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QUENCHED ASYMPTOTICS FOR BROWNIAN MOTION IN GENERALIZED GAUSSIAN POTENTIAL

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In this paper, we study the long-term asymptotics for the quenched moment

$$\mathbb{E}_x \exp\left\{\int_0^t V(B_s)\,ds\right\}$$

consisting of a d-dimensional Brownian motion $\{B_s; s \ge 0\}$ and a generalized Gaussian field V. The major progress made in this paper includes: Solution to an open problem posted by Carmona and Molchanov [*Probab. Theory Related Fields* **102** (1995) 433–453], the quenched laws for Brownian motions in Newtonian-type potentials and in the potentials driven by white noise or by fractional white noise.

1. Introduction. The classic Anderson model can be formulated as the following heat equation:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(x)u(t,x), \\ u(0,x) = 1, \end{cases} \tag{1.1}$$

where $\{V(x); x \in \mathbb{R}^d\}$ is often made as a stationary random field called potential.

Under some regularity assumption such as Hölder continuity on $V(x)$, the system has a unique solution with Feynman–Kac representation

$$u(t,x) = \mathbb{E}_x \exp\left\{\int_0^t V(B_s)\,ds\right\}, \tag{1.2}$$

where $\{B_t; t \ge 0\}$ is a d-dimensional Brownian motion independent of $V(x)$, and $\mathbb{E}_x$ is the expectation with respect to B_t given $B_0 = x$.

An important aspect in studying parabolic Anderson models is its long-term asymptotics. There are two types of asymptotics: one is labeled as quenched law concerning the limit behavior of the random field $u(t,x)$ conditioning on the random potential $V(x)$; another is known as annealed law with interest in the limit behavior of $\mathbb{E}u(t,x)$ and other deterministic moments of $u(t,x)$. In the case when

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$\{V(x); x \in \mathbb{R}^d\}$ is a mean zero stationary Gaussian field with the covariance function

$$\gamma(x) = \mathrm{Cov}\big(V(0), V(x)\big), \qquad x \in \mathbb{R}^d. \tag{1.3}$$

Carmona and Molchanov (Theorem 5.1, [5]) establish the quenched law

$$\lim_{t\to\infty} \frac{1}{t\sqrt{\log t}} \log \mathbb{E}_x \exp\left\{\int_0^t V(B_s)\,ds\right\} = \sqrt{2d\gamma(0)}, \qquad \text{a.s.} \tag{1.4}$$

under the condition $\lim_{|x|\to\infty} \gamma(x) = 0$. See [14] for the asymptotics of the second order, and [4, 6, 16, 24] and [25] for a variety of versions in literature.

This paper is concerned with the setting of the generalized Gaussian fields, in which the potential V is not defined pointwise. A typical example is when V is a white or fractional white noise. Recall that a generalized function is defined as a linear functional $\{\langle \xi, \varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ on a suitable space $\mathcal{S}(\mathbb{R}^d)$ of the functions known as the test functions. The classic notion of function is generalized in the sense that

$$\langle \xi, \varphi\rangle = \int_{\mathbb{R}^d} \xi(x)\varphi(x)\,dx, \qquad \varphi \in \mathcal{S}\big(\mathbb{R}^d\big) \tag{1.5}$$

whenever $\xi(x)$ is a "good" function defined pointwise on $\mathbb{R}^d$. We refer the book [17] by Gel'fand and Vilenkin for details.

A generalized random field V is a generalized random function. In this paper, we consider the case when $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing and infinitely smooth functions, and $\{\langle V, \varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ is a mean-zero Gaussian field satisfying the homogeneity

$$\big\{\big\langle V, \varphi(\cdot - x)\big\rangle; \varphi \in \mathcal{S}\big(\mathbb{R}^d\big)\big\} \stackrel{d}{=} \big\{\langle V, \varphi\rangle; \varphi \in \mathcal{S}\big(\mathbb{R}^d\big)\big\}, \qquad x \in \mathbb{R}^d. \tag{1.6}$$

The covariance functionals $\mathrm{Cov}(\langle V, \varphi\rangle, \langle V, \psi\rangle)$ of the generalized Gaussian fields considered in this work are continuous on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. Consequently, $\{\langle V, \varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ is continuous in probability and therefore yields a measurable version.

The classic Bochner representation can be generalized ((1), page 290, [17]) in the following way: There is a positive measure $\mu(d\lambda)$ on $\mathbb{R}^d$, known as spectral measure, such that

$$\mathrm{Cov}\big(\langle V, \varphi\rangle, \langle V, \psi\rangle\big) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(\lambda)\overline{\mathcal{F}(\psi)(\lambda)}\mu(d\lambda), \tag{1.7}$$

where $\mathcal{F}(\varphi)(\lambda)$ denotes the Fourier transform of the function $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Further, $\mu(d\lambda)$ is tempered in the sense that $(1 + |\cdot|^2)^{-p} \in \mathcal{L}(\mathbb{R}^d, \mu)$ for some $p > 0$.

In the settings considered in this paper, the notion of covariance function $\gamma(\cdot)$ defined by (1.3) can also be extended to the form

$$\mathrm{Cov}\big(\langle V, \varphi\rangle, \langle V, \psi\rangle\big) = \int_{\mathbb{R}^d} \gamma(x - y)\varphi(x)\psi(y)\,dx\,dy, \qquad \varphi, \psi \in \mathcal{S}\big(\mathbb{R}^d\big) \tag{1.8}$$

with $\gamma(x) = \delta_0(x)$ (Dirac function) or with $\gamma(x)$ being defined pointwise on $\mathbb{R}^d \setminus \{0\}$ and satisfying $\gamma(0) \equiv \lim_{x\to 0}\gamma(x) = \infty$—in both cases $\mu(d\lambda)$ is an infinite measure. As a consequence, it is impossible to make V pointwise defined through relation (1.5), for otherwise we would have to face the "Gaussian variable" $V(x)$ with $\mathrm{Var}(V(x)) = \gamma(0) = \infty$ for every $x \in \mathbb{R}^d$.

Nevertheless, representation (1.2) can be extended to the generalized setting under some suitable condition. The generalized Gaussian potentials appearing in our main theorems satisfy (Lemma A.2)

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\lambda|^2)^{1-\delta}}\mu(d\lambda) < \infty \tag{1.9}$$

for some $\delta > 0$. As a consequence (Lemma A.1), the L^2-limit

$$\int_0^t V(B_s)\,ds \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+}\int_0^t V_\varepsilon(B_s)\,ds$$

exists and, the time integral defined in this way yields a continuous version as a stochastic process, where the pointwise defined Gaussian field $V_\varepsilon(x)$ appears as a smoothed version of V; see Lemma A.1 for details. In addition, the time integral defined in this way is exponentially integrable with respect to $\mathbb{E}_x$, as pointed out in Section 3. Consequently, representation (1.2) makes sense in our settings. According to a treatment proposed on page 448 of [5], it solves the Anderson model (1.1) in some proper sense. The major goal of this work is to study the large-t behavior of the quenched exponential moment in (1.2).

In [5], Carmona and Molchanov ask what happens when the covariance function $\gamma(x)$ is defined pointwise, continuous in $\mathbb{R}^d \setminus \{0\}$ but $\gamma(0) = \infty$ with the degree of singularity measured by

$$\gamma(x) \sim c(\gamma)|x|^{-\alpha} \qquad (x \to 0) \tag{1.10}$$

for some $0 < \alpha < 2$ and $c(\gamma) > 0$. Here we point out that the restriction "$\alpha < d$" has to be added for the covariance functional $\mathrm{Cov}(\langle V, \varphi\rangle, \langle V, \psi\rangle)$ to be well-defined. Indeed, for a nonnegative $\varphi \in \mathcal{S}(\mathbb{R}^d)$ strictly positive in a neighborhood of 0, there are $C > 0$ and $\varepsilon > 0$ such that

$$\mathrm{Var}\big(\langle V, \varphi\rangle\big) \ge C^{-1}\int_{\{|x|\le\varepsilon\}\times\{|y|\le\varepsilon\}} \frac{dx\,dy}{|x-y|^\alpha}.$$

The right-hand side diverges if $\alpha \ge d$.

In their paper, Carmona and Molchanov [5] conjecture that under (1.10),

$$\log \mathbb{E}_x \exp\left\{\int_0^t V(B_s)\,ds\right\} \approx t(\log t)^{(4-\alpha)/(2-\alpha)}, \qquad \text{a.s. } (t\to\infty).$$

The following theorem tells a slightly different story.

THEOREM 1.1. *Let the covariance function $\gamma(x)$ be continuous on $\mathbb{R}^d \setminus \{0\}$ and be bounded outside every neighborhood of* 0. *Assume* (1.10) *with* $0 < \alpha < 2 \wedge d$. *Then for any* $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1}(\log t)^{-2/(4-\alpha)} \log \mathbb{E}_x \exp\left\{\int_0^t V(B_s)\,ds\right\} \\
&\qquad = \frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)} \big(2dc(\gamma)\kappa(d,\alpha)\big)^{2/(4-\alpha)}, \qquad a.s.,
\end{aligned} \tag{1.11}$$

where the constant $c(\gamma) > 0$ is given in (1.10), *and $\kappa(d,\alpha) > 0$ is the best constant of the inequality* [*see* (A.18) *in the Appendix*]

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f^2(x)f^2(y)}{|x-y|^\alpha} \le C\|f\|_2^{4-\alpha}\|\nabla f\|_2^\alpha, \qquad f \in W^{1,2}(\mathbb{R}^d)$$

with $W^{1,2}(\mathbb{R}^d)$ being defined as the Sobolev space

$$W^{1,2}(\mathbb{R}^d) = \{f \in \mathcal{L}^2(\mathbb{R}^d); \nabla f \in \mathcal{L}^2(\mathbb{R}^d)\}. \tag{1.12}$$

We now consider a special case. In light of some classical laws of physics, such as Newton's gravity law and Coulomb's electrostatics law, it makes sense to consider the potential formally given as

$$V(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^p} W(dy), \qquad x \in \mathbb{R}^d$$

in the parabolic Anderson model (1.1). Here $\{W(x); x \in \mathbb{R}^d\}$ is a standard Brownian sheet. The relevant Gaussian field

$$\langle V, \varphi\rangle = \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} \frac{\varphi(y)}{|y-x|^p}\,dy\right] W(dx), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d) \tag{1.13}$$

is well defined with the covariance function

$$\gamma(x) = C(d,p)|x|^{-(2p-d)}, \tag{1.14}$$

provided $d/2 < p < \frac{d+2}{2} \wedge d$, where

$$C(d,p) = \pi^{d/2}\frac{\Gamma^2((d-p)/2)\Gamma((2p-d)/2)}{\Gamma^2(p/2)\Gamma(d-p)}. \tag{1.15}$$

Indeed,

$$\begin{aligned}
\mathrm{Cov}(\langle V,\varphi\rangle, \langle V,\psi\rangle) &= \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} \frac{\varphi(y)\,dy}{|y-x|^p}\right]\left[\int_{\mathbb{R}^d} \frac{\psi(z)\,dy}{|z-x|^p}\right] dx \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y)\psi(z)\left[\int_{\mathbb{R}^d} \frac{dx}{|y-x|^p|z-x|^p}\right] dy\,dz \\
&= C(d,p)\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\varphi(y)\psi(z)}{|y-z|^{2p-d}}\,dy\,dz,
\end{aligned}$$

where the last step follows from (1.31) in [11] (with σ being replaced by $2p-d$).

Thus (1.10) holds with $\alpha = 2p - d < 2 \wedge d$.

COROLLARY 1.2. *In the special case given in* (1.13) *with* $d/2 < p < d \wedge \frac{d+2}{2}$,

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1}(\log t)^{-2/(4+d-2p)} \log \mathbb{E}_x \exp\left\{\theta \int_0^t V(B_s)\,ds\right\} \\
&\qquad = \frac{4+d-2p}{4}\left(\frac{2p-d}{2}\right)^{(2p-d)/(4+d-2p)} \\
&\qquad\quad \times \big(2dC(d,p)\theta^2\kappa(d,2p-d)\big)^{2/(4+d-2p)}, \qquad a.s.
\end{aligned} \tag{1.16}$$

for any $\theta > 0$, *where* $C(d,p) > 0$ *is given in* (1.15).

In the next theorem, the potential is a fractional white noise formally written as

$$V(x) = \frac{\partial^d W^H}{\partial x_1 \cdots \partial x_d}(x), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

where $W^H(x)$ ($x = (x_1, \dots, x_d) \in \mathbb{R}^d$) is a fractional Brownian sheet with Hurst index $H = (H_1, \dots, H_d)$. We assume that

$$\frac{1}{2} < H_j < 1 \qquad (j = 1, \dots, d) \quad \text{and} \quad \sum_{j=1}^d H_j > d - 1. \tag{1.17}$$

The generalized Gaussian field relevant to the problem is defined by the stochastic integral

$$\langle V, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x) W^H(dx), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{1.18}$$

In this setting,

$$\begin{aligned}
&\gamma(x) = C_H \left(\prod_{j=1}^d |x_j|^{2-2H_j}\right)^{-1} \quad \text{and} \\
&\mu(d\lambda) = \widehat{C}_H \left(\prod_{j=1}^d |\lambda_j|^{2H_j-1}\right)^{-1} d\lambda,
\end{aligned} \tag{1.19}$$

where $C_H > 0$ and $\widehat{C}_H > 0$ are two constants with

$$C_H = \prod_{j=1}^d H_j(2H_j - 1).$$

Under assumption (1.17),

$$0 < \alpha \equiv 2d - 2\sum_{j=1}^d H_j < 2 \wedge d. \tag{1.20}$$

THEOREM 1.3. *Assume* (1.17). *For any* $\theta > 0$ *and* $x \in \mathbb{R}^d$,

$$\begin{aligned} &\lim_{t\to\infty} t^{-1}(\log t)^{-2/(4-\alpha)} \log \mathbb{E}_x \exp\left\{\theta \int_0^t \frac{\partial^d W^H}{\partial x_1 \cdots \partial x_d}(B_s)\, ds\right\} \\ &\qquad = \frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)} \bigl(2dC_H\theta^2 \tilde{\kappa}(d,H)\bigr)^{2/(4-\alpha)}, \qquad a.s., \end{aligned} \tag{1.21}$$

where $\tilde{\kappa}(d,H)$ *is the best constant of the inequality* [*see* (A.30) *in the Appendix*]

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f^2(x) f^2(y)\left(\prod_{j=1}^d |x_j-y_j|^{2-2H_j}\right)^{-1} dx\,dy \le C\|f\|_2^{4-\alpha}\|\nabla f\|_2^{\alpha},$$
$$f \in W^{1,2}(\mathbb{R}^d).$$

In the next theorem, we take $d = 1$. The Gaussian potential is a white noise formally given as $V(x) = \dot{W}(x)$ where $W(x)$ $(x \in \mathbb{R})$ is a two-side Brownian motion. The relevant generalized Gaussian field is defined as

$$\langle V, \varphi\rangle = \int_{-\infty}^{\infty} \varphi(x) W(dx), \qquad \varphi \in \mathcal{S}(\mathbb{R}). \tag{1.22}$$

In this case the covariance function $\gamma(\cdot) = \delta_0(\cdot)$ is the Dirac function and the spectral measure $\mu(d\lambda) = d\lambda$ is Lebesgue measure on $(-\infty,\infty)$.

THEOREM 1.4. *For any* $\theta > 0$ *and* $x \in \mathbb{R}$,

$$\begin{aligned} &\lim_{t\to\infty} t^{-1}(\log t)^{-2/3} \log \mathbb{E}_x \exp\left\{\theta\int_0^t \dot{W}(B_s)\,ds\right\} \\ &\qquad = \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}\theta^{4/3}, \qquad a.s. \end{aligned} \tag{1.23}$$

We now comment on our main theorems. It is interesting to see that (1.11) is consistent with (1.4) when the latter is regarded as the case $\alpha = 0$, with easy and natural identifications $c(\gamma) = \gamma(0)$, $\kappa(d,0) = 1$, and the natural convention that $0^0 = 1$.

Given an integer valued symmetric simple random walk $\{X_t; t \ge 0\}$ and an independent family $\{\xi(x); x \in \mathbb{Z}\}$ of the i.i.d. standard normal random variables, by Theorem 4.1, [15], or by Theorem 2.2, [16],

$$\lim_{t\to\infty} t^{-1}(\log t)^{-1/2} \log \mathbb{E}_x \exp\left\{\theta\int_0^t \xi(X_s)\,ds\right\} = \sqrt{2}\theta, \qquad \text{a.s.}$$

Comparing this to Theorem 1.4, we witness a highly unusual difference between continuous and discrete settings.

The almost sure limits stated in our theorems are largely determined by the scaling or asymptotic scaling exponent α of the covariance function $\gamma(x)$ at $x = 0$.

The restriction $\alpha < 2$ in our theorems is essential. In connection to Theorem 1.4, notice that a Dirac function on $\mathbb{R}^d$ satisfies $\delta_0(cx) = |c|^d \delta_0(x)$. In particular, $\alpha = d$ as $\gamma(x) = \delta_0(x)$. To comply with the restriction $\alpha < 2$, the space dimension d has to be 1 in Theorem 1.4.

A challenge beyond the scope of this paper is the quenched long-term asymptotics for the time dependent potential $V(t,x)$ in connection to Theorems 1.3 and 1.4. Associated to Theorem 1.3 is the case when

$$V(t,x) = \frac{\partial^{d+1} W^H}{\partial t\, \partial x_1 \cdots \partial x_d}(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $W^H(t,x)$ is a time–space fractional Brownian sheet with some restriction on its Hurst parameter $H = (H_0, H_1, \dots, H_d)$. An interested reader is referred to the paper by Hu, Nualart and Song [19] for the Feynman–Kac representation of the solution in this system; and to the recent work [9] by Chen, Hu, Song and Xing for the annealed asymptotics in this and other time–space settings.

