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Hausdorff dimension of visible sets for well-behaved continuum percolation in the hyperbolic plane

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Abstract. Let $\mathcal{Z}$ be a so-called well-behaved percolation, that is, a certain random closed set in the hyperbolic plane, whose law is invariant under all isometries; for example, the covered region in a Poisson Boolean model. In terms of the α-value of $\mathcal{Z}$, the Hausdorff-dimension of the set of directions is determined in which visibility from a fixed point to the ideal boundary of the hyperbolic plane is possible within $\mathcal{Z}$. Moreover, the Hausdorff-dimension of the set of (hyperbolic) lines through a fixed point contained in $\mathcal{Z}$ is calculated. Thereby several conjectures raised by Benjamini, Jonasson, Schramm and Tykesson are confirmed.

1 Introduction and main result

In this note, we are interested in the fractal stochastic geometry of some well-behaved percolation models in the hyperbolic plane. Percolation in the hyperbolic plane has been considered by several authors and became an active field of research, see Benjamini et al. (2009), Benjamini and Schramm (2001), Calka and Tykesson (2011), Lalley (2011), Tykesson (2007) to name just a few. Background material on hyperbolic geometry may be found in Benedetti and Petrino (2008), Ramsay and Richtmyer (2010) and some aspects of percolation theory in the Euclidean spaces is presented in Meester and Roy (1996).

Our focus here is on the set of hyperbolic lines (bi-infinite geodesic rays) and half-lines (infinite geodesic rays) contained in the unbounded connected components of a class of continuum percolation models $\mathcal{Z}$ in the hyperbolic plane. Of course, similar problems can also be treated in higher dimensional hyperbolic spaces $\mathbb{H}^d$ or the d-dimensional Euclidean space $\mathbb{R}^d$. However, it has been shown (see Benjamini et al. (2009), Calka, Michel and Porret-Blanc (2010)) that for example 2-dimensional planes that are contained in $\mathcal{Z}$ do not exist for well behaved-percolation in $\mathbb{H}^d$ for any $d \geq 3$. Moreover, visibility to infinity in $\mathbb{R}^d$ is impossible even for $d \geq 2$. For this reason, we restrict our attention to the hyperbolic plane $\mathbb{H}^2$—in this paper we work with the standard Poincaré disc model, which is equipped with the usual hyperbolic metric $\varrho_{\mathbb{H}^2}$.

Key words and phrases. Boolean model, continuum percolation, fractal geometry, Hausdorff-dimension, hyperbolic geometry.

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To state our results, which confirm several conjectures raised by Benjamini, Jonasson, Schramm and Tykesson (Benjamini et al. (2009)), let $B(1) \subset \mathbb{H}$ be a closed disc of radius 1. A random closed set $\mathcal{Z}$ in $\mathbb{H}^2$ is called a *well-behaved percolation* if the following assumptions are satisfied (see Benjamini et al. (2009)):

(i) The law of $\mathcal{Z}$ is invariant under all isometries of $\mathbb{H}^2$.

(ii) For any two bounded increasing measurable functions g and h of $\mathcal{Z}$, the FKG-type inequality

$$\mathbb{E}\big[g(\mathcal{Z})h(\mathcal{Z})\big] \geq \mathbb{E}\big[g(\mathcal{Z})\big]\mathbb{E}\big[h(\mathcal{Z})\big]$$

is satisfied.

(iii) There is some $R_0 < \infty$ such that $\mathcal{Z}$ satisfies independence at distance R_0. That is, for every two subsets $A, B \subset \mathbb{H}^2$ with $\inf\{\varrho_{\mathbb{H}^2}(a,b) : a \in A, b \in B\} \geq R_0$, the events $\mathcal{Z} \cap A$ and $\mathcal{Z} \cap B$ are independent.

(iv) The expected number of connected components of $B(1) \setminus \mathcal{Z}$ is finite.

(v) We have $\mathbb{E}[\text{length}(B(1) \cap \partial\mathcal{Z})] < \infty$.

(vi) We have $\mathbb{P}(B(1) \subset \mathcal{Z}) > 0$.

