

Process-level large deviations for nonlinear Hawkes point processes¹

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Abstract. In this paper, we prove a process-level, also known as level-3 large deviation principle for a very general class of simple point processes, i.e. nonlinear Hawkes process, with a rate function given by the process-level entropy, which has an explicit formula.

Résumé. Dans cet article nous prouvons un principe de grandes déviations de niveau trois pour une classe très générale de processus ponctuels, c'est à dire les processus de Hawkes non-linéaires ; nous obtenons une formule explicite pour la fonctionnelle de taux, donnée par l'entropie au niveau du processus.

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1. Introduction and main results

In this paper, we study the process-level large deviations for a very general class of point processes, i.e. nonlinear Hawkes processes. The rate function is given by the process-level entropy, which has an explicit formula via the Girsanov theorem for two absolutely continuous point processes. Our methods and ideas should work for some other point processes as well and we would expect the same expression for the rate function.

Let N be a simple point process on \mathbb{R} and let $\mathcal{F}_t^{-\infty} := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t])$ be an increasing family of σ -algebras. Any nonnegative $\mathcal{F}_t^{-\infty}$ -progressively measurable process λ_t with

$$\mathbb{E}[N(a, b) | \mathcal{F}_a^{-\infty}] = \mathbb{E}\left[\int_a^b \lambda_s ds \mid \mathcal{F}_a^{-\infty}\right] \quad (1.1)$$

a.s. for all intervals $(a, b]$ is called an $\mathcal{F}_t^{-\infty}$ -intensity of N . We use the notation $N_t := N(0, t]$ to denote the number of points in the interval $(0, t]$. A point process Q is simple if $Q(\exists t: N[t-, t] \geq 2) = 0$.

A general Hawkes process is a simple point process N admitting an $\mathcal{F}_t^{-\infty}$ -intensity

$$\lambda_t := \lambda\left(\int_{-\infty}^t h(t-s)N(ds)\right), \quad (1.2)$$

where $\lambda(\cdot): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is locally integrable, left continuous, $h(\cdot): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and we always assume that $\|h\|_{L^1} = \int_0^\infty h(t) dt < \infty$. In (1.2), $\int_{-\infty}^t h(t-s)N(ds)$ stands for $\int_{(-\infty, t)} h(t-s)N(ds) = \sum_{\tau < t} h(t-\tau)$, where τ are the occurrences of the points before time t . Since we assume that $\lambda(\cdot)$ is left continuous, the intensity λ_t is predictable.

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In the literatures, $h(\cdot)$ and $\lambda(\cdot)$ are usually referred to as exciting function and rate function respectively.

A Hawkes process is linear if $\lambda(\cdot)$ is linear and it is nonlinear otherwise. The linear Hawkes process was introduced in Hawkes [8] and the nonlinear Hawkes process was introduced in Brémaud and Massoulié [3].

The Hawkes process captures both the self-exciting property and the clustering effect. You can think of the arrival times τ as “bad” events, which can be the arrivals of claims in insurance literature or the time of defaults of big firms in the real world. Hawkes process has many applications in finance. It is used for the calculation of conditional risk measures, ruin probabilities and credit default modeling. For a list of references to the applications in finance, see Liniger [9].

Hawkes processes may also have applications in cosmology, ecology, epidemiology, seismological applications, neuroscience applications and DNA modeling. For a list of references to these applications, see Bordenave and Torrisi [2].

For a short history of Hawkes processes, we refer to Liniger [9].

When $\lambda(\cdot)$ is nonlinear, Brémaud and Massoulié [3] proves that under certain conditions, there exists a unique stationary version of the nonlinear Hawkes process and Brémaud and Massoulié [3] also proves the convergence to equilibrium of a nonstationary version, both in distribution and in variation.

Throughout this paper, we assume that

- The exciting function $h(t)$ is positive, continuous and decreasing for $t \geq 0$ and $h(t) = 0$ for any $t < 0$. We also assume that $\int_0^\infty h(t) dt < \infty$.
- The rate function $\lambda(\cdot) : [0, \infty) \rightarrow \mathbb{R}^+$ is increasing and $\lim_{z \rightarrow \infty} \frac{\lambda(z)}{z} = 0$. We also assume that $\lambda(\cdot)$ is Lipschitz with constant $\alpha > 0$, i.e. $|\lambda(x) - \lambda(y)| \leq \alpha|x - y|$ for any $x, y \geq 0$.

Because of the flexibility of $\lambda(\cdot)$ and $h(\cdot)$, this will give us a very wide class of simple point processes. Later, if you go through the proofs of the process-level large deviation principle in our paper, you can see that if for any simple point process you want to obtain a process-level large deviation principle, it has to satisfy some regularities like the assumptions in our paper. We refer to Section 5 for detailed discussions.

Let Ω be the set of countable and locally finite subsets of \mathbb{R} and for any $\omega \in \Omega$ and $A \subseteq \mathbb{R}$, $\omega(A) := \omega \cap A$. For any $t \in \mathbb{R}$, we write $\omega(t) = \omega(\{t\})$. Let $N(A) = \#\omega \cap A$ denote the number of points in the set A for any $A \subset \mathbb{R}$. We also use the notation N_t to denote $N[0, t]$, the number of points up to time t starting from time 0. We define the shift operator θ_t as $\theta_t(\omega)(s) = \omega(t + s)$. We equip the sample space Ω with the topology in which the convergence $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$ is defined as

$$\sum_{\tau \in \omega_n} f(\tau) \rightarrow \sum_{\tau \in \omega} f(\tau), \quad (1.3)$$

for any continuous f with compact support.

This topology is equivalent to the vague topology for random measures. For a discussion on vague topology, random measures and point processes, see for example Grandell [7]. One can equip the space of locally finite random measures with the vague topology. The subspace of integer valued random measures is then the space of point processes. A simple point processes is a point process without multiple jumps. The space of point processes is closed. But the space of simple point processes is not closed.

Denote by $\mathcal{F}_t^s = \sigma(\omega[s, t])$ for any $s < t$, i.e. the σ -algebra generated by all the possible configurations of points in the interval $[s, t]$. Denote by $\mathcal{M}(\Omega)$ the space of probability measures on Ω . We also define $\mathcal{M}_S(\Omega)$ as the space of simple point processes that are invariant with respect to θ_t with bounded first moment, i.e. for any $Q \in \mathcal{M}_S(\Omega)$, $\mathbb{E}^Q[N[0, 1]] < \infty$. Define $\mathcal{M}_E(\Omega)$ as the set of ergodic simple point processes in $\mathcal{M}_S(\Omega)$. We define the topology of $\mathcal{M}_S(\Omega)$ as the following. For a sequence Q_n in $\mathcal{M}_S(\Omega)$ and $Q \in \mathcal{M}_S(\Omega)$, we say $Q_n \rightarrow Q$ as $n \rightarrow \infty$ if and only if

$$\int f dQ_n \rightarrow \int f dQ, \quad (1.4)$$

as $n \rightarrow \infty$ for any continuous and bounded f and

$$\int N[0, 1](\omega) Q_n(d\omega) \rightarrow \int N[0, 1](\omega) Q(d\omega), \quad (1.5)$$

as $n \rightarrow \infty$. In other words, the topology is the strengthened weak topology with the convergence of the first moment of $N[0, 1]$. For any Q_1, Q_2 in $\mathcal{M}_S(\Omega)$, one can define the metric $d(\cdot, \cdot)$ as

$$d(Q_1, Q_2) = d_p(Q_1, Q_2) + |\mathbb{E}^{Q_1}[N[0, 1]] - \mathbb{E}^{Q_2}[N[0, 1]]|, \tag{1.6}$$

where $d_p(\cdot, \cdot)$ is the usual Prokhorov metric. Because this is an unusual topology, the compactness is different from that in the usual weak topology and also, later, when we prove the exponential tightness, we need to take some extra care. See Lemma 25 and (iii) of Lemma 24.

