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THE HAUSDORFF DIMENSION OF THE CLE GASKET

BY JASON MILLER, NIKE SUN[1] AND DAVID B. WILSON

Massachussetts Institute of Technology, Microsoft Research and Stanford University

The conformal loop ensemble CLE_κ is the canonical conformally invariant probability measure on noncrossing loops in a proper simply connected domain in the complex plane. The parameter κ varies between $8/3$ and 8; $\mathrm{CLE}_{8/3}$ is empty while CLE_8 is a single space-filling loop. In this work, we study the geometry of the CLE *gasket*, the set of points not surrounded by any loop of the CLE. We show that the almost sure Hausdorff dimension of the gasket is bounded from below by $2-(8-\kappa)(3\kappa-8)/(32\kappa)$ when $4<\kappa<8$. Together with the work of Schramm–Sheffield–Wilson [*Comm. Math. Phys.* **288** (2009) 43–53] giving the upper bound for all κ and the work of Nacu–Werner [*J. Lond. Math. Soc.* (2) **83** (2011) 789–809] giving the matching lower bound for $\kappa \le 4$, this completes the determination of the CLE_κ gasket dimension for all values of κ for which it is defined. The dimension agrees with the prediction of Duplantier–Saleur [*Phys. Rev. Lett.* **63** (1989) 2536–2537] for the FK gasket.

1. Introduction. The conformal loop ensemble CLE_κ is the canonical conformally invariant measure on countably infinite collections of noncrossing loops in a proper simply connected domain D in $\mathbb{C}$ [31, 32]. It is the loop analogue of SLE_κ, the canonical conformally invariant measure on noncrossing paths. Whereas SLE_κ arises as the scaling limit of a single macroscopic interface of many two-dimensional discrete models [3–5, 17, 18, 27, 28, 33, 34], CLE_κ describes the limit of all of the interfaces simultaneously. The parameter κ varies between $8/3$ and 8; $\mathrm{CLE}_{8/3}$ is empty while CLE_8 is a single space-filling loop. CLE_κ for $\kappa \in (8/3, 4]$ consists of disjoint simple loops, while for $\kappa \in (4, 8]$ the loops intersect both themselves and each other (but are noncrossing). CLE_3 and $\mathrm{CLE}_{16/3}$ are the scaling limits of the cluster boundaries in the square lattice critical Ising spin [1] and FK-Ising [12] models, respectively, and CLE_6 is the scaling limit of the cluster boundaries in critical percolation on the triangular lattice [2, 33]. CLE_4 is the scaling limit of the level sets of the two-dimensional discrete Gaussian free field [19].

There are two different constructions of CLE_κ. In the first construction, due to Werner [35] and applicable for $\kappa \in [8/3, 4]$, the loop ensemble is given by the

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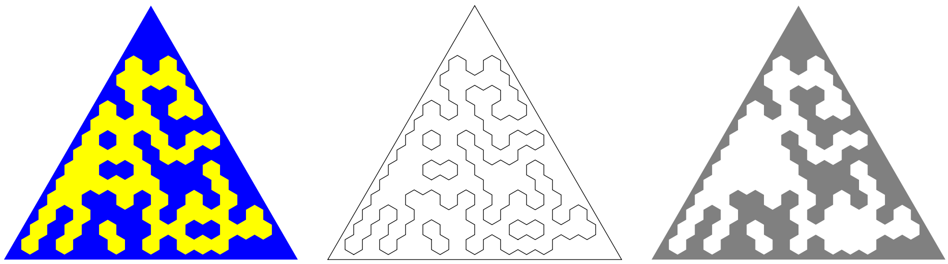


(a) Site percolation. (b) $O(n)$ model, $n = x = 1$. (c) Gasket.

FIG. 1. *Under the $O(n)$ model, a loop configuration ω has probability proportional to $x^{e(\omega)} n^{\ell(\omega)}$ where $\ell(\omega)$ is the number of loops in ω and $e(\omega)$ is the total length of all the loops. For $0 \le n \le 2$, there is a critical value $x_c \equiv x_c(n)$ at which the $O(n)$ model has a "dilute phase," believed to converge to* CLE_κ *with $n = -2\cos(4\pi/\kappa)$, $8/3 \le \kappa \le 4$. The $O(n)$ model at $x > x_c$ is in a "dense phase," again believed to converge to* CLE_κ *with $n = -2\cos(4\pi/\kappa)$, but now with $4 \le \kappa \le 8$. Critical site percolation on the triangular lattice* (*left panel*) *corresponds to the* (*dense phase*) *$O(n)$ model on the honeycomb lattice with $n = x = 1$* (*center panel*). *Its gasket* (*right panel*) *is a discretization of the* CLE_6 *gasket*.

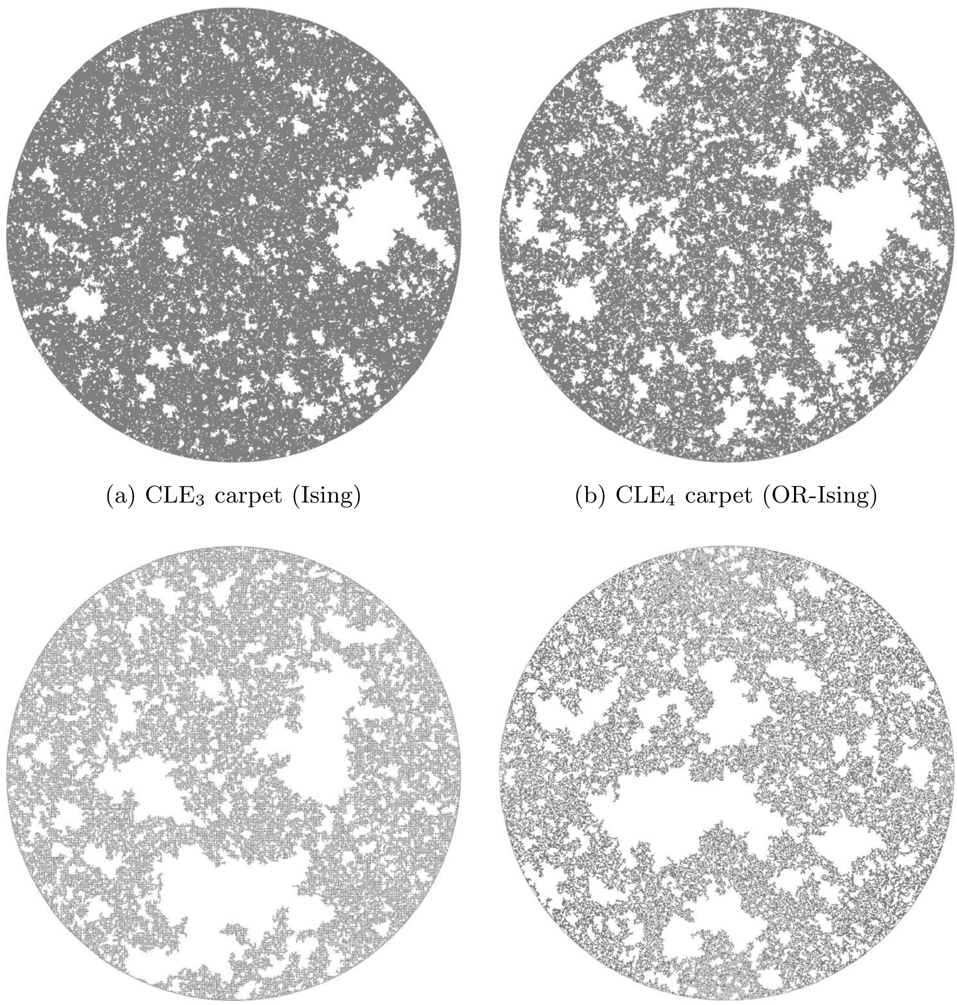
outer boundaries of Brownian loop soup clusters. In this paper, we make use of the second construction, proposed by Sheffield [31] and applicable for $\kappa \in [8/3, 8]$, based on branching $\mathrm{SLE}_\kappa(\kappa - 6)$. These constructions have been proved equivalent for $\kappa \in [8/3, 4]$ [32] (see also [37]).

Let Γ be a CLE_κ in D. The *carpet* ($\kappa \in [8/3, 4]$) or *gasket* ($\kappa \in (4, 8]$) $\mathcal{G}$ of Γ is the set of points not surrounded by any loop of Γ. (In analogy with the Sierpiński carpet and gasket, we call $\mathcal{G}$ a carpet or gasket according to whether the loops of Γ are disjoint or intersecting, although occasionally we loosely use gasket for both.) Since a.s. every neighborhood intersects a loop, $\mathcal{G}$ is given equivalently by the closure of the union of the outermost loops of Γ. Figure 1 shows the gasket for a discrete model, critical site percolation, that converges to CLE_6. Figure 2 shows discrete simulations of $\mathcal{G}$ for $\kappa = 3$ (Ising model), $\kappa = 4$ (OR of two independent Ising models, see [32], Proposition 10.2), $\kappa = 16/3$ (FK-Ising model), and $\kappa = 6$ (critical percolation). The main result of this article is the following theorem.