We denote by $f(r)$ the probability that a fixed line segment of length $r > 0$ is contained in $\mathcal{Z}$ and fix some point $o \in \mathbb{H}^2$. We recall from Benjamini et al. (2009), Lemma 3.4, that there exists a unique $\alpha \geq 0$, called the *α-value of $\mathcal{Z}$*, such that $f(r) = \Theta(e^{-\alpha r})$ for any $r \geq 0$ in the usual Landau notation, that is, $f(r)$ is bounded from above and below by a constant multiple of $e^{-\alpha r}$. In terms of its α-value, the Hausdorff-dimension of several random sets related to a well-behaved percolation $\mathcal{Z}$ in $\mathbb{H}^2$ can be determined:

Theorem 1. *Consider a well-behaved percolation $\mathcal{Z}$ in $\mathbb{H}^2$ and a fixed point $o \in \mathbb{H}^2$. Let $\mathfrak{V}$ denote the set of points z in the ideal boundary $\partial\mathbb{H}^2$ of the hyperbolic plane such that the ray $[o,z)$ is contained in $\mathcal{Z}$. If $\alpha \geq 1$ then $\mathfrak{V} = \emptyset$ with probability one. If $\alpha < 1$ then $\mathbb{P}(\mathfrak{V} \neq \emptyset) > 0$ and $\dim_H \mathfrak{V} = 1 - \alpha$ almost surely on $\mathfrak{V} \neq \emptyset$. Moreover, the union of all these rays has Hausdorff-dimension $2 - \alpha$ almost surely on $\mathfrak{V} \neq \emptyset$.*

Theorem 2. *For a well-behaved percolation $\mathcal{Z}$ in the hyperbolic plane $\mathbb{H}^2$ and fixed $o \in \mathbb{H}^2$ we have: If $\alpha \geq 1/2$ then there is no line through o contained in $\mathcal{Z}$ almost surely. If $\alpha < 1/2$ then the union of all lines in $\mathcal{Z}$ through o has Hausdorff-dimension $2 - 2\alpha$ with probability one conditioned on the event that there are such lines.*

The following random sets are *examples* to which our theory applies (see Benjamini et al. (2009), Calka and Tykesson (2011)):

Example 3. Let η_λ be an isometry-invariant Poisson point process of intensity $\lambda \in (0, \infty)$ in $\mathbb{H}^2$, $R > 0$ and define

$$\mathcal{B} := \bigcup_{x \in \eta_\lambda} B(x, R) \quad \text{and} \quad \mathcal{V} := \overline{\mathbb{H}^2 \setminus \mathcal{B}}.$$

Then, $\mathcal{B}$ and $\mathcal{V}$, the occupied and the vacant phase of the Boolean model with respect to η_λ and R, are well-behaved percolation sets. The α-value for $\mathcal{V}$ is given by $\alpha = 2\lambda \sinh R$ and the α-value for $\mathcal{B}$ is the unique solution of

$$\int_0^{2R} e^{\alpha t} H_{\lambda,R}(t)\, dt = 1,$$

where

$$H_{\lambda,R}(t) = -\exp\Big(-4\lambda \int_0^{t/2} \sinh\Big(\cosh^{-1}\Big(\frac{\cosh R}{\cosh s}\Big)\Big)\, ds\Big).$$

Example 4. Let η_λ be as above but consider the radius R of the balls in the Boolean model as random and assume that the exponential moment $\mathbb{E}[e^R]$ is finite. In this case, the α-value of the vacant phase $\mathcal{V}$ equals $\alpha = 2\lambda\mathbb{E}[\sinh R]$.

Example 5. Let η_λ be as in our previous examples and let K be a random closed convex set with a.s. finite diameter containing the origin, whose law is invariant under isometries of $\mathbb{H}^2$. We can think of $\mathbb{H}^2$ as the unit disc embedded in the complex plane and put $\varphi_x(z) = (z - x)/(1 - \overline{x}z)$, where $\bar{\cdot}$ stands for complex conjugation. Let us define

$$\mathcal{B}_K = \bigcup_{x \in \eta} \varphi_x^{-1}(K_x) \quad \text{and} \quad \mathcal{V}_K = \overline{\mathbb{H}^2 \setminus \mathcal{B}},$$

where $\{K_x : x \in \eta\}$ is an i.i.d. family of random sets indexed by the points of η_λ having the same distribution as K and are independent of η_λ. Then

$$f(r) = \exp\big(-\lambda\mathbb{E}\big[\mathrm{Area}\big(x \in \mathbb{H}^2 : \varphi_x^{-1}(K) \cap L_{o,r} \neq \varnothing\big)\big]\big)$$

for the well-behaved percolation $\mathcal{V}_K$, where $L_{o,r}$ is a line segment of length $r > 0$ starting at o. Thus, to find the α-value one has to calculate the expectation $\mathbb{E}[\mathrm{Area}(x \in \mathbb{H}^2 : \varphi_x^{-1}(K) \cap L_{o,r} \neq \varnothing)]$ that appears in the exponent.

