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THE SENETA–HEYDE SCALING FOR THE BRANCHING RANDOM WALK

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We consider the boundary case (in the sense of Biggins and Kyprianou [*Electron. J. Probab.* **10** (2005) 609–631] in a one-dimensional super-critical branching random walk, and study the additive martingale (W_n). We prove that, upon the system's survival, $n^{1/2}W_n$ converges in probability, but not almost surely, to a positive limit. The limit is identified as a constant multiple of the almost sure limit, discovered by Biggins and Kyprianou [*Adv. in Appl. Probab.* **36** (2004) 544–581], of the derivative martingale.

1. Introduction.

We consider a discrete-time one-dimensional branching random walk, whose distribution is governed by a point process Θ on the line. The system starts with an initial particle at the origin. At time 1, the particle dies, giving birth to a certain number of new particles. These new particles form the particles at generation 1. They are positioned according to the distribution of the point process Θ; it is possible that several particles share a same position. At time 2, each of these particles dies, while giving birth to new particles that are positioned (with respect to the birth place) according to the distribution of Θ. And the system goes on according to the same mechanism. At each generation, we assume that particles produce new particles independently of each other and of everything up to that generation.

We denote by $(V(x), |x| = n)$ the positions of the particles at the nth generation; so $(V(x), |x| = 1)$ is distributed as the point process Θ. The family of random variables $(V(x))$ is usually referred to as a branching random walk (Biggins [9]). Clearly, the number of particles in each generation forms a Galton–Watson process. We always assume that this Galton–Watson process is super-critical, so the system survives with positive probability.

Throughout the paper, we assume the following condition:

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)}\Big) = 1, \qquad \mathbf{E}\Big(\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\Big) = 0. \tag{1.1}$$

The branching random walk is then said to be in the boundary case (Biggins and Kyprianou [13]). Loosely speaking, under some mild integrability conditions,

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an arbitrary branching random walk can always be made to satisfy (1.1) after a suitable linear transformation, as long as either the point process Θ is not bounded from below, or if it is, $\mathbf{E}[\sum_{|x|=1} \mathbf{1}_{\{V(x)=\underline{m}\}}] < 1$, where $\underline{m}$ denotes the essential infimum of Θ. More detailed discussions on the nature of assumption (1.1) can be found in (the ArXiv version of) Jaffuel [20].

It is immediately seen that under assumption $\mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-V(x)}] = 1$,

$$W_n := \sum_{|x|=n} \mathrm{e}^{-V(x)}, \qquad n \geq 0,$$

is a martingale (with respect to its natural filtration). In the literature, (W_n) is referred to as the *additive martingale* associated with the branching random walk. Since (W_n) is nonnegative, it converges almost surely to a (finite) limit, which, under assumption $\mathbf{E}[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}] = 0$, turns out to be 0; see Biggins [7], Lyons [27]. In particular, $\min_{|x|=n} V(x) \to \infty$ almost surely on the set of nonextinction[1].

Many of the discussions in this paper are trivial if the system dies out. So let us introduce the conditional probability

$$\mathbf{P}^*(\bullet) := \mathbf{P}(\bullet|\text{nonextinction}).$$

Under (1.1), since $W_n \to 0$, $\mathbf{P}^*$-almost surely (and $\mathbf{P}$-almost surely), the martingale is not uniformly integrable. It is natural to ask at which rate W_n goes to 0; in the literature, this concerns the Seneta–Heyde norming for W_n, referring to the pioneer work on Galton–Watson processes by Seneta [34] and Heyde [18]. The study of the Seneta–Heyde norming for the branching random walk in a general context [i.e., without assuming (1.1)] goes back at least to Biggins and Kyprianou [10] and [11]. It was an open problem of Biggins and Kyprianou [13] to study the Seneta–Heyde norming under assumption (1.1). This problem was recently investigated in [19], under suitable integrability conditions.

THEOREM A ([19]). *Assume* (1.1). *If there exists* $\delta > 0$ *such that* $\mathbf{E}[(\sum_{|x|=1} 1)^{1+\delta}] < \infty$ *and that* $\mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-(1+\delta)V(x)}] + \mathbf{E}[\sum_{|x|=1} \mathrm{e}^{\delta V(x)}] < \infty$, *then there exists a deterministic sequence* (λ_n) *of positive numbers with* $0 < \liminf_{n\to\infty} \frac{\lambda_n}{n^{1/2}} \leq \limsup_{n\to\infty} \frac{\lambda_n}{n^{1/2}} < \infty$, *such that under* $\mathbf{P}^*$,

$$\lambda_n W_n \to \mathscr{W}^* \qquad \text{in distribution}, \tag{1.2}$$

where $\mathscr{W}^* > 0$ *is a positive random variable.*

Let us make a brief description of the law of $\mathscr{W}^*$. Consider the distributional equation for the nonnegative random variable Z (excluding the trivial solution

[1]In fact, according to Biggins [8], this holds as long as $\mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-V(x)}] = 1$.

$Z=0$),

$$\mathscr{L}_Z(t)=\mathbf{E}^*\Big\{\prod_{|x|=1}\mathscr{L}_Z\big(t\mathrm{e}^{-V(x)}\big)\Big\}\qquad \forall t\ge 0,$$

where $\mathscr{L}_Z(t):=\mathbf{E}^*(\mathrm{e}^{-tZ})$ denotes the Laplace transform of Z. Under assumption (1.1), it is known (Liu [26], Biggins and Kyprianou [13]) that the equation has a unique positive solution (up to multiplication by a constant), denoted by $\mathscr{W}^*$. The Laplace transform $\mathscr{L}_Z$ can be considered as a traveling wave solution to a discrete F-KPP equation.

One may wonder whether λ_n can be taken to be (a constant multiple of) $n^{1/2}$ in (1.2). Our main result, Theorem 1.1 below, will tell us that the answer is yes.

The study of the additive martingale W_n relies on analyzing another fundamental martingale. Let us define

$$D_n:=\sum_{|x|=n}V(x)\mathrm{e}^{-V(x)},\qquad n\ge 0. \tag{1.3}$$

Since $\mathbf{E}[\sum_{|x|=1}V(x)\mathrm{e}^{-V(x)}]=0$, one can easily check that (D_n) is also a martingale, with $\mathbf{E}(D_n)=0$; it is referred to in the literature as the *derivative martingale* associated with the branching random walk. Convergence of this new martingale was studied by Biggins and Kyprianou [12]. In order to state their result, we introduce the following integrability conditions:

$$\mathbf{E}\Big[\sum_{|x|=1}V(x)^2\mathrm{e}^{-V(x)}\Big]<\infty, \tag{1.4}$$

$$\mathbf{E}\big[X\log_+^2 X\big]<\infty,\qquad \mathbf{E}[\widetilde{X}\log_+\widetilde{X}]<\infty, \tag{1.5}$$

where $\log_+ y:=\max\{0,\log y\}$ and $\log_+^2 y:=(\log_+ y)^2$ for any $y\ge 0$, and

$$X:=\sum_{|x|=1}\mathrm{e}^{-V(x)},\qquad \widetilde{X}:=\sum_{|x|=1}V(x)^+\mathrm{e}^{-V(x)},$$

with $V(x)^+:=\max\{V(x),0\}$. Throughout the paper, we assume (1.1), (1.4) and (1.5). We believe that these assumptions are optimal for our results.

THEOREM B (Biggins and Kyprianou [12]). *Assuming* (1.1), (1.4) *and* (1.5), *we have*

$$D_n\to D_\infty,\qquad \mathbf{P}^*\text{-a.s.}, \tag{1.6}$$

the limit $D_\infty>0$ *having the distribution of* $\mathscr{W}^*$ *in* (1.2).

(The positiveness of D_∞ was proved in [12] under slightly stronger assumptions. To see why it is valid under current assumptions, we refer to Proposition A.3 of [2].)

It is worth mentioning that although D_n is a signed martingale, its limit D_∞ is $\mathbf{P}^*$-almost surely positive.

Our main result is as follows.

THEOREM 1.1. *Assume* (1.1), (1.4) *and* (1.5). *Under* $\mathbf{P}^*$, *we have*

$$\lim_{n\to\infty} n^{1/2} W_n = \left(\frac{2}{\pi\sigma^2}\right)^{1/2} D_\infty \qquad \textit{in probability,} \tag{1.7}$$

where $D_\infty > 0$ *is the random variable in Theorem* B, *and*

$$\sigma^2 := \mathbf{E}\left[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\right] \in (0, \infty).$$

The convergence in probability in Theorem 1.1 is optimal: it cannot be strengthened into almost sure convergence, as is shown in the following theorem.

THEOREM 1.2. *Assume* (1.1), (1.4) *and* (1.5). *We have*

$$\limsup_{n\to\infty} n^{1/2} W_n = \infty, \qquad \mathbf{P}^*\textit{-a.s.}$$

Let us say a few words about the proof of the theorems.

The first step in the proof of Theorem 1.1 consists of introducing a truncated version of the martingales W_n and D_n, denoted by $W_n^{(\alpha)}$ and $D_n^{(\alpha)}$, respectively, where $\alpha \geq 0$ is a positive parameter. The truncation argument can be traced back to Harris [17]; we use it in the context of conditional spines, following the formalism of Kyprianou [23]. Roughly speaking (for a rigorous treatment of such approximations, see Section 5), when $n \to \infty$,

$$W_n^{(\alpha)} \approx W_n, \qquad D_n^{(\alpha)} \approx c_0 D_n,$$

where $c_0 \in (0, \infty)$ is a constant depending only on the law of Θ. Moreover, $D_n^{(\alpha)}$ is a nonnegative martingale, which allows us to define a new probability, $\mathbf{Q}^{(\alpha)}$. The distribution of the branching random walk under $\mathbf{Q}^{(\alpha)}$ is characterized by Biggins and Kyprianou [12] in the form of a spinal decomposition (recalled as Fact 3.2). By means of a second moment argument, we prove in Proposition 4.1 that under $\mathbf{Q}^{(\alpha)}$,

$$n^{1/2} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \to \theta \qquad \text{in probability,}$$

where $\theta \in (0, \infty)$ is a constant. Finally, in Section 5, by taking α to be a large (but fixed) constant, we come back to the probability $\mathbf{P}^*$, and prove that under $\mathbf{P}^*$, $n^{1/2} \frac{W_n}{D_n} \to c_0 \theta = (\frac{2}{\pi\sigma^2})^{1/2}$ in probability. Together with Theorem B, this yields Theorem 1.1.

Theorem 1.2 is proved in Section 6 by studying the minimal position in the branching random walk. The main ingredient is a well-known spinal decomposition for the branching random walk (Lyons [27]). As a by-product, we give a new proof, but under assumptions we believe to be optimal, of the fact that $\liminf_{n\to\infty} \frac{1}{\log n} \min_{|x|=n} V(x) = \frac{1}{2}$, $\mathbf{P}^*$-a.s.

The rest of the paper is as follows.

- In Section 2, we introduce a one-dimensional random walk (S_n) associated with the branching random walk, and collect a few elementary properties of (S_n).
- Section 3: formalism of the truncation argument.
- Section 4: proof of convergence in probability of $n^{1/2} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$ under $\mathbf{Q}^{(\alpha)}$.
- Section 5: proof of Theorem 1.1.
- Section 6: proof of Theorem 1.2.
- In Section 7, a few questions are raised for further investigations.

Let us mention that our method allows us to prove the analogues of Theorems 1.1 and 1.2 for the branching Brownian motion. In fact, the main ingredients in our proof, namely the truncation argument and spinal decompositions, are known in the case of the branching Brownian motion. We prefer not to give any details on how to make necessary modifications to obtain the analogues of Theorems 1.1 and 1.2 for the branching Brownian motion. These modifications are more or less painless; moreover, the situation for the branching Brownian motion is often neater than for the branching random walk—for example, the analogue of the h-process whose transition probabilities are given by (3.2), is the three-dimensional Bessel process, which is a well-studied stochastic process in the literature. Instead, we close this paragraph with an anecdotal remark: the pioneering work of McKean [30] gives an important motivation of the study of the branching Brownian motion by connecting it to the Fisher–Kolmogorov–Petrovsky–Piscounov (F-KPP) differential equation. Taking the almost sure limit of a positive martingale (which is the analogue of the additive martingale W_n), McKean claims that its Laplace transform, after a simple scale change, gives a traveling wave solution to the F-KPP equation. There turns out to be a flaw in the argument, pointed out by McKean [31]. Later on, Lalley and Sellke show in [25] that the almost sure limit studied in [30] actually is 0; instead, they use another martingale (the analogue of the derivative martingale D_n), and prove that its almost sure limit, which is positive, has the Laplace transform as being a traveling wave solution. Now that we know the two martingales (with the additive martingale suitably normalized) have similar asymptotic behaviors in probability, it becomes clear that the martingale limits studied by McKean [30] and by Lalley and Sellke [25] are a.s. identical—if the additive martingale in McKean [30] is suitably normalized.

Throughout the paper, we use $a_n \sim b_n$ $(n \to \infty)$ to denote $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$; the letter c with subscript denotes a finite and positive constant. We also adopt the notation $\min_\varnothing := \infty$, $\sum_\varnothing := 0$ and $\prod_\varnothing := 1$. For $x \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, we write x^+ for $\max\{x, 0\}$.

2. One-dimensional random walks. This section collects some well-known material. We first introduce a one-dimensional random walk associated with our branching random walk, and then recall a few ingredients of fluctuation theory for one-dimensional random walks.

2.1. *An associated one-dimensional random walk.* Let $(V(x))$ be a branching random walk satisfying (1.1) and (1.4). For any vertex x, we denote by $[\![\varnothing, x]\!]$ the unique shortest path relating x to the root $\varnothing$, and x_i (for $0 \le i \le |x|$) the vertex on $[\![\varnothing, x]\!]$ such that $|x_i| = i$. Thus, $x_0 = \varnothing$ and $x_{|x|} = x$. In words, x_i (for $i < |x|$) is the ancestor of x at generation i. We also write $]\!]\varnothing, x]\!] := [\![\varnothing, x]\!] \setminus \{\varnothing\}$.

