup_{d=1}^\infty \mathbb{Q}^d$ is countable, it suffices to prove that for any $a \in \mathbb{R}^d$ and any t_1, t_2, \dots, t_d rational numbers such that $0 < t_1 < \dots < t_d \leq 1$, on a set of probability one, for any continuous and bounded function h , the convergence (4.14) holds. With this aim, for any $\ell \in \{1, \dots, d\}$, we set $t_\ell = r_\ell/s_\ell$ where r_ℓ and s_ℓ are positive integers. Let $c_d = \prod_{\ell=1}^d s_\ell$. Rewrite $t_\ell = b_\ell/c_d$. The b_ℓ 's are then positive integers such that $0 < b_1 < \dots < b_d \leq c_d$. Let m be a fixed positive integer and let $p = \lfloor n/(mc_d) \rfloor$. Notice that for any $\ell \in \{1, \dots, d\}$,

$$\lfloor nt_\ell \rfloor - mb_\ell < mpb_\ell \leq \lfloor nt_\ell \rfloor + 1.$$

Therefore for any reals a_1, \dots, a_d , with the convention that $t_0 = 0$ and $b_0 = 0$,

$$\left| \sum_{\ell=1}^d a_\ell \sum_{i=\lfloor nt_{\ell-1} \rfloor + 1}^{\lfloor nt_\ell \rfloor} X_i - \sum_{\ell=1}^d a_\ell \sum_{i=pm b_{\ell-1} + 1}^{pmb_\ell} X_i \right| \leq \sum_{\ell=1}^d |a_\ell| \sum_{i=pm b_\ell + 1}^{(p+1)mb_\ell} |X_i|.$$

Using (7.2) of Lemma 7.1, we infer that for any $\ell \in \{1, \dots, d\}$ and every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_0 \left(\frac{|a_\ell|}{\sqrt{n}} \sum_{i=pm b_\ell + 1}^{(p+1)mb_\ell} |X_i| > \varepsilon \right) = 0 \quad \text{a.s.}$$

In addition,

$$\left| \sum_{\ell=1}^d a_\ell (W_n(t_\ell) - W_n(t_{\ell-1})) - \sum_{\ell=1}^d a_\ell (S_{\lfloor nt_\ell \rfloor} - S_{\lfloor nt_{\ell-1} \rfloor}) \right| \leq 2 \sum_{\ell=1}^d |a_\ell| \max_{1 \leq i \leq n} |X_i|,$$

implying once again by (7.2) in Lemma 7.1 that

$$\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E}_0 \left(\left| \sum_{\ell=1}^d a_\ell (W_n(t_\ell) - W_n(t_{\ell-1})) - \sum_{\ell=1}^d a_\ell (S_{[nt_\ell]} - S_{[nt_{\ell-1}]}) \right| \right) = 0 \quad \text{a.s.} \quad (4.15)$$

From the preceding considerations, it remains to prove that, on a set of probability one, for any continuous and bounded function f ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E}_0 \left(f \left(n^{-1/2} \sum_{\ell=1}^d a_\ell \sum_{i=pmb_{\ell-1}+1}^{pmb_\ell} X_i \right) \right) - \mathbb{E}_0(f(\sigma_d N)) \right| = 0, \quad (4.16)$$

where $\sigma_d^2 = \eta \sum_{\ell=1}^d a_\ell^2 (t_\ell - t_{\ell-1})$ and N is a standard Gaussian random variable independent of \mathcal{F}_0 . With this aim, we write that

$$\sum_{\ell=1}^d a_\ell \sum_{i=pmb_{\ell-1}+1}^{pmb_\ell} X_i = \sum_{\ell=1}^d a_\ell \sum_{i=mb_{\ell-1}+1}^{mb_\ell} S_p^{(i)} = \sum_{k=1}^{mb_d} \lambda_{m,d,k} S_p^{(k)},$$

where $\lambda_{m,d,k} = \sum_{\ell=1}^d a_\ell \mathbf{1}_{mb_{\ell-1}+1 \leq k \leq mb_\ell}$. Hence to prove (4.16), it suffices to apply Proposition 4.2 to the random variables $Y_{n,m,i} = (mpc_d)^{-1/2} \lambda_{m,d,i} S_p^{(i)}$ and the filtration $\mathcal{G}_{n,m,i} = \mathcal{F}_{ip}$, by replacing the expectation \mathbb{E} by \mathbb{E}_0 . The conditions (4.1) and (4.4) are verified by using respectively \mathbf{C}_1 and \mathbf{C}_3 . To verify (4.2) and (4.3) with $\sigma^2 = \sigma_d^2 = \eta \sum_{\ell=1}^d a_\ell^2 (t_\ell - t_{\ell-1})$, we proceed as follows. For (4.2), we write that

$$\begin{aligned} \mathbb{E}_0 \left| \sum_{i=1}^{mb_d} \mathbb{E}_{n,m,i-1} (Y_{n,m,i+1}^2) - \sigma_d^2 \right| &= \mathbb{E}_0 \left| \frac{1}{mpc_d} \sum_{\ell=1}^d a_\ell^2 \sum_{i=mb_{\ell-1}+1}^{mb_\ell} \mathbb{E}_{(i-1)p} ((S_p^{(i+1)})^2) - \sigma_d^2 \right| \\ &\leq \sum_{\ell=1}^d a_\ell^2 \mathbb{E}_0 \left| \frac{1}{mpc_d} \sum_{i=mb_{\ell-1}+1}^{mb_\ell} \mathbb{E}_{(i-1)p} ((S_p^{(i+1)})^2) - \eta(t_\ell - t_{\ell-1}) \right|. \end{aligned}$$

Since $t_\ell = b_\ell/c_d$, we obtain that

$$\begin{aligned} \mathbb{E}_0 \left| \sum_{i=1}^{mb_d} \mathbb{E}_{n,m,i-1} (Y_{n,m,i+1}^2) - \sigma_d^2 \right| &\leq \sum_{\ell=1}^d \frac{a_\ell^2 b_\ell}{c_d} \mathbb{E}_0 \left| \frac{1}{mpb_\ell} \sum_{i=1}^{mb_\ell} \mathbb{E}_{(i-1)p} ((S_p^{(i+1)})^2) - \eta \right| \\ &\quad + \sum_{\ell=1}^d \frac{a_\ell^2 b_{\ell-1}}{c_d} \mathbb{E}_0 \left| \frac{1}{mpb_{\ell-1}} \sum_{i=1}^{mb_{\ell-1}} \mathbb{E}_{(i-1)p} ((S_p^{(i+1)})^2) - \eta \right|. \end{aligned}$$

Condition (4.2) is then proved by using the first part of \mathbf{C}_2 . Using similar arguments, we prove (4.3) by using the second part of \mathbf{C}_2 . \square

4.3. A quenched invariance principle

Let us define the maximal version of \mathbf{C}_3 . For $k \leq l$, let $\bar{S}_{k,l} = \max_{k \leq i \leq l} |S_i - S_k|$.

$$\mathbf{C}_4 \quad \text{for any } \varepsilon > 0 \quad \lim_{m \rightarrow \infty} \limsup_{p \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \frac{1}{p} \mathbb{E}_0 (\bar{S}_{(i-1)p, ip}^2 \mathbf{1}_{|\bar{S}_{(i-1)p, ip}|/\sqrt{p} > \varepsilon \sqrt{m}}) = 0 \quad \text{a.s.}$$

Proposition 4.4. *Assume that C_1 , C_2 and C_4 hold. Then, on a set of probability one, for any continuous and bounded function f from $C([0, 1])$ to \mathbb{R} ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_0(f(W_n)) = \int f(x\sqrt{\eta})W(dx),$$

where W is the distribution of a standard Wiener process.

Proof. In this proof, m will always denote a positive integer. Since C_4 implies C_3 , it follows that Proposition 4.3 holds. In what follows, we shall prove that the process $\{W_n(t), t \in [0, 1]\}$ is almost surely tight, that is, for any $\varepsilon > 0$,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_0\left(\sup_{|t-s| \leq m^{-1}} |W_n(t) - W_n(s)| > \varepsilon\right) = 0 \quad \text{almost surely.} \tag{4.17}$$

By standard arguments, (4.17) together with Proposition 4.3 imply Proposition 4.4.