THEOREM 1.1. *Fix $\kappa \in (4, 8)$ and let Γ be a* CLE_κ *in a proper simply connected domain D in $\mathbb{C}$. Then with probability one the Hausdorff dimension of the gasket $\mathcal{G}$ of Γ is*

$$2 - \frac{(8-\kappa)(3\kappa-8)}{32\kappa}. \tag{1.1}$$

The formula (1.1) was first derived in the context of the $O(n)$ model by Duplantier and Saleur [7, 8], who predicted the fractal dimension of the $O(n)$ gasket (for $n \le 2$) using nonrigorous Coulomb gas methods. The scaling limit of the $O(n)$



(a) CLE_3 carpet (Ising) (b) CLE_4 carpet (OR-Ising)

(c) $\mathrm{CLE}_{16/3}$ gasket (FK) (d) CLE_6 gasket (percolation)

FIG. 2. *Discrete simulations of the* CLE_κ *carpet* ($\kappa \in [8/3, 4]$) *or gasket* ($\kappa \in (4, 8]$) $\mathcal{G}_\kappa$ *for* $\kappa \in \{3, 4, 16/3, 6\}$. *The discretized* $\mathcal{G}_\kappa$ (*indicated in black above*) *is given by the set of points not surrounded by any cluster boundary loop of a discrete configuration sampled from a model known to converge to* CLE_κ. *Note* $\mathcal{G}_4 \subseteq \mathcal{G}_3$ *in our figures because the OR-Ising configuration used in* (a) *is the binary OR of two independent Ising configurations*, *one of which is used in* (b).

model is believed to be CLE_κ, where $n = -2\cos(4\pi/\kappa)$ ([26], Conjecture 9.7, [31], Section 2.3). There are two values of κ associated to each $n < 2$, corresponding to the "dilute" ($\kappa < 4$) and "dense" ($\kappa > 4$) phases of the $O(n)$ model. For further background see [11].

Schramm, Sheffield, and Wilson [29] showed that for all $8/3 < \kappa < 8$, (1.1) gives the *expectation dimension* of $\mathcal{G}$, the growth exponent of the expected number of balls of radius ε needed to cover $\mathcal{G}$: this (a.s.) upper bounds the Minkowski dimension which in turn upper bounds the Hausdorff dimension. (The expectation dimension for $\kappa = 6$ was derived earlier by Lawler, Schramm and Werner [16].) Nacu and Werner [23] used the Brownian loop soup construction to derive the matching lower bound for the CLE_κ carpets ($\kappa \le 4$).

A lower bound on the Hausdorff dimension of a random fractal set is obtained (by standard arguments) from a second moment estimate controlling the probability that two given points lie near the set. The complicated geometry of CLE loops prevents us from applying the second moment method directly to $\mathcal{G}$, and instead we use a "multi-scale refinement" [6]: we establish that with arbitrarily small loss in the Hausdorff dimension we can restrict to special classes of points in $\mathcal{G}$ whose correlation structure *at all scales* can be controlled.

Outline. In Section 2, we review Sheffield's branching $\mathrm{SLE}_\kappa(\kappa-6)$ construction of CLE_κ [taking $\kappa \in (4,8)$], with an emphasis on its dependency structure. In Section 3, we prove Theorem 1.1.

2. Preliminaries. In this section, we review the exploration tree construction of CLE_κ for $\kappa \in (4,8)$ given in [31] and then collect several useful estimates for conformal maps.

2.1. *The continuum exploration tree*. We begin by briefly recalling the definition of the SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes. There are many excellent surveys on the subject (e.g., [15, 36]) to which we refer the reader for a more detailed introduction. The radial Loewner evolution in the unit disk $\mathbb{D}$ is given by the differential equation

$$\dot{g}_t(z) = -g_t(z)\frac{g_t(z)+W_t}{g_t(z)-W_t}, \qquad g_0(z) = z, \tag{2.1}$$

where W_t is a continuous function which takes values in $\partial\mathbb{D}$. We refer to W_t as the *driving function* of the Loewner evolution. For $z \in \mathbb{D}$, let

$$T^z \equiv \sup\{t \ge 0 : |g_t(z)| < 1\}$$

and

$$K_t \equiv \{z \in \mathbb{D} : T^z \le t\}.$$

For each $t \ge 0$, g_t is the unique conformal transformation $\mathbb{D} \setminus K_t \to \mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$. The (random) growth process $(K_t)_{t\ge 0}$ associated with $W_t = \exp(i\sqrt{\kappa}B_t)$, where B_t is a standard Brownian motion, is the radial SLE_κ process introduced by Schramm [27]. Time is parametrized by negative log-conformal radius, that is, $g_t'(0) = e^t$. It was proved by Rohde and Schramm [26] ($\kappa \ne 8$) and Lawler, Schramm, and Werner [17] ($\kappa = 8$) that there is a *curve*

$\eta:[0,\infty)\to\overline{\mathbb{D}}$ starting at $\eta(0)=1$ such that $\mathbb{D}\setminus K_t$ is the unique connected component of $\mathbb{D}\setminus\eta[0,t]$ containing 0: we say that η *generates* the process K_t and call η the radial SLE_κ *trace*. In this setting, $W_t=\lim_{z\to\eta(t)} g_t(z)$, where the limit is taken with $z\in\mathbb{D}\setminus K_t$. For $\kappa<8$, Lawler [14] proved that $\lim_{t\to\infty}\eta(t)=0$, so $\eta:[0,\infty]\to\overline{\mathbb{D}}$ defines a curve traveling from $\eta(0)=1$ to $\eta(\infty)=0$ in $\overline{\mathbb{D}}$.

Let D be a proper simply connected domain in $\mathbb{C}$. For any conformal transformation $f:\mathbb{D}\to D$, we take the image of radial SLE_κ in $\mathbb{D}$ under f to be the definition of radial SLE_κ in D from $f(1)$ to $f(0)$, with $f(1)$ interpreted as a prime end. If f extends continuously to $\overline{\mathbb{D}}$ (equivalently if ∂D is given by a closed curve, see [25], Theorem 2.1), then radial SLE_κ in D is a.s. a continuous curve. It was proved by Garban, Rohde and Schramm [9] that radial SLE_κ with $\kappa<8$ in a *general* proper simply connected domain is a.s. continuous except possibly at its starting point.

We now describe the radial $\mathrm{SLE}_\kappa(\rho)$ processes, a natural generalization of radial SLE_κ first introduced in [13], Section 8.3. For $w,o\in\partial\mathbb{D}$, radial $\mathrm{SLE}_\kappa(\rho)$ with starting configuration (w,o) is the (random) growth process associated with the solution of (2.1) where the driving function solves the SDE

$$dW_t=-\frac{\kappa}{2}W_t\,dt+i\sqrt{\kappa}W_t\,dB_t-\frac{\rho}{2}W_t\frac{W_t+O_t}{W_t-O_t}\,dt,\qquad W_0=w \tag{2.2}$$

with $O_t=g_t(o)$, the *force point*. It is easy to see that (2.2) has a unique solution up to time $\tau_=\equiv\inf\{t\ge 0: W_t=O_t\}$.

The weight $\rho=\kappa-6$ is special because it arises as a coordinate change of ordinary chordal SLE_κ from w targeted at o. A consequence is that *radial* $\mathrm{SLE}_\kappa(\kappa-6)$ *is target invariant:* radial $\mathrm{SLE}_\kappa(\kappa-6)$ in $\mathbb{D}$ with starting configuration (w,o) and target $a\in\mathbb{D}$ has the same law (modulo time change) as an ordinary chordal SLE_κ in $\mathbb{D}$ from w to o, up to the first time the curve disconnects a and o [30].

We now explain how to construct a solution to (2.2) which is defined even after time $\tau_=$. A more detailed treatment is provided in [31], Section 3; we give here a brief summary following [29]. For $\rho>-\kappa/2-2$, there is a random continuous process θ_t taking values in $[0,2\pi]$ which evolves according to the SDE

$$d\theta_t=\sqrt{\kappa}\,dB_t+\frac{\rho+2}{2}\cot(\theta_t/2)\,dt \tag{2.3}$$

on each interval of time for which $\theta_t\notin\{0,2\pi\}$, and is instantaneously reflecting at the endpoints, that is, the set $\{t:\theta_t\in\{0,2\pi\}\}$ has Lebesgue measure zero. (This diffusion was studied in [16] for $\rho=0$.) In other words, θ_t is a random continuous process adapted to the filtration of B_t which a.s. satisfies

$$\partial_t[\theta_t-\sqrt{\kappa}B_t]=\frac{\rho+2}{2}\cot(\theta_t/2)$$

for all t for which the right-hand side is finite. The law of this process is uniquely determined by θ_0, and moreover the process is pathwise unique [31], Proposition 4.2. It then follows from the strong Markov property of Brownian motion that θ_t has the strong Markov property.

When $\rho \geq \kappa/2-2$, the θ_t process governed by SDE (2.3) is repelled so strongly by 0 and 2π that it almost surely never reaches either endpoint. When $\rho=-2$ the diffusion θ_t is simply reflected Brownian motion. When $\rho<-2$, the θ_t process is attracted to the singularity and its analysis requires more care, but it still makes sense when $\rho>-\kappa/2-2$ [29, 31]. When $\rho \leq -\kappa/2-2$, the θ_t process is attracted so strongly to the endpoints that once it hits either one it remains glued there. In the intermediate regime, $-\kappa/2-2<\rho<\kappa/2-2$, the θ_t process hits the endpoints 0 and 2π, but is instantaneously reflecting. When $\rho=\kappa-6$, this corresponds to the range $8/3<\kappa<8$.