The assumption $\mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-V(x)}] = 1$ guarantees the existence of an i.i.d. sequence of real-valued random variables $S_1, S_2 - S_1, S_3 - S_2, \ldots,$ such that for any $n \ge 1$ and any measurable function $g : \mathbb{R}^n \to [0, \infty)$,

$$\mathbf{E}\biggl\{\sum_{|x|=n} g\bigl(V(x_1), \ldots, V(x_n)\bigr)\biggr\} = \mathbf{E}\{\mathrm{e}^{S_n} g(S_1, \ldots, S_n)\}. \tag{2.1}$$

The law of S_1 is, according to (2.1), given by

$$\mathbf{E}[f(S_1)] = \mathbf{E}\biggl\{\sum_{|x|=1} \mathrm{e}^{-V(x)} f\bigl(V(x)\bigr)\biggr\},$$

for any measurable function $f : \mathbb{R} \to [0, \infty)$. Since $\mathbf{E}[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}] = 0$, we have $\mathbf{E}(S_1) = 0$. Let

$$\sigma^2 := \mathbf{E}[S_1^2] = \mathbf{E}\biggl\{\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\biggr\}. \tag{2.2}$$

Under (1.1) and (1.4), we have $0 < \sigma^2 < \infty$.

It is easy to prove (2.1) by induction on n; see, for example, Biggins and Kyprianou [11]. The presence of the new random walk (S_i) is explained via a change-of-probabilities technique as in Lyons, Pemantle and Peres [29], and Lyons [27]; see Fact 6.2 for more details. In the literature, the change-of-probabilities technique is used by many authors in various forms (see [29] for a detailed account), the idea going back at least to Kahane and Peyrière [21].

2.2. *Elementary properties of one-dimensional random walks.* Let $S_1, S_2 - S_1, S_3 - S_2, \ldots$ be an i.i.d. sequence of real-valued random variables with $\mathbf{E}(S_1) = 0$ and $\sigma^2 := \mathbf{E}[S_1^2] \in (0, \infty)$. Let $\tau^+ := \inf\{k \ge 1 : S_k \ge 0\}$, which is well defined almost surely (because $\mathbf{E}(S_1) = 0$). Let

$$R(u) := \mathbf{E}\Biggl\{\sum_{j=0}^{\tau^+ - 1} \mathbf{1}_{\{S_j \ge -u\}}\Biggr\}, \qquad u \ge 0, \tag{2.3}$$

which, according to the duality lemma, is the renewal function associated with the entrance of $(-\infty, 0)$ by the walk (S_n). More precisely, the function R can be expressed as

$$R(u) = \sum_{k=0}^{\infty} \mathbf{P}\{|H_k| \le u\}, \qquad u \ge 0, \tag{2.4}$$

where $H_0 < H_1 < H_2 < \cdots$ are the strict descending ladder heights of (S_n); that is, $H_k := S_{\tau_k^-}$, with $\tau_0^- := 0$ and $\tau_k^- := \inf\{i > \tau_{k-1}^- : S_i < \min_{0\le j\le \tau_{k-1}^-} S_j\}$, $k \ge 1$.

Throughout the paper, we regularly use the following identity:

$$R(u) = \mathbf{E}\{R(S_1 + u)\mathbf{1}_{\{S_1 \ge -u\}}\} \qquad \forall u \ge 0. \tag{2.5}$$

Conditions $\mathbf{E}[S_1^2] < \infty$ and $\mathbf{E}(S_1) = 0$ ensure that $\mathbf{E}(|H_1|) < \infty$; see, for example, [16], Theorem XVIII.5.1. The renewal theorem states that the limit

$$c_0 := \lim_{u\to\infty} \frac{R(u)}{u} \tag{2.6}$$

exists and lies in $(0, \infty)$. As a consequence, there exist constants $c_2 \ge c_1 > 0$ such that

$$c_1(1 + u) \le R(u) \le c_2(1 + u), \qquad u \ge 0. \tag{2.7}$$

The function $R(\cdot)$ describes the persistency of (S_i). In fact, if we write

$$\underline{S}_n := \min_{1\le i\le n} S_i, \qquad n \ge 1,$$

then there exists a constant $0 < \theta < \infty$ such that

$$\mathbf{P}\{\underline{S}_n \ge 0\} \sim \frac{\theta}{n^{1/2}}, \qquad n \to \infty. \tag{2.8}$$

More generally, for any $u \ge 0$,

$$\mathbf{P}\{\underline{S}_n \ge -u\} \sim \frac{\theta R(u)}{n^{1/2}}, \qquad n \to \infty. \tag{2.9}$$

See Kozlov [22], formula (12).

We will need a uniform version of (2.9) for u depending on n. Let (b_n) be a sequence of positive numbers such that $\lim_{n\to\infty} \frac{b_n}{n^{1/2}} = 0$. Then (see [3]) for any bounded continuous function $f : [0, \infty) \to \mathbb{R}$, we have, as $n \to \infty$,

$$\mathbf{E}\left\{f\left(\frac{S_n + u}{(n\sigma^2)^{1/2}}\right)\mathbf{1}_{\{\underline{S}_n \ge -u\}}\right\} = \frac{\theta R(u)}{n^{1/2}}\left(\int_0^\infty f(t)t\mathrm{e}^{-t^2/2}\,\mathrm{d}t + o(1)\right), \tag{2.10}$$

uniformly in $u \in [0, b_n]$. In particular,

$$\mathbf{P}\{\underline{S}_n \ge -u\} \sim \frac{\theta R(u)}{n^{1/2}}, \qquad n \to \infty, \tag{2.11}$$

uniformly in $u \in [0, b_n]$.

LEMMA 2.1. *Let c_0 and θ be the constants in* (2.6) *and* (2.8), *respectively. Then*

$$\theta c_0 = \left(\frac{2}{\pi\sigma^2}\right)^{1/2}. \tag{2.12}$$

PROOF. We recall from (2.4) that $R(u)$ is the mean number of strict descending ladder heights within $[-u, 0]$. By the renewal theorem (see Feller [16], Section XI.1), we have $c_0 = \frac{1}{\mathbf{E}(|H_1|)}$. On the other hand (Feller [16], Theorem XII.7.4),

$$\sum_{n\ge 1} s^n \mathbf{P}\{\underline{S}_n \ge 0\} = \exp\left(\sum_{n\ge 1} \frac{s^n}{n} \mathbf{P}\{S_n \ge 0\}\right).$$

Since $\mathbf{E}(S_1) = 0$ and $\mathbf{E}(S_1^2) < \infty$, it follows from Theorem XVIII.5.1 of Feller [16] that $c := \sum_{n\ge 1} \frac{1}{n}[\mathbf{P}\{S_n \ge 0\} - \frac{1}{2}]$ is well defined, satisfying $\mathbf{E}(|H_1|) = \frac{\sigma}{2^{1/2}} \mathrm{e}^c$. Accordingly,

$$\sum_{n\ge 1} s^n \mathbf{P}\{\underline{S}_n \ge 0\} \sim \frac{\mathrm{e}^c}{(1-s)^{1/2}}, \qquad s \uparrow 1.$$

By a Tauberian theorem (Feller [16], Theorem XIII.5.5), this yields that

$$\mathbf{P}\{\underline{S}_n \ge 0\} \sim \frac{\mathrm{e}^c}{(\pi n)^{1/2}}, \qquad n \to \infty.$$

Comparing with (2.8), we get $\theta = \frac{\mathrm{e}^c}{\pi^{1/2}} = (\frac{2}{\pi\sigma^2})^{1/2} \mathbf{E}(|H_1|) = (\frac{2}{\pi\sigma^2})^{1/2} \frac{1}{c_0}$, proving Lemma 2.1. $\square$

LEMMA 2.2. *There exists $c_3 > 0$ such that for $u > 0$, $a \ge 0$, $b \ge 0$ and $n \ge 1$,*

$$\mathbf{P}\{\underline{S}_n \ge -a, b - a \le S_n \le b - a + u\} \le c_3 \frac{(u+1)(a+1)(b+u+1)}{n^{3/2}}.$$

PROOF. The inequality is proved in [4] for a certain value of u, say 1; hence, the inequality holds for $u < 1$. The case $u > 1$ boils down to the case $u \le 1$ by splitting the interval $[b - a, b - a + u]$ into intervals of lengths ≤ 1, the number of these intervals being less than $(u + 1)$. $\square$

LEMMA 2.3. *There exists $c_4 > 0$ such that for $a \ge 0$,*

$$\sup_{n\ge 1} \mathbf{E}\big[|S_n| \mathbf{1}_{\{\underline{S}_n \ge -a\}}\big] \le c_4(a + 1).$$

PROOF. We need to check that for some $c_5 > 0$, $\mathbf{E}[S_n \mathbf{1}_{\{\underline{S}_n \ge -a\}}] \le c_5(a + 1)$, $\forall a \ge 0$, $\forall n \ge 1$.

Let $\tau_a^- := \inf\{i \ge 1 : S_i < -a\}$. Then $\{\underline{S}_n \ge -a\} = \{\tau_a^- > n\}$; thus $\mathbf{E}[S_n \mathbf{1}_{\{\underline{S}_n \ge -a\}}] = -\mathbf{E}[S_n \mathbf{1}_{\{\tau_a^- \le n\}}]$, which, by the optional sampling theorem, equals $\mathbf{E}[(-S_{\tau_a^-}) \mathbf{1}_{\{\tau_a^- \le n\}}]$. Therefore, $\sup_{n\ge 1} \mathbf{E}[S_n \mathbf{1}_{\{\underline{S}_n \ge -a\}}] = \mathbf{E}[(-S_{\tau_a^-})]$.

It remains to check that $\mathbf{E}[(-S_{\tau_a^-}) - a] \le c_6(a+1)$ for some $c_6 > 0$ and all $a \ge 0$, under the assumption $\mathbf{E}(S_1^2) < \infty$.[2] By a known trick (Lai [24]) using the sequence of strict descending ladder heights $0 =: H_0 < H_1 < H_2 < \cdots$, it boils down to proving that $\mathbf{E}[(-H_{\tau_H(-a)}) - a] \le c_7(a+1)$ for some $c_7 > 0$ and all $a \ge 0$, where $H_1, H_2 - H_1, H_3 - H_2, \ldots$, are i.i.d. *negative* random variables with $\mathbf{E}(|H_1|) < \infty$, and $\tau_H(-a) := \inf\{i \ge 1 : H_i < -a\}$. This, however, is a special case of (2.6) of Borovkov and Foss [14]. $\square$

LEMMA 2.4. *Let $0 < \lambda < 1$. There exists $c_8 > 0$ such that for $a, b \ge 0$, $0 \le u \le v$ and $n \ge 1$,*

$$\begin{aligned}\mathbf{P}\Big\{\underline{S}_{\lfloor \lambda n\rfloor} \ge -a,\ \min_{i\in[\lambda n, n]\cap\mathbb{Z}} S_i \ge b-a,\ S_n \in [b-a+u, b-a+v]\Big\}\\ \le c_8 \frac{(v+1)(v-u+1)(a+1)}{n^{3/2}}.\end{aligned} \tag{2.13}$$

PROOF. We treat λn as an integer. Let $\mathbf{P}_{(2.13)}$ denote the probability expression on the left-hand side of (2.13). Applying the Markov property at time λn, we see that $\mathbf{P}_{(2.13)} = \mathbf{E}[\mathbf{1}_{\{\underline{S}_{\lambda n} \ge -a, S_{\lambda n} \ge b-a\}} f(S_{\lambda n})]$, where $f(r) := \mathbf{P}\{\underline{S}_{n-\lambda n} \ge b-a-r, S_{n-\lambda n} \in [b-a-r+u, b-a-r+v]\}$ (for $r \ge b-a$). By Lemma 2.2, $f(r) \le c_3 \frac{(v+1)(v-u+1)(a+r-b+1)}{n^{3/2}}$ (for $r \ge b-a$). Therefore,

$$\mathbf{P}_{(2.13)} \le \frac{c_3(v+1)(v-u+1)}{n^{3/2}} \mathbf{E}\big[(S_{\lambda n} + a - b + 1)\mathbf{1}_{\{\underline{S}_{\lambda n} \ge -a, S_{\lambda n} \ge b-a\}}\big].$$

The expectation $\mathbf{E}[\cdots]$ on the right-hand side being bounded by $\mathbf{E}[|S_{\lambda n}| \times \mathbf{1}_{\{\underline{S}_{\lambda n} \ge -a\}}] + a + 1$, it suffices to apply Lemma 2.3. $\square$

LEMMA 2.5. *There exists a constant $C > 0$ such that for any sequence (b_n) of nonnegative numbers with $\limsup_{n\to\infty} \frac{b_n}{n^{1/2}} < \infty$, and any $0 < \lambda < 1$, we have*

$$\liminf_{n\to\infty} \inf_{b\in[0,b_n]} n^{3/2}\mathbf{P}\Big\{\underline{S}_{\lfloor \lambda n\rfloor} \ge 0,\ \min_{\lfloor \lambda n\rfloor < j \le n} S_j \ge b,\ b \le S_n \le b + C\Big\} > 0.$$

PROOF. The lemma is proved in [4] in the special cases $\lambda = \frac{1}{2}$ and $b = b_n$; the same proof is valid for the general case $0 < \lambda < 1$ and uniformly in $b \in [0, b_n]$. $\square$

LEMMA 2.6. *There exists a constant $c_9 > 0$ such that for any $y \ge 0$ and $z \ge 0$,*

$$\sum_{k\ge 0} \mathbf{P}\{S_k \le y - z, \underline{S}_k \ge -z\} \le c_9(1+y)\big(1 + \min\{y, z\}\big).$$

PROOF. See Lemma B.2(i) of [2]. $\square$

[2] Assuming $\mathbf{E}(|S_1|^3) < \infty$, even more is true (Mogulskii [32]): we have $\sup_{a\ge 0} \mathbf{E}[(-S_{\tau_a^-}) - a] < \infty$.

3. Truncated processes, change of probabilities. In the study of the martingales W_n and D_n, it turns out to be more convenient to work with a truncated version of the branching random walk. The truncating argument, originating from Harris [17], was formalized for the branching Brownian motion in the context of the spine conditioned to stay positive by Kyprianou [23], and was later put into the branching random walk setting by Biggins and Kyprianou [12]. It can be adapted in other situations, for example, in the study of fragmentation processes (Bertoin and Rouault [6], Berestycki, Harris and Kyprianou [5]).

