Brazilian Journal of Probability and Statistics
2014, Vol. 28, No. 3, 301–312
DOI: 10.1214/12-BJPS208

A criterion for the fuzzy set estimation of the density function

Jesús A. Fajardo

Universidad de Oriente

Abstract. In this paper we propose a criterion to estimate the density function by means of a nonparametric and fuzzy set estimator, based on n i.i.d random variable, obtaining a reduction of the integrated mean square error of the fuzzy set estimator regarding the integrated mean squared error of the classic kernel estimators. This reduction shows that the fuzzy set estimator has better performance than the kernel estimations. Also, the convergence rate of the optimal scaling factor is computed, which coincides with the convergence rate in classic kernel estimation. Finally, these theoretical findings are illustrated using a numerical example.

1 Introduction

The methods of kernel estimation are among the nonparametric methods commonly used to estimate the density function f of a random variable X, with independent samples. Nevertheless, through the theory of point processes [see e.g Reiss (1993)] we can obtain a new nonparametric estimation method. For example, the method of fuzzy set estimation introduced by Fajardo et al. (2012), which is a particular case of the method introduced by Falk and Liese (1998), is based on defining a fuzzy set estimator of the density function by means of thinned point processes [see e.g Reiss (1993), Section 2.4].

In this paper we estimate the density function by means of the nonparametric and fuzzy set estimator introduced by Fajardo et al. (2012). With the implementation of this new estimator, we can obtain a significant reduction of the integrated mean square error of the fuzzy set estimator regarding the classic kernel estimators, which implies that the fuzzy set estimator has better performance than the kernel estimations. Also, the convergence rate of the optimal scaling factor is computed, which coincides with the convergence rate in classic kernel estimation of the density function. Moreover, the function that minimizes the integrated mean square error of the fuzzy set estimator is obtained. Finally, these theoretical findings are illustrated using a numerical example estimating a density function with the fuzzy set estimator and the classic kernel estimators.

This paper is organized as follows. In Section 2, we define the fuzzy set estimator of the density function and we present its properties of convergence. In

Key words and phrases. Density estimation, fuzzy set estimation, nonparametric estimation.
Received June 2012; accepted August 2012.

Section 3, we obtain the mean square error of the fuzzy set estimator of the density function, Theorem 4, as well as the optimal scale factor and the integrated mean square error. Moreover, we establish the conditions to obtain a reduction of the constants that control the bias and the asymptotic variance regarding the classic kernel estimators, the function that minimizes the integrated mean square error of the fuzzy set estimator is also obtained. In Section 4 a simulation study was conducted to compare the performances of the fuzzy set estimator with the classical kernel estimators. Concluding remarks are given in Section 5. Appendix contains the proof of the theorem in Section 3.

2 Fuzzy set estimator of the density function and its convergence properties

In this section we define through thinned point processes a nonparametric and fuzzy set estimator of the density function, obtaining a particular case of estimator introduced by Falk and Liese (1998). Moreover, we present its properties of convergence.

The method of fuzzy set estimation introduced by Falk and Liese (1998) is based on defining a fuzzy set estimator of the density function by means of thinned point processes, a process framed inside the theory of the point processes, which is given by

$$\hat{\theta}_n = \frac{1}{na_n}\sum_{i=1}^{n} U_i,$$

where $a_n > 0$ is a scaling factor (or bandwidth) such that $a_n \to 0$ as $n \to \infty$, and the random variables U_i, $1 \leq i \leq n$, are independent with values in $\{0, 1\}$, which decides whether X_i belongs to the neighborhood of x_0 or not. Here x_0 is the point of estimation. In Falk and Liese (1998) only the asymptotic efficiency within the class of all estimators that are based on randomly selected points from the sample $X_1, \ldots, X_n$ was proved. Efficiency was established using LeCam's LAN theory. Although the almost sure, and uniform convergence properties over compact subset on $\mathbb{R}$ are not studied, the pointwise convergence in law whose distribution limit has a asymptotic variance that depends only of $f(x_0)$ is proposed. On the other hand, we observe that the random variables that define the estimator $\hat{\theta}_n$ do not possess, for example, precise functional characteristics in regards to the point of estimation. This absence of functional characteristics complicates the evaluation of the estimator using a sample. Thus, the simulations to estimate the density function will be more complicated. To overcome the difficulties presented by the estimator $\hat{\theta}_n$ we will introduce a new fuzzy set estimator of the density function, which is a particular case of the estimator $\hat{\theta}_n$.