We then set

$$\arg W_t = \arg w + \sqrt{\kappa} B_t + \frac{\rho}{2}\int_0^t \cot(\theta_s/2)\, ds. \tag{2.4}$$

That the above integral is a.s. finite follows by the comparison of $\theta_t/\sqrt{\kappa}$ [resp., $(2\pi-\theta_t)/\sqrt{\kappa}$] with a δ-dimensional Bessel process, as described above; see, for example, the proof of Lemma 3.4. We then *define* radial $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{D}$ with starting configuration (w,o) to be the solution to (2.1) with driving function W_t defined by (2.4). The force point $O_t \equiv g_t(o)$ satisfies $W_t = O_t e^{i\theta_t}$, and we interpret $\theta_t=0$ as $O_t = W_t e^{i0^-}$ ($\arg O_t$ just below $\arg W_t$) and similarly $\theta_t = 2\pi$ as $O_t = W_t e^{i0^+}$. For $\rho \geq \kappa/2-2$, the laws of radial $\mathrm{SLE}_\kappa(\rho)$ and ordinary radial SLE_κ are mutually absolutely continuous up to any fixed positive time, so $\mathrm{SLE}_\kappa(\rho)$ is a.s. generated by a curve by the result of [26]. In [20], it is established that $\mathrm{SLE}_\kappa(\rho)$ is a.s. generated by a curve for all $\rho>-2$ (see Remark 2.2); when $\rho=\kappa-6$ this corresponds to $\kappa>4$. Radial $\mathrm{SLE}_\kappa(\rho)$ in a general proper simply connected domain is defined again by conformal transformation, but the analogue of the continuity result of [9] is not known for $\rho \neq 0$.

The target invariance of radial $\mathrm{SLE}_\kappa(\kappa-6)$ processes continues to hold after time $\tau_=$, and from this we can construct a coupling of radial $\mathrm{SLE}_\kappa(\kappa-6)$ processes targeted at a countable dense subset of $\mathbb{D}$.

PROPOSITION 2.1 ([31], Proposition 3.14 and Section 4.2). *Let $(a_k)_{k\in\mathbb{N}}$ be a countable dense sequence in $\mathbb{D}$. For $4<\kappa<8$, there exists a coupling of radial $\mathrm{SLE}_\kappa(\kappa-6)$ curves η^{a_k} in $\mathbb{D}$ from 1 to a_k started from $(w,o)=(1,1e^{i0^-})$ such that for any $k,\ell\in\mathbb{N}$, η^{a_k} and η^{a_ℓ} agree a.s. (modulo time change) up to the first time that the curves separate a_k and a_ℓ and evolve independently thereafter.*

(For $8/3<\kappa<4$, the $\mathrm{SLE}_\kappa(\kappa-6)$ traces are not known to be curves, which makes the corresponding statement in this case more complicated. The case $\kappa=4$ is special, and was dealt with separately by Sheffield [31].)

From the coupling $(\eta^{a_k})_{k\in\mathbb{N}}$ defined in Proposition 2.1, we can a.s. uniquely define (modulo time change) for each $a\in\overline{\mathbb{D}}$ a curve η^a targeted at a, by considering a subsequence (a_{k_n}) converging to a. Then η^a is a radial $\mathrm{SLE}_\kappa(\kappa-6)$, and we write

θ_t^a, W_t^a, O_t^a for the corresponding processes of (2.3) and (2.4). The complete collection of curves $(\eta^a)_{a\in\overline{\mathbb{D}}}$ is the *branching* $\mathrm{SLE}_\kappa(\kappa-6)$ or *continuum exploration tree* of [31].

2.2. *Loops from exploration trees.* For $4<\kappa<8$, the CLE_κ loops $\mathcal{L}^a$ surrounding $a\in\mathbb{D}$ are defined in terms of the branch η^a of the exploration tree as follows:

1. Let $\tau_{\mathrm{ccw}}^a \equiv \inf\{t\ge 0 : \theta_t^a = 2\pi\}$, the first time η^a forms a counterclockwise loop surrounding a.
2. If $\tau_{\mathrm{ccw}}^a=\infty$, then there are no loops surrounding a and we set $\mathcal{L}^a$ to be the empty sequence. If $\tau_{\mathrm{ccw}}^a<\infty$ let $\acute{\tau}_{\mathrm{ccw}}^a \equiv \sup\{t<\tau_{\mathrm{ccw}}^a : \theta_t^a=0\}$, let $o^a\equiv \eta^a(\acute{\tau}_{\mathrm{ccw}}^a)$, and let $\widetilde{\eta}^a$ be the branch η^{o^a}, reparametrized so that $\widetilde{\eta}^a|_{[0,\tau_{\mathrm{ccw}}^a]} = \eta^a|_{[0,\tau_{\mathrm{ccw}}^a]}$. The outermost loop $\mathcal{L}_1^a$ surrounding a is defined to be $\widetilde{\eta}^a|_{[\acute{\tau}_{\mathrm{ccw}}^a,\infty]}$.

If $\mathcal{L}_1^a$ is defined, it is necessarily counterclockwise and pinned at $\eta^a(\acute{\tau}_{\mathrm{ccw}}^a)$, and for any point b surrounded by $\mathcal{L}_1^a$ we have $\mathcal{L}_1^b=\mathcal{L}_1^a$. Moreover, $\eta^a(\acute{\tau}_{\mathrm{ccw}}^a)$ lies on $\partial\mathbb{D}$ if and only if η^a has not previously made a clockwise loop around a [31], Lemma 5.2. The next loop $\mathcal{L}_2^a$ surrounding a is then defined in analogous fashion, and continuing in this way gives the full CLE_κ process Γ in $\mathbb{D}$. See Figures 3 and 4.

REMARK 2.2. For $4<\kappa<8$, assuming the conjecture that chordal $\mathrm{SLE}_\kappa(\kappa-6)$ processes are generated by continuous curves with reversible law [31], Conjecture 3.11, it was shown [31], Proposition 5.1 and Theorem 5.4, that CLE_κ loops

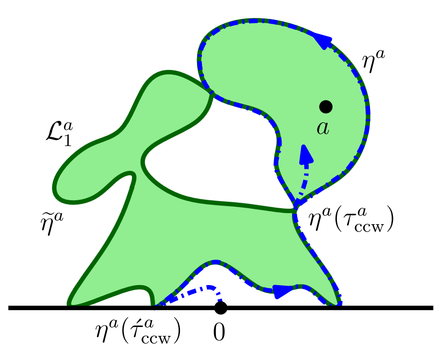


FIG. 3. *Branching* $\mathrm{SLE}_\kappa(\kappa-6)$ *construction of* CLE_κ $(4<\kappa<8)$ *process* Γ *in* $\mathbb{H}$. *For each* $a\in\mathbb{H}$, η^a *(dashed blue line) is the branch of the exploration tree targeted at* a. *It evolves as a radial* $\mathrm{SLE}_\kappa(\kappa-6)$ *which, whenever it hits the domain boundary or its past hull, continues in the complementary connected component containing* a. *Let* τ_{ccw}^a *be the first time* t *that* η^a *completes a counterclockwise loop surrounding* a; *the location of the force point at time* τ_{ccw}^a *is* $o^a\equiv\eta^a(\acute{\tau}_{\mathrm{ccw}}^a)$ *for some* $\acute{\tau}_{\mathrm{ccw}}^a<\tau_{\mathrm{ccw}}^a$. *The outermost loop* $\mathcal{L}_1^a$ *of* Γ *containing* a *is* $\eta^{o^a}|_{[\acute{\tau}_{\mathrm{ccw}}^a,\infty]}$. *Successive loops are defined in analogous fashion.* $\mathcal{L}_1^a$ *is necessarily counterclockwise and pinned at* $\eta^a(\acute{\tau}_{\mathrm{ccw}}^a)$. *It is disjoint from the domain boundary if and only if* a *is first surrounded by a clockwise loop.*

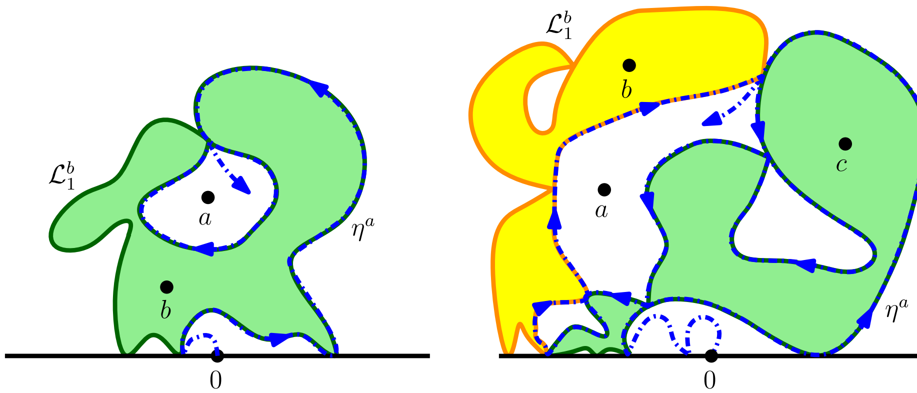


FIG. 4. *Clockwise loops of η^a (dashed blue line) are not* CLE *loops, but correspond either to complementary connected components of* CLE *loops (left panel) or complementary connected components of chains of* CLE *loops (right panel). The* CLE *process is renewed within each clockwise loop (Proposition 2.3).*

are continuous, and that the law of the full ensemble is independent of the choice of root for the exploration tree. This conjecture was proved in works of Miller and Sheffield ([20], Theorem 1.3 and [21], Theorems 1.1 and 1.2), so these properties hold. (The analogous continuity and root-invariance statements are immediate for $\kappa \in [8/3, 4]$ by the equivalence of CLE_κ and the outer boundaries of loop soups [32]; see also [37].)