Let $(V(x))$ be a branching random walk. For any vertex x, we define

$$\underline{V}(x) := \min_{y \in]\!]\varnothing, x]\!]} V(y).$$

Let $\alpha \geq 0$, and let $R(\cdot)$ be as in (2.3). Let

$$R_\alpha(u) := R(u+\alpha), \qquad u \geq -\alpha.$$

Having in mind the additive martingale (W_n) and the derivative martingale (D_n), let us introduce a new pair of processes

$$W_n^{(\alpha)} := \sum_{|x|=n} \mathrm{e}^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\alpha\}},$$

$$D_n^{(\alpha)} := \sum_{|x|=n} R_\alpha\big(V(x)\big) \mathrm{e}^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\alpha\}}.$$

Recall from (2.6) that $\lim_{u\to\infty} \frac{R(u)}{u} = c_0$. Under (1.1), we have $\inf_{|x|=n} V(x) \to \infty$, $\mathbf{P}^*$-a.s. So, it is intuively clear that if α is "sufficently large," then $W_n^{(\alpha)}$ should behave like W_n, and $D_n^{(\alpha)}$ like $c_0 D_n$. This can easily be made rigorous, and will be done in Section 5.

In Section 4, we are going to prove that for any $\alpha \geq 0$, as $n \to \infty$, $n^{1/2} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \to \theta$ in probability [θ being the constant in (2.8)], under a new probability called $\mathbf{Q}^{(\alpha)}$. To define this new probability $\mathbf{Q}^{(\alpha)}$, we first need a simple property of $D_n^{(\alpha)}$. For any n, let $\mathscr{F}_n$ denote the sigma-algebra generated by the branching random walk in the first n generations.

The following result is known, and its analogue for the branching Brownian motion is in [23].

FACT 3.1 (Biggins and Kyprianou [12]). Assume (1.1). For any $\alpha \geq 0$, $(D_n^{(\alpha)}, n \geq 0)$ is a nonnegative martingale with respect to $(\mathscr{F}_n)$, such that $\mathbf{E}(D_n^{(\alpha)}) = R_\alpha(0)$, $\forall n$.

Since $(D_n^{(\alpha)})$ is a nonnegative martingale with $\mathbf{E}(D_n^{(\alpha)}) = R_\alpha(0)$, there exists a probability measure $\mathbf{Q}^{(\alpha)}$ such that for any n,

$$\mathbf{Q}^{(\alpha)}|_{\mathscr{F}_n} := \frac{D_n^{(\alpha)}}{R_\alpha(0)} \bullet \mathbf{P}\Big|_{\mathscr{F}_n}.$$

We observe that $\mathbf{Q}^{(\alpha)}(\text{nonextinction}) = 1$, and that $\mathbf{Q}^{(\alpha)}(D_n^{(\alpha)} > 0) = 1$ for any n.

(Strictly speaking, to make our presentation mathematically rigorous, we need to work on the canonical space of branching random walks (= space of marked trees) and use the rigorous language of Neveu [33] to describe the probabilities $\mathbf{P}$ and $\mathbf{Q}^{(\alpha)}$, as well as the forthcoming spine $(w_n^{(\alpha)}, n \geq 0)$. We continue using the informal language, and referring the interested reader to Lyons [27] or Lyons and Peres [28], for a rigorous treatment. We mention that in the next paragraph, while introducing the spine $(w_n^{(\alpha)})$, we should, strictly speaking, enlarge the probability space and work on a product space.)

Recall that the positions of the particles in the first generation, $(V(x), |x| = 1)$, are distributed under $\mathbf{P}$ as the point process Θ. Fix $\alpha \geq 0$. For any real number $u \geq -\alpha$, let $\widehat{\Theta}_u^{(\alpha)}$ denote a point process whose distribution is the law of $(u + V(x), |x| = 1)$ under $\mathbf{Q}^{(u+\alpha)}$.

We now consider the distribution of the branching random walk under $\mathbf{Q}^{(\alpha)}$. The system starts with one particle, denoted by $w_0^{(\alpha)}$, at position $V(w_0^{(\alpha)}) = 0$. At each step n (for $n \geq 0$), particles of generation n die, while giving birth to point processes independently of each other: the particle $w_n^{(\alpha)}$ generates a point process distributed as $\widehat{\Theta}^{(\alpha)}_{V(w_n^{(\alpha)})}$, whereas any particle x, with $|x| = n$ and $x \neq w_n^{(\alpha)}$, generates a point process distributed as $V(x) + \Theta$. The particle $w_{n+1}^{(\alpha)}$ is chosen among the children y of $w_n^{(\alpha)}$ with probability proportional to $R_\alpha(V(y))\mathrm{e}^{-V(y)}\mathbf{1}_{\{\underline{V}(y) \geq -\alpha\}}$. The line of descent $w^{(\alpha)} := (w_n^{(\alpha)}, n \geq 0)$ is referred to as the *spine*. We denote by $\mathcal{B}^{(\alpha)}$ the family of the positions of this system.[3]

FACT 3.2 (Biggins and Kyprianou [12]). Assume (1.1). Let $\alpha \geq 0$.

(i) The branching random walk under $\mathbf{Q}^{(\alpha)}$, has the distribution of $\mathcal{B}^{(\alpha)}$.
(ii) For any n and any vertex x with $|x| = n$, we have

$$\mathbf{Q}^{(\alpha)}\{w_n^{(\alpha)} = x | \mathscr{F}_n\} = \frac{R_\alpha(V(x))\mathrm{e}^{-V(x)}\mathbf{1}_{\{\underline{V}(x) \geq -\alpha\}}}{D_n^{(\alpha)}}. \tag{3.1}$$

(iii) The spine process $(V(w_n^{(\alpha)}), n \geq 0)$ under $\mathbf{Q}^{(\alpha)}$, is distributed as the centered random walk $(S_n, n \geq 0)$ under $\mathbf{P}$ conditioned to stay in $[-\alpha, \infty)$.

Since $D_n^{(\alpha)} > 0$, $\mathbf{Q}^{(\alpha)}$-a.s., identity (3.1) makes sense $\mathbf{Q}^{(\alpha)}$-almost surely. In Fact 3.2(iii), the centered random walk (S_n) (under $\mathbf{P}$) conditioned to stay in

[3]The spine process $w^{(\alpha)}$ is, of course, part of the new system. Since working in a product space and dealing with projections and marginal laws would make the notation complicated, we feel free, by a slight abuse of notation, to identify $\mathcal{B}^{(\alpha)}$ with $(\mathcal{B}^{(\alpha)}, w^{(\alpha)})$.

$[-\alpha, \infty)$ is in the sense of Doob's h-transform: it is a Markov chain with transition probabilities given by

$$p^{(\alpha)}(u, \mathrm{d}v) := \mathbf{1}_{\{v \ge -\alpha\}} \frac{R_\alpha(v)}{R_\alpha(u)} p(u, \mathrm{d}v), \qquad u \ge -\alpha, \tag{3.2}$$

where $p(u, \mathrm{d}v) := \mathbf{P}(S_1 + u \in \mathrm{d}v)$ is the transition probability of (S_n). Fact 3.2(iii) tells that for any $n \ge 1$ and any measurable function $g : \mathbb{R}^n \to [0, \infty)$,

$$\begin{aligned} &\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\big[g\big(V\big(w_i^{(\alpha)}\big), 0 \le i \le n\big)\big] \\ &\quad = \frac{1}{R_\alpha(0)} \mathbf{E}\big[g(S_i, 0 \le i \le n) R_\alpha(S_n) \mathbf{1}_{\{\underline{S}_n \ge -\alpha\}}\big]. \end{aligned} \tag{3.3}$$

The spine decomposition will allow us, in the next section, to handle the first two moments of $\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$ under $\mathbf{Q}^{(\alpha)}$.

4. Convergence in probability of $\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$ under $\mathbf{Q}^{(\alpha)}$. The aim of this section is to prove that $\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$ converges in probability (under $\mathbf{Q}^{(\alpha)}$). We do this by estimating $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}})$ and $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}[(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}})^2]$, using Fact 3.2 and its consequence (3.3). Recall that $a_n \sim b_n$ $(n \to \infty)$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

PROPOSITION 4.1. *Assume* (1.1), (1.4) *and* (1.5). *Let* $\alpha \ge 0$. *We have*

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) \sim \frac{\theta}{n^{1/2}}, \tag{4.1}$$

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right)^2\right] \sim \frac{\theta^2}{n}, \qquad n \to \infty, \tag{4.2}$$

where $\theta \in (0, \infty)$ *is the constant in* (2.8). *As a consequence, under* $\mathbf{Q}^{(\alpha)}$,

$$\lim_{n\to\infty} n^{1/2} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} = \theta \qquad \textit{in probability}.$$

The last part (convergence in probability) of the proposition is obviously a consequence of (4.1)–(4.2) and Chebyshev's inequality.

The rest of the section is devoted to the proof of (4.1) and (4.2). The first step is to represent $\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$ as a conditional expectation. Recall that $\mathscr{F}_n$ is the sigma-algebra generated by the first n generations of the branching random walk.

LEMMA 4.2. *Assume* (1.1). *Let* $\alpha \ge 0$. *We have, for any* n,

$$\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} = \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{1}{R_\alpha(V(w_n^{(\alpha)}))}\Big|\mathscr{F}_n\right),$$

where $w_n^{(\alpha)}$ *is, as before, the element of the spine in the nth generation.*

PROOF. We have $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{1}{R_\alpha(V(w_n^{(\alpha)}))}|\mathscr{F}_n) = \sum_{|x|=n} \frac{\mathbf{Q}^{(\alpha)}\{w_n^{(\alpha)}=x|\mathscr{F}_n\}}{R_\alpha(V(x))}$, which, according to (3.1), equals $\sum_{|x|=n} \frac{\mathrm{e}^{-V(x)}}{D_n^{(\alpha)}}\mathbf{1}_{\{\underline{V}(x)\ge -\alpha\}} = \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$. $\square$

We are now able to prove the first part of Proposition 4.1, concerning $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}})$.

PROOF OF PROPOSITION 4.1: EQUATION (4.1). By Lemma 4.2, $\mathbf{E}_{\mathbf{Q}^{(\alpha)}} \times (\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}) = \mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{1}{R_\alpha(V(w_n^{(\alpha)}))})$, which, by applying (3.3) to $g(u_0, u_1, \dots, u_n) := \frac{1}{R_\alpha(u_n)}$, equals $\frac{\mathbf{P}\{\underline{S}_n \ge -\alpha\}}{R_\alpha(0)}$. By (2.9), $\mathbf{P}\{\underline{S}_n \ge -\alpha\} \sim \frac{\theta R_\alpha(0)}{n^{1/2}}$ (as $n \to \infty$), from which (4.1) follows immediately. $\square$

It remains to prove (4.2), which is done in several steps. The first step gives the correct order of magnitude of $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}[(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}})^2]$:

LEMMA 4.3. *Assume* (1.1) *and* (1.4). *Let* $\alpha \ge 0$. *We have*

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right)^2\right] = O\left(\frac{1}{n}\right), \qquad n \to \infty.$$

PROOF. By Lemma 4.2 and Jensen's inequality,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right)^2\right] \le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{1}{[R_\alpha(V(w_n^{(\alpha)}))]^2}\right).$$

The expression on the right-hand side is, by (3.3),

$$= \frac{1}{R_\alpha(0)}\mathbf{E}\left(\frac{\mathbf{1}_{\{\underline{S}_n \ge -\alpha\}}}{R_\alpha(S_n)}\right)$$
$$= \frac{1}{R_\alpha(0)}\mathbf{E}\left(\frac{\mathbf{1}_{\{\underline{S}_n \ge -\alpha\}}}{R(S_n+\alpha)}\right).$$

Recall from (2.7) that $R(u) \ge c_1(1+u)$, $\forall u \ge 0$. Therefore,

$$
\begin{aligned}
R_\alpha(0)c_1 \times \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl[\biggl(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\biggr)^2\biggr]
&\le \mathbf{E}\biggl(\frac{\mathbf{1}_{\{\underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\biggr)\\
&\le \sum_{i=0}^{\lfloor n^{1/2}\rfloor -1} \mathbf{E}\biggl(\frac{\mathbf{1}_{\{-\alpha+i\le S_n<-\alpha+i+1, \underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\biggr) + \mathbf{E}\biggl(\frac{\mathbf{1}_{\{S_n\ge -\alpha+\lfloor n^{1/2}\rfloor, \underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\biggr),
\end{aligned}
$$

which, by Lemma 2.2, is

$$
\begin{aligned}
&\le \sum_{i=0}^{\lfloor n^{1/2}\rfloor -1} \frac{1}{i+1} c_3 \frac{(\alpha+1)(i+1)}{n^{3/2}} + \frac{\mathbf{P}\{\underline{S}_n \ge -\alpha\}}{\lfloor n^{1/2}\rfloor}\\
&= \frac{\lfloor n^{1/2}\rfloor c_3(\alpha+1)}{n^{3/2}} + \frac{\mathbf{P}\{\underline{S}_n \ge -\alpha\}}{\lfloor n^{1/2}\rfloor}.
\end{aligned}
$$

By (2.9), $\mathbf{P}\{\underline{S}_n \ge -\alpha\} = O(\frac{1}{n^{1/2}})$, $n \to \infty$. The lemma follows. $\square$

Lemma 4.3 tells us that $\mathrm{Var}_{\mathbf{Q}^{(\alpha)}}(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}) = O(\frac{1}{n})$, whereas our goal is to replace $O(\frac{1}{n})$ by $o(\frac{1}{n})$. We need to do some more work.

Let E_n be an event such that $\mathbf{Q}^{(\alpha)}(E_n) \to 1$, $n \to \infty$. Let

$$
\xi_{n,E_n^c} := \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl(\frac{\mathbf{1}_{E_n^c}}{R_\alpha(V(w_n^{(\alpha)}))}\Big|\mathscr{F}_n\biggr).
$$

Since $\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} = \mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{1}{R_\alpha(V(w_n^{(\alpha)}))}|\mathscr{F}_n) = \xi_{n,E_n^c} + \mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}|\mathscr{F}_n)$, we have

$$
\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl[\biggl(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\biggr)^2\biggr] = \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl[\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\xi_{n,E_n^c}\biggr] + \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl[\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\biggr].
$$

By the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl[\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\xi_{n,E_n^c}\biggr] &\le \biggl\{\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\biggl[\biggl(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\biggr)^2\biggr]\biggr\}^{1/2} \{\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi_{n,E_n^c}^2)\}^{1/2}\\
&= O\biggl(\frac{1}{n^{1/2}}\biggr)\{\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi_{n,E_n^c}^2)\}^{1/2},
\end{aligned}
$$

the last identity being a consequence of Lemma 4.3. So (4.2) will be a straightforward consequence of the following lemmas.