Theorem 1.4 corresponds to the famous Karda–Parisi–Zhang (KPZ) model which starts from a nonlinear stochastic partial differential equation and is transformed into the parabolic Anderson equation with the potential

$$V(t,x) = \frac{\partial^2 W}{\partial t\, \partial x}(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}$$

by some renormalization treatment together with the Hopf–Cole transform. We cite the references [20] and [21] for the physical background of the problem, and [1, 2, 18] for the mathematical set-up and recent progress on the KPZ equation.

In addition, it is worth mentioning a recent work [12] by Conus et al. in which they consider a possibly nonlinear heat equation

$$\partial_t u = \frac{1}{2}\Delta u + V(t,x)\sigma(u).$$

Here $V(t,x)$ is a time–space generalized Gaussian field with the covariance function

$$\delta_0(s-t)\gamma(x-y), \qquad (s,x), (t,y) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

When the space covariance function $\gamma(x)$ satisfies (1.10) with $0 < \alpha < 2 \wedge d$, a quenched space-asymptotic law (Theorem 2.6, [12]) states that

$$C_1 \le \limsup_{|x|\to\infty} \bigl(\log|x|\bigr)^{-2/(4-\alpha)} \log u(t,x) \le C_2, \qquad \text{a.s.}$$

for any fixed $t > 0$. The exponent $2/(4-\alpha)$ seems to suggest a deep link to (1.11). In general, going from the time-independent potential to the time-dependent potential is a big step. We specially mention the work [25] by Viens and Zhang for their effort beyond the sub-additivity treatment. It is our hope that some ideas developed in the current paper may play a role in the future investigation of this direction.

We now comment on the approaches adopted in this paper and their relations to earlier works. As usual, the proof consists of two major steps: a semi-group method to associate the quenched exponential moment in (1.2) to the principal eigenvalue of random linear operator $2^{-1}\Delta + V$ with the zero boundary on $(-t, t)^d$ and asymptotic estimation of the principal eigenvalue for which a nice idea developed in [13] and [14] is adopted; see (2.27) to control the principal eigenvalue over the large domain $(-t, t)^d$ by the extreme among the principal eigenvalues over the sub-domains. On the other hand, what sets this paper apart is the singularity of our models. The following are some of the novelties appearing in this paper.

(1) Algorithm development. The algorithms existing in the literature often depend on the asymptotics of the generating function of $V(0)$. Unfortunately, this strategy does not apply here as $V(0)$ is not even defined in our models. Indeed, the appearance of $\|\nabla g\|_2$ in the constants of our main theorems is a testimony of the dynamics different from the classic settings represented by (1.4). Our approach involves a rescaling strategy that highlights the role of the diffusion part of the principal eigenvalue. Some of the ideas adopted in this paper have been used in the recent work [8] in the setting of renormalized Poissonian potential. However, there are substantial differences between these two settings that demand some new adaptations. The renormalized Poissonian potential is defined pointwise and essentially total variational in the sense that it can be decomposed as the difference of positive and negative parts under suitable truncation, while it is classic knowledge that the potentials driven by white noise or fractional white noise are not total variational.

(2) Entropy estimate. The entropy method has become an effective tool in dealing with the tail, continuity, integrability or finiteness for the random quantities given as supremum. In the case when $V(x)$ is defined pointwise, the concern is the supremum $\sup_{x\in D} V(x)$ over a compact $D \subset \mathbb{R}^d$, and the problem is to count the ε-balls that cover D. Not surprisingly, the entropy number is bounded by a polynomial of ε^{-1} if the distance is Euclidean or nearly Euclidean. On the other hand, the entropy method in the context of generalized potential is for the supremum $\sup_{g\in\mathcal{G}_d(D)}\langle V, g^2\rangle$ over (a dense set of) the unit sphere of the Sobolev space over the domain D; see Proposition 2.1. Counting the covering ε-balls in a functional space is much harder and the result is less predictable due to complexity in geometric structure.

(3) Lower bound by Slepian lemma. In the classic setting, the lower bound for (1.4) can be established by decomposing $V(x)$ into two homogeneous Gaussian fields such that the first field has finite correlation radius and the second is negligible. Under the assumptions of Theorems 1.1 and 1.3, such decomposition is not available. Our treatment is based on a famous comparison lemma by Slepian [23] and is formulated in Lemma 4.2 below.

2. Gaussian supremum. Let $D \subset \mathbb{R}^d$ be a fixed bounded open domain. We use the notation $\mathcal{S}(D)$ for the space of the infinitely smooth functions on D that vanish at the boundary of D. For convenience, we always view $\mathcal{S}(D)$ as a subspace of $\mathcal{S}(\mathbb{R}^d)$ by defining $g(x)=0$ outside D for each $g\in\mathcal{S}(D)$. Given $g\in\mathcal{S}(D)$, for example, we may alternate between the notation

$$\int_D |\nabla g(x)|^2\,dx \quad\text{and}\quad \int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx$$

according to convenience. The notation $\|\nabla g\|_2$ is used for both spaces $\mathcal{S}(D)$ and $\mathcal{S}(\mathbb{R}^d)$. Set

$$\mathcal{F}_d(D)=\{g\in\mathcal{S}(D);\ \|g\|_2^2=1\},\tag{2.1}$$

$$\mathcal{G}_d(D)=\{g\in\mathcal{S}(D);\ \|g\|_2^2+\tfrac{1}{2}\|\nabla g\|_2^2=1\}.\tag{2.2}$$

Our approach largely relies on the estimate of the supremum

$$\sup_{g\in\mathcal{F}_d(D)}\left\{\langle V,g^2\rangle-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\right\}.\tag{2.3}$$

Notice that for each $g\in\mathcal{S}(D)$, $g^2\in\mathcal{S}(D)$. Consequently, the random variable $\langle V,g^2\rangle$ is well defined and normal. On the other hand, it is not obvious whether or not the supremum is finite. When it is finite, the variation in (2.3) is the principal eigenvalue of the linear operator $(1/2)\Delta+V$ with the zero boundary condition over D. The main goal of this section is to show that the supremum in (2.3) is finite when D is bounded, and to establish a sharp almost-sure asymptotic bound as D expands to $\mathbb{R}^d$ in a suitable way. The treatment is entropy estimation.

2.1. *Entropy bounds*. Consider a pseudometric space (E,ρ) with the pseudometric $\rho(\cdot,\cdot)$. For any $\varepsilon>0$, let $N(E,\rho,\varepsilon)$ be the minimal number of the open balls of the diameter no greater than ε, which are necessary for covering E. In this section we take $E=\mathcal{G}_d(D)$ and

$$\rho(f,g)=\{\mathbb{E}[\langle V,f^2\rangle-\langle V,g^2\rangle]^2\}^{1/2},\qquad f,g\in\mathcal{G}_d(D).$$

We have that

$$\rho(f,g)=\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)\big(f^2(x)-g^2(x)\big)\big(f^2(y)-g^2(y)\big)\,dx\,dy\right\}^{1/2},\quad f,g\in\mathcal{G}_d(D).\tag{2.4}$$

Here we specially mention that $\gamma(x)=\delta_0(x)$ in the context of Theorem 1.4.

PROPOSITION 2.1. *Under the assumptions of Theorems* 1.1, 1.3 *or* 1.4,

$$\lim_{\varepsilon\to0^+}\varepsilon^\beta\log N\big(\mathcal{G}_d(D),\rho,\varepsilon\big)=0\tag{2.5}$$

whenever

$$\beta>1\vee\frac{2d}{d+2}.\tag{2.6}$$

Noticing that the right-hand side of (2.6) *is less than* 2,

$$\int_0^1 \sqrt{\log N(\mathcal{G}_d(D), \rho, \varepsilon)}\, d\varepsilon < \infty. \tag{2.7}$$

PROOF. Let $l(x) \in \mathcal{S}(\mathbb{R}^d)$ (mollifier) be a symmetric probability density function supported on $\{|x| \le 1\}$ and introduce the function $l_\varepsilon(x)$ (ε-mollifier) as

$$l_\varepsilon(x) = \varepsilon^{-d} l(\varepsilon^{-1}x), \qquad x \in \mathbb{R}^d, \varepsilon > 0. \tag{2.8}$$

In addition, we assume that $\mathcal{F}(l)(\cdot) \ge 0$. Define the operator $\mathcal{S}_\varepsilon$ on $\mathcal{S}(\mathbb{R}^d)$ as

$$\mathcal{S}_\varepsilon g(x) = \left\{\int_{\mathbb{R}^d} g^2(x-y) l_\varepsilon(y)\, dy\right\}^{1/2}, \qquad x \in \mathbb{R}^d. \tag{2.9}$$

By Fourier transform,

$$\mathbb{E}[\langle V, g^2\rangle - \langle V, \mathcal{S}_\varepsilon(g)^2\rangle]^2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |1-\mathcal{F}(l)(\varepsilon\lambda)|^2 |\mathcal{F}(g^2)(\lambda)|^2 \mu(d\lambda),$$
$$g \in \mathcal{G}_d(D).$$

Notice that $|1-\mathcal{F}(l)(\varepsilon\lambda)| \le 2$. By the mean-value theorem there is $C_\delta > 0$ such that

$$|1-\mathcal{F}(l)(\varepsilon\lambda)| \le 2^{1-\delta}|1-\mathcal{F}(l)(\varepsilon\lambda)|^\delta \le C_\delta|\varepsilon\lambda|^\delta, \qquad \lambda \in \mathbb{R}^d, \varepsilon > 0,$$

where $0 < \delta < 1$ is chosen by (1.9), in connection to Lemma A.2 in the Appendix.

Thus, there is a constant $C > 0$ independent of ε and g, such that

$$\rho(g, \mathcal{S}_\varepsilon g) \le C\varepsilon^\delta \left\{\int_{\mathbb{R}^d} |\lambda|^{2\delta} |\mathcal{F}(g^2)(\lambda)|^2 \mu(d\lambda)\right\}^{1/2}, \qquad g \in \mathcal{G}_d(D), \varepsilon > 0.$$

Notice that

$$|\mathcal{F}(g^2)(\lambda)| \le \mathcal{F}(g^2)(0) = \|g\|_2^2 \le 1, \qquad g \in \mathcal{G}_d(D).$$

In addition, for any $\lambda \in \mathbb{R}^d \setminus \{0\}$,

$$\mathcal{F}(g^2)(\lambda) = \frac{i}{d}\int_{\mathbb{R}^d}\left(\frac{\lambda}{|\lambda|^2}\cdot \nabla g^2(x)\right) e^{i\lambda\cdot x}\, dx.$$

Hence,

$$\begin{aligned}|\mathcal{F}(g^2)(\lambda)| &\le \frac{1}{d}|\lambda|^{-1}\int_{\mathbb{R}^d}|\nabla g^2(x)|\, dx = \frac{2}{d}|\lambda|^{-1}\int_{\mathbb{R}^d}|g(x)||\nabla g(x)|\, dx \\ &\le \frac{2}{d}|\lambda|^{-1}\|g\|_2\|\nabla g\|_2 \le \frac{2}{d}|\lambda|^{-1}.\end{aligned}$$

Consequently,

$$|\mathcal{F}(g^2)(\lambda)|^2 \le \left(1+\frac{2}{d}\right)\left(1 \wedge \frac{1}{|\lambda|^2}\right), \qquad g \in \mathcal{G}_d(D).$$

By (1.9), this leads to

$$\sup_{g\in\mathcal{G}_d(D)}\int_{\mathbb{R}^d}|\lambda|^{2\delta}\big|\mathcal{F}\big(g^2\big)(\lambda)\big|^2\mu(d\lambda)<\infty.$$

Summarizing our argument, there is a constant $C>0$ such that

$$\sup_{g\in\mathcal{G}_d(D)}\rho(g,\mathcal{S}_\varepsilon g)\le C\varepsilon^{\delta},\qquad \varepsilon>0. \tag{2.10}$$

Write $\phi(\varepsilon)=\varepsilon^{\delta^{-1}}$. We have that

$$\sup_{g\in\mathcal{G}_d(D)}\rho(g,\mathcal{S}_{\phi(\varepsilon)}g)\le C\varepsilon,\qquad \varepsilon>0.$$

To prove (2.5), therefore, all we need is to show that for any β satisfying (2.6),

$$\lim_{\varepsilon\to 0^+}\varepsilon^{\beta}\log N\big(\mathcal{G}_d(D),\rho_\varepsilon,\varepsilon\big)=0, \tag{2.11}$$

where the pseudometric ρ_ε is defined as $\rho_\varepsilon(f,g)=\rho(\mathcal{S}_{\phi(\varepsilon)}f,\mathcal{S}_{\phi(\varepsilon)}g)$ $(f,g\in\mathcal{G}_d(D))$. By (2.4)

$$\begin{aligned}\rho_\varepsilon(f,g)\le{}&\left(\int_{\mathbb{R}^d}\big|(\mathcal{S}_{\phi(\varepsilon)}f)^2(x)-(\mathcal{S}_{\phi(\varepsilon)}g)^2(x)\big|\,dx\right)^{1/2}\\&\times\left(\sup_{x\in D'}\left|\int_{\mathbb{R}^d}\gamma(x-y)\big\{(\mathcal{S}_{\phi(\varepsilon)}f)^2(y)-(\mathcal{S}_{\phi(\varepsilon)}g)^2(y)\big\}\,dy\right|\right)^{1/2},\end{aligned}$$

where D' is the 1-neighborhood of D. Take

$$A_\varepsilon(f)(x)=(\mathcal{S}_{\phi(\varepsilon)}f)^2(x)\quad\text{and}\quad B_\varepsilon(f)(x)=\int_{\mathbb{R}^d}\gamma(x-y)(\mathcal{S}_{\phi(\varepsilon)}f)^2(y)\,dy,$$
$$x\in D'$$

in Lemma A.3 of the Appendix. All we need is to exam that there are $p>1$ satisfying

$$\beta>\frac{2p}{2p-1}>1\vee\frac{2d}{d+2} \tag{2.12}$$

and $C>0$, $m>0$ independent of $\varepsilon>0$ such that

$$\big|(\mathcal{S}_{\phi(\varepsilon)}f)^2(x)-(\mathcal{S}_{\phi(\varepsilon)}f)^2(y)\big|\le C\varepsilon^{-m}|x-y|, \tag{2.13}$$

$$\left|\int_{\mathbb{R}^d}\{\gamma(x-z)-\gamma(y-z)\}(\mathcal{S}_{\phi(\varepsilon)}f)^2(z)\,dz\right|\le C\varepsilon^{-m}|x-y|, \tag{2.14}$$

$$\int_{\mathbb{R}^d}\big|(\mathcal{S}_{\phi(\varepsilon)}f)(z)\big|^{2p}\,dz\le C \tag{2.15}$$

and

$$\left|\int_{\mathbb{R}^d}\gamma(x-z)(\mathcal{S}_{\phi(\varepsilon)}f)^2(z)\,dz\right|\le C \tag{2.16}$$

for all $x, y \in D'$ and $f \in \mathcal{G}_d(D)$.

Indeed, by the mean value theorem

$$\begin{aligned}\big|(\mathcal{S}_{\phi(\varepsilon)} f)^2(x) - (\mathcal{S}_{\phi(\varepsilon)} f)^2(y)\big| &\le \int_{\mathbb{R}^d} \big|l_{\phi(\varepsilon)}(x+z) - l_{\phi(\varepsilon)}(y+z)\big| f^2(z)\,dz \\ &\le C\phi(\varepsilon)^{-(d+1)}|x-y| \int_{\mathbb{R}^d} f^2(z)\,dz \\ &\le C\phi(\varepsilon)^{-(d+1)}|x-y|.\end{aligned}$$

Thus (2.13) holds with $m = (d+1)\delta^{-1}$. For the same m, (2.14) follows from (2.13), the relation

$$\begin{aligned}&\int_{\mathbb{R}^d} \{\gamma(x-z) - \gamma(y-z)\}(\mathcal{S}_{\phi(\varepsilon)} f)^2(z)\,dz \\ &\qquad = \int_{\mathbb{R}^d} \gamma(z)\big\{(\mathcal{S}_{\phi(\varepsilon)} f)^2(z-x) - (\mathcal{S}_{\phi(\varepsilon)} f)^2(z-y)\big\}\,dz,\end{aligned}$$

and the fact that

$$\int_{\widetilde{D}} \big|\gamma(z)\big|\,dz < \infty$$

for $\widetilde{D} = \{z_1 + z_2 \in \mathbb{R}^d; z_1, z_2 \in D'\}$.

We now come to (2.15). First, for any $p > 1$ and by Jensen's inequality,

$$\int_{\mathbb{R}^d} \big|(\mathcal{S}_{\phi(\varepsilon)} f)(z)\big|^{2p}\,dz \le \int_{\mathbb{R}^d} \big|f(z)\big|^{2p}\,dz.$$

We claim that there is a $p > 1$ satisfying (2.12) and $p(d-2) < d$. Indeed, this is obvious when $d \le 2$ as we can make p sufficiently large. When $d \ge 3$, our assertion is secured by the facts that the quantity $2p(2p-1)^{-1}$ is strictly decreasing in p, and that the supremum of p under the constraint $p(d-2) < d$ is $b \equiv d(d-2)^{-1}$ which solves the equation

$$\frac{2b}{2b-1} = \frac{2d}{d+2}.$$

By Gagliardo–Nirenberg inequality (see, e.g., page 303, [7]), for which the restriction $p(d-2) < d$ is critically needed,

$$\int_{\mathbb{R}^d} \big|f(x)\big|^{2p}\,dx \le C\|f\|_2^{d(p-1)} \|\nabla f\|_2^{2p-d(p-1)} \le C.$$

Thus, we have proved (2.15).