Let X be a real random variable whose distribution $\mathcal{L}(X)$ has density f regarding the Lebesgue measure. For each measurable Borel function $\varphi : \mathbb{R} \to [0, 1]$ and each random variable V, uniformly on [0, 1] distributed and independent of X, the random variable $U = \mathbb{1}_{[0,\varphi(X)]}(V)$ satisfies $\varphi(x) = \mathbb{P}(U = 1|X = x)$. This simple observation allows us to construct a fuzzy set estimator of the density function f that satisfies the conditions required in Falk and Liese (1998).

Let $X_1, \dots, X_n$ be an independent random sample of X. Let $V_1, \dots, V_n$ be independent random variables uniformly on [0, 1] distributed and independent of $X_1, \dots, X_n$. Let $f_{x_0,b_n}(X_i, V_i) = \mathbb{1}_{I_i}(V_i)$ be random variables where $I_i = [0, \varphi(\frac{X_i - x_0}{b_n})]$ and $b_n > 0$ is a scaling factor (or bandwidth) such that $b_n \to 0$ as $n \to \infty$. For each $x \in \mathbb{R}$, we have

$$\varphi\left(\frac{x - x_0}{b_n}\right) = \mathbb{P}(f_{x_0,b_n}(X_i, V_i) = 1|X_i = x),$$

then $\varphi_n(x) = \varphi(\frac{x - x_0}{b_n})$ is a Markov kernel [see Reiss (1993), Section 1.4]. Thus, for independent copies (X_i, V_i), $1 \le i \le n$, of (X, V), we can define the thinned point process

$$N_n^{\varphi_n}(\cdot) = \sum_{i=1}^{n} f_{x_0,b_n}(X_i, V_i)\varepsilon_{X_i}(\cdot),$$

with underlying point process $N_n(\cdot) = \sum_{i=1}^{n} \varepsilon_{X_i}(\cdot)$ and a thinning function φ_n [see Reiss (1993), Section 2.4], where ε_X is the random Dirac measure.

Remark 1. The events $\{X_i = x\}$, $x \in \mathbb{R}$, can be described in a neighborhood of x_0 through the thinned point process $N_n^{\varphi_n}$, where $f_{x_0,b_n}(X_i, V_i)$ decides, whether X_i belongs to the neighborhood of x_0 or not. Precisely, $\varphi_n(x)$ is the probability that the observation $X_i = x$ belongs to the neighborhood of x_0. Note that this neighborhood is not explicitly defined, but it is actually a fuzzy set in the sense of Zadeh (1965), given its membership function φ_n. The thinned process $N_n^{\varphi_n}$ is therefore a fuzzy set representation of the data [see Falk and Liese (1998), Section 2].

Next, we present the fuzzy set estimator of the density function introduced in Fajardo et al. (2012), which is a particular case of the estimator proposed by Falk and Liese (1998).

Definition 1. Let φ be such that $a_n = b_n \int \varphi(x)\,dx$ and $0 < \int \varphi(x)\,dx < \infty$. Then the fuzzy set estimator of the density function f at the point $x_0 \in \mathbb{R}$ is defined as

$$\hat{\vartheta}_n(x_0) = \frac{1}{na_n} \sum_{i=1}^{n} f_{x_0,b_n}(X_i, V_i) = \frac{\tau_n(x_0)}{na_n}.$$