We denote $C(\Omega)$ the set of real-valued continuous functions on Ω . We similarly define $C(\Omega \times \mathbb{R})$. We also denote $\mathcal{B}(\mathcal{F}_t^{-\infty})$ the set of all bounded $\mathcal{F}_t^{-\infty}$ progressively measurable and $\mathcal{F}_t^{-\infty}$ predictable functions.

Before we proceed, recall that a sequence $(P_n)_{n \in \mathbb{N}}$ of probability measures on a topological space X satisfies the large deviation principle (LDP) with rate function $I : X \rightarrow \mathbb{R}$ if I is nonnegative, lower semicontinuous and for any measurable set A ,

$$-\inf_{x \in A^o} I(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_n(A) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(A) \leq -\inf_{x \in \bar{A}} I(x). \tag{1.7}$$

Here, A^o is the interior of A and \bar{A} is its closure. See Dembo and Zeitouni [5] or Varadhan [12] for general background regarding large deviations and their applications. Also Varadhan [13] has an excellent survey article on this subject.

Let us have a brief review on what is known about large deviations for Hawkes processes.

When $\lambda(\cdot)$ is linear, say $\lambda(z) = \nu + z$, then one can use immigration-birth representation, also known as Galton–Watson theory to study it. Under the immigration-birth representation, if the immigrants are distributed as Poisson process with intensity ν and each immigrant generates a cluster whose number of points is denoted by S , then N_t is the total number of points generated in the clusters up to time t . If the process is ergodic, we have

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \nu \mathbb{E}[S], \quad \text{a.s.} \tag{1.8}$$

In particular, for linear Hawkes process with rate function $\lambda(z) = \nu + z$ and exciting function $h(\cdot)$ such that $\|h\|_{L^1} < 1$, we have

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \mu, \quad \text{a.s., where } \mu := \frac{\nu}{1 - \|h\|_{L^1}}. \tag{1.9}$$

Recently, Bacry et al. [1] proved a functional central limit theorem for linear multivariate Hawkes process under certain assumptions. That includes the linear Hawkes process as a special case and they proved that

$$\frac{N_{\cdot t} - \cdot \mu t}{\sqrt{t}} \rightarrow \sigma B(\cdot), \quad \text{as } t \rightarrow \infty, \tag{1.10}$$

where $B(\cdot)$ is a standard Brownian motion. The convergence is weak convergence on $D[0, 1]$, the space of càdlàg functions on $[0, 1]$, equipped with Skorokhod topology.

Bordenave and Torrisi [2] proves that if $0 < \int_0^\infty h(t) dt < 1$ and $\int_0^\infty th(t) dt < \infty$, then $(\frac{N_t}{t} \in \cdot)$ satisfies the large deviation principle with the good rate function

$$I(x) = \begin{cases} x \log\left(\frac{x}{\nu + x \|h\|_{L^1}}\right) - x + x \|h\|_{L^1} + \nu & \text{if } x \in [0, \infty), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.11}$$

When $\lambda(\cdot)$ is linear, Zhu [14] gives an alternative proof for the large deviation principle for $(N_t/t \in \cdot)$.

Once the LDP for $(\frac{N_t}{t} \in \cdot)$ is established, it is easy to study the ruin probability. Stabile and Torrisi [11] considered risk processes with nonstationary Hawkes claims arrivals and studied the asymptotic behavior of infinite and finite horizon ruin probabilities under light-tailed conditions on the claims.

When $\lambda(\cdot)$ is nonlinear, Brémaud and Massoulié [3] studied the stability results and once you have ergodicity, the ergodic theorem automatically implies the law of large numbers. Recently, Zhu [15] proved a functional central limit theorem for nonlinear Hawkes process.

For the LDP for nonlinear Hawkes processes, [14] obtained the large deviation results for $(N_t/t \in \cdot)$ for some special cases. He proved the case when $h(\cdot)$ is exponential first, then generalizes the proof to the case when $h(\cdot)$ is a sum of exponentials and finally uses that to prove the LDP for a special class of general Hawkes process. Hawkes process is interesting, partly because it is not Markovian unless $h(\cdot)$ is exponential or sum of exponentials. The methods in Zhu [14] relies on proving the LDP for Markovian case first and then approximate the general case, i.e. general $h(\cdot)$ using the Markovian case. But it is very difficult to do so. Indeed, Zhu [14] can only prove LDP for general $h(\cdot)$ when $\lim_{z \rightarrow \infty} \frac{\lambda(z)}{z^\alpha} = 0$ for any $\alpha > 0$. Therefore, it is natural in our paper to consider proving the level-3 large deviation principle first and then use the contraction principle to obtain the LDP for $(N_t/t \in \cdot)$. We also want to point out that in the case when $\lambda(\cdot)$ is linear, even if $h(\cdot)$ is general, the LDP for $(N_t/t \in \cdot)$ can still be established because the linear case can be explicitly computed, which is not the case when $\lambda(\cdot)$ is nonlinear.

In the pioneering work by Donsker and Varadhan [6], they obtained a level-3 large deviation result for certain stationary Markov processes.

We would like to prove the large deviation principle for general Hawkes processes by proving a process-level, also known as level-3 large deviation principle first. We can then use the contraction principle to obtain the level-1 large deviation principle for $(N_t/t \in \cdot)$.

Let us define the empirical measure for the process as

$$R_{t,\omega}(A) = \frac{1}{t} \int_0^t \chi_A(\theta_s \omega_t) ds, \tag{1.12}$$

for any A , where $\omega_t(s) = \omega(s)$ for $0 \leq s \leq t$ and $\omega_t(s+t) = \omega_t(s)$ for any s . Donsker and Varadhan [6] proved that in the case when Ω is a space of càdlàg functions $\omega(\cdot)$ on $-\infty < t < \infty$ endowed with Skorohod topology and taking values in a Polish space X , under certain conditions, $P^{0,x}(R_{t,\omega} \in \cdot)$ satisfies a large deviation principle, where $P^{0,x}$ is a Markov process on Ω_0^∞ with initial value $x \in X$. The rate function $H(Q)$ is some entropy function.

Let $h(\alpha, \beta)_\Sigma$ be the relative entropy of α with respect to β restricted to the σ -algebra Σ . For any $Q \in \mathcal{M}_S(\Omega)$, let Q^{ω^-} be the regular conditional probability distribution of Q . Similarly we define P^{ω^-} .

Let us define the entropy function $H(Q)$ as

$$H(Q) = \mathbb{E}^Q[h(Q^{\omega^-}, P^{\omega^-})_{\mathcal{F}_1^0}]. \tag{1.13}$$

Notice that P^{ω^-} is the Hawkes process conditional on the past history ω^- . It has rate $\lambda(\sum_{\tau \in \omega[0,s) \cup \omega^-} h(s-\tau))$ at time $0 \leq s \leq 1$, which is well defined for almost every ω^- under Q if $\mathbb{E}^Q[N[0, 1]] < \infty$ since $\mathbb{E}^Q[\sum_{\tau \in \omega^-} h(-\tau)] = \|h\|_{L^1} \mathbb{E}^Q[N[0, 1]] < \infty$ implies $\sum_{\tau \in \omega^-} h(s-\tau) \leq \sum_{\tau \in \omega^-} h(-\tau) < \infty$ for all $0 \leq s \leq 1$.