The rest of this note is organized as follows: In Section 2, we recall some facts from fractal stochastic geometry and prove an auxiliary result on Hausdorff-dimension of random sets. In the final section, we present the proofs of our results.

2 An auxiliary result on Hausdorff dimensions of random sets

Let (E, ϱ) be a metric space, which is second countable, locally compact and has the Hausdorff property (a so-called lcscH space). Let $\mathcal{B}$ be the Borel σ-field on E

generated by ϱ, $\mathcal{F}$ be the family of closed subsets of E and let $\mathcal{M}$ be the family of Radon measures on E (recall that a *Radon measure* is a locally finite and inner regular measure on $\mathcal{B}$). We equip $\mathcal{F}$ with the σ-field $\mathfrak{F}$ generated by the usual Fell-topology [Molchanov (2005), Appendix B], on $\mathcal{F}$ and $\mathcal{M}$ with the σ-field $\mathfrak{M}$ generated by the evaluation mappings $\varphi \mapsto \varphi(B)$, $B \in \mathcal{B}$, $\varphi \in \mathcal{M}$ (cf. Chapter 1.1, Kallenberg (1983)). For $D \geq 0$ and $B \subset E$ the *D-dimensional Hausdorff-measure* $\mathcal{H}^D(B)$ is defined by

$$\mathcal{H}^D(B) := \lim_{\delta \downarrow 0} \mathcal{H}^D_\delta(B),$$

where

$$\mathcal{H}^D_\delta(B) := \inf_{\mathcal{V}} \left\{ \sum_{F \in \mathcal{V}} \varsigma(d)\big(\mathrm{diam}(F)\big)^D : B \subseteq \bigcup_{F \in \mathcal{V}} F, \mathrm{diam}(F) < \delta, F \in \mathcal{F} \right\}$$

and where the infimum is taken over all countable subfamilies $\mathcal{V}$ of $\mathcal{F}$. Moreover, we put $\varsigma(D) = \Gamma(1/2)^D / (2^D \Gamma(1 + D/2))$, where Γ denotes the usual Gamma-function. For $B \subset E$ the *Hausdorff-dimension* $\dim_H B$ of B is defined by

$$\dim_H B = \inf\{D \geq 0 : \mathcal{H}^D(B) = 0\} = \sup\{D \geq 0 : \mathcal{H}^D(B) = +\infty\}.$$

For $D \geq 0$, the $\mathcal{H}^D$*-derivative* of $\varphi \in \mathcal{M}$ at $x \in E$ is given via

$$\begin{aligned}\mathcal{D}(\varphi, D, x) &:= \limsup_{F \to x} \frac{\varphi(F)}{\varsigma(D)(\mathrm{diam}(F))^D} \\ &= \frac{1}{\varsigma(D)} \limsup_{\delta \downarrow 0} \left\{ \frac{\varphi(F)}{(\mathrm{diam}(F))^D} : x \in F, F \in \mathcal{F}, \mathrm{diam}(F) \leq \delta \right\}.\end{aligned}$$

Denote $E^{(\infty)} = E^{(\infty)}(\varphi, D) = \{x \in E : \mathcal{D}(\varphi, D, x) = +\infty\}$.

A *random measure* η on E is a $[\mathcal{M}, \mathfrak{M}]$-valued random variable defined on some probability space, cf. Kallenberg (1983). Its *second-moment measure* $\Lambda = \Lambda_\eta$ on $E \times E$ is defined by the relation

$$\Lambda\big(B \times B'\big) := \mathbb{E}\big[\eta(B)\eta\big(B'\big)\big], \qquad B, B' \in \mathcal{B}. \tag{2.1}$$

We are now in the position to rephrase a Frostman-type result, which was proved in Zähle (1984, 1988) for the special case $E = \mathbb{R}^n$. For completeness and to keep the argument below self-contained, we include a streamlined proof in our more general setting. Later on the result will be applied to subsets of the hyperbolic plane.