Remark 2. The estimator $\hat{\vartheta}_n$ can be written in terms of a fuzzy set representation of the data, since $\hat{\vartheta}_n = (na_n)^{-1} N_n^{\varphi_n}(\mathbb{R})$. This equality justifies the fuzzy term of the estimator proposed where the thinning function φ_n can be used to select points of the sample with different probabilities, in contrast to the kernel estimator, which assigns equal weight to all points of the sample. Moreover, we can observe that $\hat{\vartheta}_n$ is of easy practical implementation and the random variable $\tau_n(x_0)$ is binomial $\mathcal{B}(n, \alpha_n(x_0))$ distributed with

$$\alpha_n(x_0) = \mathbb{E}\big[f_{x_0,b_n}(X_i, V_i)\big] = \mathbb{P}\big(f_{x_0,b_n}(X_i, V_i) = 1\big) = \mathbb{E}\big[\varphi_n(X)\big]. \tag{2.1}$$

In what follows we assume that $\alpha_n(x_0) \in (0, 1)$.

Consider the following conditions:

(C1) The density function f is at least twice continuously differentiable in a neighborhood of x_0.
(C2) Sequence b_n satisfies: $b_n \to 0$ and $\frac{nb_n}{\log(n)} \to \infty$ as $n \to \infty$.
(C3) Function φ is symmetrical regarding zero, has compact support on $[-B, B]$, $B > 0$, and it is continuous at $x = 0$ with $\varphi(0) > 0$.
(C4) $nb_n^5 \to 0$ as $n \to \infty$.
(C5) Function $\varphi(\cdot)$ is monotone on the positives.
(C6) $b_n \to 0$ and $\frac{nb_n^2}{\log(n)} \to \infty$ as $n \to \infty$.
(C7) Density function f is at least twice continuously differentiable on the compact set $[-B, B]$.

Next, we present the convergence properties obtained by Fajardo et al. (2012):

Theorem 1. *Under conditions* (C1)–(C3), *we have*

$$\hat{\vartheta}_n(x_0) \to f(x_0) \qquad a.s.$$

Theorem 2. *Under conditions* (C1)–(C4), *we have*

$$\sqrt{na_n}\big(\hat{\vartheta}_n(x_0) - f(x_0)\big) \xrightarrow{\mathcal{L}} N\big(0, f(x_0)\big).$$

The "$\xrightarrow{\mathcal{L}}$" symbol denotes convergence in law.

Theorem 3. *Under conditions* (C3) *and* (C5)–(C7), *we have*

$$\sup_{a\in[-B,B]} \big|\hat{\vartheta}_n(a) - f(a)\big| = o_{\mathbb{P}}(1).$$

Remark 3. The estimator $\hat{\vartheta}_n$ has a limit distribution whose asymptotic variance depends only on the point of estimation, this does not hold to the kernel estimator. However, since $a_n = o(n^{-1/5})$ we see that the same restrictions are imposed for the smoothing parameter of the kernel estimators.

3 Statistical methodology

In this section we will obtain the mean square error of $\hat{\vartheta}_n$, as well as the optimal scale factor and the integrated mean square error. Moreover, we establish the conditions to obtain a reduction of the constants that control the bias and the asymptotic variance regarding to the classic kernel estimators. The function that minimizes the integrated mean square error of $\hat{\vartheta}_n$ is also obtained.

The following theorem provides the asymptotic representation for the mean square error (MSE) of $\hat{\vartheta}_n$.

Theorem 4. *Under conditions* (C1)–(C3), *we have*

$$\mathrm{MSE}[\hat{\vartheta}_n(x)] = \frac{f(x)}{nb_n}\frac{1}{\int \varphi(u)\,du} + b_n^4\left[\frac{f''(x)}{2}\int u^2\psi(u)\,du\right]^2 + o\left(\frac{1}{nb_n} + b_n^4\right),$$

where

$$\psi(x) = \frac{\varphi(x)}{\int \varphi(u)\,du}.$$

Next, we calculate the formula for the optimal asymptotic scale factor b_n^* to perform the estimation. The integrated mean square error (IMSE) of $\hat{\vartheta}_n$ is given by

$$\mathrm{IMSE}[\hat{\vartheta}_n] = \frac{1}{nb_n}\frac{1}{\int \varphi(u)\,du} + \frac{b_n^4}{4}\left[\int u^2\psi(u)\,du\right]^2 \int [f''(x)]^2\,dx. \tag{3.1}$$