According to Inequality (25) in Brown [3], to prove (4.17) it suffices to show that, for any $\varepsilon > 0$,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{P}_0\left(\sup_{(i-1)m^{-1} < t \leq im^{-1}} |W_n(t) - W_n((i-1)m^{-1})| > \varepsilon\right) = 0 \quad \text{a.s.} \tag{4.18}$$

Since $\sup_{t \in [0,1]} |W_n(t) - n^{-1/2}S_{[nt]}| = n^{-1/2} \max_{1 \leq i \leq n} |X_i|$, by using (7.2) of Lemma 7.1, it follows that (4.18) is equivalent to

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{P}_0\left(\sup_{(i-1)m^{-1} < t \leq im^{-1}} |S_{[nt]} - S_{[n(i-1)m^{-1}]}| > \varepsilon\sqrt{n}\right) = 0 \quad \text{a.s.} \tag{4.19}$$

Let $p = [n/m]$, and note that, for any nonnegative integer i , $[nim^{-1}] - i < pi \leq [nim^{-1}]$. It follows that, for any integer i in $[1, m]$,

$$\sup_{(i-1)m^{-1} < t \leq im^{-1}} |S_{[nt]} - S_{[n(i-1)m^{-1}]}| \leq \bar{S}_{(i-1)p,ip} + \frac{1}{\sqrt{n}} \sum_{k=[n(i-1)m^{-1}]-m}^{[n(i-1)m^{-1}]} |X_k| + \frac{1}{\sqrt{n}} \sum_{k=[nim^{-1}]-m}^{[nim^{-1}]} |X_k|.$$

Using (7.2) of Lemma 7.1, we infer that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{k=[n(i-1)m^{-1}]-m}^{[n(i-1)m^{-1}]} \mathbb{E}_0(|X_k|) = 0 \quad \text{a.s.}$$

Hence, (4.18) holds as soon as

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{P}(\bar{S}_{(i-1)p,ip} > \varepsilon\sqrt{n} | \mathcal{F}_0) = 0 \quad \text{a.s.,}$$

which holds under C_4 . □

5. Proof of Theorem 2.1 and additional comments

5.1. Proof of Theorem 2.1

We first prove that the series $\eta = \mathbb{E}(X_0^2 | \mathcal{I}) + 2 \sum_{k>0} \mathbb{E}(X_0 X_k | \mathcal{I})$ converges almost surely and in \mathbb{L}^1 . With this aim, it suffices to prove that

$$\sum_{k \geq 1} \|\mathbb{E}(X_0 X_k | \mathcal{I})\|_1 < \infty. \tag{5.1}$$

From Claim 1(b) in Dedecker and Rio [11], $\mathbb{E}(X_0 X_k | \mathcal{I}) = \mathbb{E}(\mathbb{E}(X_0 X_k | \mathcal{F}_{-\infty}) | \mathcal{I})$ almost surely, where $\mathcal{F}_{-\infty} = \bigcap_{k \in \mathbb{Z}} \mathcal{F}_k$. Hence

$$\|\mathbb{E}(X_0 X_k | \mathcal{I})\|_1 \leq \|\mathbb{E}(X_0 X_k | \mathcal{F}_{-\infty})\|_1 \leq \|X_0 \mathbb{E}_0(X_k)\|_1,$$

which proves (5.1) by using (2.1).

We turn now to the rest of the proof.

Proposition 5.1. *If (2.1) holds, then \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_4 hold, with η defined in (2.2). In addition the conclusion of Proposition 4.4 also holds for f in \mathcal{H}^* .*

Proof. We first prove that the following reinforced version of \mathbf{C}_2 holds:

\mathbf{C}_2^* there exists a T -invariant r.v. η that is \mathcal{F}_0 -measurable and such that

$$\text{for any integer } i \geq 1 \quad \lim_{n \rightarrow \infty} \mathbb{E}_0 \left| \frac{1}{n} \mathbb{E}_{(i-1)n} ((S_n^{(i)})^2) - \eta \right| = 0 \quad \text{a.s. and}$$

$$\text{for any integer } i \geq 1 \quad \lim_{n \rightarrow \infty} \mathbb{E}_0 \left| \frac{1}{n} \mathbb{E}_{(i-1)n} ((S_n^{(i)} + S_n^{(i+1)})^2) - 2\eta \right| = 0 \quad \text{a.s.}$$

More precisely, we shall prove that \mathbf{C}_2^* holds with η defined in (2.2). We shall only prove the first part of \mathbf{C}_2^* , the proof of the second part being similar. For any positive integer N ,

$$\frac{(S_n^{(i)})^2}{n} = \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 + \frac{2}{n} \sum_{j=(i-1)n+1}^{in-1} \sum_{l=1}^{(in-j) \wedge N} X_j X_{j+l} + R_{i,N}. \tag{5.2}$$

Firstly,

$$\mathbb{E}_0(|\mathbb{E}_{(i-1)n}(R_{i,N})|) \leq \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0 \left(\sum_{l>j+N} |X_j \mathbb{E}_j(X_l)| \right).$$

Let $Z_{j,N} = \sum_{l>j+N} |X_j \mathbb{E}_j(X_l)|$ and note that, by assumption, $Z_{j,N} = Z_{0,N} \circ T^j$ belongs to \mathbb{L}^1 . Applying the ergodic theorem in relation (7.1) of Lemma 7.1 we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j,N}) = \mathbb{E}(Z_{0,N} | \mathcal{I}) \quad \text{a.s.}$$

Hence,

$$\limsup_{n \rightarrow \infty} \mathbb{E}_0(|\mathbb{E}_{(i-1)n}(R_{i,N})|) \leq \mathbb{E}(Z_{0,N} | \mathcal{I}) \quad \text{a.s.}$$

and consequently

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_0(|\mathbb{E}_{(i-1)n}(R_{i,N})|) = 0 \quad \text{a.s.} \tag{5.3}$$

Next, let

$$\eta_N = \mathbb{E}(X_0^2 | \mathcal{I}) + 2 \sum_{k=1}^N \mathbb{E}(X_0 X_k | \mathcal{I})$$

and

$$\eta_{N,K} = \mathbb{E}(X_0^2 \mathbf{1}_{|X_0|^2 \leq K} | \mathcal{I}) + 2 \sum_{k=1}^N \mathbb{E}(X_0 X_k \mathbf{1}_{|X_0 X_k| \leq K} | \mathcal{I}).$$

By the ergodic theorem for stationary sequences,

$$\lim_{n \rightarrow \infty} \left| \eta_{N,K} - \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 \mathbf{1}_{|X_j|^2 \leq K} - \frac{2}{n} \sum_{j=(i-1)n+1}^{in-1} \sum_{l=1}^{(in-j) \wedge N} X_j X_{j+l} \mathbf{1}_{|X_j X_{j+l}| \leq K} \right| = 0 \quad \text{a.s.} \quad (5.4)$$

and by the ergodic theorem in relation (7.1) of Lemma 7.1 applied with $Z_j = X_j^2 \mathbf{1}_{|X_j|^2 > K}$ and with $Z_j = \sum_{l=1}^N |X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > K}$,

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_0 \left(\frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 \mathbf{1}_{|X_j|^2 > K} + \frac{2}{n} \sum_{j=(i-1)n+1}^{in-1} \sum_{l=1}^{(in-j) \wedge N} |X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > K} \right) = 0 \quad \text{a.s.} \quad (5.5)$$

Using (5.4), (5.5) and the dominated convergence theorem, it follows that

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left(\left| \eta_N - \frac{1}{n} \sum_{j=(i-1)n+1}^{in} X_j^2 - \frac{2}{n} \sum_{j=(i-1)n+1}^{in-1} \sum_{l=1}^{(in-j) \wedge N} X_j X_{j+l} \right| \right) = 0 \quad \text{a.s.} \quad (5.6)$$

The first part of condition \mathbf{C}_2^* follows from (5.2), (5.3) and (5.6), and the fact that $\lim_{N \rightarrow \infty} \eta_N = \eta$ almost surely.

Next, we prove that \mathbf{C}_1 holds. With this aim, we first notice that it suffices to prove that for any integer $i \geq 2$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_0| | \mathcal{I}) \mathbb{E}_0 \left(\left| \mathbb{E}_{(i-2)n} \left(\frac{S_n^{(i)}}{\sqrt{n}} \right) \right| \right) = 0 \quad \text{a.s.}$$

Indeed, on the invariant set where $\mathbb{E}(|X_0| | \mathcal{I}) = 0$ almost surely, the random variables X_i 's are equal to zero almost surely. Now, using the same arguments as in the proof of Lemma 7.1, we can prove that $\mathbb{E}(|X_0| | \mathcal{I}) = \mathbb{E}(\mathbb{E}(|X_0| | \mathcal{I}) | \mathcal{F}_0)$ almost surely. Hence, for any integer $i \geq 2$,

$$\mathbb{E}(|X_0| | \mathcal{I}) \mathbb{E}_0 \left(\left| \mathbb{E}_{(i-2)n} (S_n^{(i)}) \right| \right) = \mathbb{E}_0 \left(\left| \mathbb{E}_{(i-2)n} (\mathbb{E}(|X_0| | \mathcal{I}) S_n^{(i)}) \right| \right) \quad \text{a.s.}$$

Now

$$\begin{aligned} & \frac{1}{\sqrt{n}} \mathbb{E}_0 \left(\left| \mathbb{E}_{(i-2)n} (\mathbb{E}(|X_0| | \mathcal{I}) S_n^{(i)}) \right| \right) \\ & \leq \mathbb{E}_0 \left(\left| \mathbb{E}_{(i-2)n} \left(\left(\frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} |X_k| - \mathbb{E}(|X_0| | \mathcal{I}) \right) \frac{S_n^{(i)}}{\sqrt{n}} \right) \right| \right) \\ & \quad + \frac{1}{n^{3/2}} \sum_{k=(i-2)n+1}^{(i-1)n} \mathbb{E}_0 \left(\left| \mathbb{E}_{(i-2)n} (|X_k| S_n^{(i)}) \right| \right). \end{aligned} \quad (5.7)$$

Using the fact that $\mathcal{F}_0 \subseteq \mathcal{F}_{(i-2)n}$ for any $i \geq 2$, and applying Cauchy–Schwarz's inequality conditionally to \mathcal{F}_0 , the first term on right hand in (5.7) is smaller than

$$\mathbb{E}_0^{1/2} \left(\left(\frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} |X_k| - \mathbb{E}(|X_0| | \mathcal{I}) \right)^2 \right) \mathbb{E}_0^{1/2} \left(\left(\frac{S_n^{(i)}}{\sqrt{n}} \right)^2 \right). \quad (5.8)$$

By \mathbf{C}_2^* ,

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left(\left(\frac{S_n^{(i)}}{\sqrt{n}} \right)^2 \right) = \eta \quad \text{a.s.} \tag{5.9}$$

Since X_0 belongs to \mathbb{L}^2 , proceeding as in the proof of (5.6), we obtain that

$$\lim_{n \rightarrow \infty} \mathbb{E}_0 \left(\left(\frac{1}{n} \sum_{k=(i-2)n+1}^{(i-1)n} |X_k| - \mathbb{E}(|X_0||\mathcal{I}) \right)^2 \right) = 0 \quad \text{a.s.} \tag{5.10}$$

From (5.8), (5.9) and (5.10), we infer that the first term on right hand in (5.7) converges to 0 almost surely as n tends to infinity.