The CLE_κ process in a general proper simply connected domain is defined by conformal transformation, so the law of CLE_κ is conformally invariant. Moreover, conditional on the collection of all of the outermost loops, the law of the loops contained in the connected component D^a of $\mathbb{D} \setminus \mathcal{L}_1^a$ containing a is equal to that of a CLE_κ in D^a independently of the loops of Γ which are not contained in D^a. The key observation which we use to prove Theorem 1.1 is that there are additional sources of conditional independence in CLE_κ when $\kappa > 4$, in particular:

PROPOSITION 2.3. *Suppose $z \in D$ is surrounded by a clockwise loop $\mathcal{C}$ in the* $\mathrm{SLE}_\kappa(\kappa - 6)$ *exploration tree of D (as in Figure 4), allowing the domain boundary to form part of the loop $\mathcal{C}$. If U is the connected component of $D \setminus \mathcal{C}$ containing z, then the law of the* CLE_κ *loops contained within $\overline{U}$ is that of a* CLE_κ *in U, independent of the* CLE_κ *loops outside of U.*

The $\mathrm{SLE}_\kappa(\kappa - 6)$ exploration tree for $\kappa > 4$ has such clockwise loops, which are not CLE loops, and so provide additional renewal events.

2.3. *Diffusion estimate.*

PROPOSITION 2.4 ([29], equation (4)). *Suppose* $8/3 < \kappa < 8$, *and let* θ_t *be the process defined above started from* $\theta_0 = 0$, *evolving according to SDE* (2.3) *in* $(0, 2\pi)$ *and instantaneously reflecting at the endpoints* $\{0, 2\pi\}$. *Then* $\mathbb{P}[\theta_s < 2\pi\ \forall s \le t] \asymp e^{-\alpha t}$ *where*

$$\alpha \equiv \frac{(8-\kappa)(3\kappa-8)}{32\kappa}. \tag{2.5}$$

It is this diffusion exponent α which gave rise to the result of [29] that the gasket has expectation dimension $2-\alpha$, implying an upper bound of $2-\alpha$ for the Hausdorff dimension, for which Theorem 1.1 provides the matching lower bound. The actual value of α does not play a significant role in the proof of Theorem 1.1, except that we use $0 < \alpha < 2$. (Of course, $\alpha \le 2$ is a necessary condition for showing that the Hausdorff dimension is $2-\alpha$.)

2.4. *Distortion estimates.* For a proper simply connected domain D and $w \in D$, let $\mathrm{CR}(w, D)$ denote the conformal radius of D with respect to w, that is, $\mathrm{CR}(w, D) \equiv f'(0)$ for f the unique conformal map $\mathbb{D} \to D$ with $f(0) = w$ and $f'(0) > 0$. Let $\mathrm{rad}(w, D) \equiv \inf\{r : B_r(w) \supseteq D\}$ denote the out-radius of D with respect to w. By the Schwarz lemma and the Koebe one-quarter theorem,

$$\mathrm{dist}(w, \partial D) \le \mathrm{CR}(w, D) \le [4\,\mathrm{dist}(w, \partial D)] \wedge \mathrm{rad}(w, D). \tag{2.6}$$

Further (see, e.g., [25], Theorem 1.3)

$$\frac{|\zeta|}{(1+|\zeta|)^2} \le \frac{|f(\zeta) - w|}{\mathrm{CR}(w, D)} \le \frac{|\zeta|}{(1-|\zeta|)^2}.$$

As a consequence,

$$\frac{|\zeta|}{4} \le \frac{|f(\zeta) - w|}{\mathrm{CR}(w, D)} \le 4|\zeta|, \tag{2.7}$$

where the right-hand inequality holds for $|\zeta| \le 1/2$.

3. Proofs. Recall that a CLE_κ process in a general simply connected domain D is defined as the image under a conformal transformation $f : \mathbb{D} \to D$ of a CLE_κ process Γ in $\mathbb{D}$. Since $f|_{r\mathbb{D}}$ for any $0 < r < 1$ is bi-Lipschitz and so preserves Hausdorff dimension, and the Hausdorff dimension of a countable union is the supremum of the Hausdorff dimensions, we see that f preserves Hausdorff dimension, and so it suffices to prove Theorem 1.1 with $D = \mathbb{D}$. Thus, for the remainder Γ denotes a CLE_κ $(4 < \kappa < 8)$ process on $\mathbb{D}$, constructed from the collection of radial $\mathrm{SLE}_\kappa(\kappa - 6)$ curves $(\eta^z)_{z \in \mathbb{D}}$ jointly defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, as given by the remark following Proposition 2.1. In Section 3.1, we define our multi-scale refinement of the gasket $\mathcal{G}$ of Γ, and state the main result of the section, the second moment estimate Lemma 3.1 on the correlation structure of the set of "perfect points" identified by the refinement. We then use the CLE renewal property of Proposition 2.3 to reduce Lemma 3.1 to a lower bound on the probability of a single event. This bound is given by Proposition 3.3, which we prove in Section 3.2. The Hausdorff dimension lower bound follows from Lemma 3.1 by standard arguments which we give in Section 3.3, thereby concluding the proof of Theorem 1.1.

3.1. *Clockwise loops in small disks.* We now describe our multi-scale refinement of the gasket $\mathcal{G}$ which identifies a subset of "perfect points" (following the terminology of [6, 10]) in $\mathcal{G}$, satisfying a certain restriction at all scales which makes their correlation structure easy to analyze. Let $\beta > 0$ be a parameter (which we will send to ∞). Let η be any curve defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and traveling in $\overline{\mathbb{D}}$ from $\partial\mathbb{D}$ to 0. For any curve η defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and traveling in $\overline{\mathbb{D}}$ from $\partial\mathbb{D}$ to 0, define $E(\eta) \subseteq \Omega$ to be the event that

(i) the first time τ_{cw} that η closes a clockwise loop $\mathcal{C}$ surrounding 0 with $\mathcal{C} \subset e^{-\beta}\mathbb{D}$ is finite; and

(ii) η makes no counterclockwise loop surrounding 0 before time τ_{cw}.

On the event $E(\eta)$, set $D(\eta)$ to be the connected component of $\mathbb{D} \setminus \mathcal{C}$ containing the origin. See Figure 5 for an illustration.

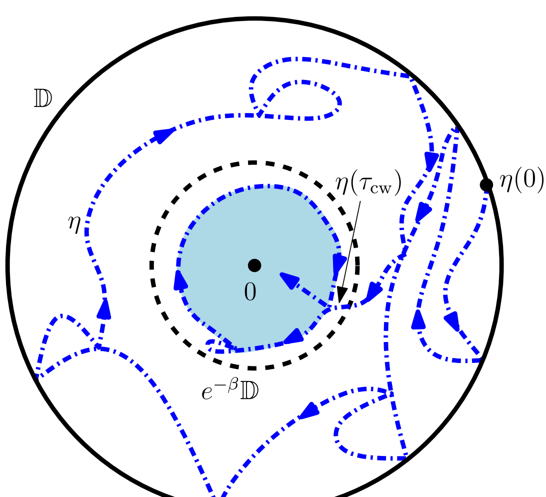


FIG. 5. *A single level of the multi-scale argument we use to prove the lower bound of Theorem 1.1. The curve η (dashed blue line) is a radial $\mathrm{SLE}_\kappa(\kappa - 6)$ targeted at zero. For $\beta > 0$, $E(\eta)$ is the event that the first time τ_{cw} that η closes a clockwise loop $\mathcal{C}$ surrounding 0 with $\mathcal{C} \subset e^{-\beta}\mathbb{D}$ is finite, and further that η makes no counterclockwise loop surrounding 0 before τ_{cw}. On the event $E(\eta)$, set $D(\eta)$ (light blue region) to be the connected component of $\mathbb{D} \setminus \eta[0, \tau_{\mathrm{cw}}]$ containing 0.*

We then define events E_j and domains $D_j \ni 0$, both nonincreasing in j for $j \geq 0$, as follows: let $(E_0, D_0) \equiv (\Omega, \mathbb{D})$, and suppose inductively that (E_j, D_j) has been defined. Let g_j be the uniformizing map $D_j \to \mathbb{D}$ with $\mathsf{g}_j(0) = 0$ and $\mathsf{g}_j'(0) > 0$, and let

$$\tau_j \equiv \inf\{t \geq 0 : \eta(t) \in D_j\}, \qquad \mathsf{g}_j\eta \equiv \big(\mathsf{g}_j\eta(\tau_j + s)\big)_{s\geq 0}.$$

We then set

$$E_{j+1}(\eta) \equiv E_j(\eta) \cap E(\mathsf{g}_j\eta), \qquad D_{j+1}(\eta) \equiv \mathsf{g}_j^{-1} D(\mathsf{g}_j\eta).$$

For $z \in \mathbb{D}$, let

$$\psi(\zeta) \equiv \psi_z(\zeta) \equiv \frac{\zeta - z}{1 - \bar{z}\zeta},$$

the conformal automorphism of $\mathbb{D}$ with $\psi(z) = 0$ and $\psi'(z) = (1 - |z|^2)^{-1} > 0$. For η^z, the branch of the $\mathrm{SLE}_\kappa(\kappa - 6)$ exploration tree targeted at z, we set

$$E_j^z \equiv E_j\big(\psi_z\eta^z\big), \qquad D_j^z \equiv \psi_z^{-1} D_j\big(\psi_z\eta^z\big). \tag{3.1}$$

The *perfect points* in the multi-scale refinement of the gasket $\mathcal{G}$ are the points $z \in \mathbb{D}$ for which $\bigcap_{j\geq 0} E_j^z$ occurs. The main estimate needed to lower bound the Hausdorff dimension is the following estimate on their correlation structure.