Proposition 1. *Let η be a random measure on E, $D \geq 0$ and $r > 0$. Suppose there exist a sequence $E_n \uparrow E$ with $E_n \in \mathcal{B}$ that satisfies*

$$\int \varrho(x, y)^{-D} \mathbf{1}[x \in E_n] \mathbf{1}\big[\varrho(x, y) < r\big] \Lambda\big(d(x, y)\big) < \infty$$

for any $n \in \mathbb{N}$. Then for $B \in \mathcal{B}$, almost surely on $\eta(B) > 0$ we have that $\dim_H B \geq D$.

Proof. The proof is divided into four steps.

Step 1: *If $\varphi \in \mathcal{M}$, then the restriction $\varphi \llcorner (E \setminus E^{(\infty)})$ of φ to $E \setminus E^{(\infty)}$ is absolutely continuous with respect to $\mathcal{H}^D$.*

Define $\psi := \varphi \llcorner (E \setminus E^{(\infty)})$, put $E^{(a)} := \{x \in E : \mathcal{D}(\varphi, D, x) \in [0, a)\}$ for $a > 0$ and let $B \subset E$ such that $\mathcal{H}^D(B) = 0$. Then 2.10.17 (3) in Federer (1969) implies

$$\psi(B) \le \psi\big(B \cap E^{(\infty)}\big) + \psi\big(B \setminus E^{(\infty)}\big) \le 0 + \lim_{a\to\infty} \varphi\big(B \cap E^{(a)}\big)$$
$$\le \lim_{a\to\infty} a\mathcal{H}^D\big(B \cap E^{(a)}\big) \le \lim_{a\to\infty} a\mathcal{H}^D(B) = 0.$$

Step 2: *If $B \subset E$, $\varphi \in \mathcal{M}$ and $\varphi(B \setminus E^{(\infty)}) > 0$, then $\dim_H B \ge D$.*

Indeed, using the result of Step 1, we see that $\varphi(B \setminus E^{(\infty)}) > 0$ implies

$$0 < \mathcal{H}^D\big(B \setminus E^{(\infty)}\big) \le \mathcal{H}^D(B).$$

Thus, $\dim_H B \ge D$, by the definition of Hausdorff-dimension.

Step 3: *If $D \ge 0$, $B \subset E$, $\varphi \in \mathcal{M}$ with $\varphi(B) > 0$ and for φ-almost all $x \in B$ there exists $r = r(x) > 0$ with $\int_{B(x,r)} \varrho(x, z)^{-D} \varphi(dz) < \infty$, where $B(x, r)$ is the ball of radius r around x, then $\dim_H B \ge D$.*

To see it, note that for φ-almost all $x \in B$ and any $F \in \mathcal{F}$ with $x \in F$ we have

$$\varsigma(D)\mathcal{D}(\varphi, D, x) = \limsup_{F\to x} \frac{\varphi(F)}{(\mathrm{diam}(F))^D} \le \limsup_{F\to x} \int_F \varrho(x, z)^{-D}\varphi(dz)$$
$$\le \int_{B(x,r)} \varrho(x, z)^{-D}\varphi(dz) < \infty.$$

Hence, $\varphi(B \cap E^{(\infty)}) = 0$ and $\varphi(B \setminus E^{(\infty)}) = \varphi(B) > 0$ and the result of Step 2 implies that $\dim_H B \ge D$.

Step 4: We use Campbell's theorem to conclude that

$$\mathbb{E} \int_{E_n} \int_{B(x,r)} \varrho(x, y)^{-D} \eta(dy)\eta(dx)$$
$$= \int \varrho(x, y)^{-D} \mathbf{1}[x \in E_n] \mathbf{1}\big[\varrho(x, y) < r\big] \Lambda\big(d(x, y)\big) < \infty.$$

This implies that for P_η-almost all η (here P_η is the distribution of η) and η-almost all $x \in \bigcup E_n = E$,

$$\int_{B(x,r)} \varrho(x, y)^{-D} \eta(dy) < \infty.$$

By the result of Step 3, we see now that, conditioned on having $\eta(B) > 0$, it almost surely holds that $\dim_H B \ge D$. □

Let us further recall that a *random closed set* in the metric space (E, ϱ) is a measurable mapping from some probability space into the measurable space $[\mathcal{F}, \mathfrak{F}]$, see Molchanov (2005).