Now, for any integer k belonging to $]i - 2n, (i - 1)n]$,

$$\begin{aligned} \frac{1}{\sqrt{n}} \mathbb{E}_0 (|\mathbb{E}(|X_k|S_n^{(i)}|\mathcal{F}_{(i-2)n})|) &\leq \frac{1}{\sqrt{n}} \mathbb{E}_0 (|\mathbb{E}(|X_k|S_n^{(i)}|\mathcal{F}_k)|) \\ &\leq \frac{1}{\sqrt{n}} \mathbb{E}_0 \left(\sum_{i=k+1}^{\infty} |X_k \mathbb{E}_k(X_i)| \right). \end{aligned}$$

Let $Z_k = \sum_{i=k+1}^{\infty} |X_k \mathbb{E}_k(X_i)|$ and note that, by assumption, $Z_k = Z_0 \circ T^k$ belongs to \mathbb{L}^1 . It follows that the second term on the right-hand side of (5.7) is smaller than $n^{-3/2} \sum_{k=(i-2)n+1}^{(i-1)n} \mathbb{E}_0(Z_k)$, which converges almost surely to 0 as n tends to infinity, by the ergodic theorem in relation (7.1) of Lemma 7.1. Hence \mathbf{C}_1 is proved.

We turn now to the proof of \mathbf{C}_4 . With this aim, we shall prove the following reinforcement of it:

$$\mathbf{C}_4^* \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \mathbb{E}_0 \left(\frac{\bar{S}_{(i-1)n, in}^2}{n} \left(1 \wedge \frac{\bar{S}_{(i-1)n, in}}{\sqrt{nk}} \right) \right) = 0 \quad \text{a.s.}$$

To prove \mathbf{C}_4^* , we shall use the following maximal inequality, which is a conditional version of the inequality given in Proposition 1(a) of Dedecker and Rio [11].

Proposition 5.2. *For any $k < l$ and $\lambda \geq 0$ let $\Gamma_{k,l}(\lambda) = \{\bar{S}_{k,l} > \lambda\}$. The following inequality holds*

$$\mathbb{E}_0((\bar{S}_{k,l} - \lambda)_+^2) \leq 8 \sum_{i=k+1}^l \mathbb{E}_0(X_i^2 \mathbf{1}_{\Gamma_{k,i}(\lambda)}) + 16 \sum_{i=k+1}^l \mathbb{E}_0(|X_i \mathbf{1}_{\Gamma_{k,i}(\lambda)} \mathbb{E}_i(S_l - S_i)|).$$

Let us continue the proof of \mathbf{C}_4^* . Note first that

$$\mathbb{E}_0 \left(\frac{\bar{S}_{(i-1)n, in}^2}{n} \left(1 \wedge \frac{\bar{S}_{(i-1)n, in}}{\sqrt{nk}} \right) \right) \leq 2\varepsilon \mathbb{E}_0 \left(\frac{\bar{S}_{(i-1)n, in}^2}{n} \right) + \frac{4}{n} \mathbb{E}_0((\bar{S}_{(i-1)n, in} - \varepsilon \sqrt{nk})_+^2).$$

From Proposition 5.2 with $\lambda = 0$, we obtain that

$$\mathbb{E}_0 \left(\frac{\bar{S}_{(i-1)n, in}^2}{n} \right) \leq \frac{8}{n} \sum_{k=(i-1)n+1}^{in} \mathbb{E}_0(X_k^2) + \frac{16}{n} \sum_{k=(i-1)n+1}^{in} \mathbb{E}_0(Z_k),$$

and, by the ergodic theorem in relation (7.1) of Lemma 7.1,

$$\limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \mathbb{E}_0 \left(\frac{\bar{S}_{(i-1)n, in}^2}{n} \right) \leq 8\mathbb{E}(X_0^2|\mathcal{I}) + 16\mathbb{E}(Z_0|\mathcal{I}) \quad \text{a.s.} \tag{5.11}$$

Hence C_4^* will be proved if, for any $\varepsilon > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{1}{n} \mathbb{E}_0((\bar{S}_{(i-1)n, in} - \varepsilon\sqrt{nk})_+^2) = 0 \quad \text{a.s.} \tag{5.12}$$

Applying Proposition 5.2, we infer that, for any positive integer N ,

$$\begin{aligned} \mathbb{E}_0((\bar{S}_{(i-1)n, in} - \varepsilon\sqrt{nk})_+^2) &\leq \frac{4}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(X_j^2 \mathbf{1}_{\Gamma_{(i-1)n, in}(\varepsilon\sqrt{nk})}) \\ &\quad + \frac{8}{n} \sum_{j=(i-1)n+1}^{in} \sum_{l=1}^{(in-j) \wedge N} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{\Gamma_{(i-1)n, in}(\varepsilon\sqrt{nk})}) \\ &\quad + \frac{8}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j, N}), \end{aligned} \tag{5.13}$$

where $Z_{j, N} = \sum_{l>j+N} |X_j \mathbb{E}_j(X_l)|$. Since by (2.1), $Z_{j, N} = Z_{0, N} \circ T^j$ belongs to \mathbb{L}^1 , the ergodic theorem in relation (7.1) of Lemma 7.1 gives: for any positive integer i ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j, N}) = \mathbb{E}(Z_{0, N} | \mathcal{I}) \quad \text{a.s.}$$

and consequently,

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(Z_{j, N}) = 0 \quad \text{a.s.} \tag{5.14}$$

Now, for any positive M and any $0 \leq l \leq N$,

$$\begin{aligned} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{\Gamma_{(i-1)n, in}(\varepsilon\sqrt{nk})}) &\leq \frac{M}{\varepsilon^2 k} \mathbb{E}_0\left(\frac{\bar{S}_{(i-1)n, in}^2}{n}\right) \\ &\quad + \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > M}). \end{aligned} \tag{5.15}$$

According to (5.11), we have that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{M}{\varepsilon^2 k} \mathbb{E}_0\left(\frac{\bar{S}_{(i-1)n, in}^2}{n}\right) = 0 \quad \text{a.s.} \tag{5.16}$$

Next, by the ergodic theorem in relation (7.1) of Lemma 7.1, for any positive integer i ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > M}) = \mathbb{E}(|X_0 X_l| \mathbf{1}_{|X_0 X_l| > M} | \mathcal{I}) \quad \text{a.s.}$$

and consequently

$$\lim_{M \rightarrow \infty} \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq i \leq k} \frac{1}{n} \sum_{j=(i-1)n+1}^{in} \mathbb{E}_0(|X_j X_{j+l}| \mathbf{1}_{|X_j X_{j+l}| > M}) = 0 \quad \text{a.s.} \tag{5.17}$$

Gathering (5.13), (5.14), (5.15), (5.16) and (5.17), we infer that (5.12) holds. This ends the proof of \mathbf{C}_4^* . Then, on a set of probability one, for any continuous and bounded function φ from $C([0, 1])$ to \mathbb{R} , (2.3) follows by applying Proposition 4.4. To prove that (2.3) also holds for φ in \mathcal{H}^* , it suffices to notice that since (2.1) implies \mathbf{C}_4^* , it entails in particular that almost surely, the sequence $(n^{-1} \max_{1 \leq k \leq n} S_k^2)_{n \geq 1}$ is uniformly integrable for the conditional expectation with respect to \mathcal{F}_0 . \square

Proof of Proposition 5.2. It is exactly the same as to get (3.12) in the paper by Dedecker and Rio [11], with the only difference that the expectation is replaced by the conditional expectation with respect to \mathcal{F}_0 . \square

5.2. Some remarks on martingale approximations

The aim of this subsection is to point out that the conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_3 are satisfied if there is an almost sure conditional martingale approximation in \mathbb{L}^2 . This is another way to see that our conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_3 lead to sharp sufficient conditions for the quenched CLT.

From the proof of Theorem 5.1, we see that, if X_1 is a martingale difference, that is $\mathbb{E}(X_1|\mathcal{F}_0) = 0$ a.s., then the conditions \mathbf{C}_1 , \mathbf{C}_2^* and \mathbf{C}_4^* are satisfied. The following claim is then easily deduced.