When $H(Q) < \infty$, $h(Q^{\omega^-}, P^{\omega^-}) < \infty$ for a.e. ω^- under Q , which implies that $Q^{\omega^-} \ll P^{\omega^-}$ on \mathcal{F}_1^0 . By the theory of absolute continuity of point processes, see for example Chapter 19 of Lipster and Shiryaev [10] or Chapter 13 of Daley and Vere-Jones [4], the compensator of Q^{ω^-} is absolutely continuous, i.e. it has some density $\hat{\lambda}$ say, such that by the Girsanov formula,

$$\begin{aligned} H(Q) &= \int_{\Omega^-} \int \left[\int_0^1 (\lambda - \hat{\lambda}) ds + \int_0^1 \log(\hat{\lambda}/\lambda) dN_s \right] dQ^{\omega^-} Q(d\omega^-) \\ &= \int_{\Omega} \left[\int_0^1 \lambda(\omega, s) - \hat{\lambda}(\omega, s) + \log\left(\frac{\hat{\lambda}(\omega, s)}{\lambda(\omega, s)}\right) \hat{\lambda} ds \right] Q(d\omega), \end{aligned} \tag{1.14}$$

where $\lambda = \lambda(\sum_{\tau \in \omega[0,s) \cup \omega^-} h(s-\tau))$. Both λ and $\hat{\lambda}$ are $\mathcal{F}_s^{-\infty}$ -predictable for $0 \leq s \leq 1$. For the equality in (1.14), we used the fact that $N_t - \int_0^t \hat{\lambda}(\omega, s) ds$ is a martingale under Q and for any $f(\omega, s)$ which is bounded, $\mathcal{F}_s^{-\infty}$ progressively measurable and predictable, we have

$$\int_{\Omega} \int_0^1 f(\omega, s) dN_s Q(d\omega) = \int_{\Omega} \int_0^1 f(\omega, s) \hat{\lambda}(\omega, s) ds Q(d\omega). \tag{1.15}$$

We will use the above fact repeatedly in our paper.

The following theorem is the main result of this paper.

Theorem 1. For any open set $G \subset \mathcal{M}_S(\Omega)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in G) \geq - \inf_{Q \in G} H(Q), \tag{1.16}$$

and for any closed set $C \subset \mathcal{M}_S(\Omega)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in C) \leq - \inf_{Q \in C} H(Q). \tag{1.17}$$

We will prove the lower bound in Section 2, the upper bound in Section 3 and the superexponential estimates that are needed in the proof of the upper bound in Section 4.

Once we establish the level-3 large deviation result, we can obtain the large deviation principle for $(N_t/t \in \cdot)$ directly by using the contraction principle.

Theorem 2. $(N_t/t \in \cdot)$ satisfies a large deviation principle with the rate function $I(\cdot)$ given by

$$I(x) = \inf_{Q \in \mathcal{M}_S(\Omega), \mathbb{E}^Q[N[0,1]]=x} H(Q). \tag{1.18}$$

Proof. Since $Q \mapsto \mathbb{E}^Q[N[0, 1]]$ is continuous, $\int_{\Omega} N[0, 1] dR_{t,\omega}$ satisfies a large deviation principle with the rate function $I(\cdot)$ by the contraction principle. (For a discussion on contraction principle, see for example Varadhan [12].)

$$\begin{aligned} \int_{\Omega} N[0, 1] dR_{t,\omega} &= \frac{1}{t} \int_0^t N[0, 1](\theta_s \omega_t) ds \\ &= \frac{1}{t} \int_0^{t-1} N[s, s+1](\omega) ds + \frac{1}{t} \int_{t-1}^t N[s, s+1](\omega_t) ds. \end{aligned} \tag{1.19}$$

Notice that

$$0 \leq \frac{1}{t} \int_{t-1}^t N[s, s+1](\omega_t) ds \leq \frac{1}{t} (N[t-1, t](\omega) + N[0, 1](\omega)), \tag{1.20}$$

and

$$\frac{1}{t} \int_0^{t-1} N[s, s+1](\omega) ds = \frac{1}{t} \left[\int_{t-1}^t N_s(\omega) ds - \int_0^1 N_s(\omega) ds \right] \leq \frac{N_t}{t}, \tag{1.21}$$

and $\frac{1}{t} \int_0^{t-1} N[s, s+1](\omega) ds \geq \frac{N_{t-1}-N_1}{t} = \frac{N_t}{t} - \frac{N[t-1,t]+N_1}{t}$. Hence,

$$\frac{N_t}{t} - \frac{N[t-1, t] + N_1}{t} \leq \int_{\Omega} N[0, 1] dR_{t,\omega} \leq \frac{N_t}{t} + \frac{N[t-1, t] + N_1}{t}. \tag{1.22}$$

For the lower bound, for any open ball $B_{\epsilon}(x)$ centered at x with radius $\epsilon > 0$,

$$\begin{aligned} P\left(\frac{N_t}{t} \in B_{\epsilon}(x)\right) &\geq P\left(\int_{\Omega} N[0, 1] dR_{t,\omega} \in B_{\epsilon/2}(x)\right) \\ &\quad - P\left(\frac{N[t-1, t]}{t} \geq \frac{\epsilon}{4}\right) - P\left(\frac{N_1}{t} \geq \frac{\epsilon}{4}\right). \end{aligned} \tag{1.23}$$

For the upper bound, for any closed set C and $C^{\epsilon} = \bigcup_{x \in C} \overline{B_{\epsilon}(x)}$,

$$\begin{aligned} P\left(\frac{N_t}{t} \in C\right) &\leq P\left(\int_{\Omega} N[0, 1] dR_{t,\omega} \in C^{\epsilon}\right) \\ &\quad + P\left(\frac{N[t-1, t]}{t} \geq \frac{\epsilon}{4}\right) + P\left(\frac{N_1}{t} \geq \frac{\epsilon}{4}\right). \end{aligned} \tag{1.24}$$

Finally, by Lemma 20, we have the following superexponential estimates

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{N[t-1, t]}{t} \geq \frac{\epsilon}{4} \right) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{N_1}{t} \geq \frac{\epsilon}{4} \right) = -\infty. \tag{1.25}$$

Hence, for the lower bound, we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{N_t}{t} \in B_\epsilon(x) \right) \geq -I(x), \tag{1.26}$$

and for the upper bound, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{N_t}{t} \in C \right) \leq - \inf_{x \in C^\epsilon} I(x), \tag{1.27}$$

which holds for any $\epsilon > 0$. Letting $\epsilon \downarrow 0$, we get the desired result. □

2. Lower bound

Lemma 3. For any $\lambda, \hat{\lambda} \geq 0, \lambda - \hat{\lambda} + \hat{\lambda} \log(\hat{\lambda}/\lambda) \geq 0$.

Proof. Write $\lambda - \hat{\lambda} + \hat{\lambda} \log(\hat{\lambda}/\lambda) = \hat{\lambda}[(\lambda/\hat{\lambda}) - 1 - \log(\lambda/\hat{\lambda})]$. Thus, it is sufficient to show that $F(x) = x - 1 - \log x \geq 0$ for any $x \geq 0$. Note that $F(0) = F(\infty) = 0$ and $F'(x) = 1 - \frac{1}{x} < 0$ when $0 < x < 1$ and $F'(x) > 0$ when $x > 1$ and finally $F(1) = 0$. Hence $F(x) \geq 0$ for any $x \geq 0$. □

Lemma 4. Assume $H(Q) < \infty$. Then,

$$\mathbb{E}^Q[N[0, 1]] \leq C_1 + C_2 H(Q), \tag{2.1}$$

where $C_1, C_2 > 0$ are some constants independent of Q .