From (3.1), we obtain the following formula for the optimal asymptotic scale factor

$$b_{n_\varphi}^* = \left[\frac{1}{n\int \varphi(u)\,du\,[\int u^2\psi(u)\,du]^2 \int [f''(u)]^2\,du}\right]^{1/5}. \tag{3.2}$$

We obtain a scaling factor of order $n^{-1/5}$, which implies a rate of optimal convergence for the $\mathrm{IMSE}^*[\hat{\vartheta}_n]$ of order $n^{-4/5}$. We observe that the optimal scaling factor order for the method of fuzzy set estimation coincides with the order of the classic kernel estimate. Moreover,

$$\mathrm{IMSE}^*[\hat{\vartheta}_n] = n^{-4/5}C_\varphi, \tag{3.3}$$

where

$$C_\varphi = \frac{5}{4}\left[\frac{[\int u^2\psi(u)\,du]^2 \int [f''(u)]^2\,du}{[\int \varphi(u)\,du]^4}\right]^{1/5}.$$

Next, we will establish the conditions to obtain a reduction of the constants that control the bias and the asymptotic variance regarding the classic kernel estimators. For it, we will consider the usual kernel density estimator

$$\hat{f}_{n_K}(x) = \frac{1}{nb_n} \sum_{i=1}^{n} K\left(\frac{x - X_i}{b_n}\right),$$

which has the mean squared error

$$\begin{aligned}\mathrm{MSE}\big[\hat{f}_{n_K}(x)\big] = {}& \frac{f(x)}{nb_n} \int K^2(u)\,du + b_n^4 \left[\frac{f''(x)}{2} \int u^2 K(u)\,du\right]^2 \\ & + o\left(\frac{1}{nb_n} + b_n^4\right).\end{aligned}$$

Moreover, the IMSE of $\hat{f}_{n_K}$ is given by

$$\mathrm{IMSE}[\hat{f}_{n_K}] = \frac{1}{nb_n} \int K^2(u)\,du + \frac{b_n^4}{4} \left[\int u^2 K(u)\,du\right]^2 \int \big[f''(u)\big]^2\,du. \qquad (3.4)$$

From (3.4), we obtain the following formula for the optimal asymptotic scale factor

$$b^*_{n_K} = \left[\frac{1}{n} \frac{\int K^2(u)\,du}{[\int u^2 K(u)\,du]^2 \int [f''(u)]^2\,du}\right]^{1/5}. \qquad (3.5)$$

Moreover,

$$\mathrm{IMSE}^*[\hat{f}_{n_K}] = n^{-4/5} C_K,$$

where

$$C_K = \frac{5}{4} \left[\left[\int K^2(u)\,du\right]^4 \left[\int u^2 K(u)\,du\right]^2 \int \big[f''(u)\big]^2\,du\right]^{1/5}.$$

The reduction of the constants that control the bias and the asymptotic variance, regarding the classic kernel estimators, are obtained if for all kernel K

$$\int \varphi(u)\,du \geq \left[\int K^2(u)\,du\right]^{-1} \quad \text{and} \quad \int u^2 \psi(u)\,du \leq \int u^2 K(u)\,du.$$