3 Proofs

3.1 Preliminaries

We recall that $f(r)$ denotes the probability that a fixed line segment of length $r > 0$ is contained in $\mathcal{Z}$. Moreover, for some fixed point $o \in \mathbb{H}^2$, A stands for a closed half-plane with o on its boundary. We define $I := A \cap \partial B(o, 1)$. Furthermore, for $r > 1$, Y_r is the set of those $x \in I$ with the property that the line segment with endpoint o through x having length r is contained in $\mathcal{Z}$. The random set Y is defined by

$$Y = \bigcap_{n \geq 1} Y_{nR_0},$$

where, recall, R_0 is the independence distance from the definition of $\mathcal{Z}$.

In Lemma 3.6 of Benjamini et al. (2009), the following has been shown.

Proposition 2. *Let $\mathcal{Z}$ be a well-behaved percolation in $\mathbb{H}^2$. Then $f(r) \leq e^{-\alpha r}$,*

$$\mathbb{E}\big[\mathrm{length}(Y_r)\big] = \mathrm{length}(I) f(r)$$

and

$$\mathbb{P}(x, y \in Y_r) \leq f(r) f\big(r + \log \varrho_{\mathbb{H}^2}(x, y) + O(1)\big), \qquad x, y \in I.$$

For the proof of Theorem 1 we will estimate the Hausdorff-dimension $\dim_H S$ of a set S from above by its upper Minkowski-dimension $\overline{\dim}_M S$. In the case that the ambient space is the boundary ∂C of a circle C and $S \subset \partial C$, the upper Minkowski-dimension of S relative to ∂C can be defined by

$$\overline{\dim}_M S = 1 - \limsup_{\delta \to 0} \frac{\log \mathrm{length}(S(\delta))}{\log \delta}.$$

Here, $S(\delta)$ stands for the δ-parallel set of S relative to ∂C, see Falconer (2003). Let us further recall the following well known inequality between Hausdorff- and upper Minkowski-dimension:

$$\dim_H S \leq \overline{\dim}_M S.$$

3.2 Proof of Theorem 1

The case $\alpha \geq 1$: It has been shown in Lemma 3.5 of Benjamini et al. (2009) that for $\alpha \geq 1$ we have $\mathfrak{V} = \varnothing$ with probability one. We can henceforth restrict our attention to the case $\alpha < 1$, where the event $\mathfrak{V} \neq \varnothing$ has positive probability.

An upper bound for the mean: We start by observing that

$$\mathbb{E}[\overline{\dim}_M Y] \leq 1 - \limsup_{r \to \infty} \frac{\log \mathbb{E}[\mathrm{length}(Y_r)]}{-r}.$$

This is because parallel sets are taken in $I \subset B(o,1)$ and because Y_r is an almost surely decreasing family of subsets of I. We now use Proposition 2, which says that $\mathbb{E}[\text{length}(Y_r)] = \text{length}(I) f(r)$. Thus, with $f(r) \le e^{-\alpha r}$ and the inequality between Hausdorff- and (upper) Minkowski-dimension, we deduce that

$$\mathbb{E}[\dim_H Y] \le \mathbb{E}[\overline{\dim}_M Y] \le 1 - \limsup_{r\to\infty} \frac{\log[\text{length}(I) f(r)]}{-r}$$
$$\le 1 - \limsup_{r\to\infty} \frac{\log[\text{length}(I) e^{-\alpha r}]}{-r} = 1-\alpha.$$

A lower bound with positive probability: Let $\mathcal{M}_I$ be the space of Radon measure on I equipped with the weak topology and define for $n \ge 1$ the random measure ν_n by

$$d\nu_n := e^{\alpha R_0 n} \mathbf{1}[\cdot \in Y_{R_0 n}]\, dx,$$

where dx stands for the element of the Lebesgue measure I and where R_0 is the independence distance from the definition of well-behaved percolation. Obviously, $\nu_n \in \mathcal{M}_I$ and $\|\nu_n\| < \infty$ with probability one. Indeed, we have from Proposition 2, $\mathbb{E}[\|\nu_n\|] \le \text{length}(I) < \infty$, which implies $\|\nu_n\| < \infty$ almost surely. Moreover, for any Borel set $B \subset I$ we have by Markov's inequality and Proposition 2,

$$\lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(\nu_n(B) > t\big) = \lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(e^{\alpha R_0 n} \text{length}(B \cap Y_{R_0 n}) > t\big)$$
$$\le \lim_{t\to\infty} \limsup_{n\to\infty} \frac{e^{\alpha R_0 n}\mathbb{E}[\text{length}(B\cap Y_r)]}{t}$$
$$\le \lim_{t\to\infty} \limsup_{n\to\infty} \frac{e^{\alpha R_0 n}\mathbb{E}[\text{length}(Y_{R_0 n})]}{t} = 0.$$