LEMMA 3.1. *For sufficiently large β there exists $\varepsilon \equiv \varepsilon(\beta) < \infty$ with $\lim_{\beta\to\infty} \varepsilon(\beta) = 0$ such that for all $z, w \in \mathbb{D}$,*

$$\frac{\mathbb{P}[E_n^z \cap E_n^w]}{\mathbb{P}[E_n^z]\mathbb{P}[E_n^w]} \leq \left(\frac{e^\beta}{|z - w|}\right)^{\alpha(1+\varepsilon)},$$

where α is given by (2.5).

In the remainder of this subsection, we reduce the proof of this lemma to a lower bound on the probability of the event E_1^0, Proposition 3.3, which we prove in Section 3.2. We begin with some easy estimates comparing the domains D_j^z to disks $B_{e^{-j\beta}}(z)$.

LEMMA 3.2. *For $\beta \geq \log 2$, $j \geq 1$, and $z \in \mathbb{D}$,*

$$\mathrm{rad}\big(z, D_j^z\big) \leq 8e^{-j\beta} \qquad \text{on } E_j^z. \tag{3.2}$$

PROOF. We first consider the domains $D_j \equiv D_j(\eta)$, defined on the event $E_j(\eta)$, for any curve η traveling in $\overline{\mathbb{D}}$ from $\partial\mathbb{D}$ to 0. [We will later take $\eta = \psi_z(\eta^z)$, where η^z is the branch of the $\mathrm{SLE}_\kappa(\kappa - 6)$ exploration tree targeted at z, and ψ_z is the Möbius transformation defined above which maps z to 0.] Recall the definition of the uniformizing map $\mathsf{g}_j : D_j \to \mathbb{D}$. By the definition of D_j and by (2.6),

$$\mathrm{CR}(0, \mathsf{g}_{j-1}D_j) \leq \mathrm{rad}(0, \mathsf{g}_{j-1}D_j) \leq e^{-\beta}. \tag{3.3}$$

Since

$$\mathrm{CR}(0, \mathsf{g}_{j-1}D_j) = \frac{1}{(\mathsf{g}_j \circ \mathsf{g}_{j-1}^{-1})'(0)} = \frac{\mathrm{CR}(0, D_j)}{\mathrm{CR}(0, D_{j-1})},$$

we have that

$$\mathrm{CR}(0, D_j) = \prod_{\ell=1}^{j} \mathrm{CR}(0, g_{\ell-1}D_\ell) \le e^{-j\beta}. \tag{3.4}$$

Applying the right-hand inequality of (2.7) with $f = \mathsf{g}_{j-1}^{-1}$ gives

$$\frac{|\zeta|}{\mathrm{CR}(0, D_{j-1})} \le 4\left|\mathsf{g}_{j-1}(\zeta)\right| \le 4e^{-\beta} \qquad \text{when } \zeta \in \partial D_j,$$

using that $\zeta \in \partial D_j$ implies $|\mathsf{g}_{j-1}(\zeta)| \le e^{-\beta} \le 1/2$. Rearranging and combining with (3.4) gives

$$|\zeta| \le 4e^{-\beta}\,\mathrm{CR}(0, D_{j-1}) \le 4e^{-j\beta} \qquad \text{when } \zeta \in \partial D_j. \tag{3.5}$$

For any $z \in \mathbb{D}$, (3.5) is satisfied with $D_j = D_j(\psi_z \eta^z) = \psi_z D_j^z$ on the event E_j^z. We have $\psi_z^{-1}(\zeta) = (z+\zeta)/(1+\bar{z}\zeta)$, so

$$\left|\psi_z^{-1}(\zeta) - z\right| = \left|\zeta \frac{1-|z|^2}{1+\bar{z}\zeta}\right| \le |\zeta| \frac{1-|z|^2}{1-|z|} = |\zeta|\big(1+|z|\big) \le 2|\zeta|,$$

giving $\mathrm{rad}(z, D_j^z) \le 2\,\mathrm{rad}(0, \psi_z D_j^z) \le 8e^{-j\beta}$ as claimed. □

Let $\mathcal{F}_j^z$ denote the σ-algebra generated by η^z up to the time τ_j^z that η^z closes the clockwise loop forming the boundary of D_j^z (if E_j^z does not occur then $\tau_j^z = \infty$). By the conformal Markov property of radial $\mathrm{SLE}_\kappa(\rho)$, for $m \le n$, we have

$$\mathbb{P}[E_n^z | \mathcal{F}_m^z]\mathbf{1}_{E_m^z} = \mathbb{P}[E_{n-m}^z]\mathbf{1}_{E_m^z} = \mathbb{P}[E_{n-m}^0]\mathbf{1}_{E_m^z},$$

and consequently

$$\mathbb{P}[E_n^z] = \mathbb{E}[\mathbb{P}[E_n^z|\mathcal{F}_{n-1}^z]\mathbf{1}_{E_{n-1}^z}] = \mathbb{P}[E_1^0]\mathbb{P}[E_{n-1}^z] = \cdots = \mathbb{P}[E_1^0]^n.$$

PROPOSITION 3.3. *There exists a constant $c > 0$ such that $\mathbb{P}[E_1^0] \ge (ce^{\alpha\beta})^{-1}$ for sufficiently large β, where α is given by* (2.5).

The proof of this proposition is deferred to Section 3.2, but we show now how to use it to deduce Lemma 3.1.

PROOF OF LEMMA 3.1. Given $z, w \in \mathbb{D}$, let $m \in \mathbb{N}$ be defined by $8e^{-m\beta} < |z-w| \le 8e^{-(m-1)\beta}$. If $E_m^z \cap E_m^w$ occurs, then Lemma 3.2 implies $w \notin D_m^z$ which in turn implies $D_m^z \cap D_m^w = \varnothing$. So for $n \geq m$, E_n^z and E_n^w are conditionally independent given $E_m^z \cap E_m^w$, and in fact

$$\mathbb{P}[E_n^z \cap E_n^w | E_m^z \cap E_m^w] = \mathbb{P}[E_{n-m}^0]^2.$$

Therefore

$$\begin{aligned}\mathbb{P}[E_n^z \cap E_n^w] &= \mathbb{P}[E_m^z \cap E_m^w]\mathbb{P}[E_{n-m}^0]^2 \\ &\leq \frac{(\mathbb{P}[E_m^0]\mathbb{P}[E_{n-m}^0])^2}{\mathbb{P}[E_m^0]} = \frac{\mathbb{P}[E_n^0]^2}{\mathbb{P}[E_1^0]^m} \leq (ce^{\alpha\beta})^m \mathbb{P}[E_n^0]^2,\end{aligned}$$

where the last inequality is by Proposition 3.3. But $|z-w| \leq 8e^{-(m-1)\beta}$ implies

$$m\beta \leq \beta + \log\frac{8}{|z-w|},$$

therefore

$$\log\left(\frac{\mathbb{P}[E_n^z \cap E_n^w]}{\mathbb{P}[E_n^0]^2}\right) \leq \alpha m\beta\left(1 + \frac{\log c}{\alpha\beta}\right) \leq \alpha\big(1 + O(1/\beta)\big)\left[\beta + \log\frac{1}{|z-w|}\right].$$

As $\beta \to \infty$, the error term goes to 0, which implies the result. $\square$

3.2. *Probability of a clockwise loop*. In this section, we prove Proposition 3.3, lower bounding the probability that a radial $\mathrm{SLE}_\kappa(\kappa-6)$ process in $\mathbb{D}$ makes a clockwise loop within the disk $e^{-\beta}\mathbb{D}$ before making any counterclockwise loop surrounding the origin.

Some notation: for $x \in [0, 2\pi]$, we write $\theta_t^{(x)}$ for the $[0, 2\pi]$-valued process of Section 2.1 started from $\theta_0^{(x)} = x$, evolving according to SDE (2.3) in $(0, 2\pi)$ and instantaneously reflecting at the endpoints $\{0, 2\pi\}$. We write $\theta_t \equiv \theta_t^{(0)}$, and for $a \in [0, 2\pi]$ we let $\sigma_a \equiv \inf\{t : \theta_t = a\}$, and set $F_t \equiv \{\sigma_{2\pi} > t\}$. For $0 < R < 1$ and $\theta_0 \in [0, 2\pi]$ let $P_R(\theta_0)$ be the probability that a radial $\mathrm{SLE}_\kappa(\kappa-6)$ in $\mathbb{D}$ with starting configuration $(w, o) = (1, e^{-i\theta_0})$ and target 0 makes a clockwise loop inside the disk $R\mathbb{D}$ surrounding 0 before making any counterclockwise loop surrounding 0. The proposition will be obtained from the following two lemmas, whose proof we defer.

LEMMA 3.4. *There exist $c_0, p_0 > 0$ such that*

$$\mathbb{P}\big[\theta_T \in [c_0, 2\pi - c_0] | F_T\big] \geq p_0 \qquad \textit{for all } T \in [1, \infty).$$

LEMMA 3.5. *For any $c_0 > 0$, we have* $\inf_{\theta_0 \in [c_0, 2\pi - c_0]} P_R(\theta_0) > 0$.

PROOF OF PROPOSITION 3.3. Recall that it is natural to parametrize the radial $\mathrm{SLE}_\kappa(\kappa-6)$ curve η^0 targeted at 0 by capacity: if U_t denotes the unique connected component of $\mathbb{D} \setminus \eta^0[0,t]$ containing 0 and g_t is the unique conformal transformation $U_t \to \mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$, then $g_t'(0) = 1/\operatorname{CR}(0, U_t) = e^t$.