Remark 4. The conditions on φ allows us to obtain a value of B such that

$$\int_{-B}^{B} \varphi(u)\,du > \left[\int K^2(u)\,du\right]^{-1}.$$

Moreover, to guarantee that

$$\int u^2 \psi(u)\,du \leq \int u^2 K(u)\,du,$$

we define the function

$$\psi(x) = \frac{\varphi(x)}{\int \varphi(u)\, du}$$

with compact support on $[-B', B'] \subset [B, B]$. Next, we guarantee the existence of B'. As

$$\frac{1}{\int \varphi(u)\, du} < \int K^2(u)\, du \quad \text{and} \quad \varphi(x) \in [0, 1],$$

we have

$$x^2 \psi(x) \leq x^2 \Big(\int K^2(u)\, du \Big). \tag{3.6}$$

Observe that for each $C \in (0, \int u^2 K(u)\, du]$ exists

$$B' = \sqrt[3]{\frac{3C}{2 \int K^2(u)\, du}}$$

such that

$$C = \int_{-B'}^{B'} \Big(\int K^2(u)\, du \Big) x^2\, dx \leq \int u^2 K(u)\, du. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\int_{-B'}^{B'} u^2 \psi(u)\, du \leq \int u^2 K(u)\, du.$$

In our case we take $B' \leq B$.

On the other hand, the criterion that we will implement to minimizing (3.3) and obtain a reduction of the constants that control the bias and the asymptotic variance regarding the classic kernel estimation, is the following

$$\text{Maximizing} \qquad \int \varphi(u)\, du$$

subject to the conditions

$$\int \varphi^2(u)\, du = \frac{5}{3}, \qquad \int u\varphi(u)\, du = 0, \qquad \int (u^2 - v)\varphi(u)\, du = 0,$$

with $u \in [-B, B]$, $\varphi(u) \in [0, 1]$, $\varphi(0) > 0$ and $v \leq \int u^2 K_E(u)\, du$, where K_E is the Epanechnikov kernel

$$K_E(x) = \tfrac{3}{4}(1 - x^2)\mathbb{1}_{[-1,1]}(x).$$

The Euler–Lagrange equation with these constraints is

$$\frac{\partial}{\partial \varphi}[\varphi + a\varphi^2 + bx\varphi + c(x^2 - v)\varphi] = 0,$$

where a, b and c the three multipliers corresponding to the three constraints. This yields

$$\varphi(x) = \left[1 - \left(\frac{16x}{25}\right)^2\right] \mathbb{1}_{[-25/16, 25/16]}(x). \tag{3.8}$$

The new conditions on φ, allows us to affirm that for all kernel K

$$\text{IMSE}^*[\hat{\vartheta}_n] \leq \text{IMSE}^*[\hat{f}_{n_K}].$$

Thus, the fuzzy set estimator has the best performance.

4 Simulations results

A simulation study was conducted to compare the performances of the fuzzy set estimator with the classical kernel estimators. For the simulation, we used the density function

$$f(x) = \begin{cases} \frac{15}{32}[x(x+2)]^2 & \text{if } -2 \leq x \leq 0, \\ \frac{15}{32}[x(x-2)]^2 & \text{if } 0 \leq x \leq 2. \end{cases}$$

In this way, we generated samples of size 100, 250 and 500. The bandwidths was computed using (3.2) and (3.5). The fuzzy set estimator and the kernel estimations were computed using (3.8), and the Epanechnikov and Gaussian kernel functions. The IMSE* values of the fuzzy set estimator and the kernel estimators are given in Table 1.

As seen from Table 1, for all sample sizes, the fuzzy set estimator using varying bandwidths have smaller IMSE* values than the kernel estimators with fixed and different bandwidth for each estimator. In each case, it is seen that the fuzzy set

Table 1 IMSE* *values of the estimations for the fuzzy set estimator and the kernel estimators*

v	n	$\text{IMSE}^*[\hat{\vartheta}_n]$	$\text{IMSE}^*[\hat{f}_{n_{K_E}}]$	$\text{IMSE}^*[\hat{f}_{n_{K_G}}]$
0.2	100	0.0149*	0.0178	0.0185
	250	0.0071*	0.0085	0.0089
	500	0.0041*	0.0049	0.0051
0.15	100	0.0133*	0.0178	0.0185
	250	0.0064*	0.0085	0.0089
	500	0.0037*	0.0049	0.0051
0.10	100	0.0113*	0.0178	0.0185
	250	0.0054*	0.0085	0.0089
	500	0.0031*	0.0049	0.0051

Minimum IMSE in each row

estimator has the best performance. Moreover, we see that the kernel estimation computed using the Epanechnikov kernel function shows a better performance than the estimations computed using the Gaussian kernel function.