Claim 5.3. *Let X_0 and d_0 be two \mathcal{F}_0 -measurable, centered and square integrable random variables with $\mathbb{E}(d_0 \circ T|\mathcal{F}_0) = 0$ a.s., and let $X_i = X_0 \circ T^i$ and $d_i = d_0 \circ T^i$. Let $S_n = X_1 + \dots + X_n$ and $M_n = d_1 + \dots + d_n$.*

1. *If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_0((S_n - M_n)^2) = 0 \quad \text{almost surely,}$$

then the conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_3 are satisfied with $\eta = \mathbb{E}(d_0^2|\mathcal{I})$.

2. *If*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_0\left(\max_{1 \leq k \leq n} (S_k - M_k)^2\right) = 0 \quad \text{almost surely,} \tag{5.18}$$

then the conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_4 are satisfied with $\eta = \mathbb{E}(d_0^2|\mathcal{I})$.

In particular, if the condition of Maxwell and Woodroffe [24] is satisfied

$$\sum_{n>0} \frac{\|\mathbb{E}_0(S_n)\|_2}{n^{3/2}} < \infty, \tag{5.19}$$

then it follows from Cuny and Merlevède [6] that (5.18) holds, so that the conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_4 are satisfied. We already know from Peligrad and Utev [29] that the Maxwell and Woodroffe condition is sharp in some sense for the FCLT, and therefore for the quenched FCLT also. This shows that the conditions \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_4 are essentially sharp for the quenched FCLT.

We mention that the Maxwell and Woodroffe condition and our condition (2.1) are of independent interests. For instance, when applied to strongly mixing sequences the condition (5.19) leads to sub-optimal results as pointed out in Merlevède et al. [25]. Obviously, the same remark is true when we apply it to α -dependent sequences as defined in Section 3. More precisely, this gives the condition: $\sum_{k \geq 0} (k + 1)^{-1/2} (\int_0^{\alpha \mathbf{y}^{(k)}} Q^2(u) du)^{1/2} < \infty$ instead of (3.2). Hence, when applied to nonnecessarily bounded functions of the Markov chain associated to the intermittent map given in Section 3.1, the criterion (5.19) is satisfied as soon as f belongs to $\mathcal{F}^*(H, \nu_\gamma)$ and H is such that $H(x) \leq Cx^{-2(1-\gamma)/(1-2\gamma)}(\ln(x))^{-b}$ for x large enough and $b > 2(1 - \gamma)/(1 - 2\gamma)$. Recall that by condition (3.2), we only need $b > (1 - \gamma)/(1 - 2\gamma)$. In addition, Point (v) of the main theorem in Durieu and Volný [16] shows that one can find a stationary sequence $(X_i)_{i \in \mathbb{Z}}$ adapted to an increasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ in such a way that the condition (5.19) holds but $X_0 \mathbb{E}_0(S_n)$ does not converge in \mathbb{L}^1 and so the condition (2.1) fails. Analyzing the examples given in their paper, one can also prove that there are stationary sequences for which (2.1) holds but (5.19)

does not. We can even say more: there are stationary sequences for which (2.1) holds but not (5.19), neither the Gordin criterion (2.4), nor the Hannan–Heyde condition are satisfied. Recall that the Hannan–Heyde condition is the following:

$$\mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0 \quad \text{a.s.} \quad \text{and} \quad \sum_{n \geq 0} \|\mathbb{E}_0(X_n) - \mathbb{E}_{-1}(X_n)\|_2 < \infty, \tag{5.20}$$

where $\mathcal{F}_{-\infty} = \bigcap_{k \in \mathbb{Z}} \mathcal{F}_k$.

In what follows $(\Omega, \mathcal{A}, \mu)$ is a probability space and $T : \Omega \rightarrow \Omega$ a bijective bimeasurable transformation preserving the measure μ . Then $(\Omega, \mathcal{A}, \mu, T)$ is called a dynamical system. We refer to Sinaï [32] for a precise definition of the entropy of a dynamical system, and for the properties of dynamical systems with positive entropy. The proof of the next proposition is given in the [Appendix](#).

Proposition 5.4. *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system with positive entropy. Let $\mathcal{F} \subset \mathcal{A}$ be a T -invariant σ -algebra, i.e. $\mathcal{F} \subset T^{-1}(\mathcal{F})$ and let $\mathcal{F}_i = T^{-i}(\mathcal{F})$. There exists a \mathcal{F}_0 -measurable and centered function f in \mathbb{L}^2 such that, setting $X_i = f \circ T^i$, the condition (2.1) is satisfied but the conditions (2.4), (5.19) and (5.20) fail.*

To be complete, note that a stationary sequence can be constructed in such a way that (2.4) holds but the condition (2.1) fails (see Section 5.2 in Durieu and Volný [16]). Moreover, a stationary sequence can be constructed in such a way that the condition (5.20) holds but the condition (2.1) fails (see Theorem 1 in Durieu [15]).

6. Normal approximation for double indexed arrays and auxiliary results

There are many situations when we are dealing with double indexed sequences of random variables. For instance at each point in the two dimensional space we start a random walk. Our motivation for this section comes from the fact that in our blocking procedure we introduce a new parameter, the number of blocks, m , that is kept fixed at the beginning.

The next theorem treats the martingale approximation for double arrays of random variables.

Theorem 6.1. *Assume that $(U_{n,m,i})_{i \geq 1}$ is an array of random variables in \mathbb{L}^2 adapted to an array $(\mathcal{G}_{n,m,i})_{i \geq 1}$ of nested sigma fields. Let $\mathbb{E}_{n,m,i}$ denote the conditional expectation with respect to $\mathcal{G}_{n,m,i}$. Suppose that*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1}(U_{n,m,i}) \right| = 0, \tag{6.1}$$

there exists $\sigma^2 \geq 0$ such that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \text{Var}(U_{n,m,i} | \mathcal{G}_{n,m,i-1}) - \sigma^2 \right| = 0, \tag{6.2}$$

and for each $\varepsilon > 0$

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{E}(U_{n,m,i}^2 \mathbf{1}_{|U_{n,m,i}| > \varepsilon}) = 0. \tag{6.3}$$

Then for any continuous and bounded function f ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left(f \left(\sum_{i=1}^m U_{n,m,i} \right) \right) - \mathbb{E}(f(\sigma N)) \right| = 0, \tag{6.4}$$

where N is a standard Gaussian variable.

Proof. For any $i \geq 1$, let $d_{n,m,i} = U_{n,m,i} - \mathbb{E}_{n,m,i-1}(U_{n,m,i})$. By condition (6.1), the theorem will follow if we can prove that (6.4) holds with $\sum_{i=1}^m d_{n,m,i}$ replacing $\sum_{i=1}^m U_{n,m,i}$. If $\sigma^2 = 0$ the theorem is trivial. So we can assume without loss of generality that $\sigma^2 = 1$. In the rest of the proof, in order to ease the notations, we shall drop the first two indexes (n, m) , keeping them only when it is necessary to avoid confusion. Let ε and M be positive reals fixed for the moment. For any $i \geq 1$, let

$$V_i = \sum_{\ell=1}^i \mathbb{E}_{\ell-1}(d_\ell^2) \quad \text{and} \quad Y_i = d_i \mathbf{1}_{|d_i| \leq \varepsilon} \mathbf{1}_{V_i \leq M}.$$

Notice first that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^m d_i \neq \sum_{i=1}^m Y_i\right) &\leq \mathbb{P}\left(\max_{1 \leq i \leq m} d_i > \varepsilon\right) + \mathbb{P}(V_m > M) \\ &\leq \frac{1}{\varepsilon^2} \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + \frac{1}{M} \left(1 + \mathbb{E}\left|\sum_{i=1}^m \text{Var}(U_i | \mathcal{G}_{i-1}) - 1\right|\right). \end{aligned}$$

Hence using Lemma 6.3, we get that

$$\mathbb{P}\left(\sum_{i=1}^m d_i \neq \sum_{i=1}^m Y_i\right) \leq \frac{12}{\varepsilon^2} \sum_{i=1}^m \mathbb{E}(U_i^2 \mathbf{1}_{|U_i| > \varepsilon/4}) + \frac{1}{M} \left(1 + \mathbb{E}\left|\sum_{i=1}^m \text{Var}(U_i | \mathcal{G}_{i-1}) - 1\right|\right).$$

Therefore using (6.3) and (6.2), it follows that for all $\varepsilon > 0$,

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sum_{i=1}^m d_{n,m,i} \neq \sum_{i=1}^m Y_{n,m,i}\right) \leq \frac{1}{M}. \tag{6.5}$$

We notice now that since $\mathbb{E}_{i-1}(d_i) = 0$ a.s. and V_i is \mathcal{G}_{i-1} -measurable,

$$\sum_{i=1}^m \mathbb{E}_{i-1}(Y_i) = \sum_{i=1}^m \mathbf{1}_{V_i \leq M} \mathbb{E}_{i-1}(d_i \mathbf{1}_{|d_i| > \varepsilon}).$$

Therefore by Lemma 6.3,

$$\mathbb{E}\left|\sum_{i=1}^m \mathbb{E}_{i-1}(Y_i)\right| \leq \frac{12}{\varepsilon} \sum_{i=1}^m \mathbb{E}(U_i^2 \mathbf{1}_{|U_i| > \varepsilon/4}),$$

implying, by using (6.3), that for all positive reals ε and M ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}\left|\sum_{i=1}^m \mathbb{E}_{i-1}(Y_i)\right| = 0. \tag{6.6}$$

Considering (6.5) and (6.6), the theorem will follow if we can show that for any continuous bounded function f ,