Assume $\beta \ge \log 8$, and let $\beta' \equiv \beta - \log 8 \ge 0$. Consider the map $g_{\beta'} : U_{\beta'} \to \mathbb{D}$. The left-hand inequality of (2.7) with $f = g_{\beta'}^{-1}$ gives $|g_{\beta'}(\zeta)| \le 4|\zeta| e^{\beta'}$ for any $\zeta \in U_{\beta'}$. In particular, $|g_{\beta'}(\zeta)| \le 1/2$ for $|\zeta| \le e^{-\beta'}/8 = e^{-\beta}$, so we can apply the right-hand inequality of (2.7) to find

$$e^{\beta'}|\zeta| \le 4\big|g_{\beta'}(\zeta)\big| \qquad \text{when } |\zeta| \le e^{-\beta}.$$

Therefore, the image of $e^{-\beta}\mathbb{D}$ under $g_{\beta'}$ contains $R\mathbb{D}$ where

$$R = \tfrac{1}{4} e^{\beta' - \beta} = \tfrac{1}{32}.$$

The curve $g_{\beta'}\eta^0$ is distributed as an $\mathrm{SLE}_\kappa(\kappa - 6)$ in $\mathbb{D}$ with starting configuration $(\tilde{W}_0, \tilde{O}_0) = (W_{\beta'}, W_{\beta'} e^{-i\theta_{\beta'}})$, so for any $c > 0$ we have

$$\mathbb{P}\big[E_1^0\big] \ge \mathbb{P}[F_{\beta'}]\mathbb{P}\big[\theta_{\beta'} \in [c, 2\pi - c] | F_{\beta'}\big] \inf_{\theta_0 \in [c, 2\pi - c]} P_R(\theta_0).$$

By Proposition 2.4, Lemmas 3.4 and 3.5 this expression is $\asymp e^{-\alpha\beta'}$, which gives the result. $\square$

The remainder of this subsection is devoted to proving the above lemmas. We will obtain Lemma 3.4 as a consequence of the following lemma.

LEMMA 3.6. *For any deterministic time $T \ge 0$,*

$$\mathbb{P}[\theta_T \le \pi | F_T] \ge 1/2.$$

PROOF. Let $S = \pm 1$ be a symmetric random sign independent of the process θ_t, and consider the event $A_T \equiv \{\theta_T < \pi\} \cup \{\theta_T = \pi, S = 1\}$. (The random sign is introduced to handle the possibility that $\theta_T = \pi$. It follows easily by comparison with Bessel processes, see, for example, the proof of Lemma 3.4, that $\mathbb{P}[F_T] > 0$ and $\mathbb{P}[\theta_T = \pi] = 0$ for all deterministic $T \ge 0$, but our proof of Lemma 3.6 can be applied to any strong Markov continuous process with reflective symmetry.) By the strong Markov property of θ_t and the reflective symmetry across π of its drift coefficient,

$$\mathbb{P}\big[A_T^c\big] = \mathbb{P}[\theta_T > \pi] + \tfrac{1}{2}\mathbb{P}[\theta_T = \pi] = \tfrac{1}{2}\mathbb{P}[\sigma_\pi \le T, \theta_T \ne \pi] + \tfrac{1}{2}\mathbb{P}[\theta_T = \pi] \le \tfrac{1}{2}.$$

By a similar argument $\mathbb{P}[A_T | F_T^c] \le 1/2$. Since $\mathbb{P}[A_T] \ge 1/2$ is a weighted average of $\mathbb{P}[A_T | F_T^c] \le 1/2$ and $\mathbb{P}[A_T | F_T]$, we conclude $\mathbb{P}[A_T | F_T] \ge 1/2$, which implies the lemma. $\square$

Recall the notation $\theta_t^{(x)}$ defined above. Using the same driving Brownian motion for any countable collection of processes $\theta_t^{(x)}$ gives a coupling under which (by continuity and pathwise uniqueness) the relative order among the processes is preserved over time.

PROOF OF LEMMA 3.4. For $T \geq 1$ and $c_0 \in (0, \pi)$, it follows from the Markov property and Lemma 3.6 that

$$\begin{aligned}\mathbb{P}[c_0 \leq \theta_T \leq 3\pi/2 | F_T] &\geq \mathbb{P}[c_0 \leq \theta_T \leq 3\pi/2, \theta_{T-1} \leq \pi | F_T] \\ &\geq \frac{\mathbb{P}[\theta_{T-1} \leq \pi | F_{T-1}] \inf_{x \leq \pi} \mathbb{P}[c_0 \leq \theta_1^{(x)} \leq 3\pi/2, F_1^{(x)}]}{\mathbb{P}[F_T | F_{T-1}]} \\ &\geq \frac{1}{2} \inf_{x \leq \pi} \mathbb{P}[\theta_1^{(x)} \in [c_0, 3\pi/2], F_1^{(x)}] \\ &\geq \frac{1}{2} \mathbb{P}\Big[\theta_1^{(0)} \geq c_0, \max_{t \leq 1} \theta_t^{(\pi)} \leq 3\pi/2\Big],\end{aligned}$$

where the last line follows by the coupling described above. Recall SDE (2.3); since $\cot(y/2)/2 \sim 1/y$ as $y \downarrow 0$, by Girsanov's theorem the process $\theta_t^{(x)}$ before hitting $3\pi/2$ has law mutually absolutely continuous with respect to that of a $\sqrt{\kappa}\mathrm{BES}^\delta$ process ($\sqrt{\kappa}$ times a δ-dimensional Bessel process) started from x, with $\delta \equiv 1 + 2(\kappa - 4)/\kappa$. Note that $\delta > 0$ since $\kappa > 8/3$. A $\sqrt{\kappa}\mathrm{BES}^\delta$ process started from $x \leq \pi$ has positive probability not to hit $3\pi/2$ by time T, so $\mathbb{P}[\max_{t \leq 1} \theta_t^{(\pi)} \leq 3\pi/2] > 0$. Meanwhile the process $\theta_t^{(0)}$ before hitting 2π is mutually absolutely continuous with respect to another $\sqrt{\kappa}\mathrm{BES}^\delta$ process (started from zero), so in particular the random variable $\theta_1^{(0)}$ does not have an atom at 0 on the event $\{\max_{t \leq 1} \theta_t^{(\pi)} \leq 3\pi/2\}$. Therefore

$$\lim_{c_0 \downarrow 0} \mathbb{P}\Big[\theta_1^{(0)} \geq c_0, \max_{t \leq 1} \theta_t^{(\pi)} \leq 3\pi/2\Big] = \mathbb{P}\Big[\max_{t \leq 1} \theta_t^{(\pi)} \leq 3\pi/2\Big] > 0,$$

which proves the existence of $c_0, p_0 > 0$ such that $\mathbb{P}[\theta_T \in [c_0, 2\pi - c_0] | F_T] \geq p_0$ for all $T \geq 1$. $\square$

PROOF OF LEMMA 3.5. Throughout the proof, let $\eta \equiv \eta_{\theta_0}$ denote a radial $\mathrm{SLE}_\kappa(\kappa - 6)$ process in $\mathbb{D}$ with starting configuration $(1, e^{-i\theta_0})$ and target 0.

We begin by comparing nearby values of θ_0. Let $o = e^{i\theta_0}$ and $o' = e^{i\theta_0'}$, where $0 < \theta_0, \theta_0' < 2\pi$. The Möbius transformation

$$f_{oo'}(\zeta) \equiv \frac{o'}{o} \frac{(o + \bar{o}' - 2)\zeta + (1 - o\bar{o}')}{(\bar{o} + o' - 2) + (1 - \bar{o}o')\zeta} = \zeta + \frac{(o' - o)(\zeta - 1)^2}{(1 - 2o + oo') + (o - o')\zeta}$$

is the automorphism of $\mathbb{D}$ sending 1 to 1, o to o', and $\bar{o}'$ to $\bar{o}$. Suppose $\theta_0, \theta_0' \in [c_0, 2\pi - c_0]$. Then $|1 - 2o + oo'| = |2 - \bar{o} - o'|$ is at least $2 - \mathrm{Re}(o + o') \geq$

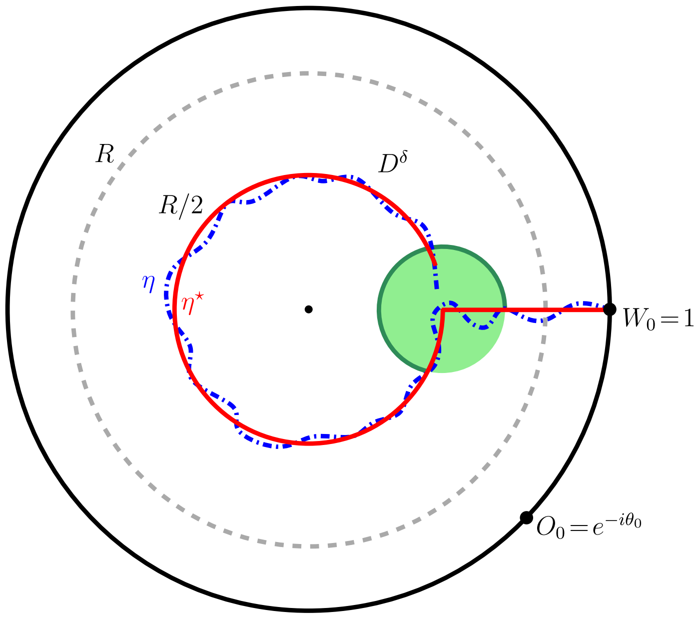


FIG. 6. *Proof of Lemma* 3.5, *Step* 1: *almost clockwise loop. A radial* $\mathrm{SLE}_\kappa(\kappa-6)$ *curve* η (*dashed blue line*) *with starting configuration* $(w,o)=(1,e^{-i\theta_0})$ ($\theta_0\in[c_0,2\pi-c_0]$) *evolves as ordinary chordal* SLE_κ *from* w *to* o, *with* (*chordal*) *driving function* W' *which is* $\sqrt{\kappa}$ *times a standard Brownian motion. Therefore*, W' *has positive probability to be uniformly close to the driving function* $W^\star$ *of the hook curve* $\eta^\star$. *On this event*, η *is close to* $\eta^\star$ *in Hausdorff distance and therefore forms an almost clockwise loop. Write* U *for the complementary connected component of the path of* η *so far which contains* z^δ. *That* η *closes the clockwise loop with positive probability is explained in Figure* 7.