It remains to establish (2.16). In the context of Theorem 1.3, by (A.29),

$$\left|\int_{\mathbb{R}^d} \gamma(x-z)(\mathcal{S}_{\phi(\varepsilon)} f)^2(z)\,dz\right| \le C\|\mathcal{S}_{\phi(\varepsilon)} f\|_2^{4-\alpha} \|\nabla \mathcal{S}_{\phi(\varepsilon)} f\|_2^{\alpha}.$$

By Jensen inequality, $\|\mathcal{S}_{\phi(\varepsilon)} f\|_2 \le \|f\|_2 \le 1$. From (2.9)

$$\begin{aligned}
&\big|\nabla \mathcal{S}_{\phi(\varepsilon)} f(x)\big| \\
&\quad = \left(\int_{\mathbb{R}^d} l_{\phi(\varepsilon)}(y) f^2(x-y)\,dy\right)^{-1/2} \left|\int_{\mathbb{R}^d} l_{\phi(\varepsilon)}(y) f(x-y)\nabla f(x-y)\,dy\right| \\
&\quad \le \left\{\int_{\mathbb{R}^d} l_{\phi(\varepsilon)}(y)\big|\nabla f(x-y)\big|^2\,dy\right\}^{1/2},
\end{aligned}$$

where the inequality follows from Cauchy–Schwarz inequality. Hence, by Fubini's theorem and translation invariance,

$$\|\nabla \mathcal{S}_{\phi(\varepsilon)} f\|_2^2 \le \int_{\mathbb{R}^d} l_{\phi(\varepsilon)}(y)\left[\int_{\mathbb{R}^d} \big|\nabla f(x-y)\big|^2\,dx\right] dy = \|\nabla f\|_2^2. \tag{2.17}$$

The right-hand side is bounded by 1. Thus (2.16) holds.

In the context of Theorem 1.1, (2.16) follows from the bound $|\gamma(z)| \le C(1 + |z|^{-\alpha})$ and a similar estimate [with (A.29) being replaced by (A.17)].

In the context of Theorem 1.4,

$$\int_{\mathbb{R}^d} \gamma(x-z)(\mathcal{S}_{\phi(\varepsilon)} f)^2(z)\,dz = (\mathcal{S}_{\phi(\varepsilon)} f)^2(x) \le \sup_{y\in\mathbb{R}} f^2(y).$$

Hence, (2.16) follows from the estimate

$$\begin{aligned}
f^2(y) &\le 2\int_{-\infty}^{\infty} \big|f(u)f'(u)\big|\,du \\
&\le 2\left\{\int_{-\infty}^{\infty} f^2(u)\,du\right\}^{1/2}\left\{\int_{-\infty}^{\infty} \big|f'(u)\big|^2\,du\right\}^{1/2} \le 2, \qquad y\in\mathbb{R}. \qquad \square
\end{aligned}$$

2.2. *Consequences of the entropy bounds*. According to the classic theory on sample path regularity (see, e.g., Appendix D, [7]), under (2.7) the supremum in (2.3) is finite, integrable and $\{\langle V, g^2\rangle; g\in\mathcal{G}_d(D)\}$ has continuous sample paths with respect to the pseudometric induced by its covariance. By the linearity of V and a standard extension argument, such sample continuity is extended to $\mathcal{S}(\mathbb{R}^d)$.

Given a generalized function ξ on $\mathbb{R}^d$, that is, a linear functional on $\mathcal{S}_d(\mathbb{R}^d)$, set

$$\lambda_\xi(D) = \sup_{g\in\mathcal{F}_d(D)}\left\{\langle \xi, g^2\rangle - \frac{1}{2}\int_D \big|\nabla g(x)\big|^2\,dx\right\}. \tag{2.18}$$

For any $\varepsilon > 0$, let D_ε be the ε-neighborhood of D. By the obvious monotonicity of $\lambda_\xi(D)$ in D, the limit

$$\lambda_\xi^+(D) \equiv \lim_{\varepsilon\to 0^+} \lambda_\xi(D_\varepsilon) \tag{2.19}$$

always exists at least as extended number. It is not clear to us whether or when $\lambda_\xi^+(D) = \lambda_\xi(D)$.

Let the ε-mollifier $l_\varepsilon(\cdot)$ be given in (2.8) and define the pointwise random field $V_\varepsilon(\cdot)$ as

$$V_\varepsilon(x) = \langle V, l_\varepsilon(\cdot - x)\rangle, \qquad x \in \mathbb{R}^d. \tag{2.20}$$

LEMMA 2.2. *Under the assumptions of Theorems 1.1, 1.3 or 1.4*

$$\lim_{\varepsilon\to 0^+} \mathbb{E} \sup_{g\in\mathcal{G}_d((-\varepsilon,\varepsilon)^d)} \langle V, g^2\rangle = 0 \tag{2.21}$$

and

$$\lambda_{\theta V}(D) \le \liminf_{\varepsilon\to 0^+} \lambda_{\theta V_\varepsilon}(D) \le \limsup_{\varepsilon\to 0^+} \lambda_{\theta V_\varepsilon}(D) \le \lambda^+_{\theta V}(D), \qquad a.s. \tag{2.22}$$

for any $\theta > 0$ *and bound domain* $D \subset \mathbb{R}^d$.

PROOF. In our view, $\mathcal{G}_d((-\varepsilon,\varepsilon)^d)$ is a subset of $\mathcal{G}_d((-1,1)^d)$ as $\varepsilon < 1$. By the continuity of the Gaussian field $\{\langle V, g^2\rangle; g \in \mathcal{G}_d((-1,1)^d)\}$ with respect to its covariance function established by Proposition 2.1,

$$\lim_{\delta\to 0^+} \mathbb{E}\sup\{\langle V, g^2\rangle; g \in \mathcal{G}_d\big((-1,1)^d\big) \text{ and } \mathbb{E}\langle V, g^2\rangle^2 \le \delta\} = 0.$$

To establish (2.21), it suffices to examine that

$$\lim_{\varepsilon\to 0^+} \sup_{g\in\mathcal{G}_d((-\varepsilon,\varepsilon)^d)} \mathbb{E}\langle V, g^2\rangle^2 = 0. \tag{2.23}$$

Indeed, in the case of Theorem 1.1,

$$\begin{aligned}\mathbb{E}\langle V, g^2\rangle^2 &= \int_{\mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y)\,dx\,dy \\ &\le C\int_{\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy \le C\varepsilon^{\alpha'-\alpha}\int_{\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^{\alpha'}}\,dx\,dy,\end{aligned}$$

where the constant $C > 0$ is different in each step but independent of g. The constant α' is chosen by the principle that $\alpha < \alpha' < 2 \wedge d$. Consequently,

$$\int_{\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^{\alpha'}}\,dx\,dy \le C_{\alpha'}\|g\|_2^{2-\alpha'}|\nabla g\|_2^{\alpha'} \le C_{\alpha'}, \qquad g \in \mathcal{G}_d\big((-\varepsilon,\varepsilon)^d\big),$$

where $C_{\alpha'}$ is given in (A.18) with α being replaced by α'. Hence, we have (2.23).

This argument applies also to the settings of Theorems 1.3 and 1.4. For Theorem 1.3, we use (A.29) instead of (A.17) and pick $2H_j - 1 < \alpha_j < 1$ $(j = 1,\dots,d)$ with $\alpha_1 + \cdots + \alpha_d < 2$.

As for Theorem 1.4, we first apply in (A.2), [3] [with $p = d = 1$, $\sigma = 1/2$ and $f(x) = g^4(x)$] that gives

$$\int_{-\infty}^{\infty} \frac{g^4(x)}{|x|^{1/2}}\,dx \le C\|g\|_8^4, \qquad g \in \mathcal{G}_d(\mathbb{R}),$$

where $C>0$ is independent of g. The right-hand side is uniformly bounded over $g\in\mathcal{G}_d(\mathbb{R})$ according to the Gagliardo–Nirenberg inequality (see, e.g., (C.1), page 303, [7])

$$\|g\|_8\le C\|g'\|_2^{3/8}\|g\|^{5/8}\le C,\qquad g\in\mathcal{G}_d(\mathbb{R}).$$

We now come to (2.22). Let $g\in\mathcal{F}_d(D)$ be fixed but arbitrary.

$$\lambda_{\theta V_\varepsilon}(D)\ge\theta\int_{\mathbb{R}^d}V_\varepsilon(x)g^2(x)\,dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx.$$

By linearity,

$$\int_{\mathbb{R}^d}V_\varepsilon(x)g^2(x)\,dx=\int_{\mathbb{R}^d}\langle V,l_\varepsilon(\cdot-x)\rangle g^2(x)\,dx=\langle V,(\mathcal{S}_\varepsilon g)^2\rangle. \tag{2.24}$$

In addition, by (2.10) and a proper normalization one can see that $\mathcal{S}_\varepsilon g$ converges to g under the covariance pseudomatric ρ given in (2.4). By the sample path continuity of the functional $\langle V,g^2\rangle$ resulting from Proposition 2.1,

$$\lim_{\varepsilon\to0^+}\langle V,(\mathcal{S}_\varepsilon g)^2\rangle=\langle V,g^2\rangle,\qquad\text{a.s.}$$

Hence,

$$\liminf_{\varepsilon\to0^+}\lambda_{\theta V_\varepsilon}(D)\ge\theta\langle V,g^2\rangle-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx,\qquad\text{a.s.}$$

Taking supremum over g on the right-hand side, we establish the lower bound needed by (2.22).

As for the upper bound, first notice that for any $g\in\mathcal{F}_d(D)$, $f\equiv\|\mathcal{S}_\varepsilon g\|_2^{-1}\mathcal{S}_\varepsilon g\in\mathcal{F}_d(D_\varepsilon)$. By (2.24) and linearity,

$$\begin{aligned}\lambda_{\theta V_\varepsilon}(D)&\le\sup_{g\in\mathcal{F}_d(D)}\left\{\langle V,(\mathcal{S}_\varepsilon g)^2\rangle-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla(\mathcal{S}_\varepsilon g)(x)|^2\,dx\right\}\\&\le\Big(\sup_{g\in\mathcal{F}_d(D)}\|\mathcal{S}_\varepsilon g\|_2^2\Big)\sup_{f\in\mathcal{F}_d(D_\varepsilon)}\left\{\langle V,f^2\rangle-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2\,dx\right\}\\&\le\lambda_{\theta V}(D_\varepsilon),\end{aligned}$$

where the last step follows from the fact $\|\mathcal{S}_\varepsilon g\|_2\le\|g\|_2=1$ [see (2.17)] for any $g\in\mathcal{F}_d(D)$.

Letting $\varepsilon\to0^+$ leads to the upper bound needed by (2.22). $\square$

In the rest of the section, we demonstrate how Proposition 2.1 (or Lemma 2.2, more precisely) is used to bound the principal eigenvalue given in (2.3).

The principal eigenvalue over a large domain can be essentially bounded by the extreme value among the principal eigenvalues of the sub-domains, according to a nice strategy developed by Gärtner and König [13]. Let $r\ge2$. By Proposition 1 in [13], also by Lemma 4.6 in [14], there is a nonnegative and continuous function $\Phi(x)$ on $\mathbb{R}^d$ whose support is contained in the 1-neighborhood of the grid $2r\mathbb{Z}^d$, such that for any $R > r$ and any generalized function ξ,

$$\lambda_{\xi-\Phi^y}(Q_R) \le \max_{z\in 2r\mathbb{Z}^d\cap Q_R} \lambda_\xi(z + Q_{r+1}), \qquad y \in Q_r, \tag{2.25}$$

where $\Phi^y(x) = \Phi(x + y)$, and we use the notation $Q_R = (-R, R)^d$ for any $R > 0$.

In addition, $\Phi(x)$ is periodic with period $2r$,

$$\Phi(x + 2rz) = \Phi(x), \qquad x \in \mathbb{R}^d, z \in \mathbb{Z}^d,$$

and there is a constant $K > 0$ independent of r such that

$$\frac{1}{(2r)^d}\int_{Q_r} \Phi(x)\,dx \le \frac{K}{r}. \tag{2.26}$$

It should be pointed out that originally, (2.25) was established for the ordinary function ξ. However, it can be extended to the generalized function without any extra effort, due to the linearity preserved by the form $\langle \xi, \varphi\rangle$ ($\varphi \in \mathcal{S}(\mathbb{R}^d)$).

Write

$$\eta(x) = \frac{1}{(2r)^d}\int_{Q_r} \Phi(x + y)\,dy = \frac{1}{(2r)^d}\int_{Q_r} \Phi^y(x)\,dy, \qquad x \in \mathbb{R}^d.$$

By periodicity, $\eta \equiv \eta(x)$ is a constant with a bound given in (2.26). Hence,

$$\begin{aligned}\lambda_\xi(Q_R) &\le \frac{K}{r} + \lambda_{\xi-\eta}(Q_R) \le \frac{K}{r} + \frac{1}{(2r)^d}\int_{Q_r} \lambda_{\xi-\Phi^y}(Q_R)\,dy \\ &\le \frac{K}{r} + \max_{z\in 2r\mathbb{Z}^d\cap Q_R} \lambda_\xi(z + Q_{r+1}),\end{aligned} \tag{2.27}$$

where the last inequality follows from (2.25), and the second inequality follows from the following steps:

$$\begin{aligned}\lambda_{\xi-\eta}(Q_R) &= \sup_{g\in\mathcal{F}_d(Q_R)}\left\{\frac{1}{(2r)^d}\int_{Q_r} \langle \xi - \Phi^y, g^2\rangle\,dy - \frac{1}{2}\int_{Q_R} |\nabla g(x)|^2\,dx\right\} \\ &= \sup_{g\in\mathcal{F}_d(Q_R)}\left\{\frac{1}{(2r)^d}\int_{Q_r}\left[\langle \xi - \Phi^y, g^2\rangle\,dy - \frac{1}{2}\int_{Q_R} |\nabla g(x)|^2\,dx\right]dy\right\} \\ &\le \frac{1}{(2r)^d}\int_{Q_r} \sup_{g\in\mathcal{F}_d(Q_R)}\left[\langle \xi - \Phi^y, g^2\rangle\,dy - \frac{1}{2}\int_{Q_R} |\nabla g(x)|^2\,dx\right]dy \\ &= \frac{1}{(2r)^d}\int_{Q_r} \lambda_{\xi-\Phi^y}(Q_R)\,dy.\end{aligned}$$

In the next lemma, we not only show that the principal eigenvalue in (2.3) is finite for any bounded domain D, but also provide sharp asymptotic bounds for the almost-sure increasing rate of the principal eigenvalue as D expands to $\mathbb{R}^d$ in a proper way.

LEMMA 2.3. *Under the assumptions of Theorems 1.1 or 1.3, for any $\theta > 0$,*

$$\limsup_{t\to\infty}(\log t)^{-2/(4-\alpha)}\lambda_{\theta V}(Q_t) \le \theta^{4/(4-\alpha)}h(d,\alpha), \qquad a.s., \tag{2.28}$$

where

$$h(d,\alpha) = \begin{cases} \dfrac{4-\alpha}{4}\left(\dfrac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\big(2dc(\gamma)\kappa(d,\alpha)\big)^{2/(4-\alpha)}, \\ \qquad \text{in the setting of Theorem 1.1}, \\ \dfrac{4-\alpha}{4}\left(\dfrac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\big(2dC_H\tilde{\kappa}(d,H)\big)^{2/(4-\alpha)}, \\ \qquad \text{in the setting of Theorem 1.3}. \end{cases} \tag{2.29}$$

Under the assumption of Theorem 1.4, for any $\theta > 0$,

$$\limsup_{t\to\infty}(\log t)^{-2/3}\lambda_{\theta V}\big((-t,t)\big) \le \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}\theta^{4/3}, \qquad a.s. \tag{2.30}$$

PROOF. Let $u > 0$ be fixed, and write

$$a(t) = \begin{cases} \sqrt{u}(\log t)^{1/(4-\alpha)}, \\ \qquad \text{in the setting of Theorems 1.1 or 1.3}, \\ \sqrt{u}(\log t)^{1/3}, \\ \qquad \text{in the setting of Theorem 1.4}. \end{cases} \tag{2.31}$$

For each $g \in \mathcal{S}(\mathbb{R}^d)$, write

$$g_t(x) = a(t)^{d/2}g\big(a(t)x\big), \qquad x \in \mathbb{R}^d. \tag{2.32}$$

By rescaling substitution $g \mapsto g_t$,

$$\lambda_{\theta V}(Q_t) = a(t)^2 \sup_{g\in\mathcal{F}_d(Q_{ta(t)})}\left\{\theta a(t)^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{Q_{ta(t)}}|\nabla g(x)|^2\,dx\right\}. \tag{2.33}$$