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left|\mathbb{E}\left(f\left(\sum_{i=1}^m d_{n,m,i}^*\right)\right) - \mathbb{E}(f(N))\right| = 0, \tag{6.7}$$

where

$$d_{n,m,i}^* = Y_{n,m,i} - \mathbb{E}_{n,m,i-1}(Y_{n,m,i}).$$

Let

$$s_{n,m}^2 = \sum_{i=1}^m \mathbb{E}(d_{n,m,i}^{*2}),$$

and notice that for any $\delta > 0$, $\mathbb{E}(|d_{n,m,i}^*|^{2+2\delta}) < \infty$, $i = 1, 2, \dots$. Hence, by the first theorem stated in Heyde and Brown [22], it follows that for any $\delta \in]0, 1]$,

$$\begin{aligned} & \sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\sum_{i=1}^m d_{n,m,i}^* \leq x \right) - \mathbb{P}(s_n N \leq x) \right| \\ & \leq K_\delta \left\{ s_{n,m}^{-2-2\delta} \left(\mathbb{E}(|d_{n,m,i}^*|^{2+2\delta}) + \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1}((d_{n,m,i}^*)^2) - s_{n,m}^2 \right|^{1+\delta} \right) \right\}^{1/(3+2\delta)}, \end{aligned} \tag{6.8}$$

where K_δ is a positive constant depending only on δ . Assume now that we can prove that there exists a δ in $]0, 1]$ such that for any positive reals ε and M ,

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}^*|^{2+2\delta}) \leq u(\varepsilon) \tag{6.9}$$

and

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{n,m,i-1}((d_{n,m,i}^*)^2) - 1 \right|^{1+\delta} \leq v(M), \tag{6.10}$$

where $u(\cdot)$ and $v(\cdot)$ are positive functions defined on \mathbb{R}^+ such that $v(\cdot)$ does not depend on ε , $\lim_{x \rightarrow 0} u(x) = 0$ and $\lim_{x \rightarrow \infty} v(x) = 0$. Then starting from (6.8) and noticing that (6.10) also implies that for any positive reals ε and M , and

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |s_{n,m}^2 - 1|^{1+\delta} \leq v(M), \tag{6.11}$$

we infer that (6.7) will hold. Indeed, by standard arguments, we will get (6.7) for every continuous function f with compact support and then (6.7) for every continuous and bounded function f by using the fact that every probability measure is tight. Hence, to end the proof of the theorem, it remains to prove that (6.9) and (6.10) hold. With this aim, we first notice that

$$\sum_{i=1}^m \mathbb{E}(|d_i^*|^{2+2\delta}) \leq 4(2\varepsilon)^{2\delta} \sum_{i=1}^m \mathbb{E}(d_i^2) \leq 4(2\varepsilon)^{2\delta} \left(1 + \mathbb{E} \left| \sum_{i=1}^m \text{Var}(U_i | \mathcal{G}_{i-1}) - 1 \right| \right).$$

Hence, using condition (6.2), (6.9) follows with $u(\varepsilon) = 4(2\varepsilon)^{2\delta}$. It remains to prove (6.10). With this aim, using the convexity inequality: $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ ($p \geq 1$, $a > 0$ and $b > 0$), we first write that

$$\mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}((d_i^*)^2) - 1 \right|^{1+\delta} \leq 2^\delta \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} + 2^\delta \mathbb{E} \left| \sum_{i=1}^m (\mathbb{E}_{i-1}(Y_i))^2 \right|^{1+\delta}. \tag{6.12}$$

Now, since $V_{n,m,i}$ is $\mathcal{G}_{n,m,i-1}$ -measurable and $\mathbb{E}_{n,m,i-1}(d_{n,m,i}) = 0$ a.s., we infer that

$$\begin{aligned} \mathbb{E} \left(\sum_{i=1}^m (\mathbb{E}_{i-1}(Y_i))^2 \right)^{1+\delta} & \leq \mathbb{E} \left(\left(\sum_{i=1}^m (\mathbb{E}_{i-1}(d_i \mathbf{1}_{|d_i| > \varepsilon}))^2 \right) \left(\sum_{k=1}^m \mathbf{1}_{V_k \leq M} \mathbb{E}_{k-1}(d_k^2) \right)^\delta \right) \\ & \leq M^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}). \end{aligned}$$

Hence by Lemma 6.3,

$$\mathbb{E} \left(\sum_{i=1}^m (\mathbb{E}_{i-1}(Y_i^2))^2 \right)^{1+\delta} \leq 12M^\delta \sum_{i=1}^m \mathbb{E}(U_i^2 \mathbf{1}_{|U_i| > \varepsilon/4}). \quad (6.13)$$

On the other hand, using again the fact that $V_{n,m,i}$ is $\mathcal{G}_{n,m,i-1}$ -measurable and also that $V_{n,m,i} \leq V_{n,m,i+1}$, we derive that

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} &\leq \mathbb{E} \left(\left(1 + \sum_{i=1}^m \mathbf{1}_{V_i \leq M} \mathbb{E}_{i-1}(d_i^2) \right)^\delta \left| 1 - \sum_{k=1}^m \mathbf{1}_{V_k \leq M} \mathbb{E}_{k-1}(d_k^2 \mathbf{1}_{|d_k| \leq \varepsilon}) \right| \right) \\ &\leq (M+1)^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + (M+1)^\delta \mathbb{E} \left| 1 - \sum_{k=1}^m \mathbf{1}_{V_k \leq M} \mathbb{E}_{k-1}(d_k^2) \right| \\ &\leq (M+1)^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + (M+1)^\delta \mathbb{E} \left| \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) - 1 \right| \\ &\quad + (M+1)^\delta \mathbb{E} \left| \mathbf{1}_{V_m > M} \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} &\leq (M+1)^\delta \sum_{i=1}^m \mathbb{E}(d_i^2 \mathbf{1}_{|d_i| > \varepsilon}) + 2(M+1)^\delta \mathbb{E} \left| \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) - 1 \right| \\ &\quad + \frac{(M+1)^\delta}{M} \left(1 + \mathbb{E} \left| \sum_{k=1}^m \mathbb{E}_{k-1}(d_k^2) - 1 \right| \right), \end{aligned}$$

which together with Lemma 6.3 and the fact that $\mathbb{E}_{k-1}(d_k^2) = \text{Var}(U_k | \mathcal{G}_{k-1})$ imply that

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^m \mathbb{E}_{i-1}(Y_i^2) - 1 \right|^{1+\delta} &\leq 12(M+1)^\delta \sum_{i=1}^m \mathbb{E}(U_i^2 \mathbf{1}_{|U_i| > \varepsilon/4}) + 2(M+1)^\delta \mathbb{E} \left| \sum_{k=1}^m \text{Var}(U_k | \mathcal{G}_{k-1}) - 1 \right| \\ &\quad + \frac{(M+1)^\delta}{M} \left(1 + \mathbb{E} \left| \sum_{k=1}^m \text{Var}(U_k | \mathcal{G}_{k-1}) - 1 \right| \right). \quad (6.14) \end{aligned}$$

Starting from (6.12) and considering the bounds (6.13) and (6.14) together with the conditions (6.2) and (6.3), we then infer that (6.10) holds for any $\delta \in]0, 1[$ with $v(M) = M^{-1}(M+1)^\delta$. This ends the proof of (6.7) and then of the theorem. \square

Lemma 6.2. Assume that $(d_{n,m,i})_{i \geq 1}$ is an array of random variables in \mathbb{L}^2 adapted to an array $(\mathcal{G}_{n,m,i})_{i \geq 1}$ of nested sigma fields, and such that for any $i \geq 1$, $\mathbb{E}_{n,m,i-1}(d_{n,m,i}) = 0$ almost surely. Suppose that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^m \mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}| > \varepsilon}) = 0 \quad \text{and} \quad \sum_{i=1}^m \mathbb{E}|d_{n,m,i}| < K \quad (6.15)$$

for some positive constant K . Then

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m d_{n,m,i} \right| = 0.$$

Proof. Let $\varepsilon > 0$, and let for any $i \geq 1$,

$$d'_{n,m,i} = d_{n,m,i} \mathbf{1}_{|d_{n,m,i}| \leq \varepsilon} \quad \text{and} \quad d''_{n,m,i} = d_{n,m,i} \mathbf{1}_{|d_{n,m,i}| > \varepsilon}.$$

With this notation and since $\mathbb{E}_{n,m,i-1}(d_{n,m,i}) = 0$ almost surely,

$$\sum_{i=1}^m d_{n,m,i} = \sum_{i=1}^m (d'_{n,m,i} - \mathbb{E}_{n,m,i-1}(d'_{n,m,i})) + \sum_{i=1}^m (d''_{n,m,i} - \mathbb{E}_{n,m,i-1}(d''_{n,m,i})).$$

Since

$$\mathbb{E} \left| (d''_{n,m,i} - \mathbb{E}_{n,m,i-1}(d''_{n,m,i})) \right| \leq 2\mathbb{E}(|d_{n,m,i}| \mathbf{1}_{|d_{n,m,i}| > \varepsilon}),$$

by using the first part of (6.15), the lemma will follow if we can prove that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \sum_{i=1}^m (d'_{n,m,i} - \mathbb{E}_{n,m,i-1}(d'_{n,m,i})) \right| = 0. \quad (6.16)$$