LEMMA 4.4. *Assume* (1.1) *and* (1.4). *Let* $\alpha \ge 0$. *For any sequence of events* (E_n) *such that* $\mathbf{Q}^{(\alpha)}(E_n) \to 1$, *we have*

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi^2_{n,E_n^c}) = o\left(\frac{1}{n}\right), \qquad n \to \infty.$$

LEMMA 4.5. *Assume* (1.1), (1.4) *and* (1.5). *Let* $\alpha \ge 0$. *There exists a sequence of events* (E_n) *such that* $\mathbf{Q}^{(\alpha)}(E_n) \to 1$, *and that*

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right] \le \frac{\theta^2}{n} + o\left(\frac{1}{n}\right), \qquad n \to \infty.$$

PROOF OF LEMMA 4.4. By Jensen's inequality,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi^2_{n,E_n^c}) \le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{\mathbf{1}_{E_n^c}}{[R_\alpha(V(w_n^{(\alpha)}))]^2}\right).$$

Consequently, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi^2_{n,E_n^c}) \\
&\quad\le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{\mathbf{1}_{E_n^c}}{[R_\alpha(V(w_n^{(\alpha)}))]^2}\mathbf{1}_{\{V(w_n^{(\alpha)}) \ge \varepsilon n^{1/2}\}}\right) + \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{\mathbf{1}_{\{V(w_n^{(\alpha)}) < \varepsilon n^{1/2}\}}}{[R_\alpha(V(w_n^{(\alpha)}))]^2}\right) \\
&\quad= \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{\mathbf{1}_{E_n^c}}{[R_\alpha(V(w_n^{(\alpha)}))]^2}\mathbf{1}_{\{V(w_n^{(\alpha)}) \ge \varepsilon n^{1/2}\}}\right) + \mathbf{E}\left(\frac{\mathbf{1}_{\{S_n < \varepsilon n^{1/2}\}}}{R_\alpha(S_n)R_\alpha(0)}\mathbf{1}_{\{\underline{S}_n \ge -\alpha\}}\right),
\end{aligned}$$

the last identity being a consequence of (3.3). Recall from (2.7) that $R_\alpha(u) = R(u+\alpha) \ge c_1(1+u+\alpha)$, $\forall u \ge -\alpha$. Hence

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi^2_{n,E_n^c}) &\le \frac{\mathbf{Q}^{(\alpha)}(E_n^c)}{c_1^2(1+\varepsilon n^{1/2}+\alpha)^2} + \frac{1}{c_1 R_\alpha(0)}\mathbf{E}\left(\frac{\mathbf{1}_{\{S_n < \varepsilon n^{1/2}, \underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\right) \\
&= o\left(\frac{1}{n}\right) + \frac{1}{c_1 R_\alpha(0)}\mathbf{E}\left(\frac{\mathbf{1}_{\{S_n < \varepsilon n^{1/2}, \underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\right),
\end{aligned}$$

the last line following from the assumption that $\mathbf{Q}^{(\alpha)}(E_n^c) \to 0$. For the expectation term on the right-hand side, we observe that, by Lemma 2.2,

$$\begin{aligned}
\mathbf{E}\left(\frac{\mathbf{1}_{\{S_n < \varepsilon n^{1/2}, \underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\right) &\le \sum_{i=0}^{\lceil \varepsilon n^{1/2}+\alpha\rceil - 1} \mathbf{E}\left(\frac{\mathbf{1}_{\{-\alpha+i \le S_n < -\alpha+i+1, \underline{S}_n \ge -\alpha\}}}{S_n+\alpha+1}\right) \\
&\le \sum_{i=0}^{\lceil \varepsilon n^{1/2}+\alpha\rceil - 1} \frac{1}{i+1} c_3 \frac{(\alpha+1)(i+1)}{n^{3/2}} \\
&= \frac{\lceil \varepsilon n^{1/2}+\alpha\rceil c_3(\alpha+1)}{n^{3/2}}.
\end{aligned}$$

We have therefore proved that

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\xi^2_{n,E_n^c}) \le o\left(\frac{1}{n}\right) + \frac{\lceil \varepsilon n^{1/2} + \alpha \rceil c_3(\alpha+1)}{n^{3/2} c_1 R_\alpha(0)}, \qquad n \to \infty.$$

Since ε can be arbitrarily small (whereas the constants c_1 and c_3 do not depend on ε), this yields Lemma 4.4. $\square$

The proof of Lemma 4.5 needs some preparation. We start by the following elementary fact. Recall that $\log_+ y := \max\{0, \log y\}$ for any $y \ge 0$.

LEMMA 4.6 ([2], Lemma B.1). *Let $X \ge 0$ and $\widetilde{X} \ge 0$ be random variables such that $\mathbf{E}[X \log_+^2 X] + \mathbf{E}[\widetilde{X} \log_+ \widetilde{X}] < \infty$. Then*

$$\mathbf{E}\big[X \log_+^2 \widetilde{X}\big] + \mathbf{E}[\widetilde{X} \log_+ X] < \infty, \tag{4.3}$$

$$\lim_{z\to\infty} \frac{1}{z} \mathbf{E}\big[X \log_+^2 (X + \widetilde{X}) \min\{\log_+(X+\widetilde{X}), z\}\big] = 0, \tag{4.4}$$

$$\lim_{z\to\infty} \frac{1}{z} \mathbf{E}\big[\widetilde{X} \log_+ (X + \widetilde{X}) \min\{\log_+(X+\widetilde{X}), z\}\big] = 0. \tag{4.5}$$

We continue our preparation for the proof of Lemma 4.5. Let $k_n < n$ be an integer such that $k_n \to \infty$ ($n \to \infty$). Recall that we defined $W_n^{(\alpha)} = \sum_{|x|=n} \mathrm{e}^{-V(x)} \times \mathbf{1}_{\{\underline{V}(x) \ge -\alpha\}}$. For each vertex x with $|x| = n$ and $x \ne w_n^{(\alpha)}$, there is a unique i with $0 \le i < n$ such that $w_i^{(\alpha)} \le x$ and that $w_{i+1}^{(\alpha)} \not\le x$. For any $i \ge 1$, let

$$\Omega(w_i^{(\alpha)}) := \{|x| = i : x > w_{i-1}^{(\alpha)}, x \ne w_i^{(\alpha)}\}.$$

[In words, $\Omega(w_i^{(\alpha)})$ stands for the set of "brothers" of $w_i^{(\alpha)}$.] Accordingly,

$$W_n^{(\alpha)} = \mathrm{e}^{-V(w_n^{(\alpha)})} \mathbf{1}_{\{\underline{V}(w_n^{(\alpha)}) \ge -\alpha\}} + \sum_{i=0}^{n-1} \sum_{y \in \Omega(w_{i+1}^{(\alpha)})} \sum_{|x|=n, x \ge y} \mathrm{e}^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \ge -\alpha\}}.$$

We write

$$W_n^{(\alpha),[0,k_n)} := \sum_{i=0}^{k_n-1} \sum_{y \in \Omega(w_{i+1}^{(\alpha)})} \sum_{|x|=n, x \ge y} \mathrm{e}^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \ge -\alpha\}},$$

$$W_n^{(\alpha),[k_n,n]} := \mathrm{e}^{-V(w_n^{(\alpha)})} \mathbf{1}_{\{\underline{V}(w_n^{(\alpha)}) \ge -\alpha\}} + \sum_{i=k_n}^{n-1} \sum_{y \in \Omega(w_{i+1}^{(\alpha)})} \sum_{|x|=n, x \ge y} \mathrm{e}^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \ge -\alpha\}},$$

so that $W_n^{(\alpha)} = W_n^{(\alpha),[0,k_n)} + W_n^{(\alpha),[k_n,n]}$. We define $D_n^{(\alpha),[0,k_n)}$ and $D_n^{(\alpha),[k_n,n]}$ similarly. Let

$$E_{n,1} := \{k_n^{1/3} \le V(w_{k_n}^{(\alpha)}) \le k_n\} \cap \bigcap_{i=k_n}^{n} \{V(w_i^{(\alpha)}) \ge k_n^{1/6}\},$$

$$E_{n,2} := \bigcap_{i=k_n}^{n-1} \left\{ \sum_{y\in\Omega(w_{i+1}^{(\alpha)})} [1 + (V(y) - V(w_i^{(\alpha)}))^+] \mathrm{e}^{-[V(y)-V(w_i^{(\alpha)})]} \le \mathrm{e}^{V(w_i^{(\alpha)})/2} \right\},$$

$$E_{n,3} := \left\{ D_n^{(\alpha),[k_n,n]} \le \frac{1}{n^2} \right\}.$$

We choose

$$E_n := E_{n,1} \cap E_{n,2} \cap E_{n,3}. \tag{4.6}$$

LEMMA 4.7. *Assume* (1.1), (1.4) *and* (1.5). *Let* $\alpha \ge 0$. *Let* k_n *be such that* $\frac{k_n}{(\log n)^6} \to \infty$ *and that* $\frac{k_n}{n^{1/2}} \to 0$, $n \to \infty$. *Let* E_n *be as in* (4.6). *Then*

$$\lim_{n\to\infty} \mathbf{Q}^{(\alpha)}(E_n) = 1, \qquad \lim_{n\to\infty} \inf_{u\in[k_n^{1/3},k_n]} \mathbf{Q}^{(\alpha)}(E_n | V(w_{k_n}^{(\alpha)}) = u) = 1.$$

PROOF. Write, for $i \ge 0$,

$$E_2^{(i)} := \left\{ \sum_{y\in\Omega(w_{i+1}^{(\alpha)})} [1 + (V(y) - V(w_i^{(\alpha)}))^+] \mathrm{e}^{-[V(y)-V(w_i^{(\alpha)})]} \le \mathrm{e}^{V(w_i^{(\alpha)})/2} \right\}.$$

(Thus $E_{n,2} = \bigcap_{i=k_n}^{n-1} E_2^{(i)}$.)

For $z \ge -\alpha$, let $\mathbf{Q}_z^{(\alpha)}$ be the law of $\mathcal{B}_\alpha$ (in Fact 3.2) when the ancestor particle is located at position z. (So $\mathbf{Q}_0^{(\alpha)} = \mathbf{Q}^{(\alpha)}$.) We claim that

$$\sum_{i\ge 0} \mathbf{Q}_z^{(\alpha)}[(E_2^{(i)})^c] < \infty \qquad \forall z \ge -\alpha, \tag{4.7}$$

$$\lim_{z\to\infty} \sum_{i\ge 0} \mathbf{Q}_z^{(\alpha)}[(E_2^{(i)})^c] = 0. \tag{4.8}$$

To check (4.7) and (4.8), we observe that by Fact 3.2, for any integer $i \ge 0$ and real number $u \ge -\alpha$,

$$\begin{aligned}
&\mathbf{Q}_z^{(\alpha)}[(E_2^{(i)})^c | V(w_i^{(\alpha)}) = u] \\
&\quad = \mathbf{Q}_u^{(\alpha)} \left\{ \sum_{x\in\Omega(w_1^{(\alpha)})} [1 + (V(x) - u)^+] \mathrm{e}^{-[V(x)-u]} > \mathrm{e}^{u/2} \right\} \\
&\quad \le \mathbf{Q}_u^{(\alpha)} \left\{ \sum_{|x|=1} [1 + (V(x) - u)^+] \mathrm{e}^{-[V(x)-u]} > \mathrm{e}^{u/2} \right\}.
\end{aligned}$$

So, if $\mathbf{E}_u$ denotes expectation with respect to the law of the branching random walk with the ancestor particle located at u, then

$$\begin{aligned}
&\mathbf{Q}_z^{(\alpha)}\big[(E_2^{(i)})^c \,|\, V(w_i^{(\alpha)}) = u\big] \\
&\quad\le \mathbf{E}_u\bigg[\frac{\sum_{|y|=1} R_\alpha(V(y))\mathrm{e}^{-V(y)}\mathbf{1}_{\{V(y)\ge -\alpha\}}}{R_\alpha(u)\mathrm{e}^{-u}} \\
&\qquad\qquad \times \mathbf{1}_{\{\sum_{|x|=1}[1+(V(x)-u)^+]\mathrm{e}^{-[V(x)-u]}>\mathrm{e}^{u/2}\}}\bigg] \\
&\quad= \mathbf{E}\bigg[\frac{\sum_{|y|=1} R_\alpha(V(y)+u)\mathrm{e}^{-[V(y)+u]}\mathbf{1}_{\{V(y)\ge -\alpha-u\}}}{R_\alpha(u)\mathrm{e}^{-u}} \\
&\qquad\qquad \times \mathbf{1}_{\{\sum_{|x|=1}[1+V(x)^+]\mathrm{e}^{-V(x)}>\mathrm{e}^{u/2}\}}\bigg].
\end{aligned}$$

By (2.7), there exists a constant $c_{10} > 0$ such that

$$\frac{R_\alpha(V(y)+u)}{R_\alpha(u)} \le c_{10}\frac{V(y)^+ + u + \alpha + 1}{u+\alpha+1} = c_{10}\bigg[1 + \frac{V(y)^+}{u+\alpha+1}\bigg];$$

thus

$$\begin{aligned}
&\mathbf{Q}_z^{(\alpha)}\big[(E_2^{(i)})^c \,|\, V(w_i^{(\alpha)}) = u\big] \\
&\quad\le c_{10}\mathbf{E}\bigg[\sum_{|y|=1} \mathrm{e}^{-V(y)}\mathbf{1}_{\{\sum_{|x|=1}[1+V(x)^+]\mathrm{e}^{-V(x)}>\mathrm{e}^{u/2}\}} \\
&\qquad\qquad + \frac{1}{u+\alpha+1}\sum_{|y|=1} V(y)^+\mathrm{e}^{-V(y)}\mathbf{1}_{\{\sum_{|x|=1}[1+V(x)^+]\mathrm{e}^{-V(x)}>\mathrm{e}^{u/2}\}}\bigg] \\
&\quad= c_{10}\mathbf{E}\bigg[X\mathbf{1}_{\{X+\widetilde{X}>\mathrm{e}^{u/2}\}} + \frac{\widetilde{X}\mathbf{1}_{\{X+\widetilde{X}>\mathrm{e}^{u/2}\}}}{u+\alpha+1}\bigg],
\end{aligned}$$

where $X := \sum_{|y|=1} \mathrm{e}^{-V(y)}$ and $\widetilde{X} := \sum_{|y|=1} V(y)^+\mathrm{e}^{-V(y)}$. Consequently,

$$\mathbf{Q}_z^{(\alpha)}\big[(E_2^{(i)})^c\big] \le c_{10}\big(\mathbf{E}\otimes\mathbf{E}_z^{(\alpha)}\big)\bigg[X\mathbf{1}_{\{X+\widetilde{X}>\mathrm{e}^{S_i/2}\}} + \frac{\widetilde{X}\mathbf{1}_{\{X+\widetilde{X}>\mathrm{e}^{S_i/2}\}}}{S_i+\alpha+1}\bigg],$$

where, on the right-hand side, we assume that $(X, \widetilde{X})$ and S_i are independent, the expectation $\mathbf{E}$ being for $(X, \widetilde{X})$, while the expectation $\mathbf{E}_z^{(\alpha)}$ for S_i. Here, $\mathbf{E}_z^{(\alpha)}$ stands for the expectation with respect to $\mathbf{P}_z^{(\alpha)}$, the law of the h-process of (S_i) starting from z and conditioned to stay in $[-\alpha, \infty)$; the transition probabilities of this h-process being given in (3.2).