Proof. If $H(Q) < \infty$, then $h(Q^{\omega^-}, P^{\omega^-})_{\mathcal{F}_1^0} < \infty$ for a.e. ω^- under Q , which implies that $Q^{\omega^-} \ll P^{\omega^-}$ and thus $\hat{A}_t \ll A_t$, where \hat{A}_t and A_t are the compensators of N_t under Q^{ω^-} and P^{ω^-} respectively. (For the theory of absolute continuity of point processes and Girsanov formula, see for example Lipster and Shiryaev [10] or Daley and Vere-Jones [4].) Since $A_t = \int_0^t \lambda(\omega, s) ds$, we have $\hat{A}_t = \int_0^t \hat{\lambda}(\omega, s) ds$ for some $\hat{\lambda}$. By the Girsanov formula,

$$H(Q) = \mathbb{E}^Q \left[\int_0^1 \lambda - \hat{\lambda} + \log(\hat{\lambda}/\lambda) \hat{\lambda} ds \right]. \tag{2.2}$$

Notice that $\mathbb{E}^Q[N[0, 1]] = \int \int_0^1 \hat{\lambda} ds dQ$.

$$\begin{aligned} \int \int_0^1 \lambda ds dQ &\leq \epsilon \int \int_0^1 \sum_{\tau < s} h(s - \tau) ds dQ + C_\epsilon \\ &\leq \epsilon \int h(0) N[0, 1] dQ + \epsilon \int \sum_{\tau < 0} h(-\tau) dQ + C_\epsilon \\ &= \epsilon (h(0) + \|h\|_{L^1}) \mathbb{E}^Q[N[0, 1]] + C_\epsilon \\ &= \epsilon (h(0) + \|h\|_{L^1}) \int \int_0^1 \hat{\lambda} ds dQ + C_\epsilon. \end{aligned} \tag{2.3}$$

Therefore, we have

$$\int \int_0^1 \hat{\lambda} \cdot 1_{\hat{\lambda} < K\lambda} \, ds \, dQ \leq K\epsilon(h(0) + \|h\|_{L^1}) \int \int_0^1 \hat{\lambda} \, ds \, dQ + KC_\epsilon. \tag{2.4}$$

On the other hand, by Lemma 3,

$$\begin{aligned} H(Q) &\geq \int \int_0^1 [\lambda - \hat{\lambda} + \hat{\lambda} \log(\hat{\lambda}/\lambda)] \cdot 1_{\hat{\lambda} \geq K\lambda} \, ds \, dQ \\ &\geq (\log K - 1) \int \int_0^1 \hat{\lambda} \cdot 1_{\hat{\lambda} \geq K\lambda} \, ds \, dQ. \end{aligned} \tag{2.5}$$

Thus,

$$\int \int_0^1 \hat{\lambda} \, ds \, dQ \leq K\epsilon(h(0) + \|h\|_{L^1}) \int \int_0^1 \hat{\lambda} \, ds \, dQ + KC_\epsilon + \frac{H(Q)}{\log K - 1}. \tag{2.6}$$

Choose $K > e$ and $\epsilon < \frac{1}{K(h(0) + \|h\|_{L^1})}$. We get

$$\mathbb{E}^Q[N[0, 1]] \leq \frac{KC_\epsilon}{1 - K\epsilon(h(0) + \|h\|_{L^1})} + \frac{H(Q)}{(\log K - 1)K\epsilon(h(0) + \|h\|_{L^1})}. \tag{2.7}$$

□

Lemma 5. *We have the following alternative expression for $H(Q)$.*

$$H(Q) = \sup_{f(\omega, s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) \, ds + \int_0^1 f \, dN_s \right]. \tag{2.8}$$

Proof. $\mathbb{E}^Q[N[0, 1]] < \infty$ implies that $\mathbb{E}^{Q^{\omega^-}}[N[0, 1]] < \infty$ for almost every ω^- under Q , which also implies that $\sum_{\tau \in \omega^-} h(-\tau) < \infty$ since $\mathbb{E}^Q[\sum_{\tau \in \omega^-} h(-\tau)] = \|h\|_{L^1} \mathbb{E}^Q[N[0, 1]] < \infty$. Thus,

$$\begin{aligned} \mathbb{E}^{P^{\omega^-}}[N[0, 1]] &= \mathbb{E}^{P^{\omega^-}} \left[\int_0^1 \lambda \left(\sum_{\tau \in \omega(0, s) \cup \omega^-} h(s - \tau) \right) \, ds \right] \\ &\leq C_\epsilon + \epsilon h(0) \mathbb{E}^{P^{\omega^-}}[N[0, 1]] + \epsilon \sum_{\tau \in \omega^-} h(-\tau) < \infty. \end{aligned} \tag{2.9}$$

Pick up $\epsilon < \frac{1}{h(0)}$, we have $\mathbb{E}^{P^{\omega^-}}[N[0, 1]] < \infty$.

By the theory of absolute continuity of point processes, see for example Chapter 13 of Daley and Vere-Jones [4], if $\mathbb{E}^{Q^{\omega^-}}[N[0, 1]], \mathbb{E}^{P^{\omega^-}}[N[0, 1]] < \infty$, then, $Q^{\omega^-} \ll P^{\omega^-}$ if and only if $\hat{A}_t \ll A_t$, where \hat{A}_t and $A_t = \int_0^t \lambda(\omega^-, \omega, s) \, ds$ are the compensators of N_t under Q^{ω^-} and P^{ω^-} respectively. If that's the case, we can write $\hat{A}_t = \int_0^t \hat{\lambda}(\omega^-, \omega, s) \, ds$ for some $\hat{\lambda}$ and there is Girsanov formula

$$\log \frac{dQ^{\omega^-}}{dP^{\omega^-}} \Big|_{\mathcal{F}_1^0} = \int_0^1 (\lambda - \hat{\lambda}) \, ds + \int_0^1 \log(\hat{\lambda}/\lambda) \, dN_s, \tag{2.10}$$

which implies that

$$H(Q) = \mathbb{E}^Q \left[\int_0^1 \lambda - \hat{\lambda} + \log(\hat{\lambda}/\lambda) \hat{\lambda} \, ds \right]. \tag{2.11}$$

For any f , $\hat{\lambda} f + (1 - e^f)\lambda \leq \hat{\lambda} \log(\hat{\lambda}/\lambda) + \lambda - \hat{\lambda}$ and equality is achieved when $f = \log(\hat{\lambda}/\lambda)$. Thus, clearly, we have

$$\sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) ds + \int_0^1 f dN_s \right] \leq H(Q). \tag{2.12}$$

On the other hand, we can always find a sequence f_n convergent to $\log(\hat{\lambda}/\lambda)$ and by Fatou's lemma, we get the other direction.

Now, assume that we do not have $Q^{\omega^-} \ll P^{\omega^-}$ for a.e. ω^- under Q . That implies that $H(Q) = \infty$. We want to show that

$$\sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) ds + \int_0^1 f dN_s \right] = \infty. \tag{2.13}$$

Let us assume that

$$\sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) ds + \int_0^1 f dN_s \right] < \infty. \tag{2.14}$$

We want to prove that $H(Q) < \infty$.

Let $P_\epsilon^{\omega^-}$ be the point process on $[0, 1]$ with compensator $A_t + \epsilon \hat{A}_t$. Clearly $\hat{A}_t \ll A_t + \epsilon \hat{A}_t$ and $Q^{\omega^-} \ll P_\epsilon^{\omega^-}$. For any f ,

$$\begin{aligned} & \mathbb{E}^Q \left[\int_0^1 (1 - e^f) d(A_s + \epsilon \hat{A}_s) + f d\hat{A}_s \right] \\ &= \mathbb{E}^Q \left[\int_0^1 (1 - e^f) \chi_{f < 0} d(A_s + \epsilon \hat{A}_s) + f \chi_{f < 0} d\hat{A}_s \right] \\ & \quad + \mathbb{E}^Q \left[\int_0^1 (1 - e^f) \chi_{f \geq 0} d(A_s + \epsilon \hat{A}_s) + f \chi_{f \geq 0} d\hat{A}_s \right] \\ &\leq \mathbb{E}^Q \left[\int_0^1 d(A_s + \epsilon \hat{A}_s) \right] + \mathbb{E}^Q \left[\int_0^1 (1 - e^f) \chi_{f \geq 0} dA_s + f \chi_{f \geq 0} d\hat{A}_s \right] \\ &= \mathbb{E}^Q \left[\int_0^1 d(A_s + \epsilon \hat{A}_s) \right] + \mathbb{E}^Q \left[\int_0^1 (1 - e^{f \chi_{f \geq 0}}) dA_s + f \chi_{f \geq 0} d\hat{A}_s \right] \\ &\leq C_\delta + \delta(h(0) + \|h\|_{L^1}) \mathbb{E}^Q [N[0, 1]] \\ & \quad + \sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) ds + \int_0^1 f dN_s \right] < \infty. \end{aligned} \tag{2.15}$$

Therefore, we have

$$\begin{aligned} & \infty > \liminf_{\epsilon \downarrow 0} \sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 (1 - e^f) d(A_s + \epsilon \hat{A}_s) + f d\hat{A}_s \right] \\ &= \liminf_{\epsilon \downarrow 0} \sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \left(1 - e^f + f \cdot \frac{d\hat{A}_s}{d(A_s + \epsilon \hat{A}_s)} \right) d(A_s + \epsilon \hat{A}_s) \right] \\ &= \liminf_{\epsilon \downarrow 0} \mathbb{E}^Q [h(Q^{\omega^-}, P_\epsilon^{\omega^-})_{\mathcal{F}_1^0}] \\ &= \mathbb{E}^Q [h(Q^{\omega^-}, P^{\omega^-})_{\mathcal{F}_1^0}] = H(Q), \end{aligned} \tag{2.16}$$

by lower semicontinuity of the relative entropy $h(\cdot, \cdot)$ and Fatou's lemma and the fact that $P_\epsilon^{\omega^-} \rightarrow P^{\omega^-}$ weakly as $\epsilon \downarrow 0$. Hence $H(Q) < \infty$. \square

Lemma 6. $H(Q)$ is lower semicontinuous and convex in Q .