With this aim, it suffices to notice that

$$\mathbb{E} \left(\sum_{i=1}^m (d'_{n,m,i} - \mathbb{E}_{n,m,i-1}(d'_{n,m,i})) \right)^2 \leq \sum_{i=1}^m \mathbb{E} (d'_{n,m,i})^2 \leq \varepsilon \sum_{i=1}^m \mathbb{E} |d_{n,m,i}|,$$

showing that (6.16) holds under (6.15). \square

Lemma 6.3. Let X be a real random variable and \mathcal{F} a sigma-field. For any $p \geq 1$ and any $\varepsilon > 0$,

$$\mathbb{E}(|X|^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon}) \leq 2\mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon}), \quad (6.17)$$

and setting $Y = X - \mathbb{E}(X|\mathcal{F})$,

$$\mathbb{E}(|X|^p \mathbf{1}_{|Y| > 3\varepsilon}) \leq 2\mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon}) \quad \text{and} \quad \mathbb{E}(|Y|^p \mathbf{1}_{|Y| > 4\varepsilon}) \leq 3 \times 2^p \mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon}). \quad (6.18)$$

Proof. We first write that

$$|X|^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon} \leq |X|^p \mathbf{1}_{|X| > \varepsilon} + \varepsilon^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon}. \quad (6.19)$$

Notice now that $\{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon\} \subseteq \{|\mathbb{E}(X \mathbf{1}_{|X| > \varepsilon})| > \varepsilon\}$, implying that

$$\varepsilon^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon} \leq |\mathbb{E}(X \mathbf{1}_{|X| > \varepsilon})|^p \leq \mathbb{E}(|X|^p \mathbf{1}_{|X| > \varepsilon} | \mathcal{F}), \quad (6.20)$$

starting from (6.19), using (6.20) and taking the expectation, (6.17) follows. To prove the first part of (6.18), we start by writing that

$$|X|^p \mathbf{1}_{|Y| > 3\varepsilon} \leq |X|^p \mathbf{1}_{|X| > \varepsilon} + \varepsilon^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon},$$

and we use (6.20). To prove the second part of (6.18), we first notice that for any positive reals a, b and ε , $(a + b)^p \mathbf{1}_{a+b > 4\varepsilon} \leq 2^p a^p \mathbf{1}_{a > 2\varepsilon} + 2^p b^p \mathbf{1}_{b > 2\varepsilon}$. Therefore

$$\mathbb{E}(|Y|^p \mathbf{1}_{|Y| > 4\varepsilon}) \leq 2^p \mathbb{E}(|X|^p \mathbf{1}_{|X| > 2\varepsilon}) + 2^p \mathbb{E}(|X|^p \mathbf{1}_{|\mathbb{E}(X|\mathcal{F})| > 2\varepsilon}).$$

The second part of (6.18) then follows by using (6.17). \square

7. Ergodic theorem

We gather below the ergodic theorems used in this paper. We keep the notations of Section 2 and we define $\mathcal{F}_\infty = \bigvee_{k \in \mathbb{Z}} \mathcal{F}_k$.

Lemma 7.1. *Let Z be a \mathcal{F}_∞ -measurable real-valued random variable in \mathbb{L}^1 . Define $Z_k = Z \circ T^k$ for any k in \mathbb{Z} . Then*

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(Z_i) \rightarrow \mathbb{E}(Z|\mathcal{I}) \quad \text{almost surely and in } \mathbb{L}^1, \tag{7.1}$$

and

$$\frac{1}{n} \mathbb{E}_0\left(\max_{1 \leq i \leq n} |Z_i|\right) \rightarrow 0 \quad \text{almost surely and in } \mathbb{L}^1. \tag{7.2}$$

Proof. By definition of the operator K (see the beginning of Section 2),

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(Z_i) = \frac{1}{n} \sum_{i=1}^n K^i(Z).$$

Applying the Dunford–Schwartz ergodic theorem (see for instance Krengel [23]) we obtain that $(K(Z) + \dots + K^n(Z))/n$ converges almost surely and in \mathbb{L}^1 to some $g \in \mathbb{L}^1$. We prove now that $g = \mathbb{E}(Z|\mathcal{I})$. Let $N \in \mathbb{N}$. Define $Z_{0,N} = \mathbb{E}(Z|\mathcal{F}_N)$ and $Z_{k,N} = Z_{0,N} \circ T^k$ for any k in \mathbb{Z} . From the stationarity of the sequence $(Z_{k,N})_{k \in \mathbb{Z}}$ and the invariance of $\mathbb{E}(Z_{0,N}|\mathcal{I})$, we have

$$\left\| \mathbb{E}(Z_{0,N}|\mathcal{I}) - \frac{1}{n} \sum_{k=1}^n Z_{k,N} \right\|_1 = \left\| \mathbb{E}(Z_{0,N}|\mathcal{I}) - \frac{1}{n} \sum_{k=1}^{-N} Z_{k,N} \right\|_1.$$

Both this equality and the \mathbb{L}^1 -ergodic theorem imply that $\mathbb{E}(Z_{0,N}|\mathcal{I})$ is the limit in \mathbb{L}^1 of a sequence of \mathcal{F}_0 -measurable random variables. Hence $\mathbb{E}(\mathbb{E}(Z_{0,N}|\mathcal{I})|\mathcal{F}_0) = \mathbb{E}(Z_{0,N}|\mathcal{I})$ almost surely. Therefore, noticing that for any $i \in \mathbb{N}$, $\mathbb{E}_0(Z_i) = \mathbb{E}_0(Z_{i,N})$ and using, once again, the \mathbb{L}^1 -ergodic theorem, we derive that

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(Z_i) - \mathbb{E}(Z_{0,N}|\mathcal{I}) \right\|_1 \leq \limsup_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{i=1}^n Z_{i,N} - \mathbb{E}(Z_{0,N}|\mathcal{I}) \right\|_1 = 0.$$

Hence the proof will be complete if we show that $\lim_{N \rightarrow \infty} \|\mathbb{E}(Z_{0,N}|\mathcal{I}) - \mathbb{E}(Z|\mathcal{I})\|_1 = 0$. Notice that

$$\|\mathbb{E}(Z_{0,N}|\mathcal{I}) - \mathbb{E}(Z|\mathcal{I})\|_1 \leq \|\mathbb{E}(Z|\mathcal{F}_N) - Z\|_1.$$

Therefore since $(\mathbb{E}(Z|\mathcal{F}_N))_{N \geq 1}$ is an uniformly integrable martingale, and Z is \mathcal{F}_∞ -measurable, the desired convergence follows by the martingale convergence theorem.

We turn now to the proof of (7.2). With this aim, we notice that for any $N > 0$,

$$\frac{1}{n} \mathbb{E}_0\left(\max_{1 \leq i \leq n} |Z_i|\right) \leq \frac{N}{n} + \frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(|Z_i| \mathbf{1}_{|Z_i| > N}).$$

By using (7.1), $n^{-1} \sum_{i=1}^n \mathbb{E}_0(|Z_i| \mathbf{1}_{|Z_i| > N})$ converges to $\mathbb{E}(|Z| \mathbf{1}_{|Z| > N}|\mathcal{I})$ almost surely and in \mathbb{L}^1 , as n tends to infinity. Therefore

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}_0(|Z_i| \mathbf{1}_{|Z_i| > N}) = 0 \quad \text{almost surely and in } \mathbb{L}^1,$$

which ends the proof of (7.2). □

Appendix

This section is devoted to the proof of Proposition 5.4. We shall see that it follows from a slight modification of the example given in Section 5.4 in Durieu and Volný [16].

We consider the ergodic dynamical system $(\Omega, \mathcal{A}, \mu, T)$ with positive entropy, the sequence $(e_i)_{i \in \mathbb{Z}}$ of independent identically distributed (i.i.d.) Rademacher random variables with parameter $1/2$, and the σ -algebra \mathcal{F}_0 as described at the beginning of Section 4.1 in Durieu and Volný [16]. Now, for any positive integer k , we define

$$N_k = 4^k, \quad \rho_k = \frac{1}{4^k}, \quad \theta_k = \frac{1}{k2^k}, \quad \varepsilon_k = \frac{1}{k24^{3k}}, \tag{A.1}$$

and we consider mutually disjoint sets $(A_k)_{k \in \mathbb{Z}}$ by using their Lemma 2 with $2N_k$ instead of N_k , and the sequences (ρ_k) and (ε_k) defined above. In addition to being disjoint, the sets $(A_k)_{k \in \mathbb{N}^*}$ are such that

- (i) $\frac{2}{3}\rho_k \leq \mu(A_k) \leq \rho_k$ for all $k \in \mathbb{N}^*$;
- (ii) for all $k \in \mathbb{N}^*$ and all $i, j \in \{0, \dots, 2N_k\}$, $\mu(T^{-i}A_k \Delta T^{-j}A_k) \leq \varepsilon_k$.

The function f is then defined as

$$f = \sum_{k \geq 1} f_k \mathbf{1}_{A_k} \quad \text{with} \quad f_k = \theta_k \sum_{j=N_k+1}^{2N_k} e_{-j}. \tag{A.2}$$

The function f defined in (A.2) is centered, \mathcal{F}_0 -measurable and, since $\sum_{k \geq 1} \theta_k^2 N_k \rho_k < \infty$, it belongs to \mathbb{L}^2 (see Proposition 7 in Durieu and Volný [16]).