Let us consider the expression on the right-hand side. We first take the expectation for S_i with respect to $\mathbf{E}_z^{(\alpha)}$. The event $\{X + \widetilde{X} > \mathrm{e}^{S_i/2}\}$ can be written as

$S_i < 2\log(X+\widetilde{X})$. Therefore, by the definition of $\mathbf{E}_z^{(\alpha)}$, for any $x \ge 0$ and $\widetilde{x} \ge 0$,

$$\begin{aligned}
&\mathbf{E}_z^{(\alpha)}\biggl[x\mathbf{1}_{\{x+\widetilde{x}>e^{S_i/2}\}} + \frac{\widetilde{x}\mathbf{1}_{\{x+\widetilde{x}>e^{S_i/2}\}}}{S_i+\alpha+1}\biggr] \\
&\quad = \frac{1}{R_\alpha(z)}\mathbf{E}\biggl[R_\alpha(S_i+z)\mathbf{1}_{\{\underline{S}_i \ge -z-\alpha\}} \\
&\qquad\qquad \times \biggl(x\mathbf{1}_{\{S_i+z<2\log(x+\widetilde{x})\}} + \frac{\widetilde{x}\mathbf{1}_{\{S_i+z<2\log(x+\widetilde{x})\}}}{S_i+z+\alpha+1}\biggr)\biggr],
\end{aligned}$$

which, by (2.6), is[4]

$$\begin{aligned}
&\le \frac{c_2}{R_\alpha(z)}\mathbf{E}\biggl[(S_i+z+\alpha+1)\mathbf{1}_{\{\underline{S}_i \ge -z-\alpha\}} \\
&\qquad \times \biggl(x\mathbf{1}_{\{S_i+z<2\log(x+\widetilde{x})\}} + \frac{\widetilde{x}\mathbf{1}_{\{S_i+z<2\log(x+\widetilde{x})\}}}{S_i+z+\alpha+1}\biggr)\biggr] \\
&\le \frac{c_{11}[x(1+\log_+(x+\widetilde{x}))+\widetilde{x}]}{R_\alpha(z)}\mathbf{P}\bigl\{\underline{S}_i \ge -z-\alpha,\ S_i+z<2\log(x+\widetilde{x})\bigr\}.
\end{aligned}$$

Applying Lemma 2.6 yields that

$$\begin{aligned}
&\sum_{i\ge 0}\mathbf{E}_z^{(\alpha)}\biggl[x\mathbf{1}_{\{x+\widetilde{x}>e^{S_i/2}\}} + \frac{\widetilde{x}\mathbf{1}_{\{x+\widetilde{x}>e^{S_i/2}\}}}{S_i+\alpha+1}\biggr] \\
&\quad \le \frac{c_{12}[x(1+\log_+(x+\widetilde{x}))+\widetilde{x}][1+\log_+(x+\widetilde{x})][1+\min\{\log_+(x+\widetilde{x}),z\}]}{R_\alpha(z)}.
\end{aligned}$$

Taking expectation for $(X,\widetilde{X})$, using (4.3)–(4.5) in Lemma 4.6 [which we are entitled to apply, in view of assumption (1.5)], and recalling from (2.6) that $R_\alpha(z)$ grows linearly when $z\to\infty$, we obtain (4.7) and (4.8).

We now prove that $\mathbf{Q}^{(\alpha)}(E_n)\to 1$, $n\to\infty$. Since $E_n = E_{n,1}\cap E_{n,2}\cap E_{n,3}$, let us check that $\lim_{n\to\infty}\mathbf{Q}^{(\alpha)}(E_{n,\ell}) = 1$, for $\ell = 1$ and 2, and that $\lim_{n\to\infty}\mathbf{Q}^{(\alpha)}(E_{n,3}^c\cap E_{n,1}\cap E_{n,2}) = 0$.

For $E_{n,1}$: Fact 3.2 says that $(V(w_n^{(\alpha)}), n\ge 0)$ under $\mathbf{Q}^{(\alpha)}$ is the centered random walk (S_n) conditioned to stay in $[-\alpha,\infty)$; so it is clear that $\mathbf{Q}^{(\alpha)}(E_{n,1})\to 1$, $n\to\infty$.

For $E_{n,2}$: this follows from (4.7) (by taking $z=0$ there).

For $E_{n,3}$: Let $\mathcal{G}_\infty := \sigma\{V(w_k^{(\alpha)}), V(z), z\in\Omega(w_{k+1}^{(\alpha)}), k\ge 0\}$ be the sigma-algebra generated by the positions of the spine and its brothers. We know that the branching random walk rooted at $z\in\Omega(w_i^{(\alpha)})$ has the same law under $\mathbf{P}$ and

[4]The constant c_{11}, as well as the forthcoming c_{12} and c_{13}, may depend on α. This, however, makes no trouble as α will ultimately be a large (but fixed) constant.

under $\mathbf{Q}^{(\alpha)}$. Therefore,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}[D_n^{(\alpha),[k_n,n]}|\mathcal{G}_\infty] = R_\alpha(V(w_n^{(\alpha)}))\mathrm{e}^{-V(w_n^{(\alpha)})} + \sum_{i=k_n}^{n-1} \sum_{z\in\Omega(w_{i+1}^{(\alpha)})} R_\alpha(V(z))\mathrm{e}^{-V(z)}.$$

For $z \in \Omega(w_{i+1}^{(\alpha)})$, we have $R_\alpha(V(z)) \le c_{13}[1+\alpha+V(w_i^{(\alpha)})][1+(V(z)-V(w_i^{(\alpha)}))^+]$. Therefore,

$$\mathbf{1}_{E_{n,1}\cap E_{n,2}}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}[D_n^{(\alpha),[k_n,n]}|\mathcal{G}_\infty] = O(n\mathrm{e}^{-k_n^{1/6}/3}), \qquad n\to\infty, \tag{4.9}$$

where the $O(n\mathrm{e}^{-k_n^{1/6}/3})$ term on the right-hand side represents a deterministic expression. Since $\frac{k_n}{(\log n)^6}\to\infty$, it follows from the Markov inequality that $\mathbf{Q}^{(\alpha)}(E_{n,3}^c\cap E_{n,1}\cap E_{n,2})\to 0$, $n\to\infty$.

It remains to check that $\mathbf{Q}^{(\alpha)}(E_n|V(w_{k_n}^{(\alpha)})=u)\to 1$ uniformly in $u\in[k_n^{1/3},k_n]$.

By (4.8), $\mathbf{Q}^\alpha(E_{n,2}^c|V(w_{k_n}^{(\alpha)})=u)\to 0$ uniformly in $u\in[k_n^{1/3},k_n]$, whereas according to (4.9), $\mathbf{1}_{E_{n,1}\cap E_{n,2}}\mathbf{Q}^{(\alpha)}(E_{n,3}^c|\mathcal{G}_\infty)$ is bounded by a deterministic expression which goes to 0 when $n\to\infty$. Therefore, we only have to check that $\mathbf{Q}^{(\alpha)}(E_{n,1}|V(w_{k_n}^{(\alpha)})=u)\to 1$, uniformly in $u\in[k_n^{1/3},k_n]$. By Fact 3.2 and (3.2),

$$\mathbf{Q}^{(\alpha)}(E_{n,1}|V(w_{k_n}^{(\alpha)})=u) = \frac{1}{R_\alpha(u)}\mathbf{E}[R_\alpha(S_{n-k_n}+u)\mathbf{1}_{\{\underline{S}_{n-k_n}\ge k_n^{1/6}-u\}}].$$

Let, as before, $c_0 := \lim_{t\to\infty}\frac{R_\alpha(t)}{t}$, and let $\eta\in(0,c_0)$. Let $f_\eta(t) := (c_0-\eta)\min\{t,\frac{1}{\eta}\}$. Then $R_\alpha(t)\ge bf_\eta(\frac{t}{b})$ for all sufficiently large t and uniformly in $b>0$. We take $b:=(n-k_n)^{1/2}\sigma$ (with $\sigma^2:=\mathbf{E}[S_1^2]$ as before), to see that for all sufficiently large n and uniformly in $u>k_n^{1/6}$,

$$\begin{aligned}
&\mathbf{Q}^{(\alpha)}(E_{n,1}|V(w_{k_n}^{(\alpha)})=u)\\
&\quad\ge \frac{(n-k_n)^{1/2}\sigma}{R_\alpha(u)}\mathbf{E}\left[f_\eta\left(\frac{S_{n-k_n}+u}{(n-k_n)^{1/2}\sigma}\right)\mathbf{1}_{\{\underline{S}_{n-k_n}\ge k_n^{1/6}-u\}}\right]\\
&\quad\ge \frac{(n-k_n)^{1/2}\sigma}{R_\alpha(u)}\mathbf{E}\left[f_\eta\left(\frac{S_{n-k_n}+u-k_n^{1/6}}{(n-k_n)^{1/2}\sigma}\right)\mathbf{1}_{\{\underline{S}_{n-k_n}\ge k_n^{1/6}-u\}}\right].
\end{aligned}$$

Since $\frac{k_n}{n^{1/2}}\to 0$, we can apply (2.10) to see that, as $n\to\infty$,

$$\mathbf{E}\left[f_\eta\left(\frac{S_{n-k_n}+u-k_n^{1/6}}{(n-k_n)^{1/2}\sigma}\right)\mathbf{1}_{\{\underline{S}_{n-k_n}\ge k_n^{1/6}-u\}}\right]\sim\frac{\theta R(u-k_n^{1/6})}{(n-k_n)^{1/2}}\int_0^\infty t\mathrm{e}^{-t^2/2}f_\eta(t)\,\mathrm{d}t,$$

uniformly in $u\in[k_n^{1/6},k_n]$. Consequently,

$$\liminf_{n\to\infty}\inf_{u\in[k_n^{1/3},k_n]}\mathbf{Q}^{(\alpha)}(E_{n,1}|V(w_{k_n}^{(\alpha)})=u)\ge\theta\sigma\int_0^\infty t\mathrm{e}^{-t^2/2}f_\eta(t)\,\mathrm{d}t.$$

Note that $\int_0^\infty t\mathrm{e}^{-t^2/2} f_\eta(t)\,\mathrm{d}t \ge (c_0-\eta)\int_0^{1/\eta} t^2\mathrm{e}^{-t^2/2}\,\mathrm{d}t$. Letting $\eta \to 0$ gives

$$\liminf_{n\to\infty} \inf_{u\in[k_n^{1/3},k_n]} \mathbf{Q}^{(\alpha)}\big(E_{n,1} \,|\, V\big(w_{k_n}^{(\alpha)}\big)=u\big) \ge c_0\theta\sigma\left(\frac{\pi}{2}\right)^{1/2} = 1,$$

the last identity following from (2.12). Consequently, $\mathbf{Q}^{(\alpha)}(E_n|V(w_{k_n}^{(\alpha)})=u) \to 1$ uniformly in $u \in [k_n^{1/3}, k_n]$. Lemma 4.7 is proved. $\square$

We now proceed to prove Lemma 4.5.

PROOF OF LEMMA 4.5. Let k_n be such that $k_n \to \infty$ and that $\frac{k_n}{n^{1/2}} \to 0$, $n \to \infty$. Let E_n be the event in (4.6). By Lemma 4.7, $\mathbf{Q}^{(\alpha)}(E_n) \to 1$, $n \to \infty$.

On E_n, we have $D_n^{(\alpha),[k_n,n]} \le \frac{1}{n^2}$; in particular, since $W_n^{(\alpha),[k_n,n]} \le D_n^{(\alpha),[k_n,n]}$, we have $W_n^{(\alpha),[k_n,n]} \le \frac{1}{n^2}$ on E_n. On the other hand, $R_\alpha(V(w_n^{(\alpha)})) \ge 1$, so

$$\begin{aligned} &\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha),[k_n,n]}}{D_n^{(\alpha)}} \frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right] \\ &\quad\le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{1/n^2}{D_n^{(\alpha)}}\right] = \mathbf{E}\left[\frac{1/n^2}{R_\alpha(0)}\right] = o\left(\frac{1}{n}\right). \end{aligned} \tag{4.10}$$

It remains to treat $\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha)}} \frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}$. Since $D_n^{(\alpha)} \ge D_n^{(\alpha),[0,k_n)}$, we have[5]

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha)}} \frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right] \le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}} \frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right].$$

Therefore, by Fact 3.2,

$$\begin{aligned} &\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha)}} \frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right] \\ &\quad\le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}} \mathbf{1}_{\{V(w_{k_n}^{(\alpha)})\in[k_n^{1/3},k_n]\}}\right) \sup_{u\in[k_n^{1/3},k_n]} \mathbf{E}_u^{(\alpha)}\left(\frac{1}{R_\alpha(S_{n-k_n})}\right). \end{aligned} \tag{4.11}$$

For any $u \ge -\alpha$ and $j \ge 1$, we have $\mathbf{E}_u^{(\alpha)}(\frac{1}{R_\alpha(S_j)}) = \frac{1}{R_\alpha(u)}\mathbf{P}\{\underline{S}_j \ge -\alpha - u\}$, which yields, by (2.11),

$$\sup_{u\in[k_n^{1/3},k_n]} \mathbf{E}_u^{(\alpha)}\left(\frac{1}{R_\alpha(S_{n-k_n})}\right) \sim \frac{\theta}{(n-k_n)^{1/2}} \sim \frac{\theta}{n^{1/2}}, \qquad n \to \infty.$$

[5]Notation: $\frac{0}{0} := 0$ for the ratio $\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}$; noting that if $D_n^{(\alpha),[0,k_n)} = 0$, then $W_n^{(\alpha),[0,k_n)} = 0$.