A similar argument also shows that

$$\inf_B \limsup_{n\to\infty} \mathbb{P}\big(\nu_n(B^C) > \varepsilon\big) = 0, \qquad \varepsilon > 0,$$

where the infimum is taken over all Borel sets $B \subset I$. We can now apply Lemmas 4.5 and 4.11 in Kallenberg (1983) to conclude that the sequence (ν_n) is relatively compact with respect to the weak topology on $\mathcal{M}_I$. Thus, any sequence (n) contains a subsequence (n') such that $\nu_{n'}$ converges weakly to some limit measure, which is almost surely bounded. Moreover, the second-moment estimate $\mathbb{E}[\|\nu_n\|^2] = O(1)(\mathbb{E}[\|\nu_n\|])^2$ has been shown in Benjamini et al. (2009). Thus, there exists $\varepsilon > 0$ such that $\mathbb{P}(\|\nu_n\| > \varepsilon) > 0$ for all n. Hence, with positive probability we can extract a subsequence ν_{n_k}, such that $\|\nu_{n_k}\| > \varepsilon$ for all k. Moreover, by a compactness argument we can pass to a further subsequence that converges weakly to some limit measure ν satisfying $\|\nu\| > 0$. For this reason the measure ν can be regarded as a mass distribution on the intersection $Y = \bigcap_{n>1} Y_{R_0 n}$, provided Y is not empty.

We consider now the second-moment measure $\Lambda = \Lambda_\nu$ of ν, which can be defined as in (2.1). By Fatou's lemma and Fubini's theorem, we find that

$$\Lambda\big(B \times B'\big) \le \lim_{n\to\infty} \mathbb{E} \int_B \int_{B'} e^{2\alpha R_0 n} \mathbf{1}[x, y \in Y_{R_0 n}]\, dx\, dy$$
$$= \lim_{n\to\infty} \int_B \int_{B'} e^{2\alpha R_0 n} \mathbb{P}(x, y \in Y_{R_0 n})\, dx\, dy$$

for Borel sets $B, B' \subset I$. Furthermore, from the second-moment estimate in Proposition 2 it follows that

$$\mathbb{P}(x, y \in Y_{R_0 n}) \le f(R_0 n) f\big(R_0 n + \log \varrho_{\mathbb{H}^2}(x, y) + O(1)\big)$$
$$\le e^{-2\alpha R_0 n} \varrho_{\mathbb{H}^2}(x, y)^{-\alpha} O(1)$$

for any $n \ge 1$, whence

$$\Lambda\big(B \times B'\big) \le O(1) \int_B \int_{B'} \frac{dx\, dy}{\varrho_{\mathbb{H}^2}(x, y)^\alpha}.$$

We now observe that

$$\int_Y \int_Y \frac{\Lambda(d(x, y))}{\varrho_{\mathbb{H}^2}(x, y)^D} \le O(1) \int_I \int_I \frac{dx\, dy}{\varrho_{\mathbb{H}^2}(x, y)^{D+\alpha}}$$

is finite whenever $D + \alpha < 1$, or equivalently, if $D < 1 - \alpha$. Hence, together with Proposition 1 we see that there is positive probability for the event $\dim_H Y \ge 1-\alpha$.

A lower bound with probability one: It remains to show that we have $\dim_H Y \ge 1 - \alpha$ with probability one on $Y \ne \emptyset$. To this end assume $Y \ne \emptyset$, denote by $\mathcal{F}_n$ the σ-field generated by $Y_{R_0 n}$ and observe that all these σ-fields are independent, because of the definition of R_0. Define further $\mathcal{A}_n$ as the σ-field generated by the family $\{\mathcal{F}_m : m \ge n\}$ and put $\mathcal{T} := \bigcap_{n\ge 1} \mathcal{A}_n$. It is easily checked that $\{\dim_H Y \ge s\} \in \mathcal{T}$ for any $s \in [0, \infty)$. Thus the 0–1-law, Theorem 3.13 in Kallenberg (2002), implies that the event $\{\dim_H Y \ge s\}$ has probability 0 or 1. On the other hand, we have shown that $\dim_H Y \ge 1 - \alpha$ holds on $Y \ne \emptyset$ with positive probability, which allows us to conclude $\dim_H Y \ge 1 - \alpha$ holds almost surely on $Y \ne \emptyset$.