Let $\{\langle V_t,\varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ be the generalized Gaussian field defined as $\langle V_t,\varphi\rangle\rangle = \langle V,\tilde{\varphi}_t\rangle$, where $\tilde{\varphi}(x) = a(t)^d\varphi(a(t)x)$ [notice that this is different from the definition in (2.32)]. Then we have $\langle V, g_t^2\rangle = \langle V_t, g^2\rangle$. Taking $\xi = \theta a(t)^{-2}V_t$ in (2.27), by (2.33) we have that

$$\lambda_{\theta V}(Q_t) \le a(t)^2\left\{\frac{K}{r} + \max_{z\in 2r\mathbb{Z}^d\cap Q_{ta(t)}} X_z(t)\right\} \tag{2.34}$$

for any $r \ge 2$, where, by homogeneity of the Gaussian field $\{\langle V,\varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R}^d)\}$, the stochastic processes

$$X_z(t) \equiv \sup_{g\in\mathcal{F}_d(z+Q_{r+1})}\left\{\theta a(t)^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{z+Q_{r+1}}|\nabla g(x)|^2\,dx\right\},$$

$$z \in 2r\mathbb{Z}^d \cap Q_{ta(t)}$$

are identically distributed. Thus

$$\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^d\cap Q_{ta(t)}} X_z(t) > 1\Big\} \le \#\{2r\mathbb{Z}^d \cap Q_{ta(t)}\}\mathbb{P}\{X_0(t) > 1\}.$$

By linearity, for any $g \in \mathcal{F}_d(Q_{r+1})$,

$$\begin{aligned}&\theta a(t)^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{z+Q_{r+1}} |\nabla g(x)|^2\,dx \\ &\quad\le \theta a(t)^{-2}\Big(\sup_{f\in\mathcal{G}_d(Q_{r+1})}\langle V, f_t^2\rangle\Big)\Big(1+\frac{1}{2}\|\nabla g\|_2^2\Big) - \frac{1}{2}\|\nabla g\|_2^2.\end{aligned}$$

Here we recall that the class $\mathcal{G}_d(D)$ is defined in (2.2). Taking supremum over g,

$$X_0(t) \le \sup_{g\in\mathcal{F}_d(Q_{r+1})}\Big\{\theta a(t)^{-2}\Big(\sup_{f\in\mathcal{G}_d(Q_{r+1})}\langle V, f_t^2\rangle\Big)\Big(1+\frac{1}{2}\|\nabla g\|_2^2\Big) - \frac{1}{2}\|\nabla g\|_2^2\Big\}.$$

Consequently,

$$\{X_0(t) \ge 1\} \subset \Big\{\sup_{f\in\mathcal{G}_d(Q_{r+1})}\langle V, f_t^2\rangle \ge \theta^{-1}a(t)^2\Big\}.$$

Summarizing our argument,

$$\begin{aligned}&\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^d\cap Q_{ta(t)}} X_z(t) > 1\Big\} \\ &\quad\le \#\{2r\mathbb{Z}^d \cap Q_{ta(t)}\}\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(Q_{r+1})}\langle V, g_t^2\rangle \ge \theta^{-1}a(t)^2\Big\}.\end{aligned} \tag{2.35}$$

Notice that for each $g \in \mathcal{G}_d(Q_{r+1})$, $(1 + a(t)^2\|\nabla g\|_2^2)^{-1/2}g_t(\cdot) \in \mathcal{G}_d(Q_{(r+1)a(t)^{-1}})$. By linearity,

$$\begin{aligned}\mathbb{E}\sup_{g\in\mathcal{G}_d(Q_{r+1})}\langle V, g_t^2\rangle &\le (1+a(t)^2)\mathbb{E}\sup_{f\in\mathcal{G}_d(Q_{(r+1)a(t)^{-1}})}\langle V, f^2\rangle \\ &= o(a(t)^2) \qquad (t\to\infty),\end{aligned}$$

where the last step follows from (2.21) in Lemma 2.2.

By the concentration inequality for Gaussian field (see, e.g., (5.152), Theorem 5.4.3, page 219, [22], in connection to Corollary 5.4.5, page 224, [22]),

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(Q_{r+1})}\langle V, g_t^2\rangle > \theta^{-1}a(t)^2\Big\} \\ &\quad= \mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(Q_{r+1})}\langle V, g_t^2\rangle - \mathbb{E}\sup_{g\in\mathcal{G}_d(Q_{r+1})}\langle V, g_t^2\rangle > (1+o(1))\theta^{-1}a(t)^2\Big\} \\ &\quad\le \exp\Big\{-(1+o(1))\frac{a(t)^4}{2\theta^2\sigma_t^2}\Big\},\end{aligned} \tag{2.36}$$

where

$$\sigma_t^2 = \sup_{g\in\mathcal{G}_d(Q_{r+1})} \mathrm{Var}(\langle V, g_t^2\rangle).$$

In the setting of Theorem 1.1, by (1.10) and other assumptions on $\gamma(x)$,

$$\begin{aligned}
\sigma_t^2 &= \sup_{g\in\mathcal{G}_d(Q_{r+1})} \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y) g_t^2(x) g_t^2(y)\,dx\,dy \\
&= \sup_{g\in\mathcal{G}_d(Q_{r+1})} \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma\big(a(t)^{-1}(x-y)\big) g^2(x) g^2(y)\,dx\,dy \\
&\sim c(\gamma)a(t)^\alpha \sup_{g\in\mathcal{G}_d(Q_{r+1})} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy \qquad (t\to\infty).
\end{aligned}$$

Notice that

$$\begin{aligned}
\sup_{g\in\mathcal{G}_d(Q_{r+1})} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy &\le \sigma^2(d,\alpha) \\
&= \left(\frac{4-\alpha}{4}\right)^{(4-\alpha)/2} \left(\frac{\alpha}{2}\right)^{\alpha/2} \kappa(d,\alpha),
\end{aligned}$$

where $\sigma(d,\alpha)$ is the variation defined in (A.21) and the last step follows from (A.23) of Lemma A.4 in the Appendix.

In view of (2.31),

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(Q_{r+1})} \langle V, g_t^2\rangle > \theta^{-1}a(t)^2\Big\} \\
&\quad\le \exp\left\{-\big(1+o(1)\big)\left(\frac{4}{4-\alpha}\right)^{(4-\alpha)/2}\left(\frac{2}{\alpha}\right)^{\alpha/2} \frac{a(t)^{4-\alpha}}{2c(\gamma)\theta^2\kappa(d,\alpha)}\right\} \qquad (2.37)\\
&\quad\le \exp\{-(d+v)\log t\}
\end{aligned}$$

for some $v>0$, whenever t is large and the constant u [appearing in (2.31)] satisfies $u > \theta^{4/(4-\alpha)} h(d,\alpha)$.

The asymptotic bound (2.37) also holds in the setting of Theorem 1.3 by the same calculation of σ_t^2, where (A.23) in Lemma A.4 is replaced by (A.35) in Lemma A.6.

By (2.35), for large t there is $v'>0$ such that

$$\mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^d\cap Q_{2ta(t)+2r}} X_z(t) > 1\Big\} \le \exp\{-v'\log t\}.$$

Consequently,

$$\sum_k \mathbb{P}\Big\{\max_{z\in 2r\mathbb{Z}^d\cap Q_{2t_k a(t_k)+2r}} X_z(t_k) > 1\Big\} < \infty$$

for $t_k = 2^k$ $(k = 1, 2, \ldots)$. By Borel–Cantelli lemma,

$$\limsup_{k\to\infty} \max_{z\in 2r\mathbb{Z}^d\cap Q_{2t_k a(t_k)+2r}} X_z(t_k) \le 1, \qquad \text{a.s.}$$

In view of (2.31) and (2.34),

$$\limsup_{k\to\infty} (\log t_k)^{-2/(4-\alpha)} \lambda_{\theta V}(Q_{t_k}) \le \left(\frac{K}{r} + 1\right) u, \qquad \text{a.s.}$$

for any $u > \theta^{4/(4-\alpha)} h(d, \alpha)$. Thus, (2.28) follows from the facst that $\lambda_{\theta V}(Q_t)$ is monotonic in t, $K > 0$ is independent of r, r can be arbitrarily large and u can be arbitrarily close to $\theta^{4/(4-\alpha)} h(d, \alpha)$.

Based on the same argument, to establish (2.30) all we need is to show that

$$\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_1(Q_{r+1})} \langle V, g_t^2\rangle \ge \theta^{-1} a(t)^2\Big\} \le \exp\{-(1+v)\log t\} \tag{2.38}$$

for some $v > 0$, whenever t is large and and $u > \frac{1}{2}(\frac{3}{2})^{2/3}\theta^{4/3}$.

Indeed,

$$\sigma_t^2 = \sup_{g\in\mathcal{G}_1(Q_{r+1})} \int_{-(r+1)}^{r+1} \big(g_t^2(x)\big)^2\,dx \le a(t) \sup_{g\in\mathcal{G}_1(\mathbb{R})} \int_{-\infty}^{\infty} g^4(x)\,dx = \frac{3}{4}\left(\frac{1}{2}\right)^{3/2} a(t),$$

where the last step follows from (A.37) in Lemma A.7. By (2.36), therefore,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_1(Q_{r+1})} \langle V, g_t^2\rangle \ge \theta^{-1} a(t)^2\Big\} \\
&\quad \le \exp\left\{-(1+o(1))\left(\frac{2}{3}\right) 2^{3/2}\theta^{-2} a(t)^3\right\} \\
&\quad = \exp\left\{-(1+o(1))\left(\frac{2}{3}\right) 2^{3/2}\theta^{-2} u^{3/2}\log t\right\},
\end{aligned}$$

which leads to (2.38). $\square$

REMARK. Clearly, (2.28) and (2.30) still hold when $\lambda_{\theta V}(Q_t)$ is replaced by $\lambda_{\theta V}^{+}(Q_t)$. Further, they can be improved into equalities where the limsup can be strengthened into limit. The needed lower bounds will be given in Lemma 4.1 below.

3. Upper bounds. In this section we establish the upper bounds needed for Theorems 1.1, 1.3 and 1.4. Thanks to the homogeneity of the potential, the distribution of the quenched moment in our theorems does not depends on B_0. Therefore, we may take $B_0 = 0$ in the proof. In other words, we prove that for any $\theta > 0$,

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-1}(\log t)^{-2/(4-\alpha)} \log \mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} \\
&\qquad \le \theta^{4/(4-\alpha)} h(d, \alpha), \qquad \text{a.s.}
\end{aligned} \tag{3.1}$$

in the context of Theorems 1.1 or 1.3, where $h(d,\alpha)$ is defined in (2.29) and

$$\limsup_{t\to\infty} t^{-1}(\log t)^{-2/3}\log \mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} \le \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}\theta^{4/3}, \qquad \text{a.s.} \tag{3.2}$$

in the context of Theorem 1.4.

First, in all settings,

$$\mathbb{E}\otimes\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} < \infty, \qquad t > 0. \tag{3.3}$$

Consequently,

$$\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} < \infty, \qquad \text{a.s. } t > 0.$$

Here we recall our notation that "$\mathbb{E}$," "$\mathbb{P}$" are used for the expectation and probability with respect to the Gaussian potential, and that "$\mathbb{E}_0$," "$\mathbb{P}_0$" are used for the expectation and probability with respect to the Brownian motion starting at 0.

Indeed, by the (conditional) Gaussian property stated in Lemma A.1,

$$\mathbb{E}\otimes\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} = \mathbb{E}_0 \exp\left\{\frac{\theta^2}{2}\int_0^t\int_0^t \gamma(B_u - B_v)\,du\,dv\right\}.$$

Therefore, (3.3) follows from Theorem 4.3, [5] in the setting of Theorem 1.1; from (A.28) below in the setting of Theorem 1.3; and from Theorem 4.2.1, page 103, [7] in the setting of Theorem 1.4.

For any open domain $D \in \mathbb{R}^d$, set the exit time

$$\tau_D = \inf\{s \ge 0;\, B_s \notin D\}.$$

Recall the notation $Q_R = (-R, R)^d$.

In light of Lemma 2.3, our strategy for both upper and lower bounds can be roughly outlined by the following asymptotic relation:

$$\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} \approx \exp\{t\lambda_{\theta V}(Q_{R(t)})\}, \tag{3.4}$$

where the principal eigenvalue is introduced in (2.18), and square radius $R(t)$ is nearly linear and carefully chosen according to the context. To implement the upper bound, we consider the decomposition

$$\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\}$$
$$= \mathbb{E}_0\left[\exp\left\{\theta\int_0^t V(B_s)\,ds\right\}; \tau_{Q_{R_1}} \ge t\right]$$

$$+\sum_{k=1}^{\infty}\mathbb{E}_0\biggl[\exp\biggl\{\theta\int_0^t V(B_s)\,ds\biggr\};\tau_{Q_{R_k}}<t\le\tau_{Q_{R_{k+1}}}\biggr]$$

$$\le\mathbb{E}_0\biggl[\exp\biggl\{\theta\int_0^t V(B_s)\,ds\biggr\};\tau_{Q_{R_1}}\ge t\biggr]$$

$$+\sum_{k=1}^{\infty}\bigl(\mathbb{P}_0\{\tau_{Q_{R_k}}<t\}\bigr)^{1/2}\biggl\{\mathbb{E}_0\biggl[\exp\biggl\{2\theta\int_0^t V(B_s)\,ds\biggr\};\tau_{Q_{R_{k+1}}}\ge t\biggr]\biggr\}^{1/2},$$

where

$$R_k=\begin{cases}\bigl(Mt(\log t)^{1/(4-\alpha)}\bigr)^k, & \\ \qquad\text{in the context of Theorems 1.1 or 1.3,} & \\ \bigl(Mt(\log t)^{1/3}\bigr)^k, & \\ \qquad\text{in the context of Theorem 1.4,} & \end{cases}\qquad k=1,2,\ldots$$

and the constant $M>0$ is fixed (for a while at least), but arbitrary.

The first term in the above decomposition is the dominating term and is estimated in the following. Let $p,q>1$ with $p^{-1}+q^{-1}=1$ with p close to 1. By Lemma 4.3 [(4.5), with $\delta=1$ and (α,β) being replaced by (p,q)] in [8], we have for any $\varepsilon>0$,

$$\mathbb{E}_0\biggl[\exp\biggl\{\theta\int_0^t V_\varepsilon(B_s)\,ds\biggr\};\tau_{Q_{R_1}}\ge t\biggr]$$

$$\le\biggl(\mathbb{E}_0\exp\biggl\{\theta q\int_0^1 V_\varepsilon(B_s)\,ds\biggr\}\biggr)^{1/q}$$

$$\times\biggl\{\frac{1}{(2\pi)^{d/2}}\int_{Q_{R_1}}\mathbb{E}_x\biggl[\exp\biggl\{p\theta\int_0^{t-1}V_\varepsilon(B_s)\,ds\biggr\};\tau_{Q_{R_1}}\ge t-1\biggr]dx\biggr\}^{1/p},$$

where the Gaussian field $V_\varepsilon(\cdot)$ is defined in (2.20).

The purpose of taking the above steps is to localize the Brownian range and to re-shuffle the starting point of the Brownian motion uniformly over Q_{R_1}. The Brownian motion reaches anywhere of a super-linear (in t) distance from the origin with a super-exponentially small probability which is negligible in comparison to the essentially linear deviation scales shown in our main theorems. The reason behind re-shuffling is the explicit bounds (see, e.g., Lemmas 4.1 and 4.2 in [8]) between the principal eigenvalues appearing in Lemma 2.3 and the exponential moment of the Brownian occupation time, in the case when the Brownian motion has a uniformly distributed starting point. Indeed, according to Lemma 4.1 in [8],

$$\int_{Q_{R_1}}\mathbb{E}_x\biggl[\exp\biggl\{p\theta\int_0^{t-1}V_\varepsilon(B_s)\,ds\biggr\};\tau_{Q_{R_1}}\ge t-1\biggr]dx$$

$$\le|Q_{R_1}|\exp\bigl\{(t-1)\lambda_{p\theta V_\varepsilon}(Q_{R_1})\bigr\}.$$

Hence,

$$\mathbb{E}_0\left[\exp\left\{\theta\int_0^t V_\varepsilon(B_s)\,ds\right\}; \tau_{Q_{R_1}} \geq t\right] dx$$
$$\leq \left(\frac{2R_1^2}{\pi}\right)^{d/(2p)} \left(\mathbb{E}_0 \exp\left\{q\theta\int_0^1 V_\varepsilon(B_s)\,ds\right\}\right)^{1/q} \exp\{(t-1)\lambda_{\theta p V_\varepsilon}(Q_{R_1})\}.$$

The reason for considering V_ε instead of V is that Lemmas 4.3 and 4.1 in [8] were designed only for the pointwise defined functions. To pass the above inequality from V_ε to V, we let $\varepsilon \to 0^+$ on the both sides. First notice that for any fixed t, by comparing the variance between V_ε and V, we have that

$$\mathbb{E}\otimes\mathbb{E}_0 \exp\left\{\theta\int_0^t V_\varepsilon(B_s)\,ds\right\}$$
$$\leq \mathbb{E}\otimes\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\}$$

and by (3.3), the right-hand side is finite for arbitrary $\theta > 0$. Hence, a standard argument by uniform integrability together with Lemma A.1 leads to

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\otimes\mathbb{E}_0\left|\exp\left\{\theta\int_0^t V_\varepsilon(B_s)\,ds\right\} - \exp\left\{\theta\int_0^t V(B_s)\,ds\right\}\right| = 0. \tag{3.5}$$

Applying Fatou's lemma and (2.22) in Lemma 2.2 to the inequality,

$$\mathbb{E}_0\left[\exp\left\{\theta\int_0^t V(B_s)\,ds\right\}; \tau_{Q_{R_1}} \geq t\right] dx$$
$$\leq \left(\frac{2R_1^2}{\pi}\right)^{d/(2p)} \left(\mathbb{E}_0 \exp\left\{q\theta\int_0^1 V(B_s)\,ds\right\}\right)^{1/q} \exp\{(t-1)\lambda^+_{\theta p V}(Q_{R_1})\}, \quad \text{a.s.}$$

By a similar argument with $p = q = 2$,

$$\mathbb{E}_0\left[\exp\left\{2\theta\int_0^t V(B_s)\,ds\right\}; \tau_{Q_{R_{k+1}}} \geq t\right]$$
$$\leq \left(\frac{2R_{k+1}^2}{\pi}\right)^{d/4} \left(\mathbb{E}_0 \exp\left\{4\theta\int_0^1 V(B_s)\,ds\right\}\right)^{1/2} \exp\{(t-1)\lambda^+_{4\theta V}(Q_{R_{k+1}})\}, \quad \text{a.s.}$$

for $k = 1, 2, \ldots$.