The graphs of the real density function and the estimations of the density functions are computed over a sample of 500, using 200 points and $v = 0.2$, are illustrated in Figures 1 and 2.

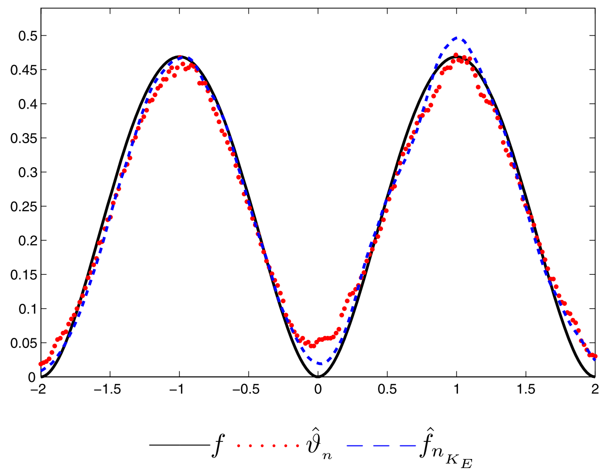

Figure 1 *Estimation of f with $\hat{\vartheta}_n$ and $\hat{f}_{n_{K_E}}$.*

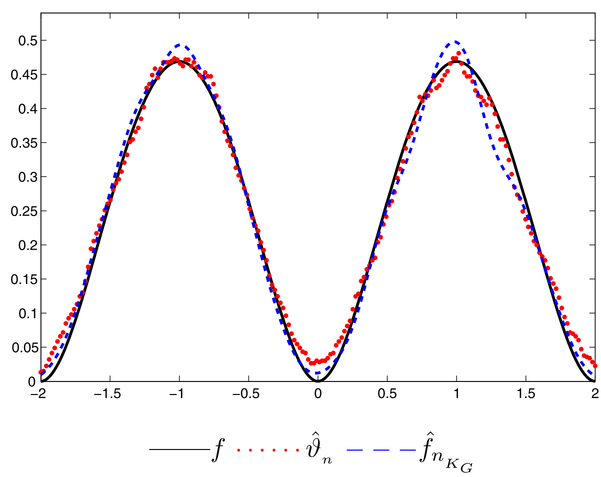

Figure 2 *Estimation of f with $\hat{\vartheta}_n$ and $\hat{f}_{n_{K_G}}$.*

5 Concluding remarks

In this paper, we have presented a criterion to estimate the density function by means of the fuzzy set estimator introduced by Fajardo et al. (2012). In practice, the method depends on a scaling factor and a membership function, in the context of the Fuzzy Set Theory. This shows a parallelism with the classic kernel estimator, which depends on a bandwidth and a kernel. It is proved that the new estimator has the same asymptotic behavior than the classic kernel estimator. The optimal scaling factor is of order $n^{-1/5}$ and the rate of optimal convergence for the IMSE* is of order $n^{-4/5}$. Nevertheless, the results showed that the thinning function or membership function can be taken in such a way that the IMSE* is less than that of the classic kernel estimators. This reduction allows us to affirm that the fuzzy set estimator provides better estimates than the classic kernel estimators. Moreover, it is important to emphasize that the thinning function can be used to select points of the sample with different probabilities, in contrast to the classic kernel estimators, which assigns equal weight to all points of the sample.