The ideal boundary: So far, we have proved that

$$\mathbb{E}[\dim_H Y] \le 1 - \alpha \quad \text{and that} \quad \mathbb{P}(\dim_H Y \ge 1 - \alpha | Y \ne \emptyset) = 1,$$

which clearly implies $\dim_H Y = 1 - \alpha$ with probability one on $Y \ne \emptyset$. But this value is independent of the choice of the defining half-plane A, which implies in view of the invariance of $\mathcal{Z}$ that the random set $Y' \subset B(o, 1)$ of those x for which the hyperbolic half-line (ray) through x starting at o is fully contained in $\mathcal{Z}$ has also Hausdorff-dimension $1 - \alpha$ with probability one on $Y' \ne \emptyset$. However, this is obviously the same as the set $\mathfrak{Y}$ of points z on the ideal boundary $\partial\mathbb{H}^2$ for which $[o, z) \subset \mathcal{Z}$, which proves

$$\mathbb{P}(\dim_H \mathfrak{Y} = 1 - \alpha | \mathfrak{Y} \ne \emptyset) = 1.$$

The set of rays: We denote by $\mathcal{R}_o$ the set of hyperbolic rays $[o, z)$ with $z \in \partial\mathbb{H}^2$ and the property that $[o, z) \subset \mathcal{Z}$. Defining $\mathcal{R}'_o := Y' \times [0, 1]$ with Y' as in the previous paragraph, standard fractal geometry (see Corollary 7.4 in Falconer (2003)) implies that

$$\dim_H \mathcal{R}'_o = \dim_H Y' + \dim_H [0, 1] = 2 - \alpha$$

almost surely on $Y' \neq \varnothing$. It is readily verified that this implies

$$\dim_H \mathcal{R}_o = \dim_H \mathfrak{V} + 1 = 2 - \alpha \qquad \text{a.s. on } \mathcal{R}_o \neq \varnothing,$$

which finally completes the proof.

Remark 1. Let μ be a Radon measure on some lcscH space E as in Section 2 and define the *lower and upper pointwise dimension of* μ *at* $x \in E$ as

$$\underline{\mathrm{d}}\mu(x) = \liminf_{r \to 0} \frac{\ln \mu(B(x, r))}{\ln r} \quad \text{and} \quad \overline{\mathrm{d}}\mu(x) = \limsup_{r \to 0} \frac{\ln \mu(B(x, r))}{\ln r},$$

respectively. Moreover, the *lower and upper Hausdorff-dimension of* μ are given by

$$\underline{\mathrm{d}}_H \mu = \operatorname*{ess\,inf}_{x \in E} \underline{d}\mu(x) \quad \text{and} \quad \overline{\mathrm{d}}_H \mu = \operatorname*{ess\,sup}_{x \in E} \overline{\mathrm{d}}\mu(x),$$

respectively. The measure μ is said to have *carrying dimension* β, this is cardim $\mu = \beta$, if $\underline{\mathrm{d}}_H \mu = \overline{\mathrm{d}}_H \mu = \beta$. Our proof above also shows that that the random limit measure ν fulfills cardim $\nu \geq 1 - \alpha$ with conditional probability one. Moreover, an upper-bound technique due to Dawson and Hochberg (see Dawson and Hochberg (1979), Zähle (1988)) can easily be applied in our setting to show that also the reverse inequality holds true. Thus,

$$\mathbb{P}(\text{cardim } \nu = 1 - \alpha | \nu \neq \text{null-measure}) = 1.$$

3.3 Proof of Theorem 2

This follows as in Theorem 1 together from a modified Proposition 2. The latter can be obtained by following the lines of the proofs of Lemmas 3.5 and 3.6 in Benjamini et al. (2009). For these reasons the details are omitted.

Remark 2. In Remark 3.7 in Benjamini et al. (2009) it has been conjectured that the Hausdorff-dimension in Theorem 2 should be $1 - 2\alpha$ instead of $2 - 2\alpha$, where $\alpha \in (0, 1/2)$. However, being a union of lines, the random set under discussion must have Hausdorff-dimension at least 1 so that the conjecture cannot be correct, because $1 - 2\alpha \in (0, 1)$, whereas $2 - 2\alpha \in (1, 2)$ as expected.

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