Going back to (4.11), we obtain

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha)}}\frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right] \le \frac{\theta+o(1)}{n^{1/2}}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{\{V(w_{k_n}^{(\alpha)})\in[k_n^{1/3},k_n]\}}\right).$$

We claim that

$$\limsup_{n\to\infty} n^{1/2}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{\{V(w_{k_n}^{(\alpha)})\in[k_n^{1/3},k_n]\}}\right) \le \theta. \tag{4.12}$$

Then we will have

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left[\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha)}}\frac{\mathbf{1}_{E_n}}{R_\alpha(V(w_n^{(\alpha)}))}\right] \le \frac{\theta^2}{n}+o\left(\frac{1}{n}\right),$$

which, together with (4.10) and remembering $W_n^{(\alpha)} = W_n^{(\alpha),[0,k_n)} + W_n^{(\alpha),[k_n,n]}$, will complete the proof of Lemma 4.5.

It remains to check (4.12). By Fact 3.2,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{E_n}\right) \ge \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{\{V(w_{k_n}^{(\alpha)})\in[k_n^{1/3},k_n]\}}\right)\inf_{u\in[k_n^{1/3},k_n]}\mathbf{Q}^{(\alpha)}\big(E_n|V\big(w_{k_n}^{(\alpha)}\big)=u\big).$$

By Lemma 4.7, $\inf_{u\in[k_n^{1/3},k_n]}\mathbf{Q}^{(\alpha)}(E_n|V(w_{k_n}^{(\alpha)})=u)\to 1$. Therefore, as $n\to\infty$,

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{\{V(w_{k_n}^{(\alpha)})\in[k_n^{1/3},k_n]\}}\right) \le \big(1+o(1)\big)\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{E_n}\right).$$

Since $D_n^{(\alpha),[0,k_n)} \ge W_n^{(\alpha),[0,k_n)}$, we have

$$\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{E_n}\right) \le \mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{E_n}\mathbf{1}_{\{D_n^{(\alpha)}>\frac{1}{n}\}}\right)+\mathbf{Q}^{(\alpha)}\left(D_n^{(\alpha)}\le\frac{1}{n}\right).$$

Let $0<\eta_1<1$. By the Markov inequality, we see that $\mathbf{Q}^{(\alpha)}(D_n^{(\alpha)}\le\frac{1}{n}) \le \frac{1}{n}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{1}{D_n^{(\alpha)}}) = \frac{1}{nR_\alpha(0)}$. On the other hand, we already noticed that $D_n^{(\alpha),[k_n,n]}\mathbf{1}_{E_n}$

is bounded by a deterministic $o(\frac{1}{n})$. Therefore, for all sufficiently large n, $D_n^{(\alpha),[k_n,n]} \le \eta_1 D_n^{(\alpha)}$ on $E_n \cap \{D_n^{(\alpha)} > \frac{1}{n}\}$. Accordingly, for all sufficiently large n,

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{E_n}\right) &\le \frac{1}{1-\eta_1}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha)}}\mathbf{1}_{E_n\cap\{D_n^{(\alpha)}>\frac{1}{n}\}}\right) + \frac{1}{nR_\alpha(0)} \\ &\le \frac{1}{1-\eta_1}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) + \frac{1}{nR_\alpha(0)}.\end{aligned}$$

On the right-hand side, $\mathbf{E}_{\mathbf{Q}^{(\alpha)}}(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}) \sim \frac{\theta}{n^{1/2}}$; see (4.1). It follows that

$$\limsup_{n\to\infty} n^{1/2}\mathbf{E}_{\mathbf{Q}^{(\alpha)}}\left(\frac{W_n^{(\alpha),[0,k_n)}}{D_n^{(\alpha),[0,k_n)}}\mathbf{1}_{\{V(w_{k_n}^{(\alpha)})\in[k_n^{1/3},k_n]\}}\right) \le \frac{\theta}{1-\eta_1}.$$

Sending $\eta_1 \to 0$ gives (4.12), and completes the proof of Lemma 4.5. □

PROOF OF PROPOSITION 4.1. Equation (4.2) follows from Lemmas 4.4 and 4.5. □

5. Proof of Theorem 1.1.

Assume (1.1), (1.4) and (1.5). Let $\alpha \ge 0$. By Proposition 4.1, under $\mathbf{Q}^{(\alpha)}$, $n^{1/2}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}$ converges, as $n \to \infty$, in probability to θ. Therefore, for any $0 < \varepsilon < 1$,

$$\mathbf{Q}^{(\alpha)}\left\{\left|n^{1/2}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} - \theta\right| > \theta\varepsilon\right\} \to 0, \qquad n \to \infty,$$

that is,

$$\mathbf{E}\big[D_n^{(\alpha)}\mathbf{1}_{\{|n^{1/2}(W_n^{(\alpha)}/D_n^{(\alpha)})-\theta|>\theta\varepsilon\}}\big] \to 0, \qquad n \to \infty.$$

Recall that $\mathbf{P}^*(\bullet) := \mathbf{P}(\bullet|\text{nonextinction})$. By Biggins [8], condition $\mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-V(x)}) = 1$ in (1.1) implies that $\inf_{|x|=n} V(x) \to \infty$, $\mathbf{P}^*$-a.s.; thus $\inf_{|x|\ge 0} V(x) > -\infty$, $\mathbf{P}^*$-a.s.

Let $\Omega_k := \{\inf_{|x|\ge 0} V(x) \ge -k\} \cap \{\text{nonextinction}\}$. Then $(\Omega_k, k \ge 1)$ is a sequence of nondecreasing events such that $\mathbf{P}^*(\bigcup_{k\ge 1}\Omega_k) = \mathbf{P}^*(\text{nonextinction}) = 1$. Let $\eta > 0$. There exists $k_0 = k_0(\eta)$ such that $\mathbf{P}^*(\Omega_{k_0}) \ge 1 - \eta$.

Since $\mathbf{1}_{\Omega_{k_0}} \le 1$, we have

$$\mathbf{E}\big[D_n^{(\alpha)}\mathbf{1}_{\{|n^{1/2}(W_n^{(\alpha)}/D_n^{(\alpha)})-\theta|>\theta\varepsilon\}}\mathbf{1}_{\Omega_{k_0}}\big] \to 0, \qquad n \to \infty.$$

Because $D_n^{(\alpha)} \ge 0$, this is equivalent to say that, under $\mathbf{P}$,

$$D_n^{(\alpha)}\mathbf{1}_{\{|n^{1/2}(W_n^{(\alpha)}/D_n^{(\alpha)})-\theta|>\theta\varepsilon\}}\mathbf{1}_{\Omega_{k_0}} \to 0 \qquad (5.1)$$

in $L^1(\mathbf{P})$, a fortiori in probability.

On Ω_{k_0}, we have $W_n^{(\alpha)} = W_n$ for all n and all $\alpha \ge k_0$. For the behavior of $D_n^{(\alpha)}$, we observe that according to (2.6), there exists a constant $M = M(\varepsilon) > 0$ sufficiently large such that

$$c_0(1-\varepsilon)u \le R(u) \le c_0(1+\varepsilon)u \qquad \forall u \ge M.$$

We fix our choice of α from now on: $\alpha := k_0 + M$. Since $R_\alpha(u) = R(u+\alpha)$, we have, on Ω_{k_0}, $0 < c_0(1-\varepsilon)(V(x)+\alpha) \le R_\alpha(V(x)) \le c_0(1+\varepsilon)(V(x)+\alpha)$ (for all vertices x), so that on Ω_{k_0},

$$0 < c_0(1-\varepsilon)(D_n + \alpha W_n) \le D_n^{(\alpha)} \le c_0(1+\varepsilon)(D_n + \alpha W_n) \qquad \forall n.$$

(We insist on the fact that on Ω_{k_0}, $D_n + \alpha W_n > 0$ for all n.)

Recall that $D_n \to \mathscr{W}^* > 0$, $\mathbf{P}^*$-a.s., and that $W_n \to 0$, $\mathbf{P}^*$-a.s. Therefore, on the one hand, $\liminf_{n\to\infty} D_n^{(\alpha)} \ge c_0(1-\varepsilon)\mathscr{W}^* > 0$, $\mathbf{P}^*$-a.s. on Ω_{k_0}; on the other hand, on Ω_{k_0},

$$A_n \subset \left\{ \left| n^{1/2} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} - \theta \right| > \theta\varepsilon \right\} \qquad \forall n,$$

where

$$A_n := \left\{ n^{1/2} \frac{W_n}{D_n + \alpha W_n} > (1+\varepsilon)^2 c_0 \theta \right\} \cup \left\{ n^{1/2} \frac{W_n}{D_n + \alpha W_n} < (1-\varepsilon)^2 c_0 \theta \right\}.$$

In view of (5.1), we obtain that, under $\mathbf{P}^*$,

$$\mathbf{1}_{A_n} \mathbf{1}_{\Omega_{k_0}} \to 0 \qquad \text{in probability},$$

that is, $\mathbf{P}^*(A_n \cap \Omega_{k_0}) \to 0$, $n \to \infty$. Since $\mathbf{P}^*(\Omega_{k_0}) \ge 1 - \eta$, this implies

$$\limsup_{n\to\infty} \mathbf{P}^*(A_n) \le \eta.$$

In other words, $n^{1/2} \frac{W_n}{D_n}$ converges in probability (under $\mathbf{P}^*$) to $c_0\theta$, which is $(\frac{2}{\pi\sigma^2})^{1/2}$ according to (2.12). Theorem 1.1 now follows by an application of Theorem B in the Introduction.

6. Proof of Theorem 1.2. We first study the minimal displacement in a branching random walk. Recall that $\mathbf{P}^*(\bullet) := \mathbf{P}(\bullet | \text{nonextinction})$.

THEOREM 6.1. *Assume* (1.1), (1.4) *and* (1.5). *We have*

$$\liminf_{n\to\infty} \left(\min_{|x|=n} V(x) - \frac{1}{2}\log n \right) = -\infty, \qquad \mathbf{P}^*\text{-}a.s.$$

REMARK. Although we are not going to use it, we mention that $\min_{|x|=n} V(x)$ behaves typically like $\frac{3}{2}\log n$: if conditions (1.1), (1.4) and (1.5) hold, then under $\mathbf{P}^*$, $\frac{1}{\log n}\min_{|x|=n} V(x) \to \frac{3}{2}$ in probability; see [19], [1] or [4] for proofs under some additional assumptions. A proof assuming only (1.1), (1.4) and (1.5) can be found in [2]. In particular, we cannot replace "lim inf" in Theorem 6.1 by "lim."

By admitting Theorem 6.1 for the time being, we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. By definition, $W_n = \sum_{|x|=n} \mathrm{e}^{-V(x)} \geq \exp[-\min_{|x|=n} V(x)]$, so Theorem 1.2 is a consequence of Theorem 6.1. $\square$

The rest of the section is devoted to the proof of Theorem 6.1. We use once again a change-of-probabilities technique. This time, however, we only need the well-known change-of-probabilities setting in Lyons [27]: Under (1.1), (W_n) is a nonnegative martingale, so we can define a probability $\mathbf{Q}$ such that for any n,

$$\mathbf{Q}|_{\mathscr{F}_n} := W_n \bullet \mathbf{P}|_{\mathscr{F}_n}. \tag{6.1}$$

Recall that the positions of the particles in the first generation, $(V(x), |x| = 1)$, are distributed under $\mathbf{P}$ as the point process Θ; let $\widehat{\Theta}$ denote a point process whose distribution is the law of $(V(x), |x| = 1)$ under $\mathbf{Q}$.

Lyons's spinal decomposition describes the distribution of the branching random walk under $\mathbf{Q}$; it involves a spine process denoted by $(w_n, n \geq 0)$: We take $w_0 := \varnothing$, and the system starts at the initial position $V(w_0) = 0$. At time 1, w_0 gives birth to the point process $\widehat{\Theta}$. We choose w_1 at step 1 among the offspring x with probability proportional to $\mathrm{e}^{-V(x)}$. The particle w_1 gives birth to particles distributed as $\widehat{\Theta}$ [with respect to their birth position, $V(w_1)$], while all other particles in the first generation, $\{x : |x| = 1, x \neq w_1\}$ generate independent copies of Θ (with respect to their birth positions). The process goes on. The new system is denoted by $\mathcal{B}$.

FACT 6.2 (Lyons [27]). Assume (1.1). The branching random walk under $\mathbf{Q}$, has the distribution of $\mathcal{B}$. For any $|x| = n$, we have

$$\mathbf{Q}(w_n = x \,|\, \mathscr{F}_n) = \frac{\mathrm{e}^{-V(x)}}{W_n}. \tag{6.2}$$

The spine process $(V(w_n))_{n\geq 0}$ under $\mathbf{Q}$ has the distribution of $(S_n)_{n\geq 0}$ introduced in Section 2.

We mention that the analogue of Fact 6.2 for the branching Brownian motion was known to Chauvin and Rouault [15].

Fact 6.2 is useful in the proof of the following probabilistic estimate.