Appendix: Proof of Theorem 4

Let us consider the following decomposition

$$\mathrm{MSE}[\hat{\vartheta}_n(x)] = \mathrm{Var}[\hat{\vartheta}_n(x)] + \big(\mathbb{E}[\hat{\vartheta}_n(x) - f(x)]\big)^2.$$

Next, we will present two equivalent expressions for the terms to the right in the above decomposition. The combination of Definition 1 and (2.1), allows us to write

$$\mathrm{Var}[\hat{\vartheta}_n(x)] = \frac{1}{nb_n^2(\int \varphi(u)\,du)^2}\alpha_n(x)\big(1-\alpha_n(x)\big).$$

Now, if we combine condition (C1), which allows us to make a Taylor expansion of the density function f on the neighborhood of x_0, with condition (C3), we can write (2.1) as follows

$$\alpha_n(x) = b_n \int \varphi(u)\,du \left\{ f(x) + \frac{b_n^2 \int u^2 \varphi(u) f''(x+\beta u b_n)\,du}{2\int \varphi(u)\,du} \right\}, \tag{A.1}$$

where $\beta \in (0, 1)$. Moreover, conditions (C1) and (C3) imply that

$$\int u^2 \varphi(u)[f''(x+\beta u b_n)\,du - f''(x)]\,du = o(1). \tag{A.2}$$

Now, we can write (A.1) as

$$\alpha_n(x) = b_n \int \varphi(u)\,du \left\{ f(x) + \frac{b_n^2 (f''(x) \int u^2 \varphi(u)\,du + o(1))}{2\int \varphi(u)\,du} \right\}.$$

Thus,

$$\operatorname{Var}[\hat{\vartheta}_n(x)] = \frac{f(x)}{nb_n}\frac{1}{\int \varphi(u)} + \frac{b_n(f''(x)\int u^2\varphi(u)\,du + o(1))}{n\int \varphi(u)} + \frac{b_n^2}{n}\left(f(x) + \frac{b_n^2(f''(x)\int u^2\varphi(u)\,du + o(1))}{2\int \varphi(u)\,du}\right)^2.$$

By condition (C2)

$$\operatorname{Var}[\hat{\vartheta}_n(x)] = \frac{f(x)}{nb_n}\frac{1}{\int \varphi(u)} + o\left(\frac{1}{nb_n}\right). \tag{A.3}$$

On the other hand, (2.1) and (A.1) imply that

$$\mathbb{E}[\hat{\vartheta}_n(x) - f(x)] = \frac{b_n^2}{2\int \varphi(u)\,du}\int u^2\varphi(u) f''(x + \beta u b_n)\,du.$$

By (A.2)

$$b_n^2 \int u^2\varphi(u)[f''(x + \beta u b_n)\,du - f''(x)]\,du = o(b_n^2).$$

Thus,

$$\mathbb{E}[\hat{\vartheta}_n(x) - f(x)] = \frac{b_n^2}{2\int \varphi(u)\,du} f''(x)\int u^2\varphi(u)\,du + o(b_n^2).$$

Therefore,

$$\left(\mathbb{E}[\hat{\vartheta}_n(x) - f(x)]\right)^2 = b_n^4\left(\frac{f''(x)\int u^2\varphi(u)\,du}{2\int \varphi(u)\,du}\right)^2 + o(b_n^4). \tag{A.4}$$

The assertion now follows from (A.3) and (A.4).

Acknowledgments

The author wants to especially thank the referees for their valuable suggestions and revisions. I also thank Henry Lezama for proofreading and editing the english text.

References

Fajardo, J., Ríos, R. and Rodríguez, L. (2012). Properties of convergence of an fuzzy set estimator of the density function. *Brazilian Journal of Probability and Statistics* **26**, 208–217. MR2880906

Falk, M. and Liese, F. (1998). Lan of thinned empirical processes with an application to fuzzy set density estimation. *Extremes* **1**, 323–349. MR1814708

Reiss, R. D. (1993). *A Course on Point Processes*. *Springer Series in Statistics*. New York: Springer. MR1199815

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control* **8**, 338–353. MR0219427

Universidad de Oriente
Núcleo de Sucre
Escuela de Ciencias
Departamento de Matemáticas
Venezuela
Cumaná 6101
E-mail: jfajardogonzalez@gmail.com