Let now $X_i = f \circ T^i$ for any $i \in \mathbb{Z}$. This sequence is adapted to the stationary and nondecreasing sequence of σ -algebras $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ where $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let us first prove that the sequence $(X_i)_{i \in \mathbb{Z}}$ satisfies the condition (2.1). With this aim, we first emphasize some additional important properties of $(e_i)_{i \in \mathbb{Z}}$ and of $(A_k)_{k \in \mathbb{Z}}$. First, the sequence $(e_i)_{i \in \mathbb{Z}}$ is adapted to $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ and $\mathbb{E}(e_i | \mathcal{F}_0) = e_i \mathbf{1}_{i \leq 0}$ almost surely. Second, for all k and i , $\mathbf{1}_{A_k} \circ T^i$ is \mathcal{F}_0 -measurable. Finally, the e_i 's and the $\mathbf{1}_{A_k}$'s are independent for all i and k . As in relation (4) in Durieu and Volný [16], we then write that for any $i \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}(X_i | \mathcal{F}_0) &= \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{T^{-i}(A_k)} \mathbf{1}_{i \leq j} \\ &= \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{A_k} \mathbf{1}_{i \leq j} + \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} (\mathbf{1}_{T^{-i}(A_k) \setminus A_k} - \mathbf{1}_{A_k \setminus T^{-i}(A_k)}) \mathbf{1}_{i \leq j}. \end{aligned} \tag{A.3}$$

Using item (ii) above, and the fact that the e_j 's are bounded by one, we obtain

$$\begin{aligned} &\sum_{i \geq 0} \left\| \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} (\mathbf{1}_{T^{-i}(A_k) \setminus A_k} - \mathbf{1}_{A_k \setminus T^{-i}(A_k)}) \right\|_2 \\ &\leq \sum_{k \geq 1} \sum_{i=0}^{N_k} \theta_k N_k (\mu(T^{-i}(A_k) \Delta A_k))^{1/2} + \sum_{k \geq 1} \theta_k \sum_{i=N_k+1}^{2N_k} \sum_{j=i}^{2N_k} (\mu(T^{-i}(A_k) \Delta A_k))^{1/2} \\ &\leq 2 \sum_{k \geq 1} \theta_k N_k (N_k + 1) \sqrt{\varepsilon_k}. \end{aligned} \tag{A.4}$$

Since, by (A.1), $\sum_{k \geq 1} \theta_k N_k^2 \sqrt{\varepsilon_k} < \infty$, in order to prove that (2.1) holds, it is enough to show that

$$\sum_{i \geq 0} \left\| f \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{A_k} \mathbf{1}_{i \leq j} \right\|_1 < \infty. \tag{A.5}$$

By disjointness of the A_k 's,

$$\begin{aligned} \sum_{i \geq 0} \left\| f \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{A_k} \mathbf{1}_{i \leq j} \right\|_1 &= \sum_{i \geq 0} \left\| \sum_{k \geq 1} \theta_k^2 \left(\sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} \right) \left(\sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right) \mathbf{1}_{A_k} \right\|_1 \\ &= \sum_{i \geq 0} \left\| \sum_{k \geq 1} \mathbf{1}_{i \leq N_k} \theta_k^2 \left(\sum_{j=N_k+1}^{2N_k} e_{i-j} \right) \left(\sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right) \mathbf{1}_{A_k} \right\|_1 \\ &\quad + \sum_{i \geq 0} \left\| \sum_{k \geq 1} \mathbf{1}_{N_k+1 \leq i \leq 2N_k} \theta_k^2 \left(\sum_{j=i}^{2N_k} e_{i-j} \right) \left(\sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right) \mathbf{1}_{A_k} \right\|_1. \end{aligned} \tag{A.6}$$

Now, by independence between the e_i 's and the $\mathbf{1}_{A_k}$'s,

$$\begin{aligned} \sum_{i \geq 0} \left\| \sum_{k \geq 1} \mathbf{1}_{i \leq N_k} \theta_k^2 \left(\sum_{j=N_k+1}^{2N_k} e_{i-j} \right) \left(\sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right) \mathbf{1}_{A_k} \right\|_1 \\ \leq \sum_{i \geq 0} \sum_{k \geq 1} \mathbf{1}_{i \leq N_k} \theta_k^2 \left\| \left(\sum_{j=N_k+1}^{2N_k} e_{i-j} \right) \left(\sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right) \right\|_1 \mu(A_k). \end{aligned}$$

Since the e_i 's are i.i.d., centered and with variance one, we have

$$\left\| \left(\sum_{j=N_k+1}^{2N_k} e_{i-j} \right) \left(\sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right) \right\|_1 \leq \left\| \sum_{j=N_k+1}^{2N_k} e_{i-j} \right\|_2 \left\| \sum_{\ell=N_k+1}^{2N_k} e_{-\ell} \right\|_2 \leq N_k.$$

The second term in the right-hand side of (A.6) can be handled similarly. So overall, we infer that

$$\sum_{i \geq 0} \left\| f \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{A_k} \mathbf{1}_{i \leq j} \right\|_1 \leq 2 \sum_{k \geq 1} \theta_k^2 N_k^2 \rho_k,$$

which is finite according to (A.1). This ends the proof of (A.5) and then of the fact that the sequence $(X_i)_{i \in \mathbb{Z}}$ satisfies the condition (2.1).

Let us prove now that the condition (2.4) fails for the sequence $(X_i)_{i \in \mathbb{Z}}$ defined above. With this aim, we shall prove that

$$\sup_{K \in \mathbb{N}} \left\| \mathbb{E}_0(S_{N_K}) \right\|_1 = \infty. \tag{A.7}$$

Starting from (A.3) and using (A.4), it suffices to prove that

$$\sup_{K \in \mathbb{N}} \left\| \sum_{i=1}^{N_K} \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} \mathbf{1}_{A_k} \right\|_1 = \infty. \tag{A.8}$$

Let $K \geq 3$. By disjointness of the A_k 's and by independence between the e_i 's and the $\mathbf{1}_{A_k}$'s, it follows that

$$\left\| \sum_{i=1}^{N_K} \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} \mathbf{1}_{A_k} \right\|_1 = \sum_{k \geq 1} \theta_k \mu(A_k) \mathbb{E} \left| \sum_{j=N_k+1}^{2N_k} \sum_{i=1}^{N_K} e_{i-j} \mathbf{1}_{i \leq j} \right|.$$

Therefore

$$\left\| \sum_{i=1}^{N_K} \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} \mathbf{1}_{A_k} \right\|_1 \geq \sum_{k=1}^{K-1} \theta_k \mu(A_k) \mathbb{E} \left| \sum_{\ell=(N_k+1-N_K) \vee 0}^{2N_k-1} e^{-\ell} \sum_{i=1}^{N_K} \mathbf{1}_{i \geq N_k+1-\ell} \mathbf{1}_{i \leq 2N_k-\ell} \right|.$$

Notice now that for any $k \in \{1, \dots, K-1\}$, $N_k + 1 - N_K \leq N_{K-1} + 1 - N_K \leq 0$ and for any $\ell \geq 0$, $2N_k - \ell \leq N_K$ (since $2N_k - N_K \leq 2N_{K-1} - N_K \leq 0$). Hence

$$\left\| \sum_{i=1}^{N_K} \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} \mathbf{1}_{A_k} \right\|_1 \geq \sum_{k=1}^{K-1} \theta_k \mu(A_k) \mathbb{E} \left| N_k \sum_{\ell=0}^{N_k} e^{-\ell} + \sum_{\ell=N_k+1}^{2N_k-1} (2N_k - \ell) e^{-\ell} \right|.$$

Next, by using the Marcinkiewicz–Zygmund’s inequality together with item (i) above, we get that there exists a positive constant A such that

$$\begin{aligned} \left\| \sum_{i=1}^{N_K} \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} e_{i-j} \mathbf{1}_{i \leq j} \mathbf{1}_{A_k} \right\|_1 &\geq A \sum_{k=1}^{K-1} \theta_k \rho_k \mathbb{E} \left(N_k^2 \sum_{\ell=0}^{N_k} e^{-\ell} + \sum_{\ell=N_k+1}^{2N_k-1} (2N_k - \ell)^2 e^{-\ell} \right)^{1/2} \\ &\geq A \sum_{k=1}^{K-1} \theta_k \rho_k N_k^{3/2} \geq A \ln(K-1), \end{aligned}$$

which proves (A.8) and therefore (A.7).

Let us prove now that the condition (5.19) fails for the sequence $(X_i)_{i \in \mathbb{Z}}$ defined above. Following the computations page 339 in Durieu and Volný [16], it suffices to prove that

$$\sum_{n \geq 1} \frac{1}{n^{3/2}} \left(\sum_{k \geq 1} \mathbf{1}_{2N_k \leq n} \theta_k^2 N_k^3 \rho_k \right)^{1/2} = \infty. \tag{A.9}$$

Since $N_k = 4^k$, using (A.1), we get that

$$\sum_{k \geq 1} \mathbf{1}_{2N_k \leq n} \theta_k^2 N_k^3 \rho_k = \sum_{k=1}^{[(\ln n - \ln 2)/(2 \ln 2)]} \frac{4^k}{k^2} \geq C \frac{n}{(\ln n)^2},$$

where C is a positive constant. This shows (A.9) and then that (5.19) fails.