LEMMA 6.3. *Assume* (1.1), (1.4) *and* (1.5). *Let $C > 0$ be the constant in Lemma* 2.5. *There exists a constant $c_{14} > 0$ such that for all sufficiently large n*,

$$\mathbf{P}\big\{\exists x : n \leq |x| \leq 2n, \tfrac{1}{2}\log n \leq V(x) \leq \tfrac{1}{2}\log n + C\big\} \geq c_{14}.$$

PROOF OF LEMMA 6.3. The proof of the lemma borrows an idea from [2]; see (6.5) below. We fix n and let

$$a_i = a_i(n) := \begin{cases} 0, & \text{if } 0 \le i \le \frac{n}{2}, \\ \frac{1}{2}\log n, & \text{if } \frac{n}{2} < i \le 2n \end{cases}$$

and for $n < k \le 2n$,

$$b_i^{(k)} = b_i^{(k)}(n) := \begin{cases} i^{1/12}, & \text{if } 0 \le i \le \frac{n}{2}, \\ (k-i)^{1/12}, & \text{if } \frac{n}{2} < i \le k. \end{cases}$$

For any vertex y, let, as before, y_i denote the ancestor of y at generation i (for $0 \le i \le |y|$, with $y_{|y|} := y$), and $\Omega(y)$ the set of brothers of y. We consider

$$Z^{(n)} := \sum_{k=n+1}^{2n} Z_k^{(n)},$$

$$Z_k^{(n)} := \#(E_k \cap F_k),$$

where

$$E_k := \{y : |y| = k, V(y_i) \ge a_i, \forall 0 \le i \le k, V(y) \le \tfrac{1}{2}\log n + C\},$$

$$F_k := \Bigg\{y : |y| = k, \sum_{v \in \Omega(y_{i+1})} [1 + (V(v) - a_i)^+] \mathrm{e}^{-(V(v)-a_i)} \le c_{15} \mathrm{e}^{-b_i^{(k)}}, \quad \forall 0 \le i \le k-1\Bigg\}.$$

[So if $x \in E_k$, then $\frac{1}{2}\log n \le V(x) \le \frac{1}{2}\log n + C$. The set E_k here has nothing to do with the event E_n in (4.6).] The constant c_{15} in the definition of F_k is positive and will be set later on. We make use of the new probability measure $\mathbf{Q}$ introduced in (6.1): for $n < k \le 2n$,

$$\mathbf{E}[Z_k^{(n)}] = \mathbf{E_Q}\bigg[\frac{Z_k^{(n)}}{W_k}\bigg] = \mathbf{E_Q}\bigg[\sum_{|x|=k} \frac{\mathbf{1}_{\{x \in E_k \cap F_k\}}}{W_k}\bigg],$$

which, by (6.2), is

$$= \mathbf{E_Q}\bigg[\sum_{|x|=k} \mathbf{1}_{\{x \in E_k \cap F_k\}} \mathrm{e}^{V(x)} \mathbf{1}_{\{w_k = x\}}\bigg] = \mathbf{E_Q}[\mathrm{e}^{V(w_k)} \mathbf{1}_{\{w_k \in E_k \cap F_k\}}].$$

Thus,

$$\mathbf{E}[Z_k^{(n)}] \ge n^{1/2} \mathbf{Q}(w_k \in E_k \cap F_k). \tag{6.3}$$

We need to estimate $\mathbf{Q}(w_k \in E_k \cap F_k)$. By Fact 6.2, the process $(V(w_n))_{n\geq 0}$ has the law of $(S_n)_{n\geq 0}$. Therefore, for $k \in (n, 2n] \cap \mathbb{Z}$,

$$\begin{aligned}\mathbf{Q}(w_k \in E_k) &= \mathbf{P}\left\{S_i \geq a_i, \forall 0 \leq i \leq k, S_k \leq \frac{1}{2}\log n + C\right\} \\ &\in \left[\frac{c_{16}}{n^{3/2}}, \frac{c_{17}}{n^{3/2}}\right],\end{aligned} \tag{6.4}$$

by Lemmas 2.4 and 2.5. We now use Lemma C.1 of [2], stating that for any $\varepsilon > 0$, it is possible to choose the constant c_{15} (appearing in the definition of F_k) sufficiently large such that for all large n,

$$\max_{k: n<k\leq 2n} \mathbf{Q}(w_k \in E_k, w_k \notin F_k) \leq \frac{\varepsilon}{n^{3/2}}. \tag{6.5}$$

(The uniformity in $k \in (n, 2n] \cap \mathbb{Z}$ is not stated in [2], but the same proof holds.) In particular, choosing $\varepsilon := \frac{c_{16}}{2}$ [c_{16} being in (6.4)] leads to the existence of c_{15} such that for all large n,

$$\mathbf{Q}(w_k \in E_k, w_k \in F_k) \geq \frac{c_{16}}{2n^{3/2}}.$$

It follows from (6.3) that for all sufficiently large n,

$$\mathbf{E}[Z^{(n)}] \geq \sum_{k=n+1}^{2n} n^{1/2} \frac{c_{16}}{2n^{3/2}} \geq c_{18}. \tag{6.6}$$

We now estimate the second moment of $Z^{(n)}$. By definition,

$$\mathbf{E}[(Z^{(n)})^2] = \sum_{k=n+1}^{2n} \sum_{\ell=n+1}^{2n} \mathbf{E}[Z_k^{(n)} Z_\ell^{(n)}] \leq 2 \sum_{k=n+1}^{2n} \sum_{\ell=n+1}^{k} \mathbf{E}[Z_k^{(n)} Z_\ell^{(n)}].$$

Using again the probability $\mathbf{Q}$, we have for $n < \ell \leq k \leq 2n$,

$$\begin{aligned}\mathbf{E}[Z_k^{(n)} Z_\ell^{(n)}] &= \mathbf{E}_{\mathbf{Q}}\left[Z_\ell^{(n)} \frac{Z_k^{(n)}}{W_k}\right] = \mathbf{E}_{\mathbf{Q}}\left[Z_\ell^{(n)} \sum_{|x|=k} \frac{\mathbf{1}_{\{x \in E_k \cap F_k\}}}{W_k}\right] \\ &= \mathbf{E}_{\mathbf{Q}}[Z_\ell^{(n)} \mathrm{e}^{V(w_k)} \mathbf{1}_{\{w_k \in E_k \cap F_k\}}]\end{aligned}$$

by (6.2), and thus is bounded by $\mathrm{e}^C n^{1/2} \mathbf{E}_{\mathbf{Q}}[Z_\ell^{(n)} \mathbf{1}_{\{w_k \in E_k \cap F_k\}}]$. Therefore,

$$\mathbf{E}[(Z^{(n)})^2] \leq 2\mathrm{e}^C n^{1/2} \sum_{k=n+1}^{2n} \sum_{\ell=n+1}^{k} \mathbf{E}_{\mathbf{Q}}[Z_\ell^{(n)} \mathbf{1}_{\{w_k \in E_k \cap F_k\}}].$$

We now estimate $\mathbf{E}_{\mathbf{Q}}[Z_\ell^{(n)} \mathbf{1}_{\{w_k \in E_k \cap F_k\}}]$ on the right-hand side. It will be more convenient to work with $Y_\ell^{(n)} := \sum_{|x|=\ell} \mathbf{1}_{\{x \in E_\ell\}}$ which is greater than $Z_\ell^{(n)}$. Decomposing the sum $Y_\ell^{(n)}$ (for $n<\ell\le 2n$) along the spine yields that

$$Y_\ell^{(n)} = \mathbf{1}_{\{w_\ell\in E_\ell\}} + \sum_{i=1}^{\ell}\sum_{y\in\Omega(w_i)} Y_\ell^{(n)}(y),$$

where $\Omega(w_i)$ is, as before, the set of the brothers of w_i, and $Y_\ell^{(n)}(y) := \#\{x : |x| = \ell, x\ge y, x\in E_\ell\}$ the number of descendants x of y at generation ℓ such that $x\in E_\ell$. By Fact 6.2, the branching random walk emanating from $y\in\Omega(w_i)$ has the same law under $\mathbf{Q}$ and under $\mathbf{P}$. Therefore, conditioning on $\mathscr{G}_\infty := \sigma\{V(w_j), w_j, \Omega(w_j), (V(y))_{y\in\Omega(w_j)}, j\ge 0\}$, we have, for $y\in\Omega(w_i)$,

$$\mathbf{E}_{\mathbf{Q}}\big[Y_\ell^{(n)}|\mathscr{G}_\infty\big] = \varphi_{i,\ell}\big(V(y)\big),$$

where, for $r\in\mathbb{R}$,

$$\varphi_{i,\ell}(r) := \mathbf{E}\biggl[\sum_{|x|=\ell-i}\mathbf{1}_{\{r+V(x_j)\ge a_{j+i},\forall 0\le j\le \ell-i, r+V(x)\le (1/2)\log n+C\}}\biggr].$$

Consequently,

$$\begin{aligned}\mathbf{E}\big[\big(Z^{(n)}\big)^2\big] &\le 2\mathrm{e}^C n^{1/2}\sum_{k=n+1}^{2n}\sum_{\ell=n+1}^{k}\mathbf{Q}\{w_k\in E_k\cap F_k, w_\ell\in E_\ell\}\\ &\quad + 2\mathrm{e}^C n^{1/2}\sum_{k=n+1}^{2n}\sum_{\ell=n+1}^{k}\sum_{i=1}^{\ell}\mathbf{E}_{\mathbf{Q}}\biggl[\mathbf{1}_{\{w_k\in E_k\cap F_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}\big(V(y)\big)\biggr].\end{aligned}$$

In the first double sum on the right-hand side, if $\ell=k$, we simply argue that $\mathbf{Q}\{w_k\in E_k\cap F_k, w_\ell\in E_\ell\}\le \mathbf{Q}\{w_k\in E_k\}\le \frac{c_{17}}{n^{3/2}}$ [by (6.4)], so that $\sum_{k=n+1}^{2n}\mathbf{Q}\{w_k\in E_k\cap F_k, w_k\in E_k\}\le \sum_{k=n+1}^{2n}\frac{c_{17}}{n^{3/2}} = \frac{c_{17}}{n^{1/2}}$. This leads to

$$\begin{aligned}\mathbf{E}\big[\big(Z^{(n)}\big)^2\big] &\le 2\mathrm{e}^C c_{17} + 2\mathrm{e}^C n^{1/2}\sum_{k=n+2}^{2n}\sum_{\ell=n+1}^{k-1}\mathbf{Q}\{w_k\in E_k\cap F_k, w_\ell\in E_\ell\}\\ &\quad + 2\mathrm{e}^C n^{1/2}\sum_{k=n+1}^{2n}\sum_{\ell=n+1}^{k}\sum_{i=1}^{\ell}\mathbf{E}_{\mathbf{Q}}\biggl[\mathbf{1}_{\{w_k\in E_k\cap F_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}\big(V(y)\big)\biggr].\end{aligned}$$

Recall from (6.6) that $\mathbf{E}[Z^{(n)}]\ge c_{18}$. Since $\mathbf{P}(Z^{(n)}>0)\ge \frac{\{\mathbf{E}[Z^{(n)}]\}^2}{\mathbf{E}[(Z^{(n)})^2]}$, the proof of Lemma 6.3 is reduced to showing the following estimates: for some constants $c_{19}>0$ and $c_{20}>0$ and all sufficiently large n,

$$\sum_{k=n+2}^{2n}\sum_{\ell=n+1}^{k-1}\mathbf{Q}\{w_k\in E_k, w_\ell\in E_\ell\}\le \frac{c_{19}}{n^{1/2}}, \tag{6.7}$$

$$\sum_{k=n+1}^{2n}\sum_{\ell=n+1}^{k}\sum_{i=1}^{\ell}\mathbf{E}_{\mathbf{Q}}\biggl[\mathbf{1}_{\{w_k\in E_k\cap F_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}\big(V(y)\big)\biggr]\le \frac{c_{20}}{n^{1/2}}. \tag{6.8}$$

Let us first prove (6.7). By Fact 6.2, for $n < \ell < k \le 2n$,

$$\begin{aligned}
&\mathbf{Q}\{w_k \in E_k, w_\ell \in E_\ell\} \\
&\quad = \mathbf{P}\big\{S_i \ge a_i, \forall 0 \le i \le k, S_\ell \le \tfrac{1}{2}\log n + C, S_k \le \tfrac{1}{2}\log n + C\big\} \\
&\quad = \mathbf{E}\big\{\mathbf{1}_{\{S_i \ge a_i, \forall 0 \le i \le \ell, S_\ell \le \frac{1}{2}\log n + C\}} p_{k,\ell}(S_\ell)\big\},
\end{aligned}$$

where[6] $p_{k,\ell}(r) := \mathbf{P}\{r + S_j \ge \frac{1}{2}\log n, \forall 1 \le j \le k-\ell, r + S_{k-\ell} \le \frac{1}{2}\log n + C\}$ (for $r \ge \frac{1}{2}\log n$). Applying Lemma 2.2 to $a := r - \frac{1}{2}\log n$ and $b := 0$, we obtain, for $r \ge \frac{1}{2}\log n$,

$$p_{k,\ell}(r) \le c_{21}\frac{r - (1/2)\log n + 1}{(k-\ell)^{3/2}},$$

which leads to

$$\begin{aligned}
&\mathbf{Q}\{w_k \in E_k, w_\ell \in E_\ell\} \\
&\quad \le \frac{c_{21}}{(k-\ell)^{3/2}}\mathbf{E}\left\{\mathbf{1}_{\{S_i \ge a_i, \forall 0 \le i \le \ell, S_\ell \le (1/2)\log n + C\}}\left(S_\ell - \frac{1}{2}\log n + 1\right)\right\} \\
&\quad \le \frac{(C+1)c_{21}}{(k-\ell)^{3/2}}\mathbf{P}\left\{S_i \ge a_i, \forall 0 \le i \le \ell, S_\ell \le \frac{1}{2}\log n + C\right\} \\
&\quad \le \frac{(C+1)c_{21}}{(k-\ell)^{3/2}}\frac{c_{22}}{n^{3/2}},
\end{aligned}$$

the last inequality following from Lemma 2.4. This readily yields (6.7).

It remains to check (6.8). By (2.1),

$$\begin{aligned}
&\varphi_{i,\ell}(r) \\
&\quad = \mathbf{E}\big[\mathrm{e}^{S_{\ell-i}}\mathbf{1}_{\{r+S_j \ge a_{j+i}, \forall 0 \le j \le \ell-i, r+S_{\ell-i} \le (1/2)\log n + C\}}\big] \qquad (6.9)\\
&\quad \le n^{1/2}\mathrm{e}^{C-r}\mathbf{P}\big[r + S_j \ge a_{j+i}, \forall 0 \le j \le \ell - i, r + S_{\ell-i} \le \tfrac{1}{2}\log n + C\big].
\end{aligned}$$

From here, we bound $\varphi_{i,\ell}(r)$ differently depending on whether $i \le \frac{n}{2}$ or $i > \frac{n}{2}$.