Proof. By Lemma 5, we can rewrite $H(Q)$ as

$$\begin{aligned} H(Q) &= \sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) + \hat{\lambda}f \, ds \right] \\ &= \sup_{f(\omega,s) \in \mathcal{B}(\mathcal{F}_s^{-\infty}) \cap C(\Omega \times \mathbb{R}), 0 \leq s \leq 1} \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^f) \, ds + \int_0^1 f \, dN_s \right]. \end{aligned} \tag{2.17}$$

Assume $Q_n \rightarrow Q$, then $\mathbb{E}^{Q_n}[N[0, 1]] \rightarrow \mathbb{E}^Q[N[0, 1]]$ and $Q_n \rightarrow Q$ weakly. Since $f(\omega, s) \in C(\Omega \times \mathbb{R}) \cap \mathcal{B}(\mathcal{F}_s^{-\infty})$, $\int_0^1 f(\omega, s) \, dN_s$ is continuous on Ω and since f is uniformly bounded, $\int_0^1 f(\omega, s) \, dN_s \leq \|f\|_{L^\infty} N[0, 1]$. Hence,

$$\mathbb{E}^{Q_n} \left[\int_0^1 f(\omega, s) \, dN_s \right] \rightarrow \mathbb{E}^Q \left[\int_0^1 f(\omega, s) \, dN_s \right]. \tag{2.18}$$

Let $\lambda^M = \lambda(\sum_{\tau < s} h^M(s - \tau))$, where $h^M(s) = h(s)\chi_{s \leq M}$. Therefore, $\lambda^M(\omega, s) \in C(\Omega \times \mathbb{R})$ and thus $\int_0^1 \lambda^M(1 - e^{f(\omega,s)}) \, ds \in C(\Omega)$. Also, $\int_0^1 \lambda^M(1 - e^{f(\omega,s)}) \, ds \leq K(1 + e^{\|f\|_{L^\infty}})N[-M, 1]$, where $K > 0$ is some constant. Therefore,

$$\mathbb{E}^{Q_n} \left[\int_0^1 \lambda^M(1 - e^{f(\omega,s)}) \, ds \right] \rightarrow \mathbb{E}^Q \left[\int_0^1 \lambda^M(1 - e^{f(\omega,s)}) \, ds \right] \tag{2.19}$$

as $n \rightarrow \infty$. Next, notice that

$$\begin{aligned} &\left| \mathbb{E}^Q \left[\int_0^1 \lambda^M(1 - e^{f(\omega,s)}) \, ds \right] - \mathbb{E}^Q \left[\int_0^1 \lambda(1 - e^{f(\omega,s)}) \, ds \right] \right| \\ &\leq \mathbb{E}^Q(1 + e^{\|f\|_{L^\infty}}) \alpha \mathbb{E}^Q[N[0, 1]] \int_M^\infty h(s) \, ds \rightarrow 0 \end{aligned} \tag{2.20}$$

as $M \rightarrow \infty$. Similarly, we have

$$\limsup_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbb{E}^{Q_n} \left[\int_0^1 \lambda^M(1 - e^{f(\omega,s)}) \, ds \right] - \mathbb{E}^{Q_n} \left[\int_0^1 \lambda(1 - e^{f(\omega,s)}) \, ds \right] \right| = 0. \tag{2.21}$$

Hence,

$$\mathbb{E}^{Q_n} \left[\int_0^1 \lambda(\omega, s)(1 - e^{f(\omega,s)}) \, ds \right] \rightarrow \mathbb{E}^Q \left[\int_0^1 \lambda(\omega, s)(1 - e^{f(\omega,s)}) \, ds \right]. \tag{2.22}$$

The supremum is taken over a linear functional of Q , which is continuous in Q , therefore the supremum over these linear functionals will be lower semicontinuous. Similarly, since in the variational formula expression of $H(Q)$ in Lemma 5, the supremum is taken over a linear functional of Q , $H(Q)$ is convex in Q . \square

Lemma 7. $H(Q)$ is linear in Q .

Proof. It is in general true that the process-level entropy function $H(Q)$ is linear in Q . Following the arguments in Donsker and Varadhan [6], there exists a subset $\Omega_0 \subset \Omega$ which is $\mathcal{F}_0^{-\infty}$ measurable and a $\mathcal{F}_0^{-\infty}$ measurable map $\hat{Q} : \Omega_0 \rightarrow \mathcal{M}_E(\Omega)$ such that $Q(\Omega_0) = 1$ for all $Q \in \mathcal{M}_S(\Omega)$ and $Q(\omega : \hat{Q} = Q) = 1$ for all $Q \in \mathcal{M}_E(\Omega)$. Therefore,

there exists a universal version, say \hat{Q}^{ω^-} independent of Q such that $\int \hat{Q}^{\omega^-} Q(d\omega^-) = Q$. Since it is true for all $Q \in \mathcal{M}_E(\Omega)$, it also holds for $Q \in \mathcal{M}_S(\Omega)$. Hence, we have

$$H(Q) = \mathbb{E}^Q[h(Q^{\omega^-}, P^{\omega^-})_{\mathcal{F}_1^0}] = \mathbb{E}^Q[h(\hat{Q}^{\omega^-}, P^{\omega^-})_{\mathcal{F}_1^0}], \tag{2.23}$$

which implies that $H(Q)$ is linear in Q . □

In our paper, we are proving the large deviation principle for Hawkes processes started with empty history, i.e. with probability measure P^\emptyset . But when time elapses by t , the Hawkes process generates points and that create a new history up to time t . We need to understand how the history that was created would affect the future. What we want to prove is some uniform estimates that if given the past history which is well controlled, then the new history will also be well controlled. This is essentially what the following Lemma 9 says. Consider the configuration of points starting from time 0 up to time t . We shift it by t and denote it by w_t such that $w_t \in \Omega^-$, where Ω^- is Ω restricted to \mathbb{R}^- . These notations will be used in Lemma 9.