Let us prove now that the Hannan–Heyde condition (5.20) fails for the sequence $(X_i)_{i \in \mathbb{Z}}$ defined above. With this aim, we first notice that

$$\mathbb{E}(X_i | \mathcal{F}_0) - \mathbb{E}(X_i | \mathcal{F}_{-1}) = e_0 \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} \mathbf{1}_{T^{-i}(A_k)} \mathbf{1}_{i=j}.$$

Proceeding as in (A.3) and since $\sum_{k \geq 1} \theta_k N_k \sqrt{\varepsilon_k} < \infty$, it suffices to prove that

$$\sum_{i \geq 1} \left\| \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} \mathbf{1}_{i=j} \mathbf{1}_{A_k} \right\|_2 = \infty. \tag{A.10}$$

But, by disjointness of the A_k ’s,

$$\sum_{i \geq 1} \left\| \sum_{k \geq 1} \theta_k \sum_{j=N_k+1}^{2N_k} \mathbf{1}_{i=j} \mathbf{1}_{A_k} \right\|_2 = \sum_{i \geq 1} \left(\sum_{k \geq 1} \theta_k^2 \left(\sum_{j=N_k+1}^{2N_k} \mathbf{1}_{i=j} \right) \mu(A_k) \right)^{1/2}.$$

Therefore, since $N_k = 2^{2k}$,

$$\begin{aligned} \sum_{i \geq 1} \left\| \sum_{k \geq 1} \theta_k \sum_{j=N_{k+1}}^{2N_k} \mathbf{1}_{i=j} \mathbf{1}_{A_k} \right\|_2 &= \sum_{\ell \geq 0} \sum_{i=2^{2\ell+1}}^{2^{2(\ell+1)}} \left(\sum_{k \geq 1} \theta_k^2 \left(\sum_{j=2^{2k+1}}^{2^{2k+1}} \mathbf{1}_{i=j} \right)^2 \mu(A_k) \right)^{1/2} \\ &\geq \sum_{\ell \geq 0} \sum_{i=2^{2\ell+1}}^{2^{2\ell+1}} \left(\sum_{k \geq 1} \theta_k^2 \left(\sum_{j=2^{2k+1}}^{2^{2k+1}} \mathbf{1}_{i=j} \right)^2 \mu(A_k) \right)^{1/2} \\ &= \sum_{\ell \geq 0} \sum_{i=2^{\ell+1}}^{2^{2\ell+1}} (\theta_\ell^2 \mu(A_\ell))^{1/2} \geq \frac{\sqrt{2}}{\sqrt{3}} \sum_{\ell \geq 0} 2^{2\ell} \theta_\ell \sqrt{\rho_\ell}, \end{aligned}$$

which does not converge according to (A.1). This ends the proof of (A.10) and then of the fact that the sequence $(X_i)_{i \in \mathbb{Z}}$ does not satisfy the Hannan–Heyde condition (5.20).

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References

- [1] A. N. Borodin and I. A. Ibragimov. *Limit Theorems for Functionals of Random Walks*. *Trudy Mat. Inst. Steklov.* **195**. Nauka, St. Petersburg, 1994. Transl. into English: *Proc. Steklov Inst. Math.* **195**. Amer. Math. Soc., Providence, RI, 1995. [MR1368394](#)
- [2] R. C. Bradley. On quantiles and the central limit question for strongly mixing sequences. Dedicated to Murray Rosenblatt. *J. Theoret. Probab.* **10** (1997) 507–555. [MR1455156](#)
- [3] B. M. Brown. Martingale central limit theorems. *Ann. Math. Statist.* **42** (1971) 59–66. [MR0290428](#)
- [4] X. Chen. Limit theorems for functionals of ergodic Markov chains with general state space. *Mem. Amer. Math. Soc.* **139** (1999) xiv+203. [MR1491814](#)
- [5] C. Cuny. Pointwise ergodic theorems with rate and application to limit theorems for stationary processes. *Stoch. Dyn.* **11** (2011) 135–155. [MR2771346](#)
- [6] C. Cuny and F. Merlevède. On martingale approximations and the quenched weak invariance principle. *Ann. Probab.* **42** (2014) 760–793. [MR3178473](#)
- [7] C. Cuny and M. Peligrad. Central limit theorem started at a point for additive functional of reversible Markov Chains. *J. Theoret. Probab.* **25** (2012) 171–188. [MR2886384](#)
- [8] C. Cuny and D. Volný. A quenched invariance principle for stationary processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **10** (2013) 107–115. [MR3083921](#)
- [9] J. Dedecker, S. Gouëzel and F. Merlevède. Some almost sure results for unbounded functions of intermittent maps and their associated Markov chains. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010) 796–821. [MR2682267](#)
- [10] J. Dedecker and F. Merlevède. Necessary and sufficient conditions for the conditional central limit theorem. *Ann. Probab.* **30** (2002) 1044–1081. [MR1920101](#)
- [11] J. Dedecker and E. Rio. On the functional central limit theorem for stationary processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **36** (2000) 1–34. [MR1743095](#)
- [12] Y. Derriennic and M. Lin. The central limit theorem for Markov chains with normal transition operators, started at a point. *Probab. Theory Related Fields* **119** (2001) 508–528. [MR1826405](#)
- [13] Y. Derriennic and M. Lin. The central limit theorem for Markov chains started at a point. *Probab. Theory Related Fields* **125** (2003) 73–76. [MR1952457](#)
- [14] P. Doukhan, P. Massart and E. Rio. The functional central limit theorem for strongly mixing processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **30** (1994) 63–82. [MR1262892](#)
- [15] O. Durieu. Independence of four projective criteria for the weak invariance principle. *ALEA Lat. Am. J. Probab. Math. Stat.* **5** (2009) 21–26. [MR2475604](#)
- [16] O. Durieu and D. Volný. Comparison between criteria leading to the weak invariance principle. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (2008) 324–340. [MR2446326](#)
- [17] C. G. Esseen and S. Janson. On moment conditions for normed sums of independent variables and martingale differences. *Stochastic Process. Appl.* **19** (1985) 173–182.
- [18] M. I. Gordin. The central limit theorem for stationary processes. *Soviet Math. Dokl.* **10** (1969) 1174–1176.

- [19] M. I. Gordin. Abstracts of communication, T.1: A-K. In *International Conference on Probability Theory, Vilnius*, 1973.
- [20] M. I. Gordin and B. A. Lifšic. The central limit theorem for stationary Markov processes. *Soviet Math. Dokl.* **19** (1978) 392–394.
- [21] S. Gouëzel. Central limit theorem and stable laws for intermittent maps. *Probab. Theory Related Fields* **128** (2004) 82–122. [MR2027296](#)
- [22] C. C. Heyde and B. M. Brown. On the departure from normality of a certain class of martingales. *Ann. Math. Statist.* **41** (1970) 2161–2165. [MR0293702](#)
- [23] U. Krengel. *Ergodic Theorems. de Gruyter Studies in Mathematics* **6**. de Gruyter, Berlin, 1985. [MR0797411](#)
- [24] M. Maxwell and M. Woodroffe. Central limit theorem for additive functionals of Markov chains. *Ann. Probab.* **28** (2000) 713–724. [MR1782272](#)
- [25] F. Merlevède, M. Peligrad and S. Utev. Recent advances in invariance principles for stationary sequences. *Probab. Surv.* **3** (2006) 1–36. [MR2206313](#)
- [26] F. Merlevède, C. Peligrad and M. Peligrad. Almost sure invariance principles via martingale approximation. *Stochastic Process. Appl.* **122** (2012) 170–190. [MR2860446](#)
- [27] F. Merlevède and E. Rio. Strong approximation of partial sums under dependence conditions with application to dynamical systems. *Stochastic Process. Appl.* **122** (2012) 386–417. [MR2860454](#)
- [28] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability. Communications and Control Engineering Series*. Springer-Verlag, London, 1993. [MR1287609](#)
- [29] M. Peligrad and S. Utev. Central limit theorem for stationary linear processes. *Ann. Probab.* **34** (2006) 1608–1622. [MR2257658](#)
- [30] Y. Pomeau and P. Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.* **74** (1980) 189–197. [MR0576270](#)
- [31] M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proc. Natl. Acad. Sci. USA* **42** (1956) 43–47. [MR0074711](#)
- [32] Ja. G. Sinaĭ. A weak isomorphism of transformations with invariant measure. *Dokl. Akad. Nauk SSSR* **147** (1962) 797–800. [MR0161960](#)
- [33] D. Volný and P. Samek. On the invariance principle and the law of iterated logarithm for stationary processes. In *Mathematical Physics and Stochastic Analysis* 424–438. World Sci. Publishing, River Edge, 2000. [MR1893125](#)
- [34] D. Volný and M. Woodroffe. An example of non-quenched convergence in the conditional central limit theorem for partial sums of a linear process. In *Dependence in Analysis, Probability and Number Theory (The Phillip memorial volume)* 317–323. Kendrick Press, Heber City, UT, 2010. [MR2731055](#)
- [35] D. Volný and M. Woodroffe. Quenched central limit theorems for sums of stationary processes. Preprint, 2010. Available at [arXiv:1006.1795](#).
- [36] W.-B. Wu and M. Woodroffe. Martingale approximations for sums of stationary processes. *Ann. Probab.* **32** (2004) 1674–1690. [MR2060314](#)
- [37] O. Zhao and M. Woodroffe. Law of the iterated logarithm for stationary processes. *Ann. Probab.* **36** (2008) 127–142. [MR2370600](#)