First case: $i \le \frac{n}{2}$. By considering the $j = 0$ term, we get $\varphi_{i,\ell}(r) = 0$ for $r < 0$. For $r \ge 0$, we have, by (6.9) and Lemma 2.4,

$$\begin{aligned}
\varphi_{i,\ell}(r) &\le n^{1/2}\mathrm{e}^{C-r}c_{23}\frac{r+1}{n^{3/2}} \\
&= \frac{\mathrm{e}^C c_{23}}{n}\mathrm{e}^{-r}(r+1),
\end{aligned} \qquad (6.10)$$

[6] Since $\ell > n$, we have, by definition, $a_i = \frac{1}{2}\log n$ for $i \ge \ell$.

so that writing $c_{24} := \mathrm{e}^C c_{23}$ and $\mathbf{E_Q}[k,i,\ell] := \mathbf{E_Q}[\mathbf{1}_{\{w_k\in E_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}(V(y))]$ for brevity,

$$\mathbf{E_Q}[k,i,\ell] \le \frac{c_{24}}{n}\mathbf{E_Q}\Big[\mathbf{1}_{\{w_k\in E_k\cap F_k\}}\sum_{y\in\Omega(w_i)}\mathbf{1}_{\{V(y)\ge 0\}}\mathrm{e}^{-V(y)}\big(V(y)+1\big)\Big]$$
$$\le \frac{c_{24}}{n}\mathbf{E_Q}\Big[\mathbf{1}_{\{w_k\in E_k\cap F_k\}}\sum_{y\in\Omega(w_i)}\mathrm{e}^{-V(y)}\big(V(y)^+ +1\big)\Big].$$

By definition, we have $\sum_{y\in\Omega(w_i)}\mathrm{e}^{-V(y)}(V(y)^+ +1) \le c_{15}\mathrm{e}^{-(i-1)^{1/12}}$ when $w_k\in F_k$. It yields that

$$\mathbf{E_Q}[k,i,\ell] \le \frac{c_{24}c_{15}}{n}\mathrm{e}^{-(i-1)^{1/12}}\mathbf{Q}(w_k\in E_k) \le \frac{c_{24}c_{15}c_{17}}{n^{5/2}}\mathrm{e}^{-(i-1)^{1/12}}$$

by (6.4). As a consequence,

$$\sum_{k=n+1}^{2n}\sum_{\ell=n+1}^{k}\sum_{1\le i\le n/2}\mathbf{E_Q}\Big[\mathbf{1}_{\{w_k\in E_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}\big(V(y)\big)\Big] \le \frac{c_{25}}{n^{1/2}}. \tag{6.11}$$

Second (and last) case: $\frac{n}{2} < i \le \ell$. This time, we bound $\varphi_{i,\ell}(r)$ slightly differently. Let us go back to (6.9). Since $i > \frac{n}{2}$, we have $a_{j+i} = \frac{1}{2}\log n$ for all $0\le j\le \ell-i$, thus $\varphi_{i,\ell}(r) = 0$ for $r < \frac{1}{2}\log n$, whereas for $r \ge \frac{1}{2}\log n$, we have, by Lemma 2.2,

$$\varphi_{i,\ell}(r) \le n^{1/2}\mathrm{e}^{C-r}\frac{c_{26}}{(\ell-i+1)^{3/2}}\Big(r-\frac{1}{2}\log n+1\Big).$$

This is the analogue of (6.10); noting that the factor $\frac{1}{n}$ becomes $\frac{n^{1/2}}{(\ell-i+1)^{3/2}}$ now. From here, we can proceed as in the first case: writing again $\mathbf{E_Q}[k,i,\ell] := \mathbf{E_Q}[\mathbf{1}_{\{w_k\in E_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}(V(y))]$ for brevity, we have

$$\mathbf{E_Q}[k,i,\ell] \le \frac{c_{26}\mathrm{e}^C n^{1/2}}{(\ell-i+1)^{3/2}}$$
$$\times\mathbf{E_Q}\Big[\mathbf{1}_{\{w_k\in E_k\cap F_k\}}\sum_{y\in\Omega(w_i)}\mathrm{e}^{-V(y)}\Big[\Big(V(y)-\frac{1}{2}\log n\Big)^+ +1\Big]\Big]$$
$$\le \frac{c_{26}\mathrm{e}^C c_{15}n^{1/2}}{(\ell-i+1)^{3/2}}\frac{\mathrm{e}^{-(k-i+1)^{1/12}}}{n^{1/2}}\mathbf{Q}(w_k\in E_k)$$
$$\le \frac{c_{27}}{(\ell-i+1)^{3/2}n^{3/2}}\mathrm{e}^{-(k-i+1)^{1/12}},$$

where the last inequality comes from (6.4). Consequently,

$$\sum_{k=n+1}^{2n}\sum_{\ell=n+1}^{k}\sum_{\frac{n}{2}<i\le\ell}\mathbf{E_Q}\Big[\mathbf{1}_{\{w_k\in E_k\}}\sum_{y\in\Omega(w_i)}\varphi_{i,\ell}\big(V(y)\big)\Big] \le \frac{c_{28}}{n^{1/2}}.$$

Together with (6.11), this yields (6.8), and completes the proof of Lemma 6.3. $\square$

We have now all the ingredients for the proof of Theorem 6.1.

PROOF OF THEOREM 6.1. Assume (1.1), (1.4) and (1.5). Let $K > 0$.

The system being super-critical, assumption (1.1) ensures $\mathbf{P}\{\min_{|x|=1} V(x) < 0\} > 0$. Therefore, there exists an integer $L = L(K) \geq 1$ such that

$$c_{29} := \mathbf{P}\Big\{\min_{|x|=L} V(x) \leq -K\Big\} > 0.$$

Let $n_k := (L+2)^k$, $k \geq 1$, so that $n_{k+1} \geq 2n_k + L$, $\forall k$. For any k, let

$$T_k := \inf\Big\{i \geq n_k : \min_{|x|=i} V(x) \leq \frac{1}{2}\log n_k + C\Big\},$$

where $C > 0$ is the constant in Lemma 6.3. If $T_k < \infty$, let x_k be such that $|x_k| = T_k$ and that $V(x) \leq \frac{1}{2}\log n_k + C$. (If there are several such x_k, any one of them will do the job, e.g., the one with the smallest Harris–Ulam index.) Let

$$G_k := \{T_k \leq 2n_k\} \cap \Big\{\min_{|y|=L}[V(x_k y) - V(x_k)] \leq -K\Big\},$$

where $x_k y$ is the concatenation of the words x_k and y. For any pair of positive integers $j < \ell$,

$$\mathbf{P}\Big\{\bigcup_{k=j}^{\ell} G_k\Big\} = \mathbf{P}\Big\{\bigcup_{k=j}^{\ell-1} G_k\Big\} + \mathbf{P}\Big\{\bigcap_{k=j}^{\ell-1} G_k^c \cap G_\ell\Big\}. \tag{6.12}$$

On $\{T_\ell < \infty\}$, we have

$$\mathbf{P}\{G_\ell | \mathscr{F}_{T_\ell}\} = \mathbf{1}_{\{T_\ell \leq 2n_\ell\}}\mathbf{P}\Big\{\min_{|x|=L} V(x) \leq -K\Big\} = c_{30}\mathbf{1}_{\{T_\ell \leq 2n_\ell\}}.$$

Since $\bigcap_{k=j}^{\ell-1} G_k^c$ is $\mathscr{F}_{T_\ell}$-measurable, we obtain

$$\begin{aligned}\mathbf{P}\Big\{\bigcap_{k=j}^{\ell-1} G_k^c \cap G_\ell\Big\} &= c_{30}\mathbf{P}\Big\{\bigcap_{k=j}^{\ell-1} G_k^c \cap \{T_\ell \leq 2n_\ell\}\Big\} \\ &\geq c_{30}\mathbf{P}\{T_\ell \leq 2n_\ell\} - c_{30}\mathbf{P}\Big\{\bigcup_{k=j}^{\ell-1} G_k\Big\}.\end{aligned}$$

Recall that $\mathbf{P}\{T_\ell \leq 2n_\ell\} \geq c_{14}$ (Lemma 6.3; for large ℓ, say $\ell \geq j_0$). Combining this with (6.12) yields that

$$\mathbf{P}\Big\{\bigcup_{k=j}^{\ell} G_k\Big\} \geq (1 - c_{30})\mathbf{P}\Big\{\bigcup_{k=j}^{\ell-1} G_k\Big\} + c_{14}c_{30}, \qquad j_0 \leq j < \ell.$$

Iterating the inequality leads to

$$\mathbf{P}\left\{\bigcup_{k=j}^{\ell} G_k\right\} \geq (1-c_{30})^{\ell-j}\mathbf{P}\{G_j\} + c_{14}c_{30}\sum_{i=0}^{\ell-j-1}(1-c_{30})^i$$

$$\geq c_{14}c_{30}\sum_{i=0}^{\ell-j-1}(1-c_{30})^i.$$

This yields $\mathbf{P}\{\bigcup_{k=j}^{\infty} G_k\} \geq c_{14}$, $\forall j \geq j_0$. As a consequence, $\mathbf{P}(\limsup_{k\to\infty} G_k) \geq c_{14}$.

On the event $\limsup_{k\to\infty} G_k$, there are infinitely many vertices x such that $V(x) \leq \frac{1}{2}\log|x| + C - K$. Therefore,

$$\mathbf{P}\left\{\liminf_{n\to\infty}\left(\min_{|x|=n} V(x) - \frac{1}{2}\log n\right) \leq C - K\right\} \geq c_{14}.$$

The constant $K > 0$ being arbitrary, we obtain

$$\mathbf{P}\left\{\liminf_{n\to\infty}\left(\min_{|x|=n} V(x) - \frac{1}{2}\log n\right) = -\infty\right\} \geq c_{14}.$$

Let $0 < \varepsilon < 1$. Let $J_1 \geq 1$ be an integer such that $(1-c_{14})^{J_1} \leq \varepsilon$. Under $\mathbf{P}^*$, the system survives almost surely; so there exists a positive integer J_2 sufficiently large such that $\mathbf{P}^*\{\sum_{|x|=J_2} 1 \geq J_1\} \geq 1-\varepsilon$. By applying what we have just proved to the sub-trees of the vertices at generation J_2, we obtain

$$\mathbf{P}^*\left\{\liminf_{n\to\infty}\left(\min_{|x|=n} V(x) - \frac{1}{2}\log n\right) = -\infty\right\} \geq 1 - (1-c_{14})^{J_1} - \varepsilon \geq 1 - 2\varepsilon.$$

Sending ε to 0 completes the proof of Theorem 6.1. $\square$

Theorem 6.1 leads to the following result for the lower limits of $\min_{|x|=n} V(x)$, which was proved in [19] under stronger assumptions (namely, $\mathbf{E}[(\sum_{|x|=1} 1)^{1+\delta}] + \mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-(1+\delta)V(x)}] + \mathbf{E}[\sum_{|x|=1} \mathrm{e}^{\delta V(x)}] < \infty$ for some $\delta > 0$, and (1.1)). Recall that $\mathbf{P}^*(\bullet) := \mathbf{P}(\bullet|\text{nonextinction})$.

THEOREM 6.4. *Assume* (1.1), (1.4) *and* (1.5). *We have*

$$\liminf_{n\to\infty}\frac{1}{\log n}\min_{|x|=n} V(x) = \frac{1}{2}, \qquad \mathbf{P}^*\text{-}a.s.$$

PROOF. In view of Theorem 6.1, we only need to check that $\liminf_{n\to\infty}\frac{1}{\log n}\min_{|x|=n} V(x) \geq \frac{1}{2}$, $\mathbf{P}^*$-a.s.

Let $k > 0$ and $a < \frac{1}{2}$. By formula (2.1) and in its notation,

$$\mathbf{E}\left(\sum_{|x|=n}\mathbf{1}_{\{\underline{V}(x)>-k\}}\mathbf{1}_{\{V(x)\leq a\log n\}}\right) = \mathbf{E}\left(\mathrm{e}^{S_n}\mathbf{1}_{\{\underline{S}_n>-k\}}\mathbf{1}_{\{S_n\leq a\log n\}}\right)$$

$$\leq n^a\mathbf{P}(\underline{S}_n > -k, S_n \leq a\log n),$$

which, according to Lemma 2.2, is bounded by a constant multiple of $n^a \frac{(\log n)^2}{n^{3/2}}$, and which is summable in n if $a < \frac{1}{2}$. Therefore, as long as $a < \frac{1}{2}$, we have

$$\sum_{n\ge 1}\sum_{|x|=n} \mathbf{1}_{\{\underline{V}(x)>-k\}}\mathbf{1}_{\{V(x)\le a\log n\}} < \infty, \qquad \mathbf{P}\text{-a.s.}$$

By Biggins [8], condition $\mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-V(x)}) = 1$ in (1.1) implies that $\inf_{|x|=n} V(x) \to \infty$, $\mathbf{P}^*$-a.s.; thus $\inf_{|x|\ge 0} V(x) > -\infty$, $\mathbf{P}^*$-a.s. Consequently, $\liminf_{n\to\infty} \frac{1}{\log n} \min_{|x|=n} V(x) \ge a$, $\mathbf{P}^*$-a.s., for any $a < \frac{1}{2}$. □

7. Some questions. Let $(V(x))$ be a branching random walk satisfying (1.1), (1.4) and (1.5). Let, as before, $\mathbf{P}^*(\bullet) := \mathbf{P}(\bullet|\text{nonextinction})$. Theorem 6.1 tells us that $\liminf_{n\to\infty}[\min_{|x|=n} V(x) - \frac{1}{2}\log n] = -\infty$, $\mathbf{P}^*$-a.s., but it does not give us any quantitative information about how this "lim inf" expression goes to $-\infty$. This leads to our first open question.

QUESTION 7.1. *Is there a deterministic sequence* (a_n) *with* $\lim_{n\to\infty} a_n = \infty$ *such that*

$$-\infty < \liminf_{n\to\infty} \frac{1}{a_n}\Bigl(\min_{|x|=n} V(x) - \frac{1}{2}\log n\Bigr) < 0, \qquad \mathbf{P}^*\text{-}a.s.?$$

Our second question concerns the additive martingale W_n. In Theorem 1.2, we have proved that $\limsup_{n\to\infty} n^{1/2} W_n = \infty$, $\mathbf{P}^*$-a.s., but the rate at which this "lim sup" goes to infinity remains unknown.

QUESTION 7.2. *Study the rate at which the upper limits of* $n^{1/2}W_n$ *go to infinity* $\mathbf{P}^*$*-almost surely.*

Questions 7.1 and 7.2 are obviously related via the inequality $W_n \ge \exp[-\min_{|x|=n} V(x)]$. It is, however, not clear whether answering one of the questions will necessarily lead to answering the other.

About the lower limits of W_n, we have a conjecture.

CONJECTURE 7.3. *We would have*

$$\liminf_{n\to\infty} n^{1/2} W_n = \Bigl(\frac{2}{\pi\sigma^2}\Bigr)^{1/2} D_\infty, \qquad \mathbf{P}^*\text{-}a.s.,$$

where $\sigma^2 := \mathbf{E}[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}]$.

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