$2(1-\cos c_0)$, thus bounded away from 0. From this, it is clear that if $|\theta_0-\theta_0'|$ is sufficiently small, then the image of $R\mathbb{D}$ under $f_{oo'}$ will contain the disk $(R-\varepsilon)\mathbb{D}$. It follows that $P_R(\theta_0)\geq P_{R-\varepsilon}(\theta_0')$ [using the target invariance of Proposition 2.1 since $f_{oo'}(\eta_{\theta_0})$ has target $f_{oo'}(0)\neq 0$]. This reduces the problem of showing $\inf_{\theta_0\in[c_0,2\pi-c_0]}P_R(\theta_0)>0$ to that of showing $P_R(\theta_0)>0$ for each fixed choice of $0<R<1$ and $\theta_0\in[c_0,2\pi-c_0]$.

We prove $P_R(\theta_0)>0$ in two steps which are informally explained in Figures 6 and 7.

Step 1: *Almost clockwise loop.* The function

$$f(\zeta)=\frac{i(\zeta-1)(o-1)}{2(\zeta-o)},$$

conformally maps $\mathbb{D}$ to $\mathbb{H}$ sending $W_0=1$ to 0 and $O_0=o$ to ∞. Observe

$$|f'(\zeta)|=\left|-\frac{i(o-1)^2}{2(\zeta-o)^2}\right|\geq\frac{|o^{1/2}-o^{-1/2}|^2}{8}\geq\frac{\sin^2(c_0/2)}{2}\qquad\text{for }\zeta\in\mathbb{D},$$

so the inverse transformation $f^{-1}:\mathbb{H}\to\mathbb{D}$ is Lipschitz.

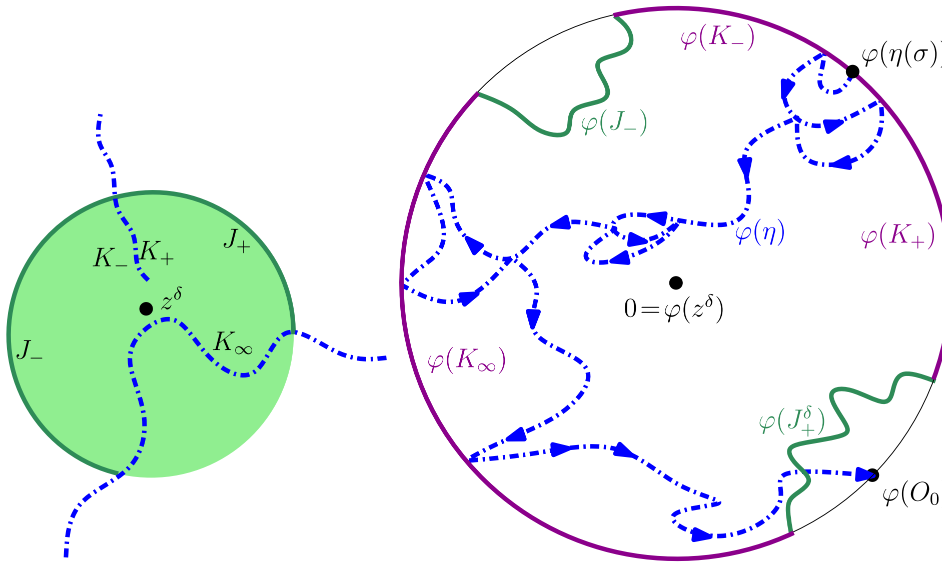


FIG. 7. *Proof of Lemma* 3.5, *Step* 2: *loop closure. The left panel shows U in a neighborhood of z^δ* (*see Figure* 6 *for the notation*). *The right panel shows the image of U under the conformal map $\varphi : U \to \mathbb{D}$ with $\varphi(z^\delta) = 0$, $\varphi'(z^\delta) > 0$. By conformal invariance of Brownian motion, it follows from consideration of hitting probabilities of Brownian motion started from z^δ in Figure* 6 *that as $\delta \downarrow 0$, $\varphi(J_\pm)$ converge to points on $\partial\mathbb{D}$ bounded away from one another, and from the image of the tip of η under φ. Loop closure occurs if $\varphi(\eta)$ crosses to the opposing arc $\varphi(K_\infty)$ before reaching $\varphi(O_0)$; this has positive probability for sufficiently small δ since* SLE_κ $(4 < \kappa < 8)$ *is boundary-intersecting but not boundary-filling*.

Recall from Section 2.1 that up to the stopping time $\tau_= \equiv \inf\{t \geq 0 : W_t = O_t\}$, η coincides (modulo time change) with the exploration tree branch η^o, which is an ordinary chordal SLE_κ in $\mathbb{D}$ from $W_0 = 1$ to $O_0 = o$. That is, $f(\eta^o(u))_{u\geq 0}$ is a standard chordal SLE_κ in $\mathbb{H}$ with associated chordal Loewner driving function $W'_u = \sqrt{\kappa} B_u$ for B_u a standard Brownian motion, and $\eta(t(u)) = \eta^o(u)$ for $t(u) \leq \tau_=$.

For $0 < \delta \ll c_0$, consider the curve $\eta^\star$ in $\mathbb{D}$ which travels in a straight line from 1 to $R/2$, then travels clockwise along the circle $\{\zeta : |\zeta| = R/2\}$ until it reaches $z^\delta \equiv (R/2)e^{i\delta/2}$. Let $W^\star_u$ be the driving function for $f(\eta^\star)$ viewed as a chordal Loewner evolution in $\mathbb{H}$, defined up to the half-plane capacity $T^\star < \infty$ of $f(\eta^\star)$. By [15], Lemma 4.2, $W^\star_u$ is continuous in u, hence uniformly continuous on $[0, T^\star]$ and thus uniformly approximable by a piecewise linear function (with finitely many pieces). Since W'_u is a Brownian motion, for any $\delta' > 0$ the event

$$\sup_{u\leq T^\star} |W'_u - W^\star_u| \leq \delta' \tag{3.6}$$

occurs with positive probability. By [15], Proposition 4.47, there exists $\delta' > 0$ such that if (3.6) occurs, then $f(\eta^o[0, T^\star])$ is within Hausdorff distance $\delta^2 \sin^2(c_0/2)/2$

of $f(\eta^\star)$, so $\eta^o[0,T^\star]$ is within Hausdorff distance δ^2 of $\eta^\star$. For sufficiently small δ this implies $t(T^\star) < \tau_=$, therefore $\eta[0,t(T^\star)]$ coincides with $\eta^o[0,T^\star]$. Thus, if we define stopping times

$$\sigma \equiv \inf\{t \geq 0 : \arg\eta(t) = \delta\}, \qquad \acute{\sigma} \equiv \inf\{t \geq 0 : \operatorname{dist}(\eta(t), \eta^\star) \geq \delta^2\},$$

then we will have $\sigma < \acute{\sigma}$ on the event (3.6).

Step 2: *Loop closure.* On the event $\{\sigma < \acute{\sigma}\}$, let τ be the first time that η closes a clockwise loop inside the disk $R\mathbb{D}$, and $\acute{\tau}$ the first time after σ that η exits $B_{R/4}(R/2)$; the result will follow by showing that

$$\liminf_{\delta\downarrow 0} \mathbb{P}[\tau < \acute{\tau} | \sigma < \acute{\sigma}] > 0. \tag{3.7}$$

Let U denote the unique connected component of $\mathbb{D} \setminus \eta[0,\sigma]$ whose closure contains both 0 and O_0. Recall that $z^\delta \equiv (R/2)e^{i\delta/2} \in U$, and let $\varphi \equiv \varphi^\delta$ denote the uniformizing map $U \to \mathbb{D}$ with $\varphi(z^\delta) = 0$ and $\varphi'(z^\delta) > 0$. Let J_+ (resp., J_-) denote the unique connected component of $U \cap \partial B_{R/4}(R/2)$ containing the point $(R/2) + e^{i\pi/4}(R/4)$ (resp., $R/4$): the $J_\pm$ are disjoint crosscuts[2] of U, and we write G for the connected component of $U \setminus (J_+ \cup J_-)$ containing z^δ. The boundary ∂G has a parametrization as a closed curve $b : [0, 2\pi] \to \partial G$, oriented counterclockwise with $b(0) = b(2\pi) = \eta(\sigma)$.[3] We then define times $0 < t_1 < t_2 < t_3 < t_4 < 2\pi$ such that $b(t_1,t_2) = J_-$ and $b(t_3,t_4) = J_+$, and write $K_- \equiv b[0,t_1]$, $K_+ \equiv b[t_4, 2\pi]$, $K_\infty \equiv b[t_2,t_3]$.