Remark 8. *At the very beginning of the paper, we defined ω_t . It should not be confused with w_t in this section.*

Lemma 9. *For any $Q \in \mathcal{M}_E(\Omega)$ such that $H(Q) < \infty$ and any open neighborhood N of Q , there exists some K_ℓ^- such that $\emptyset \in K_\ell^-$ and $Q(K_\ell^-) \rightarrow 1$ as $\ell \rightarrow \infty$ and*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \inf_{w_0 \in K_\ell^-} \log P^{w_0}(R_t, \omega \in N, w_t \in K_\ell^-) \geq -H(Q). \tag{2.24}$$

Proof. Let us abuse the notations a bit by defining

$$\lambda(\omega^-) = \lambda\left(\sum_{\tau \in \omega^-, \tau \in \omega[0, s)} h(s - \tau)\right). \tag{2.25}$$

For any $t > 0$, since $\lambda(\cdot) \geq c > 0$ and $\lambda(\cdot)$ is Lipschitz with constant α , we have

$$\begin{aligned} \log \frac{dP^{\omega^-}}{dP^{w_0}} \Big|_{\mathcal{F}_t^0} &= \int_0^t \lambda(w_0) - \lambda(\omega^-) ds + \int_0^t \log\left(\frac{\lambda(\omega^-)}{\lambda(w_0)}\right) dN_s \\ &\leq \int_0^t |\lambda(w_0) - \lambda(\omega^-)| ds + \int_0^t \log\left(1 + \frac{|\lambda(w_0) - \lambda(\omega^-)|}{\lambda(w_0)}\right) dN_s \\ &\leq \int_0^t \alpha \sum_{\tau \in \omega^- \cup w_0} h(s - \tau) ds + \int_0^t \frac{\alpha}{c} \sum_{\tau \in \omega^- \cup w_0} h(s - \tau) dN_s. \end{aligned} \tag{2.26}$$

Define

$$K_\ell^- = \{\omega: N[-t, 0](\omega) \leq \ell(1 + t), \forall t > 0\}. \tag{2.27}$$

By maximal ergodic theorem, letting $[t]$ denote the largest integer less or equal to t , we have

$$\begin{aligned} Q((K_\ell^-)^c) &= Q\left(\sup_{t > 0} \frac{N[-t, 0]}{t + 1} > \ell\right) \\ &\leq Q\left(\sup_{t > 0} \frac{N[-([t] + 1), 0]}{[t] + 1} > \ell\right) \\ &= Q\left(\sup_{n \geq 1, n \in \mathbb{N}} \frac{N[-n, 0]}{n} > \ell\right) \\ &\leq \frac{\mathbb{E}^Q[N[0, 1]]}{\ell} \rightarrow 0 \end{aligned} \tag{2.28}$$

as $\ell \rightarrow \infty$. Thus $Q(K_\ell^-) \rightarrow 1$ as $\ell \rightarrow \infty$.

For any $s > 0$ and $\omega^- \in K_\ell^-$, since h is decreasing, $h' \leq 0$, by integration by parts,

$$\begin{aligned} \sum_{\tau \in \omega^-} h(s - \tau) &= \int_0^\infty h(s + \sigma) dN[-\sigma, 0] \\ &= - \int_0^\infty N[-\sigma, 0] h'(s + \sigma) d\sigma \\ &\leq - \int_0^\infty \ell(1 + \sigma) h'(s + \sigma) d\sigma \\ &= \ell h(s) + \ell \int_0^\infty h(s + \sigma) d\sigma \\ &= \ell h(s) + \ell H(s), \end{aligned} \tag{2.29}$$

where $H(t) = \int_t^\infty h(s) ds$.

Therefore, uniformly for $\omega_-, w_0 \in K_\ell^-$,

$$\int_0^t \alpha \sum_{\tau \in \omega^- \cup w_0} h(s - \tau) ds \leq 2\ell\alpha \|h\|_{L^1} + 2\ell\alpha u(t), \tag{2.30}$$

where $u(t) = \int_0^t H(s) ds$ and

$$\int_0^t \frac{\alpha}{c} \sum_{\tau \in \omega^- \cup w_0} h(s - \tau) dN_s \leq \frac{2\ell\alpha}{c} \int_0^t (h(s) + H(s)) dN_s. \tag{2.31}$$

Define

$$K_{\ell,t}^+ = \left\{ \omega: \frac{2\ell\alpha}{c} \int_0^t (h(s) + H(s)) dN_s \leq \ell^2 (\|h\|_{L^1} + u(t)) \right\}. \tag{2.32}$$

Then, uniformly in $t > 0$,

$$Q((K_{\ell,t}^+)^c) \leq \frac{2\alpha \mathbb{E}^Q[N[0, 1]]}{c \cdot \ell} \rightarrow 0, \tag{2.33}$$

as $\ell \rightarrow \infty$. Thus $\inf_{t>0} Q(K_{\ell,t}^+) \rightarrow 1$ as $\ell \rightarrow \infty$.

Hence, uniformly for $\omega_-, w_0 \in K_\ell^-$ and $\omega \in K_{\ell,t}^+$,

$$\begin{aligned} \log \frac{dP^{\omega^-}}{dP^{w_0}} \Big|_{\mathcal{F}_t^0} &\leq 2\ell\alpha \|h\|_{L^1} + 2\ell\alpha u(t) + \ell^2 (\|h\|_{L^1} + u(t)) \\ &= C_1(\ell) + C_2(\ell)u(t), \end{aligned} \tag{2.34}$$

where $C_1(\ell) = 2\ell\alpha \|h\|_{L^1} + \ell^2 \|h\|_{L^1}$ and $C_2(\ell) = 2\ell\alpha + \ell^2$.

Observe that

$$\limsup_{t \rightarrow \infty} \frac{u(t)}{t} = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t H(s) ds = 0. \tag{2.35}$$

Let $D_t = \{R_{t,\omega} \in N, w_t \in K_\ell^-\}$.

Uniformly for $w_0 \in K_{\ell,t}^-$,

$$\begin{aligned}
 & P^{w_0}(D_t) \\
 & \geq e^{-t(H(Q)+\epsilon)-C_1(\ell)-C_2(\ell)u(t)} \\
 & \quad \cdot Q \left[D_t \cap \left\{ \frac{1}{t} \log \frac{dP^{\omega^-}}{dQ^{\omega^-}} \Big|_{\mathcal{F}_t^0} \leq H(Q) + \epsilon \right\} \cap \left\{ \log \frac{dP^{\omega^-}}{dP^{w_0}} \Big|_{\mathcal{F}_t^0} \leq C_1(\ell) + C_2(\ell)u(t) \right\} \right] \\
 & \geq e^{-t(H(Q)+\epsilon)-C_1(\ell)-C_2(\ell)u(t)} \\
 & \quad \cdot Q \left[D_t \cap \left\{ \frac{1}{t} \log \frac{dP^{\omega^-}}{dQ^{\omega^-}} \Big|_{\mathcal{F}_t^0} \leq H(Q) + \epsilon \right\} \cap \{K_{\ell,t}^+ \cap K_{\ell,t}^-\} \right].
 \end{aligned} \tag{2.36}$$

Since $Q \in \mathcal{M}_E(\Omega)$, by ergodic theorem,

$$\lim_{t \rightarrow \infty} Q(R_{t,\omega} \in N) = 1, \tag{2.37}$$

and by observing that for $\psi(\omega, t) = \log \frac{dQ^\omega}{dP^\omega} \Big|_{\mathcal{F}_t^0}$,

$$\psi(\omega, t+s) = \psi(\omega, t) + \psi(\theta_t \omega, s), \quad \mathbb{E}^Q[\psi(\omega, t)] = tH(Q), \tag{2.38}$$

for almost every ω^- under Q ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{dP^{\omega^-}}{dQ^{\omega^-}} \Big|_{\mathcal{F}_t^0} = H(Q). \tag{2.39}$$

Since Q is stationary, $Q(w_t \in K_\ell^-) \geq Q(K_\ell^-) \rightarrow 1$ as $\ell \rightarrow \infty$. Also, $Q(K_{\ell,t}^+) \geq \inf_{t>0} Q(K_{\ell,t}^+) \rightarrow 1$ as $\ell \rightarrow \infty$. Remember that $\limsup_{t \rightarrow \infty} \frac{u(t)}{t} = 0$. By choosing ℓ big enough, we conclude that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \inf_{w_0 \in K_\ell^-} \log P^{w_0}(R_{t,\omega} \in N, w_t \in K_\ell^-) \geq -H(Q) - \epsilon. \tag{2.40}$$

Since it holds for any $\epsilon > 0$, we get the desired result. □

Theorem 10 (Lower bound). For any open set G ,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in G) \geq - \inf_{Q \in G} H(Q). \tag{2.41}$$