Summarizing our estimate,

$$\mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\}$$
$$\leq \left(\frac{2R_1^2}{\pi}\right)^{d/(2p)} \left(\mathbb{E}_0 \exp\left\{\theta q\int_0^1 V(B_s)\,ds\right\}\right)^{1/q} \exp\{(t-1)\lambda^+_{\theta p V}(Q_{R_1})\}$$

$$+\left(\mathbb{E}_0 \exp\left\{4\theta \int_0^1 V(B_s)\,ds\right\}\right)^{1/2}$$
$$\times \sum_{k=1}^{\infty}\left(\frac{2R_{k+1}^2}{\pi}\right)^{d/4} (\mathbb{P}_0\{\tau_{Q_{R_k}} < t\})^{1/2} \exp\{(t-1)\lambda_{4\theta V}^{+}(Q_{R_{k+1}})\}, \qquad \text{a.s.}$$

By the classic fact on the Gaussian tail,

$$(\mathbb{P}_0\{\tau_{Q_{R_k}} < t\})^{1/2} \le \exp\{-cR_k^2/t\} = \exp\{-cM^{2k}t^{2k-1}(\log t)^{2k/(4-\alpha)}\}.$$

Consequently, (3.1) and (3.2) follow from Lemma 2.3. Indeed, by (2.28) or (2.30) (depending on the context), the second term (in the form of infinite series) on the right-hand side of the established bound is almost surely bounded when M is sufficiently large, and the first term contributes essentially up to the bound given in (3.1) or (3.2) as $p > 1$ can be made arbitrarily close to 1.

4. Lower bounds. In this section we establish the lower bounds needed for Theorems 1.1, 1.3 and 1.4. In other words, we prove that for any $\theta > 0$,

$$\begin{aligned}(4.1)\qquad &\liminf_{t\to\infty} t^{-1}(\log t)^{-2/(4-\alpha)} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t V(B_s)\,ds\right\}\\ &\qquad \ge \theta^{4/(4-\alpha)} h(d,\alpha), \qquad \text{a.s.}\end{aligned}$$

in the context of Theorems 1.1 or 1.3, where $h(d,\alpha)$ is defined in (2.29) and

$$\begin{aligned}(4.2)\qquad &\liminf_{t\to\infty} t^{-1}(\log t)^{-2/3} \log \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)\,ds\right\}\\ &\qquad \ge \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}\theta^{4/3}, \qquad \text{a.s.}\end{aligned}$$

in the context of Theorem 1.4.

Our treatment consists of two parts: Implementation of (3.4) for its lower bounds and establishment of the lower bounds for the principal eigenvalues which correspond to the upper bounds given in Lemma 2.3.

All notation used in Sections 2 and 3 is adopted here. Let $p, q > 1$ satisfy $p^{-1} + q^{-1} = 1$ with p being close to 1, and let $0 < b < 1$ be close to 1. For each $\varepsilon > 0$, let the pointwise defined potential $V_\varepsilon(x)$ be given as (2.20). Taking $\alpha = p$ and $q = \beta$, $\delta = t^b$ in (4.6), Lemma 4.3, [8] we have

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\theta \int_0^t V_\varepsilon(B_s)\,ds\right\}\\ &\qquad \ge \left(\mathbb{E}_0 \exp\left\{-\frac{q}{p}\theta \int_0^{t^b} V_\varepsilon(B_s)\,ds\right\}\right)^{-p/q}\end{aligned}$$

$$\times \left(\int_{Q_{t^b}} p_{t^b}(x) \mathbb{E}_x \exp\left\{ \frac{\theta}{p} \int_0^{t-t^b} V_\varepsilon(B_s)\, dx \right\} \right)^p$$

$$\geq \left(\mathbb{E}_0 \exp\left\{ -\frac{q}{p} \theta \int_0^{t^b} V_\varepsilon(B_s)\, ds \right\} \right)^{-p/q}$$

$$\times \left(\frac{e^{-ct^b}}{(2\pi t^b)^{d/2}} \int_{Q_{t^b}} \mathbb{E}_x \exp\left\{ \frac{\theta}{p} \int_0^{t-t^b} V_\varepsilon(B_s) \right\} dx \right)^p,$$

where $p_{t^b}(x)$ is the probability density of B_{t^b}.

Taking $\delta = t^b$ again and replacing t, α and β by $t - t^b$, p and q, respectively, in Lemma 4.2, [8],

$$\int_{Q_{t^b}} \mathbb{E}_x \exp\left\{ \frac{\theta}{p} \int_0^{t-t^b} V_\varepsilon(B_s) \right\} dx$$

$$\geq (2\pi)^{pd/2} (t - t^b)^{db/2} (t - t^b)^{pd/(2q)} (t - t^b)^{-2db}$$

$$\times \exp\left\{ -\frac{p}{q} (t - t^b)^b \lambda_{(p/q)\theta V_\varepsilon}(Q_{t^b}) \right\} \exp\{ pt \lambda_{\theta V_\varepsilon/p}(Q_{t^b}) \}.$$

Noticing that $\lambda_{\theta V_\varepsilon/p}(Q_{t^b}), \lambda_{(p/q)\theta V_\varepsilon}(Q_{t^b}) \geq 0$, and replacing e^{-ct^b} by e^{-Ct^b} for a larger C to absorb all bounded-by-polynomial quantities,

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^t V_\varepsilon(B_s)\, ds \right\} \geq e^{-Ct^b} \left(\mathbb{E}_0 \exp\left\{ -\frac{q}{p} \theta \int_0^{t^b} V_\varepsilon(B_s)\, ds \right\} \right)^{-p/q}$$

$$\times \exp\left\{ -\frac{p^2}{q} t^b \lambda_{(p/q)\theta V_\varepsilon}(Q_{t^b}) \right\} \exp\{ t \lambda_{\theta V_\varepsilon/p}(Q_{t^b}) \}.$$

Letting $\varepsilon \to 0^+$ and taking the relation $V \overset{d}{=} -V$ into account, by (3.5) and (2.22) in Lemma 2.2,

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^t V(B_s)\, ds \right\}$$

$$\geq e^{-Ct^b} \left(\mathbb{E}_0 \exp\left\{ -\frac{q}{p} \theta \int_0^{t^b} V(B_s)\, ds \right\} \right)^{-p/q}$$

$$\times \exp\left\{ -\frac{p^2}{q} t^b \lambda^+_{(p/q)\theta V}(Q_{t^b}) \right\} \exp\{ t \lambda_{\theta V/p}(Q_{t^b}) \}, \qquad \text{a.s.}$$

Here we try to explain the strategy used in the above steps. The Brownian motion is allowed to re-shuffle its starting point uniformly over Q_{t^b} within the affordable price e^{-Ct^b}. We take $b < 1$ to make sure that the energy spent by the Brownian motion during the "relocation period" $[0, t^b]$ is insignificant. Indeed, replacing V

by $-V$ and t by t^b in (3.1) or in (3.2),

$$\log \mathbb{E}_0 \exp\left\{-\frac{q}{p}\theta \int_0^{t^b} V(B_s)\,ds\right\} = o(t), \qquad \text{a.s. } (t \to \infty).$$

In addition, by Lemma 2.1,

$$\frac{p^2}{q} t^b \lambda^+_{(p/q)\theta V}(Q_{t^b}) = o(t), \qquad \text{a.s.}$$

under $b < 1$.

On the other hand, we make b close to 1 to give the Brownian motion a decent chance to reach any location (within the period $[0, t^b]$) up to the distance $t^b \approx t$ where the energy is rich to the degree requested by the lower bounds in (4.1) and (4.2).

By the fact that $p > 1$ and $b < 1$ can be made arbitrarily close to 1 [In particular, $\lambda_{\theta V/p}(Q_{t^b}) \approx \lambda_{\theta V}(Q_t)$.], the lower bounds (4.1) and (4.2) follow from the next lemma which states another side of the story stated in Lemma 2.3.

LEMMA 4.1. *Under the assumptions of Theorems 1.1 or 1.3, for any $\theta > 0$,*

$$\liminf_{t\to\infty} (\log t)^{-2/(4-\alpha)} \lambda_{\theta V}(Q_t) \ge \theta^{4/(4-\alpha)} h(d, \alpha), \qquad a.s., \tag{4.3}$$

where $h(d, \alpha)$ is given in (2.29).

Under the assumption of Theorem 1.4, for any $\theta > 0$

$$\liminf_{t\to\infty} (\log t)^{-2/3} \lambda_{\theta V}\big((-t, t)\big) \ge \frac{1}{2}\left(\frac{3}{2}\right)^{2/3} \theta^{4/3}, \qquad a.s. \tag{4.4}$$

PROOF. Recall that $a(t)$ and $g_t(x)$ are defined in (2.31) and (2.32), respectively. Let the constant $r > 0$ be fixed but arbitrary, and set $\mathcal{N}_t = 2r\mathbb{Z}^d \cap Q_{t-r}$. By (2.33) and by the monotonicity of $\lambda_{\theta V}(D)$ in the set $D \subset \mathbb{R}^d$,

$$\lambda_{\theta V}(Q_t) \ge a(t)^2 \max_{z\in\mathcal{N}_t} \sup_{g\in\mathcal{F}_d(a(t)z+Q_r)} \left\{\theta a(t)^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{a(t)z+Q_r} |\nabla g(x)|^2\,dx\right\}.$$

For any $g \in \mathcal{G}_d(Q_r)$ and $z \in \mathcal{N}_t$, notice that $g^z(\cdot) \equiv g(\cdot - a(t)z) \in \mathcal{F}_d(a(t)z + Q_r)$, and by translation invariance,

$$\int_{a(t)z+Q_r} |\nabla g^z(x)|^2\,dx = \int_{Q_r} |\nabla g(x)|^2\,dx, \qquad z \in \mathcal{N}_t.$$

Consequently,

$$\lambda_{\theta V}(Q_t) \ge a(t)^2 \left\{\theta a(t)^{-2} \max_{z\in\mathcal{N}_t} \langle V, (g^z)_t^2\rangle - \frac{1}{2}\int_{Q_r} |\nabla g(x)|^2\,dx\right\} \tag{4.5}$$

for any $g \in \mathcal{F}_d(Q_r)$. In the following argument $g \in \mathcal{F}_d(Q_r)$ is fixed but arbitrary. Set $t_k = 2^k$ $(k = 1, 2, \ldots)$. Our next step is to show that

$$\liminf_{k\to\infty} a(t_k)^{-2} \max_{z\in\mathcal{N}_{t_k}} \langle V, (g^z)^2_{t_k}\rangle \geq \sigma(g), \qquad \text{a.s.} \tag{4.6}$$

whenever

$$\begin{cases} u < \big(2dc(\gamma)\big)^{2/(4-\alpha)}, & \text{in the context of Theorem 1.1,} \\ u < (2dC_H)^{2/(4-\alpha)}, & \text{in the context of Theorem 1.3,} \\ u < 2^{2/3}, & \text{in the context of Theorem 1.4,} \end{cases} \tag{4.7}$$

where

$$\sigma(g) = \begin{cases} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^p}\,dx\,dy\right)^{1/2}, \\ \qquad \text{in Theorem 1.1,} \\ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} g^2(x)g^2(y)\left(\prod_{j=1}^d |x_j-y_j|^{2-2H_j}\right)^{-1} dx\,dy\right)^{1/2}, \\ \qquad \text{in Theorem 1.3,} \\ \left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2}, \\ \qquad \text{in Theorem 1.4.} \end{cases}$$

The proof of (4.6) in the setting of Theorem 1.4 is easy due to the fact that the sequence

$$\langle V, (g^z)^2_t\rangle, \qquad z \in \mathcal{N}_t$$

is an i.i.d. family with the common distribution $N(0, a(t)\sigma^2(g))$. Consequently,

$$\begin{aligned} &\mathbb{P}\left\{\max_{z\in\mathcal{N}_t}\langle V, (g^z)^2_t\rangle \leq a(t)^2\left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2}\right\} \\ &\quad = \left(1 - \mathbb{P}\left\{\langle V, (g^0)^2_t\rangle > a(t)^2\left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2}\right\}\right)^{\#(\mathcal{N}_t)}. \end{aligned}$$

By the classic tail estimate for normal distribution,

$$\begin{aligned} &\mathbb{P}\left\{\langle V, (g^0)^2_t\rangle > a(t)^2\left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2}\right\} \\ &\quad = \exp\left\{-(1+o(1))\frac{a(t)^3}{2}\right\} = \exp\left\{-(1+o(1))\frac{u^{3/2}\log t}{2}\right\}. \end{aligned}$$

By the fact that $\#(\mathcal{N}_t) \sim (2r)^{-1}t$ as $t \to \infty$, we have

$$\mathbb{P}\left\{\max_{z\in\mathcal{N}_t}\langle V, (g^z)^2_t\rangle \leq a(t)^2\left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2}\right\} \leq \exp\{-t^\beta\} \tag{4.8}$$

for some $\beta > 0$, whenever $u < 2^{2/3}$. Consequently,

$$\sum_k \mathbb{P}\left\{\max_{z\in\mathcal{N}_{t_k}} \langle V, (g^z)^2_{t_k}\rangle \le a(t_k)^2\left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2}\right\} < \infty.$$

Hence, (4.6) follows from Borel–Cantelli lemma.

In the settings of Theorems 1.1 and 1.3, the proof of (4.6) is harder due to lack of independence. Our approach relies on the control of the covariance. Write $\xi_z(t) = \langle V, (g^z)^2_t\rangle$. For each $z, z' \in \mathcal{N}_t$,

$$\begin{aligned}
&\mathrm{Cov}(\xi_z, \xi_{z'})\\
&\quad= \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y)\big(g^z\big)^2_t(x)\big(g^{z'}\big)^2_t(y)\,dx\,dy\\
&\quad= \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma\big(x-y+\big(z-z'\big)\big)g^2_t(x)g^2_t(y)\,dx\,dy\\
&\quad= \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma\big(a(t)^{-1}(x-y)+\big(z-z'\big)\big)g^2(x)g^2(y)\,dx\,dy, \qquad z, z' \in \mathcal{N}_t.
\end{aligned}$$

Taking $z = z'$ in the setting of Theorem 1.1,

$$\begin{aligned}
\mathrm{Var}\big(\xi_0(t)\big) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma\big(a(t)^{-1}(x-y)\big)g^2(x)g^2(y)\,dx\,dy\\
&\sim c(\gamma)\sigma^2(g)a(t)^\alpha \qquad (t\to\infty),
\end{aligned} \tag{4.9}$$

where the last step follows from (1.10).

Using (1.19) instead of (1.10), we can see that in the setting of Theorem 1.3,

$$\mathrm{Var}\big(\xi_0(t)\big) = C_H\sigma^2(g)a(t)^\alpha \qquad (t > 0). \tag{4.10}$$

We now claim that in both settings,

$$R_t \equiv \max_{\substack{z,z'\in\mathcal{N}_t\\ z\ne z'}} \big|\mathrm{Cov}\big(\xi_z(t), \xi_{z'}(t)\big)\big| = o\big(a(t)^\alpha\big) \qquad (t\to\infty). \tag{4.11}$$

By the assumption that $\gamma(x)$ is bounded on $\{|x| \ge 1\}$, $\mathrm{Cov}(\xi_z(t), \xi_{z'}(t))$ is bounded uniformly over the pairs (z, z') with $z \ne z'$ and over t in the setting of Theorem 1.1. In particular, (4.11) holds.

The proof of (4.11) is a little trickier when it comes to Theorem 1.3. That is the reason why we cannot have a constant bound for $\mathrm{Cov}(\xi_z(t), \xi_{z'}(t))$ with $z \ne z'$. More precisely, $\mathrm{Cov}(\xi_z(t), \xi_{z'}(t)) \to \infty$ as $t \to \infty$ when $z_j = z'_j$ for some $1 \le j \le d$. Here we use the notation $z = (z_1, \dots, z_d)$. Write

$$J\big(z, z'\big) = \big\{1 \le j \le d; z_j = z'_j\big\}, \qquad z, z' \in \mathcal{N}_t.$$

By (1.19),

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma\big(a(t)^{-1}(x-y)+(z-z')\big)g^2(x)g^2(y)\,dx\,dy\\&\quad\sim C_H\Big(\prod_{j\notin J(z,z')}|z_j-z_j'|^{2-2H_j}\Big)^{-1}a(t)^{\alpha(z,z')}\\&\qquad\times\int_{\mathbb{R}^d\times\mathbb{R}^d}g^2(x)g^2(y)\Big(\prod_{j\in J(z,z')}|x_j-y_j|^{2-2H_j}\Big)^{-1}dx\,dy\qquad(t\to\infty),\end{aligned}$$

where

$$\alpha\big(z,z'\big)=\sum_{j\in J(z,z')}(2-2H_j).$$

By the fact that $|z_j-z_j'|\ge 2r$ for $j\notin J(z,z')$, the above asymptotic equivalence can be developed into the uniform bound

$$\max_{\substack{z,z'\in\mathcal{N}_t\\ z\neq z'}}\big|\mathrm{Cov}\big(\xi_z(t),\xi_{z'}(t)\big)\big|\le Ca(t)^{\alpha'},$$

where

$$\alpha'\equiv\max_{\substack{z,z'\in\mathcal{N}_t\\ z\neq z'}}\alpha\big(z,z'\big)<\alpha.$$

So (4.11) holds.