By the conformal Markov property, the probability of $\{\tau < \acute{\tau}\}$, conditioned on the path η up to time σ on the event $\{\sigma < \acute{\sigma}\}$, is given by the probability that a chordal SLE$_\kappa$ traveling in $\mathbb{D}$ from $\varphi(\eta(\sigma))$ to $\varphi(O_0)$ hits $\varphi(K_\infty)$ before hitting $\varphi(J_\pm)$.[4] It follows from consideration of hitting probabilities of Brownian motion traveling in U started from z^δ (using, e.g., the Beurling estimate [15], Theorem 3.76) that as $\delta \downarrow 0$, the diameters of the $\varphi(J_\pm)$ tend to zero while the boundary arcs $\varphi(K_\infty)$ and $\varphi(K_\pm)$ are all of sizes bounded away from zero. Since SLE$_\kappa$ ($4 < \kappa < 8$) is a.s. boundary-intersecting but not boundary-filling (see [15], Proposition 6.8) it follows that for sufficiently small δ this probability is positive. □

[2]A *crosscut* J of a domain D is an open Jordan arc in D such that $\overline{J} = J \cup \{a,b\}$ with $a, b \in \partial D$; $a = b$ is allowed. A crosscut separates the domain into exactly two components [25], Proposition 2.12, and if φ is a conformal map $D \to \mathbb{D}$ then φJ is a crosscut of $\mathbb{D}$ [25], Proposition 2.14.

[3]The set $A = \partial\mathbb{D} \cup \eta[0,\sigma] \cup J_\pm$ is compact, connected, and (since it is a finite union of curves defined on compact intervals) locally connected. By Torhorst's theorem (see [24], page 285, Problem 1 or [38], page 106, Theorem 2.2), for such A, each connected component of $\hat{\mathbb{C}} \setminus A$ has a locally connected boundary. In particular, ∂G is locally connected, so has a parametrization as a closed curve by the Hahn–Mazurkiewicz theorem.

[4]Here we abuse notation and write φS for the pre-image of S under the map $\varphi^{-1} : \mathbb{D} \to U$ which has a continuous extension to $\overline{\mathbb{D}}$.

3.3. *Hausdorff dimension*. In this section, we use the second moment estimate Lemma 3.1 and the lower bound Proposition 3.3 to deduce the main result Theorem. 1.1. The argument is standard (see, e.g., [10], Lemma 3.4) but we give some details here for completeness.

The γ*-energy* of a Borel measure μ on a metric space (E, d) is

$$I_\gamma(\mu) = \int_E \int_E d(x, y)^{-\gamma} \, d\mu(x) \, d\mu(y).$$

If there exists a positive Borel measure on E with finite γ-energy, then E has Hausdorff dimension bounded below by γ (see, e.g., [22], Theorem 4.27).

PROOF OF THEOREM 1.1. Following the proof of [10], Lemma 3.4, we first show that for any fixed $\varepsilon > 0$, there exists with positive probability a nonzero Borel measure supported on the CLE_κ gasket with finite $[2 - \alpha(1 + 2\varepsilon)]$-energy, where α is given by (2.5).

For the full range of κ we have $\alpha < 2$, so we may assume ε is sufficiently small that $\alpha(1 + \varepsilon) < 2$. Fix β large such that $e^\beta/2$ is an integer and $\varepsilon(\beta) \le \varepsilon$, with $\varepsilon(\beta)$ as in the statement of Lemma 3.1, and $c \le e^{\varepsilon\alpha\beta}$, with c as in the statement of Proposition 3.3.

For $z \in \mathbb{C}$ let $S_r(z) \equiv z + [-\frac{r}{2}, \frac{r}{2}) \times [-\frac{r}{2}, \frac{r}{2})$ denote the box with side length r centered at z, and write $H \equiv S_1(0) \subset \mathbb{D}$. For $n \ge 0$ let $S_n^z \equiv S_{e^{-n\beta}}(z)$. Since $e^\beta/2$ is an integer, H can be expressed as the disjoint union

$$H = \bigsqcup_{z \in H_n} S_n^z, \qquad H_n \equiv \{e^{-n\beta}(\mathbb{Z} + 1/2)\}^2 \cap H.$$

Recall the events E_n^z defined in (3.1). Define a random measure μ_n on H by

$$\mu_n(A) = \int_A \sum_{z \in H_n} \frac{\mathbf{1}\{E_n^z\}}{\mathbb{P}[E_n^z]} \mathbf{1}\{z' \in S_n^z\} \, dz', \qquad A \subseteq H.$$

Then $\mathbb{E}[\mu_n(H)] = 1$, and

$$\mathbb{E}[(\mu_n(H))^2] = e^{-4n\beta} \sum_{z, w \in H_n} \frac{\mathbb{P}[E_n^z \cap E_n^w]}{\mathbb{P}[E_n^z]\mathbb{P}[E_n^w]}.$$

The sum over off-diagonal terms is, by Lemma 3.1,

$$e^{-4n\beta} \sum_{\substack{z, w \in H_n \\ z \ne w}} \frac{\mathbb{P}[E_n^z \cap E_n^w]}{\mathbb{P}[E_n^z]\mathbb{P}[E_n^w]} \le e^{-2n\beta} e^{\alpha\beta(1+\varepsilon)} \sum_{\substack{w \in (e^{-n\beta}\mathbb{Z})^2 \\ |w| < \sqrt{2}}} \frac{1}{|w|^{\alpha(1+\varepsilon)}} \preccurlyeq e^{\alpha\beta(1+\varepsilon)},$$

using $\alpha(1 + \varepsilon) < 2$. By Proposition 3.3, the sum over diagonal terms is

$$e^{-4n\beta} \sum_{z \in H_n} \frac{1}{\mathbb{P}[E_n^z]} \le e^{-2n\beta} (ce^{\alpha\beta})^n \le e^{-n\beta[2-\alpha(1+\varepsilon)]} \le 1,$$

therefore $\mathbb{E}[\mu_n(H)^2] \preccurlyeq e^{\alpha\beta(1+\varepsilon)}$. Similarly,

$$
\begin{aligned}
&\mathbb{E}\big[I_{2-\alpha(1+2\varepsilon)}(\mu_n)\big] \\
&\quad = \sum_{z,w\in H_n} \frac{\mathbb{P}[E_n^z \cap E_n^w]}{\mathbb{P}[E_n^z]\mathbb{P}[E_n^w]} \int_{S_n^z}\int_{S_n^w} \frac{1}{|z'-w'|^{2-\alpha(1+2\varepsilon)}}\,dw'\,dz' \\
&\quad \preccurlyeq e^{-4n\beta} \sum_{z,w\in H_n} \frac{\mathbb{P}[E_n^z \cap E_n^w]}{\mathbb{P}[E_n^z]\mathbb{P}[E_n^w]} \bigg\{ e^{n\beta[2-\alpha(1+2\varepsilon)]} \wedge \frac{1}{\operatorname{dist}(S_n^z, S_n^w)^{2-\alpha(1+2\varepsilon)}} \bigg\} \\
&\quad \preccurlyeq e^{\alpha\beta(1+\varepsilon)} \bigg(e^{-n\beta\alpha\varepsilon} + e^{-4n\beta} \sum_{z\neq w\in H_n} \frac{1}{|z-w|^{2-\alpha\varepsilon}} \bigg) \preccurlyeq e^{\alpha\beta(1+\varepsilon)}.
\end{aligned}
$$

The argument of [10], Lemma 3.4, then implies that the CLE_κ gasket has Hausdorff dimension $\geq 2-\alpha(1+2\varepsilon)$ with positive probability.

To go from positive probability to probability one, we again make use of conditional independence in the CLE_κ process. Recall the construction of $\mathcal{L}_1^a$, illustrated in Figure 3 and described in Section 2.2. At the first time τ_{ccw}^a that a is surrounded by a counterclockwise loop, the loop $\mathcal{L}_1^a$ is formed from $\eta^a|_{[\acute{\tau}_{\mathrm{ccw}}^a, \tau_{\mathrm{ccw}}^a]}$ together with an ordinary chordal SLE_κ curve $\widetilde{\eta}^a|_{[\acute{\tau}_{\mathrm{ccw}}^a,\infty]}$ from $\eta^a(\tau_{\mathrm{ccw}}^a)$ to $\eta^a(\acute{\tau}_{\mathrm{ccw}}^a)$ in the unique connected component U of $\mathbb{D}\setminus\eta^a[0,\tau_{\mathrm{ccw}}^a]$ that has both these points on its boundary. Since $\kappa > 4$, this chordal SLE_κ hits both boundary segments (between the start and the target) infinitely often, and there are infinitely many connected components of $U \setminus \widetilde{\eta}^a[\acute{\tau}_{\mathrm{ccw}}^a, \infty]$ which are to the right of the chordal SLE_κ. Each connected component is surrounded by a clockwise loop formed from a segment of $\widetilde{\eta}^a$, so by Proposition 2.3 the components are filled in by conditionally independent CLE_κ processes. Since none of the components is surrounded by a loop of the original CLE_κ, the gasket of each small CLE_κ is contained within the gasket of the original CLE_κ. Since each of these infinitely many conditionally independent small gaskets has Hausdorff dimension $\geq 2-\alpha(1+2\varepsilon)$ with positive probability, the original gasket has Hausdorff dimension $\geq 2-\alpha(1+2\varepsilon)$ almost surely.

Taking $\varepsilon \downarrow 0$ implies the theorem. $\square$

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J. MILLER
DEPARTMENT OF MATHEMATICS
MASSACHUSSETTS INSTITUTE OF TECHNOLOGY
77 MASSACHUSETTS AVE
CAMBRIDGE, MASSACHUSETTS 02139
USA

N. SUN
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STAFORD, CALIFORNIA 94305
USA
URL: http://www-stat.stanford.edu/~nikesun/

D. B. WILSON
MICROSOFT RESEARCH
ONE MICROSOFT WAY
REDMOND, WASHINGTON 98052
USA
URL: http://dbwilson.com