Proof. It is sufficient to prove that for any $Q \in \mathcal{M}_S(\Omega)$, $H(Q) < \infty$, for any neighborhood N of Q , $\liminf_{t \rightarrow \infty} \frac{1}{t} \times \log P(R_{t,\omega} \in N) \geq -H(Q)$. Since for every invariant measure $P \in \mathcal{M}_S$, there exists a probability measure μ_P on the space \mathcal{M}_E of ergodic measures such that $P = \int_{\mathcal{M}_E} Q \mu_P(dQ)$, for any $Q \in \mathcal{M}_S(\Omega)$ such that $H(Q) < \infty$, without loss of generality, we can assume that $Q = \sum_{j=1}^\ell \alpha_j Q_j$, where $\alpha_j \geq 0$, $1 \leq j \leq \ell$ and $\sum_{j=1}^\ell \alpha_j = 1$. By linearity of $H(\cdot)$, $H(Q) = \sum_{j=1}^\ell \alpha_j H(Q_j)$. Divide the interval $[0, t]$ into subintervals of length $\alpha_j t$ and let t_j , $1 \leq j \leq \ell$, be the right hand endpoints of these subintervals and let $t_0 = 0$. For each Q_j , take K_M^- as in Lemma 9. We have $\min_{1 \leq j \leq \ell} Q_j(K_M^-) \rightarrow 1$, as $M \rightarrow \infty$. Choose neighborhoods N_j of Q_j , $1 \leq j \leq \ell$ such that $\bigcup_{j=1}^\ell \alpha_j N_j \subseteq N$. We have

$$\begin{aligned}
 P^\varnothing(R_{t,\omega} \in N) & \geq P^\varnothing(R_{t_1,\omega} \in N_1, w_{t_1} \in K_M^-) \\
 & \quad \cdot \prod_{j=2}^\ell \inf_{w_0 \in K_{t_{j-1}-t_{j-2}}^-} P^{w_0}(R_{t_j-t_{j-1},\omega} \in N_j, w_{t_j-t_{j-1}} \in K_M^-).
 \end{aligned} \tag{2.42}$$

Now, applying Lemma 9 and the linearity of $H(\cdot)$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P^\varnothing(R_{t,\omega} \in N) \geq - \sum_{j=1}^{\ell} \alpha_j H(Q_j) = -H(Q). \quad (2.43)$$

□

3. Upper bound

Remark 11. By following the same argument as in Donsker and Varadhan [6], if $\omega^- \mapsto P^{\omega^-}$ is weakly continuous, then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in A) \leq - \inf_{Q \in A} H(Q), \quad (3.1)$$

for any compact A . If the Hawkes process has finite range of memory, i.e. $h(\cdot)$ has compact support and is continuous, then, for any $a < b$, if $\omega_n^- \rightarrow \omega^-$, we have

$$\begin{aligned} & \left| \int_a^b \lambda(\omega_n^-, \omega, s) ds - \int_a^b \lambda(\omega^-, \omega, s) ds \right| \\ & \leq \alpha \int_a^b \left| \sum_{\tau \in \omega_n^-} h(s - \tau) - \sum_{\tau \in \omega^-} h(s - \tau) \right| ds \rightarrow 0, \end{aligned} \quad (3.2)$$

as $n \rightarrow \infty$, which implies that $P^{\omega_n^-} \rightarrow P^{\omega^-}$.

If the Hawkes process does not have finite range of memory, then, we should use the specific features of the Hawkes process to obtain the upper bound.

Before we proceed, let us prove an easy but very useful lemma that we will use repeatedly in the proofs of the estimates in this paper.

Lemma 12. Let $f(\omega, s)$ be $\mathcal{F}_s^{-\infty}$ progressively measurable and predictable. Then,

$$\mathbb{E} \left[e^{\int_0^t f(\omega, s) dN_s} \right] \leq \mathbb{E} \left[e^{\int_0^t (e^{2f(\omega, s)} - 1) \lambda(\omega, s) ds} \right]^{1/2}. \quad (3.3)$$

Proof. Since $\exp\{\int_0^t 2f(\omega, s) dN_s - \int_0^t (e^{2f(\omega, s)} - 1) \lambda(\omega, s) ds\}$ is a martingale, by Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E} \left[e^{\int_0^t f(\omega, s) dN_s} \right] &= \mathbb{E} \left[e^{(1/2) \int_0^t 2f(\omega, s) dN_s - (1/2) \int_0^t (e^{2f(\omega, s)} - 1) \lambda(\omega, s) ds + (1/2) \int_0^t (e^{2f(\omega, s)} - 1) \lambda(\omega, s) ds} \right] \\ &\leq \mathbb{E} \left[e^{\int_0^t (e^{2f(\omega, s)} - 1) \lambda(\omega, s) ds} \right]^{1/2}. \end{aligned} \quad (3.4)$$

□

Let us define \mathcal{C}_T as

$$\begin{aligned} \mathcal{C}_T = & \left\{ F(\omega) := \int_0^T f(\omega, s) dN_s - \int_0^T (e^{f(\omega, s)} - 1) \lambda(\omega, s) ds, \right. \\ & \left. f(\omega, s) \in \mathcal{B}(\mathcal{F}_s^0) \cap C(\Omega \times \mathbb{R}) \right\}. \end{aligned} \quad (3.5)$$

Here $\lambda(\omega, s)$ is $\mathcal{F}_s^{-\infty}$ progressively measurable, predictable and $f(\omega, s) \in \mathcal{B}(\mathcal{F}_s^0) \cap C(\Omega \times \mathbb{R})$ means that it is \mathcal{F}_s^0 progressively measurable, predictable and bounded and continuous.

Lemma 13. For any $T > 0$ and $F \in \mathcal{C}_T$, we have, for any $t > 0$,

$$\mathbb{E}^{P^\varnothing} \left[e^{1/T \int_0^t F(\theta_s \omega) ds} \right] \leq 1. \quad (3.6)$$

Proof. For any $t > 0$, writing $\psi(s) = \sum_{k:s+kT \leq t} F(\theta_{s+kT} \omega)$,

$$\begin{aligned} \mathbb{E}^{P^\varnothing} \left[e^{(1/T) \int_0^t F(\theta_s \omega) ds} \right] &= \mathbb{E}^{P^\varnothing} \left[e^{(1/T) \int_0^T \psi(s) ds} \right] \\ &\leq \frac{1}{T} \int_0^T \mathbb{E}^{P^\varnothing} \left[e^{\psi(s)} \right] ds = 1, \end{aligned} \quad (3.7)$$

by Jensen's inequality and the fact that $\mathbb{E}^{P^\varnothing} [e^{\psi(s)}] = 1$ by iteratively conditioning since $\mathbb{E}^{P^{\omega^-}} [e^{F(\omega)}] = 1$ for any ω^- . \square

Remark 14. Under P^\varnothing , the $\mathcal{F}_s^{-\infty}$ progressively measurable rate function λ is well defined since it only creates a history between time 0 and time t . Similarly, in the proof in Lemma 13, $\mathbb{E}^{P^{\omega^-}} [e^{F(\omega)}] = 1$ for any ω^- should be interpreted as the expectation is 1 given any history created between time 0 and t , which is well defined.

Next, we need to compare $\frac{1}{T} \int_0^t F(\theta_s \omega_t) ds$ and $\frac{1}{T} \int_0^t F(\theta_s \omega) ds$.