Given a small but fixed $v>0$, taking $A=\sigma(g)a(t)^2$ and $B=v\sigma(g)a(t)^2$ in Lemma 4.2 below,

$$\begin{aligned}&\mathbb{P}\Big\{\max_{z\in\mathcal{N}_t}\xi_z(t)\le\sigma(g)a(t)^2\Big\}\\&\quad\le\bigg(\mathbb{P}\bigg\{\xi_0(t)\le(1+v)\sigma(g)a(t)^2\sqrt{\frac{2R_t+\mathrm{Var}(\xi_0(t))}{\mathrm{Var}(\xi_0(t))}}\bigg\}\bigg)^{\#\mathcal{N}_t}\\&\qquad+\mathbb{P}\big\{U\ge v\sigma(g)a(t)^2/\sqrt{2R_t}\big\},\end{aligned}$$

where U is a standard normal random variable.

For the second term on the right-hand side,

$$\mathbb{P}\big\{U\ge v\sigma(g)a(t)^2\big\}=\exp\bigg\{-\big(1+o(1)\big)\frac{v^2a(t)^4\sigma^2(g)}{4R_t}\bigg\}\le\exp\{-2\log t\}$$

for large t, where the last step follows from (4.11).

As for the first term, by (4.9) and (4.10) the algorithm used in (4.8) shows that it is bounded by e^{-t^β} for some $\beta>0$ when t is large, v is small and u satisfies (4.7).

Summarizing our computation, we obtain a bound that leads to

$$\sum_k \mathbb{P}\Big\{\max_{z\in\mathcal{N}_{t_k}} \xi_z(t_k) \le \sigma(g)a(t_k)^2\Big\} < \infty.$$

So (4.6) follows from Borel–Cantelli lemma.

In view of (4.5), (4.6) implies that for every $g \in \mathcal{F}_d(Q_r)$,

$$\begin{aligned}\liminf_{k\to\infty}&(\log t_k)^{-2/(4-\alpha)}\lambda_{\theta V}(Q_{t_k})\\ &\ge \big(2dc(\gamma)\big)^{2/(4-\alpha)}\Big\{\theta\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy\Big)^{1/2}\\ &\qquad -\frac{1}{2}\int_{\mathbb{R}^d}\big|\nabla g(x)\big|^2\,dx\Big\}, \qquad \text{a.s.}\end{aligned} \tag{4.12}$$

in the setting of Theorem 1.1, that

$$\begin{aligned}\liminf_{k\to\infty}&(\log t_k)^{-2/(4-\alpha)}\lambda_{\theta V}(Q_{t_k})\\ &\ge (2dC_H)^{2/(4-\alpha)}\Big\{\theta\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} g^2(x)g^2(y)\Big(\prod_{j=1}^d |x_j-y_j|^{2-2H_j}\Big)^{-1}\,dx\,dy\Big)^{1/2}\\ &\qquad -\frac{1}{2}\int_{\mathbb{R}^d}\big|\nabla g(x)\big|^2\,dx\Big\},\\ &\qquad \text{a.s.}\end{aligned} \tag{4.13}$$

in the setting of Theorem 1.3, and that

$$\begin{aligned}\liminf_{k\to\infty}&(\log t_k)^{-2/3}\lambda_{\theta V}(Q_{t_k})\\ &\ge 2^{2/3}\Big\{\theta\Big(\int_{-\infty}^{\infty} g^4(x)\,dx\Big)^{1/2} - \frac{1}{2}\int_{-\infty}^{\infty}\big|g'(x)\big|^2\,dx\Big\}, \qquad \text{a.s.}\end{aligned} \tag{4.14}$$

in the setting of Theorem 1.4.

By the monotonicity of $\lambda_{\theta V}(Q_t)$ in t, the liminf along the sub-sequence t_k in (4.12), (4.13) and (4.14) can be extended into the liminf along the continuous time t.

Recall that $W^{1,2}(\mathbb{R}^d)$ is the Sobolev space defined in (1.12). Consistently with (2.1), we define

$$\mathcal{F}_d(\mathbb{R}^d) = \{g \in W^{1,2}(\mathbb{R}^d);\ \|g\|_2 = 1\}.$$

We now prove that the functions g on the right-hand sides of (4.12), (4.13) and (4.14) can be extended from $\mathcal{F}_d(Q_r)$ to $\mathcal{F}_d(\mathbb{R}^d)$, and complete the proof of Lemma 4.1.

We start with (4.12). The right-hand side can be extended to all $g \in \mathcal{F}_d(\mathbb{R}^d)$ for the following two reasons: First, the infinitely smooth, rapidly decreasing and locally supported functions are dense in the Sobolev space $W^{1,2}(\mathbb{R}^d)$ under the Sobolev norm

$$\|g\|_{W^{1,2}(\mathbb{R}^d)} \equiv \sqrt{\|g\|_2^2 + \frac{1}{2}\|\nabla g\|_2^2}$$

and $r > 0$ in (4.12) is arbitrary. Second, by (A.18) the functional

$$\mathcal{F}(g) = \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha} \, dx \, dy \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^d} \left| \nabla g(x) \right|^2 dx$$

is continuous under the Sobolev norm $\| \cdot \|_{W^{1,2}(\mathbb{R}^d)}$.

Taking supremum over $g \in \mathcal{F}_d(\mathbb{R}^d)$ on the right-hand side of (4.12) we obtain the lower bound

$$\begin{aligned} &\liminf_{t\to\infty} (\log t)^{-2/(4-\alpha)} \lambda_{\theta V}(Q_t) \\ &\quad \geq \left(2dc(\gamma)\right)^{2/(4-\alpha)} M_{d,\alpha}(\theta) \\ &\quad = \frac{4-\alpha}{4} \left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)} \left(2dc(\gamma)\theta^2 \kappa(d,\alpha)\right)^{2/(4-\alpha)}, \qquad \text{a.s.} \end{aligned}$$

in the setting of Theorem 1.1, where $M_{d,\alpha}(\theta)$ is defined in (A.20), and the last step follows from the variation identity (A.22).

Using (A.34) (with $\alpha_j = 2 - 2H_j$) instead of (A.22), by the same argument, from (4.13) we derive that

$$\begin{aligned} &\liminf_{t\to\infty} (\log t)^{-2/(4-\alpha)} \lambda_{\theta V}(Q_t) \\ &\quad \geq (2dC_H)^{2/(4-\alpha)} \widetilde{M}_{d,\alpha}(\theta) \\ &\quad = \frac{4-\alpha}{4} \left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)} \left(2dc(\gamma)\theta^2 \tilde{\kappa}(d,H)\right)^{2/(4-\alpha)}, \qquad \text{a.s.} \end{aligned}$$

in the setting of Theorem 1.3.

In the same way, by (A.36) and (4.14) we have

$$\begin{aligned} &\liminf_{t\to\infty} (\log t)^{2/3} \lambda_{\theta V}(Q_t) \\ &\quad \geq 2^{2/3} \sup_{g\in\mathcal{F}_1(\mathbb{R})} \left\{ \theta \left(\int_{-\infty}^{\infty} g^4(x)\,dx \right)^{1/2} - \frac{1}{2} \int_{-\infty}^{\infty} |g'(x)|^2 \, dx \right\} \\ &\quad = \frac{1}{2} \left(\frac{3}{2}\right)^{2/3} \theta^{4/3}, \qquad \text{a.s.} \end{aligned}$$

in the setting of Theorem 1.4. $\square$

We end this section with the following lemma.

LEMMA 4.2. *Let* $(\xi_1, \dots, \xi_n)$ *be a mean-zero Gaussian vector with identically distributed components. Write*

$$R = \max_{i \neq j} \left| \mathrm{Cov}(\xi_i, \xi_j) \right|$$

and assume that $\mathrm{Var}(\xi_1) \geq 2R$. *Then for any* $A, B > 0$,

$$\mathbb{P}\Big\{ \max_{k \leq n} \xi_k \leq A \Big\} \leq \left(\mathbb{P}\Big\{ \xi_1 \leq \sqrt{\frac{2R + \mathrm{Var}(\xi_1)}{\mathrm{Var}(\xi_1)}}(A + B) \Big\} \right)^n + \mathbb{P}\{U \geq B/\sqrt{2R}\},$$

where U *is a standard normal random variable.*

PROOF. Let $\eta_1, \dots, \eta_n$ be an i.i.d. sequence independent of U. Assume that $\eta_1 \overset{d}{=} \xi_1$ and write

$$\zeta_k = \sqrt{\frac{\mathrm{Var}(\xi_1)}{2R + \mathrm{Var}(\xi_1)}}(\eta_k + \sqrt{2R}U).$$

With the assumption $\mathrm{Var}(\xi_1) \geq 2R$, it is straightforward to exam that

$$\mathrm{Var}(\xi_k) = \mathrm{Var}(\zeta_k) \quad \text{and} \quad \mathrm{Cov}(\xi_i, \xi_j) \leq \mathrm{Cov}(\zeta_i, \zeta_j), \qquad i, j, k = 1, \dots, n.$$

By Slepian's lemma ([23], see also Lemma 5.5.1, [22]),

$$\mathbb{P}\Big\{ \max_{k \leq n} \xi_k \leq A \Big\} \leq \mathbb{P}\Big\{ \max_{k \leq n} \zeta_k \leq A \Big\}.$$

Notice that

$$\max_{k \leq n} \zeta_k = \sqrt{\frac{2R\, \mathrm{Var}(\xi_1)}{2R + \mathrm{Var}(\xi_1)}} U + \sqrt{\frac{\mathrm{Var}(\xi_1)}{2R + \mathrm{Var}(\xi_1)}} \max_{k \leq n} \eta_k.$$

By the triangle inequality,

$$\mathbb{P}\Big\{ \max_{k \leq n} \xi_k \leq A \Big\} \leq \mathbb{P}\Big\{ \max_{k \leq n} \eta_k \leq \sqrt{\frac{2R + \mathrm{Var}(\xi_1)}{\mathrm{Var}(\xi_1)}}(A + B) \Big\} + \mathbb{P}\{U \leq -B/\sqrt{2R}\}.$$

The conclusion follows from the symmetry of U and the independence of $\{\eta_k\}$. $\square$

APPENDIX

A.1. Brownian integral as a limit. In this subsection, $\langle V, \varphi \rangle$ ($\varphi \in \mathcal{S}(\mathbb{R}^d)$) is a mean-zero generalized Gaussian field with homogeneity defined in (1.6). Let $\mu(dx)$ be the spectral measure of $\langle V, \varphi \rangle$ and let the pointwise defined Gaussian field $V_\varepsilon(x)$ ($x \in \mathbb{R}^d$) be given in (2.20). The main goal here is to prove

LEMMA A.1. *Assume that*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\mu(d\lambda) < \infty. \tag{A.1}$$

Under the product law $\mathbb{P}\otimes\mathbb{P}_x$, *the* $\mathcal{L}^2$*-limit*

$$\int_0^t V(B_s)\,ds \stackrel{def}{=} \lim_{\varepsilon\to 0^+}\int_0^t V_\varepsilon(B_s)\,ds \tag{A.2}$$

exists for every $t \ge 0$. *In addition, there is a modification of the limiting process in* (A.2) *that is* $(\frac{1}{2}-u)$*-Hölder continuous for any* $u > 0$. *Further, conditioned on the Brownian motion, the process*

$$\int_0^t V(B_s)\,ds, \qquad t \ge 0 \tag{A.3}$$

is mean-zero Gaussian with the (*conditional*) *variance*

$$\mathbb{E}\left\{\int_0^t V(B_s)\,ds\right\}^2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\lambda\cdot B_s}\,ds\right|^2\mu(d\lambda), \qquad t \ge 0. \tag{A.4}$$

PROOF. First notice that conditioned on the Brownian motion, the process

$$I_\varepsilon(t) = \int_0^t V_\varepsilon(B_s)\,ds, \qquad t \ge 0$$

is Gaussian with the conditional variance

$$\mathbb{E}I_\varepsilon^2(t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left|\int_0^t e^{i\lambda\cdot B(u)}\,du\right|^2 \big|\mathcal{F}(l)(\varepsilon\lambda)\big|^2\mu(d\lambda). \tag{A.5}$$

We claim that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^d}\mathbb{E}_x\left|\int_0^t e^{i\lambda\cdot B(u)}\,du\right|^2\mu(d\lambda) \le C\big(t\vee t^2\big), \qquad t \ge 0. \tag{A.6}$$

Indeed,

$$\begin{aligned}\mathbb{E}_x\left|\int_0^t e^{i\lambda\cdot B(u)}\,du\right|^2 &= \int_0^t\int_0^t \mathbb{E}_x e^{i\lambda\cdot(B_u-B_v)}\,du\,dv\\ &= \int_0^t\int_0^t \exp\left\{-\frac{|\lambda|^2}{2}|u-v|\right\}du\,dv.\end{aligned}$$

The right-hand side is equal to

$$\frac{4}{|\lambda|^2}\left[t - \frac{2}{|\lambda|^2}\big(1-e^{-t|\lambda|^2/2}\big)\right],$$

which yields a bound $4t/|\lambda|^2$ for $|\lambda| \ge 1$. As for $|\lambda| \le 1$, we use the trivial bound

$$\int_0^t\int_0^t \exp\left\{-\frac{|\lambda|^2}{2}|u-v|\right\}du\,dv \le t^2.$$

Thus

$$\int_{\mathbb{R}^d} \mathbb{E}_x \left| \int_0^t e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda) \le 4t \int_{\{|\lambda|\ge 1\}} \frac{1}{|\lambda|^2} \mu(d\lambda) + t^2 \int_{\{|\lambda|\le 1\}} \mu(d\lambda).$$

Hence, (A.6) follows from (A.1).

To prove the $\mathcal{L}^2$-convergence described in (A.2), all we need is to establish the existence of the limit $\lim_{\varepsilon,\varepsilon'\to 0^+} \mathbb{E}_x \otimes \mathbb{E}(I_{\varepsilon'}(t) I_\varepsilon(t))$.

Indeed, similar to (A.5),

$$\mathbb{E}_x \otimes \mathbb{E}\big(I_{\varepsilon'}(t) I_\varepsilon(t)\big) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}_x \left| \int_0^t e^{i\lambda\cdot B(u)}\, du \right|^2 \mathcal{F}(l)(\varepsilon\lambda) \overline{\mathcal{F}(l)\big(\varepsilon'\lambda\big)} \mu(d\lambda).$$

By (A.6), the fact that

$$\big|\mathcal{F}(l)(\varepsilon\lambda)\big| \le 1 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \mathcal{F}(l)(\varepsilon\lambda) = 1,$$

and by the dominant convergence theorem we obtain

$$\lim_{\varepsilon,\varepsilon'\to 0^+} \mathbb{E}_x \otimes \mathbb{E}\big(I_{\varepsilon'}(t) I_\varepsilon(t)\big) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}_x \left| \int_0^t e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda).$$

Write $I_0(t) = \lim_{\varepsilon\to 0^+} I_\varepsilon(t)$ as the $\mathcal{L}^2(\mathbb{P}_x \otimes \mathbb{P})$-limit. Recall the classical fact that the $\mathcal{L}^2$-limit of Gaussian process remains Gaussian. Conditioned on the Brownian motion, $\{I_0(t); t \ge 0\}$ is Gaussian with zero mean and the conditional variance

$$\mathbb{E} I_0^2(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \int_0^t e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda). \tag{A.7}$$

Strictly speaking, $\{I_0(t); t \ge 0\}$ exists as a family of equivalent classes. In the following we try to find a continuous modification of this family. For any $s, t \ge 0$ with $s < t$, notice that $I_0(t) - I_0(s)$ is conditionally normal with the variance

$$\begin{aligned} \mathbb{E}\big[I_0(t) - I_0(s)\big]^2 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \int_s^t e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda) \\ &\stackrel{d}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \int_0^{t-s} e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda). \end{aligned}$$

Thus, for any integer $m \ge 1$,

$$\begin{aligned} &\mathbb{E}_x \otimes \mathbb{E}\big[I_0(t) - I_0(s)\big]^{2m} \\ &\quad = (2m-1)!!\mathbb{E}_x \left(\int_{\mathbb{R}^d} \left| \int_0^{t-s} e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda) \right)^m. \end{aligned} \tag{A.8}$$

To estimate the right-hand side, we consider the nonnegative, continuous process

$$Z_t = \left\{ \int_{\mathbb{R}^d} \left| \int_0^t e^{i\lambda\cdot B(u)}\, du \right|^2 \mu(d\lambda) \right\}^{1/2}, \qquad t \ge 0.$$

By the triangle inequality,

$$Z_{s+t} \le Z_t + Z'_s, \qquad s,t \ge 0, \tag{A.9}$$

where

$$Z'_s = \left\{ \int_{\mathbb{R}^d} \left| \int_t^{t+s} e^{i\lambda\cdot B(u)}\,du \right|^2 \mu(d\lambda) \right\}^{1/2}$$

is independent of $\{B_u; 0 \le u \le t\}$ and equal in law to Z_s. By (1.3.7), page 21 in [7], for any $t, a, b > 0$,

$$\mathbb{P}_x\{Z_t \ge a+b\} \le \mathbb{P}_x\{Z_t \ge a\}\mathbb{P}_x\{Z_t \ge b\}.$$

Consequently,

$$\mathbb{P}_x\{Z_t \ge Mn\sqrt{t}\} \le \left(\mathbb{P}_x\{Z_t \ge M\sqrt{t}\}\right)^n, \qquad n = 1, 2, \dots.$$

By (A.6), one can take $M > 0$ sufficiently large so

$$\sup_{0<t\le 1} \mathbb{P}_x\{Z_t \ge M\sqrt{t}\} \le e^{-2}.$$

Hence,

$$\sup_{0<t\le 1} \mathbb{E}_x \exp\{M^{-1} Z_t/\sqrt{t}\} < \infty. \tag{A.10}$$

Replacing t by $t-s$ and applying it to (A.8), we obtain

$$\mathbb{E}\otimes\mathbb{E}\left|I_0(t) - I_0(s)\right|^{2m} \le C_m |t-s|^m \qquad \text{for all } s,t \ge 0 \text{ with } |t-s| \le 1.$$

By the classic result on chaining (see, e.g., Lemma 9, [10]), there is a modification of $\{I_0(t); t \ge 0\}$ that is $(\frac{1}{2} - u)$-Hölder continuous for any $u > 0$. $\square$

LEMMA A.2. *Under the assumptions in Theorems 1.1, 1.3 or 1.4, (1.9) holds for some $\delta > 0$. In particular, the Brownian integral in (A.3) is well-defined as stated in Lemma A.1.*

PROOF. We first consider the setting of Theorem 1.1. By the fact that μ is tempered, all we need to show is

$$\int_{\{|\lambda|\ge 1\}} \frac{1}{|\lambda|^{2(1-\delta)}} \mu(d\lambda) < \infty.$$

Let φ be the density of the standard normal distribution on $\mathbb{R}^d$. By Fourier transform

$$\begin{aligned} 2^{kd} \int_{\mathbb{R}^d} \gamma(x)\varphi(2^k x)\,dx &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left\{-\frac{|2^{-k}\lambda|^2}{2}\right\}\mu(d\lambda) \\ &\ge c\mu\{2^{k-1} \le |\lambda| \le 2^k\}. \end{aligned}$$

On the other hand, by (1.10)

$$2^{kd}\int_{\mathbb{R}^d}\gamma(x)\varphi(2^k x)\,dx=\int_{\mathbb{R}^d}\gamma(2^{-k}x)\varphi(x)\,dx$$
$$\sim c(\gamma)2^{\alpha k}\int_{\mathbb{R}^d}\frac{\varphi(x)}{|x|^\alpha}\,dx \qquad (k\to\infty).$$

Hence, there is a constant $C>0$ such that

$$\mu\{2^{k-1}\le|\lambda|\le 2^k\}\le C2^{\alpha k},\qquad k=1,2,\dots.$$

Thus

$$\int_{\{|\lambda|\ge 1\}}\frac{1}{|\lambda|^{2(1-\delta)}}\mu(d\lambda)\le C\sum_{k=1}^\infty 2^{-2(1-\delta)(k-1)}\mu\{2^{k-1}\le|\lambda|\le 2^k\}<\infty$$

for any $\delta<\frac{2-\alpha}{2}$.

In the setting of Theorem 1.4, where $\mu(d\lambda)=d\lambda$ is the 1-dimensional Lebesgue measure, the validity of (1.9) can be directly verified with any $\delta<1$.

As for the setting of Theorem 1.3, by (1.19) and spherical substitution,

$$\int_{\mathbb{R}^d}\frac{1}{(1+|\lambda|^2)^{1-\delta}}\mu(d\lambda)=\widehat{C}_H\int_{\mathbb{R}^d}\left(\prod_{j=1}^d|\lambda_j|^{2H_j-1}\right)^{-1}\frac{1}{(1+|\lambda|^2)^{1-\delta}}\,d\lambda$$
$$=C\int_0^\infty r^{-(d-\alpha)}\frac{r^{d-1}}{(1+r^2)^{1-\delta}}\,dr$$
$$=C\int_0^\infty\frac{r^{\alpha-1}}{(1+r^2)^{1-\delta}}\,dr<\infty$$

as $\delta<\frac{2-\alpha}{2}$. $\square$

A.2. Counting the covering balls. Let $D, D'\subset\mathbb{R}^d$ be two domains in $\mathbb{R}^d$ and $\mathcal{Q}(D)$ be a class of functions on D. Assume that D' is bounded. For each $\varepsilon>0$, let $\rho_\varepsilon(f,g)$ be a pseudometric on $\mathcal{Q}(D)$ such that

$$\rho_\varepsilon(f,g)\le\left(\int_{D'}\left|A_\varepsilon(f)(x)-A_\varepsilon(g)(x)\right|dx\right)^{1/2}\left(\sup_{x\in D'}\left|B_\varepsilon(f)(x)-B_\varepsilon(g)(x)\right|\right)^{1/2},$$

where A_ε and B_ε are two (possibly nonlinear) maps from $\mathcal{Q}(D)$ to the space $\mathrm{Lip}(D')$ of Lipschitz functions on D'. Assume further that there are constants $C>0$, $p>1$, $m\ge 1$ such that

$$\text{(A.11)}\qquad \begin{gathered}\left|B_\varepsilon(g)(x)\right|\le C\quad\text{and}\\ \left|B_\varepsilon(g)(x)-B_\varepsilon(g)(y)\right|\le C\varepsilon^{-m}|x-y|,\qquad x,y\in D',\end{gathered}$$

$$\text{(A.12)}\qquad \begin{gathered}\int_{D'}\left|A_\varepsilon(g)(x)\right|^p dx\le C\quad\text{and}\\ \left|A_\varepsilon(g)(x)-A_\varepsilon(g)(y)\right|\le C\varepsilon^{-m}|x-y|,\qquad x,y\in D'\end{gathered}$$

uniformly for all $g \in \mathcal{Q}(D)$ and sufficiently small $\varepsilon > 0$.

LEMMA A.3. *Under the above assumptions*,

$$\log N(\mathcal{Q}(D), \rho_\varepsilon, \varepsilon) = O\left(\varepsilon^{-2p/(2p-1)} \log \frac{1}{\varepsilon}\right) \qquad (\varepsilon \to 0^+).$$

PROOF. Notice that

$$\hat{\rho}_\varepsilon(f, g) = \int_{D'} |A_\varepsilon(f)(x) - A_\varepsilon(g)(x)|\, dx \quad \text{and}$$
$$\rho^*_\varepsilon(f, g) = \sup_{x \in D'} |B_\varepsilon(f)(x) - B_\varepsilon(g)(x)|$$

define two pseudometrics on $\mathcal{Q}(D)$. We now claim that for any $u, v > 0$ with $\sqrt{uv} = \varepsilon$,

$$N(\mathcal{Q}(D), \rho_\varepsilon, \varepsilon) \le N(\mathcal{Q}(D), \rho^*_\varepsilon, u) N(\mathcal{Q}(D), \hat{\rho}_\varepsilon, v). \tag{A.13}$$

Indeed, we first cover $\mathcal{Q}(D)$ by $N(\mathcal{Q}(D), \rho^*_\varepsilon, u)$ ρ^*_ε-balls with the diameter smaller than u. For each such ball, it can be covered by at most $N(\mathcal{Q}(D), \hat{\rho}_\varepsilon, v)$ of $\hat{\rho}_\varepsilon$-balls with the diameter smaller than v. In this way, the set $\mathcal{Q}(D)$ is covered by at most $N(\mathcal{Q}(D), \rho^*, u) N(\mathcal{Q}(D), \hat{\rho}_\varepsilon, v)$ of its nonempty subsets. For f, g coming from same subset, $\hat{\rho}_\varepsilon(f, g) < u$ and $\rho^*_\varepsilon(f, g) < v$. Hence

$$\rho_\varepsilon(f, g) \le \sqrt{\hat{\rho}_\varepsilon(f, g) \rho^*(f, g)} < \sqrt{uv} = \varepsilon.$$

Hence, (A.13) holds.

With (A.13), it is sufficient to establish

$$N(\mathcal{Q}(D), \rho^*_\varepsilon, \varepsilon^{2p/(2p-1)}) = \exp\left\{O\left(\varepsilon^{-2p/(2p-1)} \log \frac{1}{\varepsilon}\right)\right\} \qquad (\varepsilon \to 0^+) \tag{A.14}$$

$$N(\mathcal{Q}(D), \hat{\rho}_\varepsilon, \varepsilon^{(2(p-1))/(2p-1)}) = \exp\left\{O\left(\varepsilon^{-2p/(2p-1)} \log \frac{1}{\varepsilon}\right)\right\} \qquad (\varepsilon \to 0^+). \tag{A.15}$$

Indeed, applying (A.13) with

$$u(\varepsilon) = \varepsilon^{2p/(2p-1)} \quad \text{and} \quad v(\varepsilon) = \varepsilon^{(2(p-1))/(2p-1)}, \tag{A.16}$$

and using (A.14) and (A.15) we have

$$\begin{aligned} N(\mathcal{Q}(D), \rho_\varepsilon, \varepsilon) &\le N(\mathcal{Q}(D), \rho^*_\varepsilon, u(\varepsilon)) N(\mathcal{Q}(D), \hat{\rho}_\varepsilon, v(\varepsilon)) \\ &= \exp\left\{O\left(\varepsilon^{-2p/(2p-1)} \log \frac{1}{\varepsilon}\right)\right\} \qquad (\varepsilon \to 0^+). \end{aligned}$$

We first prove (A.14). Let $u(\varepsilon)$ be defined in (A.16). Define the map

$$B_\varepsilon^*: \mathcal{Q}(D) \longrightarrow \big((\sqrt{d}C)^{-1}\varepsilon^m u(\varepsilon)\mathbb{Z}^d \cap D'\big)^{u(\varepsilon)\mathbb{Z}\cap[-C,C]}$$

as $B_\varepsilon^* f(x) = u(\varepsilon)[u(\varepsilon)^{-1}B_\varepsilon(f)(x_0)]$ whenever

$$x \in \big(x_0 - (2\sqrt{d}C)^{-1}\varepsilon^m u(\varepsilon), x_0 + (2\sqrt{d}C)^{-1}\varepsilon^m u(\varepsilon)\big]^d$$

for some $x_0 \in (\sqrt{d}C)^{-1}\varepsilon^m u(\varepsilon)\mathbb{Z}^d \cap D'$, where $[\cdot]$ is the integer-part function.

By (A.11)

$$\sup_{x\in D'} \big|B_\varepsilon g(x) - B_\varepsilon^* g(x)\big| < \frac{u(\varepsilon)}{2}, \qquad g \in \mathcal{Q}(D).$$

Consequently, for any $f, g \in \mathcal{Q}(D)$ with $B_\varepsilon^* f = B_\varepsilon^* g$,

$$\rho_\varepsilon^*(f,g) = \sup_{x\in D'} \big|B_\varepsilon f(x) - B_\varepsilon g(x)\big| < u(\varepsilon).$$

Hence,

$$\begin{aligned} N\big(\mathcal{Q}(D), \rho_\varepsilon^*, u(\varepsilon)\big) &\le \#\big\{\big((\sqrt{d}C)^{-1}\varepsilon^m u(\varepsilon)\mathbb{Z}^d \cap D'\big)^{u(\varepsilon)\mathbb{Z}\cap[-C,C]}\big\} \\ &= \exp\left\{O\left(\varepsilon^{-2p/(2p-1)}\log\frac{1}{\varepsilon}\right)\right\} \qquad (\varepsilon \to 0^+). \end{aligned}$$

It remains to establish (A.15). Let $v(\varepsilon)$ be given in (A.16), and write

$$M_\varepsilon = \big(8Cv(\varepsilon)^{-1}\big)^{(p-1)^{-1}}.$$

Define the map

$$A_\varepsilon^*: \mathcal{Q}(D) \longrightarrow \big((4|D'|\sqrt{d}C)^{-1}\varepsilon^m v(\varepsilon)\mathbb{Z}^d \cap D'\big)^{(8|D'|)^{-1}v(\varepsilon)\mathbb{Z}\cap[-M_\varepsilon,M_\varepsilon]}$$

as $A_\varepsilon^* g(x) = \{(8|D'|)^{-1}v(\varepsilon)[8|D'|v(\varepsilon)^{-1}A_\varepsilon g(x_0)] \wedge M_\varepsilon\} \vee (-M_\varepsilon)$, whenever

$$x \in \big(x_0 - (8|D'|\sqrt{d}C)^{-1}\varepsilon^m v(\varepsilon), x_0 + (8|D'|\sqrt{d}C)^{-1}\varepsilon^m v(\varepsilon)\big]^d$$

for some $x_0 \in (4|D'|\sqrt{d}C)^{-1}\varepsilon^m v(\varepsilon)\mathbb{Z}^d \cap D'$.

By (A.12),

$$\begin{aligned} &\sup_{g\in\mathcal{Q}(D)} \int_{D'} \big|A_\varepsilon(g)(x) - A_\varepsilon^*(g)(x)\big|\,dx \\ &\qquad \le \frac{1}{4}v(\varepsilon) + 2\sup_{g\in\mathcal{Q}(D)} \int_{\{|A_\varepsilon(g)|>M_\varepsilon\}} \big|A_\varepsilon(g)(x)\big|\,dx \\ &\qquad \le \frac{1}{4}v(\varepsilon) + 2M_\varepsilon^{-(p-1)}C \le \frac{1}{2}v(\varepsilon). \end{aligned}$$

Consequently, for $f, g \in \mathcal{Q}(D)$ with $A_\varepsilon^* f = A_\varepsilon^* g$, $\hat{\rho}_\varepsilon(f,g) < v(\varepsilon)$ for small ε. Hence,

$$\begin{aligned} N\big(\mathcal{Q}(D), \hat{\rho}_\varepsilon, v(\varepsilon)\big) &\le \#\big\{\big((4|D'|\sqrt{d}C)^{-1}\varepsilon^m v(\varepsilon)\mathbb{Z}^d \cap D'\big)^{(8|D'|)^{-1}v(\varepsilon)\mathbb{Z}\cap[-M_\varepsilon,M_\varepsilon]}\big\} \\ &= \exp\left\{O\left(\varepsilon^{-2p/(2p-1)}\log\frac{1}{\varepsilon}\right)\right\} \qquad (\varepsilon \to 0^+). \qquad \square \end{aligned}$$

A.3. Variations. In this section we establish some Sobolev-type inequalities and validate the variations used in the paper. Recall that $W^{1,2}(\mathbb{R}^d)$ is the Sobolev space defined in (1.12) and

$$\mathcal{F}_d(\mathbb{R}^d) = \{g \in W^{1,2}(\mathbb{R}^d); \|g\|_2 = 1\}.$$

Similar to (2.2), define

$$\mathcal{G}_d(\mathbb{R}^d) = \{g \in W^{1,2}(\mathbb{R}^d); \|g\|_2^2 + \tfrac{1}{2}\|g\|_2^2 = 1\}.$$

Recall (Lemma 7.2, [8]) that for any $0 \le \alpha < 2 \wedge d$ there is $C_\alpha > 0$ such that

$$\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^\alpha} dx \le C_\alpha \|f\|_2^{2-\alpha} \|\nabla f\|_2^\alpha, \qquad f \in W^{1,2}(\mathbb{R}^d). \tag{A.17}$$

A simple trick by translation invariance, show that (A.17) remains true with the same constant C_α if the left-hand side is replaced by

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f^2(x)}{|x-y|^\alpha} dx.$$

Immediately,

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f^2(x) f^2(y)}{|x-y|^\alpha} dx\, dy &= \int_{\mathbb{R}^d} f^2(y) \left[\int_{\mathbb{R}^d} \frac{f^2(x)}{|x-y|^\alpha} dx \right] dy \\ &\le C_\alpha \|f\|_2^{4-\alpha} \|\nabla f\|_2^\alpha \end{aligned} \tag{A.18}$$

for every $f \in W^{1,2}(\mathbb{R}^d)$.

As a consequence, the constant

$$\begin{aligned} \kappa(d,\alpha) = \inf\biggl\{ C > 0; \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f^2(x) f^2(y)}{|x-y|^\alpha} dx\, dy \\ \le C \|f\|_2^{4-\alpha} \|\nabla f\|_2^\alpha \ \forall f \in W^{1,2}(\mathbb{R}^d) \biggr\} \end{aligned} \tag{A.19}$$

is finite.

Other variations relevant to Theorem 1.1 are

$$\begin{aligned} M_{d,\alpha}(\theta) = \sup_{g \in \mathcal{F}_d(\mathbb{R}^d)} \biggl\{ \theta \biggl(\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{g^2(x) g^2(y)}{|x-y|^\alpha} dx\, dy \biggr)^{1/2} \\ - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \biggr\}, \qquad \theta > 0, \end{aligned} \tag{A.20}$$

$$\sigma(d,\alpha) = \sup_{g \in \mathcal{G}_d(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{g^2(x) g^2(y)}{|x-y|^\alpha} dx\, dy \right\}^{1/2}. \tag{A.21}$$

By (A.17), one can easily show that $M_{d,\alpha}(\theta)$ and $\sigma(d,\alpha)$ are finite under the assumption $0 \le \alpha < 2 \wedge d$.

LEMMA A.4. *Under* $\alpha < 2 \wedge d$,

$$M_{d,\alpha}(\theta) = \frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)} \kappa(d,\alpha)^{2/(4-\alpha)}\theta^{4/(4-\alpha)}, \tag{A.22}$$

$$\sigma(d,\alpha) = \left(\frac{4-\alpha}{4}\right)^{(4-\alpha)/4}\left(\frac{\alpha}{2}\right)^{\alpha/4} \kappa(d,\alpha)^{1/2}. \tag{A.23}$$

PROOF. Let $f \in \mathcal{F}(\mathbb{R}^d)$ be fixed but arbitrary, and let $C_f > 0$ satisfy

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f^2(x)f^2(y)}{|x-y|^\alpha}\,dx\,dy = C_f\|\nabla f\|_2^\alpha.$$

Given $\beta > 0$ let $g(x) = \beta^{d/2} f(\beta x)$. Then $\|\nabla g\|_2 = \beta\|\nabla f\|_2$ and therefore

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy = \beta^\alpha \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f^2(x)f^2(y)}{|x-y|^\alpha}\,dx\,dy = C_f\beta^\alpha\|\nabla f\|_2^\alpha.$$

By the fact that $g \in \mathcal{F}_d(\mathbb{R}^d)$,

$$M_{d,\alpha}(\theta) \ge \theta C_f^{1/2}\beta^{\alpha/2}\|\nabla f\|_2^{\alpha/2} - \tfrac{1}{2}\|\nabla g\|_2^2 = \theta C_f^{1/2}\beta^{\alpha/2}\|\nabla f\|_2^{\alpha/2} - \tfrac{1}{2}\beta^2\|\nabla f\|_2^2.$$

Notice $\beta\|\nabla f\|_2$ runs over all positive numbers. So we have

$$M_{d,\alpha}(\theta) \ge \sup_{x>0}\left\{\theta C_f^{1/2}x^{\alpha/2} - \frac{1}{2}x^2\right\} = \frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)} C_f^{2/(4-\alpha)}\theta^{4/(4-\alpha)}.$$

Take supremum over f on the right-hand side. Noticing that $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$, by space homogeneity we have established the relation "$\ge$" for (A.22).

On the other hand, for any $g \in \mathcal{F}_d(\mathbb{R}^d)$,

$$\begin{aligned}
&\theta\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy\right)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx \\
&\quad\le \theta\kappa(d,\alpha)^{1/2}\|\nabla g\|_2^{\alpha/2} - \frac{1}{2}\|\nabla g\|_2^2 \le \sup_{x>0}\left\{\theta\kappa(d,\alpha)^{1/2}x^{\alpha/2} - \frac{1}{2}x^2\right\} \\
&\quad= \frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\kappa(d,\alpha)^{2/(4-\alpha)}\theta^{4/(4-\alpha)}.
\end{aligned}$$

Taking supremum over $g \in \mathcal{F}_d(\mathbb{R}^d)$ on the left-hand side, we reach the relation "$\le$" for (A.22).

For any $g \in \mathcal{F}_d(\mathbb{R}^d)$ by space homogeneity,

$$\begin{aligned}
&\frac{1}{\sigma(d,\alpha)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g^2(x)g^2(y)}{|x-y|^\alpha}\,dx\,dy\right)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx \\
&\quad\le \frac{1}{\sigma(d,\alpha)}\sigma(d,\alpha)\big(1 + \|\nabla g\|_2^2\big) - \frac{1}{2}\|\nabla g\|_2^2 = 1.
\end{aligned}$$

Taking supremum over g,

$$M_{d,\alpha}\left(\frac{1}{\sigma(d,\alpha)}\right) \le 1.$$

Combining this with (A.22) we have proved the "$\ge$" half for (A.23).

On the other hand, for any $f \in W^{1,2}(\mathbb{R}^d)$,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f^2(x)f^2(y)}{|x-y|^\alpha}\,dx\,dy\right)^{1/2} \\
&\quad\le \kappa(d,\alpha)^{1/2}\|f\|_2^{(4-\alpha)/2}\|\nabla f\|_2^{\alpha/2} \\
&\quad= \kappa(d,\alpha)^{1/2}\left(\frac{2\alpha}{4-\alpha}\right)^{\alpha/4}(\|f\|_2^2)^{(4-\alpha)/4}\left(\frac{4-\alpha}{2\alpha}\|\nabla f\|_2^2\right)^{\alpha/4} \\
&\quad\le \kappa(d,\alpha)^{1/2}\left(\frac{2\alpha}{4-\alpha}\right)^{\alpha/4}\frac{4-\alpha}{4}\left(\|f\|_2^2+\frac{1}{2}\|\nabla f\|_2^2\right),
\end{aligned}$$

where the last step follows from the Hölder inequality $ab \le p^{-1}a^p + q^{-1}b^q$ with $p = 4(4-\alpha)^{-1}$ and $q = 4/\alpha$. This leads to the "$\le$" half for (A.23). $\square$

We need an inequality comparable to the one in (A.17) for formulating and proving Theorem 1.3, but could not find it in literature. We establish it in the following.

Let the real numbers $\alpha_1, \dots, \alpha_d$ satisfy $0 \le \alpha_j < 1$ and $\alpha \equiv \alpha_1 + \cdots + \alpha_d < 2$.

LEMMA A.5. *For any $\theta > 0$,*

$$\sup_{g\in\mathcal{F}_d(\mathbb{R}^d)}\left\{\theta\int_{\mathbb{R}^d}\left(\prod_{j=1}^d |x_j|^{-\alpha_j}\right)g^2(x)\,dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\right\} < \infty. \tag{A.24}$$

PROOF. Define the function

$$K(x) = \prod_{j=1}^d |x_j|^{-\alpha_j}, \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

The fact that $K(x)$ blows up at every coordinate plane make the problem harder comparing to setting of the Newtonian kernel $|x|^{-\alpha}$ which blows up only at 0. The fact that $\alpha_1, \dots, \alpha_d$ are allowed to be different posts an extra challenge. The proof provided here is probabilistic.

Let the linear Brownian motions $B_1(s), \dots, B_d(s)$ be the independent components of the d-dimensional Brownian motion B_s and define the process

$$\eta_t = \int_0^t K(B_s)\,ds = \int_0^t \left(\prod_{j=1}^d |B_j(s)|^{-\alpha_j}\right)ds, \qquad t > 0. \tag{A.25}$$

This process is well defined under our assumption on $\alpha_1, \dots, \alpha_d$. Indeed, it is not hard to see that for each $t > 0$, $\mathbb{E}_0 \eta_t < \infty$. Further, we now prove that there is a $b > 0$ such that

$$\mathbb{E}_0 \exp\{b\eta_1^{2/\alpha}\} < \infty. \tag{A.26}$$

We point out that (A.26) is a strengthened version of the exponential integrability for η_1 obtained by Hu, Nualart and Song (Lemma A.5, [19]) and the approach for (A.26) presented here is modified from theirs.

Given the integer $m \ge 1$,

$$\begin{aligned}\mathbb{E}_0 \eta_t^m &= \int_{[0,t]^m} ds_1 \cdots ds_m \prod_{j=1}^d \mathbb{E}_0 \prod_{k=1}^m |B_1(s_k)|^{-\alpha_j} \\ &= m! \int_{[0,t]_<^m} ds_1 \cdots ds_m \prod_{j=1}^d \mathbb{E}_0 \prod_{k=1}^m |B_1(s_k)|^{-\alpha_j},\end{aligned}$$

where the multi-dimensional time set $[0,t]_<^m$ is defined as

$$[0,t]_<^m = \{(s_1, \dots s_m) \in [0,t]^m; s_1 < s_2 \cdots < s_m\}.$$

Let $(s_1, \dots, s_m) \in [0,t]_<^m$ be fixed for a while and $\mathcal{A}_s = \sigma\{B_1(u); 0 \le u \le s\}$ be the filtration generated by the linear Brownian motion $B_1(t)$. Write

$$\begin{aligned}\mathbb{E}_0\{|B_1(s_k)|^{-\alpha_j} | \mathcal{A}_{s_{k-1}}\} &= \int_0^\infty \mathbb{P}_0\{|B_1(s_k)|^{-\alpha_j} \ge a | \mathcal{A}_{s_{k-1}}\}\, da \\ &= \int_0^\infty \mathbb{P}_0\{|B_1(s_k)| \le a^{-1/\alpha_j} | \mathcal{A}_{s_{k-1}}\}\, da.\end{aligned}$$

By Anderson's inequality,

$$\begin{aligned}&\mathbb{P}_0\{|B_{s_k}| \le a^{-1/\alpha_j} | \mathcal{A}_{s_{k-1}}\} \\ &\quad = \mathbb{P}_0\{|B_1(s_{k-1}) + (B_1(s_k) - B_1(s_{k-1}))| \le a^{-1/\alpha_j} | \mathcal{A}_{s_{k-1}}\} \\ &\quad \le \mathbb{P}_0\{|B_1(s_k) - B_1(s_{k-1})| \le a^{-1/\alpha_j} | \mathcal{A}_{s_{k-1}}\} = \mathbb{P}_0\{|B_1(s_k - s_{k-1})|^{-\alpha_j} \ge a\}.\end{aligned}$$

So we have

$$\begin{aligned}\mathbb{E}_0 \prod_{k=1}^m |B_1(s_k)|^{-\alpha_j} &\le \prod_{k=1}^m \mathbb{E}_0 |B_1(s_k - s_{k-1})|^{-\alpha_j} \\ &= \{\mathbb{E}_0 |B_1(1)|^{-\alpha_j}\}^m \prod_{k=1}^m (s_k - s_{k-1})^{-\alpha_j}, \qquad j = 1, \dots, d.\end{aligned}$$

Here the convention $s_0 = 0$ is adopted.

Summarizing our computation,

$$\mathbb{E}_0\eta_t^m \le m!\left(\prod_{j=1}^d \mathbb{E}_0\big|B_1(1)\big|^{-\alpha_j}\right)^m \int_{[0,t]_<^m}\prod_{k=1}^m (s_k - s_{k-1})^{-\alpha/2}\,ds_1\cdots ds_m.$$

Let τ be an exponential time with parameter 1 such that τ is independent of B_t. By Fubini's theorem

$$\begin{aligned}
&\mathbb{E}^\tau \otimes \mathbb{E}_0\eta_\tau^m \\
&\quad\le m!\left(\prod_{j=1}^d \mathbb{E}_0\big|B_1(1)\big|^{-\alpha_j}\right)^m \\
&\qquad\times\int_0^\infty e^{-t}\left[\int_{[0,t]_<^m}\prod_{k=1}^m (s_k - s_{k-1})^{-\alpha/2}\,ds_1\cdots ds_m\right]dt \qquad\text{(A.27)}\\
&\quad= m!\left(\prod_{j=1}^d \mathbb{E}_0\big|B_1(1)\big|^{-\alpha_j}\right)^m \left(\int_0^\infty t^{-\alpha/2}e^{-t}\,dt\right)^m \\
&\quad= m!\left(\Gamma\left(\frac{2-\alpha}{2}\right)\prod_{j=1}^d \mathbb{E}_0\big|B_1(1)\big|^{-\alpha_j}\right)^m
\end{aligned}$$

for $m = 1, 2, \ldots$.

On the other hand, notice that $\eta_t \stackrel{d}{=} t^{(2-\alpha)/2}\eta_1$. So we have

$$\mathbb{E}^\tau \otimes \mathbb{E}_0\eta_\tau^m = \big(\mathbb{E}^\tau \tau^{((2-\alpha)/2)m}\big)\mathbb{E}_0\eta_1^m = \Gamma\left(1 + \frac{2-\alpha}{2}m\right)\mathbb{E}_0\eta_1^m.$$

Combining this with (A.27), by Stirling formula we conclude that there is a constant $C > 0$ such that

$$\mathbb{E}_0\eta_1^m \le (m!)^{\alpha/2}C^m, \qquad m = 1, 2, \ldots.$$

This implies (A.26) with $b < C^{-2/\alpha}$.

We now claim that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\{\theta\eta_t\} < \infty \qquad \forall\theta > 0. \tag{A.28}$$

Indeed, by scaling,

$$\begin{aligned}
\mathbb{E}_0\exp\{\theta\eta_t\} &= \mathbb{E}_0\exp\big\{\theta t^{(2-\alpha)/2}\eta_1\big\} \\
&\le \mathbb{E}_0\exp\big\{b\eta_1^{2/\alpha}\big\} + \mathbb{E}_0\big\{\exp\big\{\theta t^{(2-\alpha)/2}\eta_1\big\}; \eta_1 \le \big(\theta b^{-1}\big)^{2/(2-\alpha)}t^{\alpha/2}\big\} \\
&\le \mathbb{E}_0\exp\big\{b\eta_1^{2/\alpha}\big\} + \exp\big\{\big(\theta b^{-1}\big)^{2/(2-\alpha)}t\big\}.
\end{aligned}$$

Hence, (A.28) follows from (A.26).

Given $N > 0$,

$$\eta_t \geq \int_0^t \big(K(B_s) \wedge N\big)\, ds.$$

On the other hand, applying Theorem 4.1.6, [7] to the bounded, continuous function $K(x) \wedge N$ gives

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{\int_0^t \big(K(B_s) \wedge N\big)\, ds\right\} \\
&\qquad = \sup_{g\in\mathcal{F}_d(\mathbb{R}^d)} \left\{\int_{\mathbb{R}^d} \big(K(x) \wedge N\big) g^2(x)\, dx - \frac{1}{2}\int_{\mathbb{R}^d} \big|\nabla g(x)\big|^2\, dx\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{g\in\mathcal{F}_d(\mathbb{R}^d)} \left\{\int_{\mathbb{R}^d} \big(K(x) \wedge N\big) g^2(x)\, dx - \frac{1}{2}\int_{\mathbb{R}^d} \big|\nabla g(x)\big|^2\, dx\right\} \\
&\qquad \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\{\theta \eta_t\}.
\end{aligned}$$

Letting $N \to \infty$ on the left-hand side, by (A.28) we have (A.24). $\square$

With (A.24), an obvious modification of the argument for (A.22) shows that there is a constant $\widetilde{C}_\alpha > 0$ such that the inequality

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{-\alpha_j}\right) f^2(x)\, dx \leq \widetilde{C}_\alpha \|f\|_2^{2-\alpha} \|\nabla f\|_2^\alpha, \qquad f \in W^{1,2}(\mathbb{R}^d) \tag{A.29}$$

holds. Recall our discussion based on the inequality (A.17). Replacing (A.17) by (A.29) and copying the same derivation we obtain a parallel system of inequalities and relations among variations that are summarized in the following.

First, we have the inequality

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\prod_{j=1}^d |x_j - y_j|^{-\alpha_j}\right) f^2(x) f^2(y)\, dx\, dy \\
&\qquad \leq \widetilde{C}_\alpha \|f\|_2^{4-\alpha} \|\nabla f\|_2^\alpha, \qquad f \in W^{1,2}(\mathbb{R}^d).
\end{aligned} \tag{A.30}$$

Consequently, the best consequence

$$\begin{aligned}
\tilde{\kappa}(d,\alpha) = \inf\Bigg\{C > 0;\ &\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\prod_{j=1}^d |x_j - y_j|^{-\alpha_j}\right) f^2(x) f^2(y)\, dx\, dy \\
&\leq C \|f\|_2^{4-\alpha} \|\nabla f\|_2^\alpha\ \forall f \in W^{1,2}(\mathbb{R}^d)\Bigg\}
\end{aligned} \tag{A.31}$$

is finite.

Second, the quantities defined through the variations

$$\widetilde{M}_{d,\alpha}(\theta) = \sup_{g\in\mathcal{F}_d(\mathbb{R}^d)}\left\{\theta\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\prod_{j=1}^d |x_j-y_j|^{-\alpha_j}\right)g^2(x)g^2(y)\,dx\,dy\right)^{1/2} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\right\}, \quad \theta>0, \tag{A.32}$$

$$\tilde{\sigma}(d,\alpha) = \sup_{f\in\mathcal{G}_d(\mathbb{R}^d)}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\prod_{j=1}^d |x_j-y_j|^{-\alpha_j}\right)f^2(x)f^2(y)\,dx\,dy\right\}^{1/2} \tag{A.33}$$

are finite

Third, these variations are co-related according to the following lemma.

LEMMA A.6. *Under* $0\le\alpha_j<1$ $(j=1,\dots,d$ *and* $\alpha_1+\cdots+\alpha_d<2$,

$$\widetilde{M}_{d,\alpha}(\theta) = \frac{4-\alpha}{4}\left(\frac{\alpha}{2}\right)^{\alpha/(4-\alpha)}\tilde{\kappa}(d,\alpha)^{2/(4-\alpha)}\theta^{4/(4-\alpha)}, \tag{A.34}$$

$$\tilde{\sigma}(d,\alpha) = \left(\frac{4-\alpha}{4}\right)^{(4-\alpha)/4}\left(\frac{\alpha}{2}\right)^{\alpha/4}\tilde{\kappa}(d,\alpha)^{1/2}. \tag{A.35}$$

The next lemma is related to Theorem 1.4.

LEMMA A.7.

$$\sup_{g\in\mathcal{F}_1(\mathbb{R})}\left\{\theta\left(\int_{-\infty}^{\infty} g^4(x)\,dx\right)^{1/2} - \frac{1}{2}\int_{-\infty}^{\infty}|f'(x)|^2\,dx\right\} = \frac{1}{2}\left(\frac{3}{4}\right)^{2/3}\theta^{4/3} \qquad (\theta>0), \tag{A.36}$$

$$\sup_{g\in\mathcal{G}_1(\mathbb{R})}\int_{-\infty}^{\infty} g^4(x)\,dx = \frac{3}{4}\left(\frac{1}{2}\right)^{3/2}. \tag{A.37}$$

PROOF. The identity (A.36) is given in Theorem C.4, page 307, [7]. This theorem also claims the Sobolev inequality

$$\|f\|_4 \le 3^{-1/8}\|f\|_2^{3/4}\|f'\|_2^{3/4}, \qquad f\in W^{1,2}(\mathbb{R}^d)$$

with $3^{-1/8}$ as the best constant. A natural modification of the proof for (A.23) leads to (A.37). $\square$

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