Lemma 15. For any $q > 0$, $T > 0$ and $F \in \mathcal{C}_T$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[\exp \left\{ q \left| \frac{1}{T} \int_0^t F(\theta_s \omega_t) ds - \frac{1}{T} \int_0^t F(\theta_s \omega) ds \right| \right\} \right] = 0. \quad (3.8)$$

Proof.

$$\begin{aligned} &\left| \frac{1}{T} \int_0^t F(\theta_s \omega_t) ds - \frac{1}{T} \int_0^t F(\theta_s \omega) ds \right| \\ &\leq \left| \frac{1}{T} \int_0^t \int_0^T f(u, \theta_s \omega) dN_u ds - \frac{1}{T} \int_0^t \int_0^T f(u, \theta_s \omega_t) dN_u ds \right| \\ &\quad + \left| \frac{1}{T} \int_0^t \int_0^T (e^{f(u, \theta_s \omega)} - 1) \lambda(\theta_s \omega, u) du ds \right. \\ &\quad \left. - \frac{1}{T} \int_0^t \int_0^T (e^{f(u, \theta_s \omega_t)} - 1) \lambda(\theta_s \omega_t, u) du ds \right|. \end{aligned} \quad (3.9)$$

It is easy to see that $\int_0^T f(u, \theta_s \omega) dN_u ds$ is \mathcal{F}_{s+T}^s -measurable and

$$\int_0^T f(u, \theta_s \omega) dN_u ds = \int_0^T f(u, \theta_s \omega_t) dN_u ds \quad (3.10)$$

for any $0 \leq s \leq t - T$. Hence,

$$\begin{aligned} &\left| \frac{1}{T} \int_0^t \int_0^T f(u, \theta_s \omega) dN_u ds - \frac{1}{T} \int_0^t \int_0^T f(u, \theta_s \omega_t) dN_u ds \right| \\ &\leq \frac{1}{T} \int_{t-T}^t \int_0^T |f(u, \theta_s \omega)| dN_u ds + \frac{1}{T} \int_{t-T}^t \int_0^T |f(u, \theta_s \omega_t)| dN_u ds \\ &\leq \frac{\|f\|_{L^\infty}}{T} \int_{t-T}^t N[s, s+T](\omega) ds + \frac{\|f\|_{L^\infty}}{T} \int_{t-T}^t N[s, s+T](\omega_t) ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|f\|_{L^\infty}}{T} [N[t - T, t + T](\omega) + N[t - T, t + T](\omega_t)] \\ &= \frac{\|f\|_{L^\infty}}{T} [N[t - T, t + T](\omega) + N[t - T, T](\omega) + N[0, T](\omega)]. \end{aligned} \tag{3.11}$$

By Hölder’s inequality and Lemma 20, we have

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[e^{|(1/T) \int_0^t \int_0^T f(u, \theta_s \omega) dN_u ds - (1/T) \int_0^t \int_0^T f(u, \theta_s \omega_t) dN_u ds|} \right] \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[e^{(\|f\|_{L^\infty} T) [N[t-T, t+T](\omega) + N[t-T, T](\omega) + N[0, T](\omega)]} \right] = 0. \end{aligned} \tag{3.12}$$

Furthermore,

$$\begin{aligned} &\left| \frac{1}{T} \int_0^t \int_0^T (e^{f(u, \theta_s \omega)} - 1) \lambda(\theta_s \omega, u) du ds - \frac{1}{T} \int_0^t \int_0^T (e^{f(u, \theta_s \omega_t)} - 1) \lambda(\theta_s \omega_t, u) du ds \right| \\ &\leq \frac{1}{T} \int_0^t \int_0^T |e^{f(u, \theta_s \omega)} - e^{f(u, \theta_s \omega_t)}| \lambda(\theta_s \omega, u) du ds \\ &\quad + \frac{1}{T} \int_0^t \int_0^T (e^{f(u, \theta_s \omega_t)} - 1) |\lambda(\theta_s \omega_t, u) - \lambda(\theta_s \omega, u)| du ds. \end{aligned} \tag{3.13}$$

For the first term

$$\begin{aligned} &\frac{1}{T} \int_0^t \int_0^T |e^{f(u, \theta_s \omega)} - e^{f(u, \theta_s \omega_t)}| \lambda(\theta_s \omega, u) du ds \\ &= \frac{1}{T} \int_{t-T}^t \int_0^T |e^{f(u, \theta_s \omega)} - e^{f(u, \theta_s \omega_t)}| \lambda(\theta_s \omega, u) du ds \\ &\leq \frac{2e^{\|f\|_{L^\infty}}}{T} \int_{t-T}^t \int_0^T \lambda(\theta_s \omega, u) du ds \\ &= \frac{2e^{\|f\|_{L^\infty}}}{T} \int_{t-T}^t \int_0^T \lambda \left(\sum_{\tau \in \omega[0, u+s]} h(u + s - \tau) \right) du ds \\ &\leq 2e^{\|f\|_{L^\infty}} TC_\epsilon + \frac{2e^{\|f\|_{L^\infty}}}{T} \epsilon \int_{t-T}^t \int_0^T \sum_{\tau \in \omega[0, u+s]} h(u + s - \tau) du ds \\ &\leq 2e^{\|f\|_{L^\infty}} TC_\epsilon + \frac{2e^{\|f\|_{L^\infty}}}{T} \epsilon \int_{t-T}^t \int_0^T N[0, u + s] h(0) du ds \\ &\leq 2e^{\|f\|_{L^\infty}} TC_\epsilon + 2e^{\|f\|_{L^\infty}} \epsilon \int_{t-T}^t N[0, s + T] h(0) ds \\ &\leq 2e^{\|f\|_{L^\infty}} TC_\epsilon + 2e^{\|f\|_{L^\infty}} \epsilon T (N[0, t] + N[t, t + T]) h(0). \end{aligned} \tag{3.14}$$

Therefore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[e^{(1/T) \int_0^t \int_0^T |e^{f(u, \theta_s \omega)} - e^{f(u, \theta_s \omega_t)}| \lambda(\theta_s \omega, u) du ds} \right] \leq c(\epsilon), \tag{3.15}$$

where $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Since it is bounded above for any $c(\epsilon)$, it is bounded above by 0.

For the second term,

$$\begin{aligned}
 & \frac{1}{T} \int_0^t \int_0^T (e^{f(u, \theta_s \omega_t)} - 1) |\lambda(\theta_s \omega_t, u) - \lambda(\theta_s \omega, u)| \, du \, ds \\
 & \leq \frac{e^{\|f\|_{L^\infty}} + 1}{T} \int_0^t \int_0^T \alpha \left| \sum_{\tau \in \omega_t[0, u+s) \cup (\omega_t)^-} h(u+s-\tau) - \sum_{\tau \in \omega[0, u+s)} h(u+s-\tau) \right| \, du \, ds \\
 & \leq \frac{e^{\|f\|_{L^\infty}} + 1}{T} \int_{t-T}^t \int_0^T \alpha \sum_{\tau \in \omega_t[0, u+s)} h(u+s-\tau) \, du \, ds \\
 & \quad + \frac{e^{\|f\|_{L^\infty}} + 1}{T} \int_{t-T}^t \int_0^T \alpha \sum_{\tau \in \omega[0, u+s)} h(u+s-\tau) \, du \, ds \\
 & \quad + \frac{e^{\|f\|_{L^\infty}} + 1}{T} \int_0^t \int_0^T \alpha \sum_{\tau \in (\omega_t)^-} h(u+s-\tau) \, du \, ds. \tag{3.16}
 \end{aligned}$$

Assume that $h(\cdot)$ is decreasing and $\lim_{z \rightarrow \infty} \frac{\lambda(z)}{z} = 0$. By applying Jensen's inequality twice, we can estimate the second term above,

$$\begin{aligned}
 & \mathbb{E}^{P^\varnothing} \left[e^{(e^{\|f\|_{L^\infty}} + 1)/T} \alpha \int_{t-T}^t \int_0^T \int_0^{u+s} h(u+s-v) \, dN_v \, du \, ds \right] \\
 & \leq \frac{1}{T} \int_{t-T}^t \mathbb{E}^{P^\varnothing} \left[e^{(e^{\|f\|_{L^\infty}} + 1)\alpha} \int_0^T \int_0^{u+s} h(u+s-v) \, dN_v \, du \right] \, ds \\
 & \leq \frac{1}{T^2} \int_{t-T}^t \int_0^T \mathbb{E}^{P^\varnothing} \left[e^{(e^{\|f\|_{L^\infty}} + 1)\alpha T} \int_0^{u+s} h(u+s-v) \, dN_v \right] \, du \, ds \\
 & \leq \frac{1}{T^2} \int_{t-T}^t \int_0^T \mathbb{E}^{P^\varnothing} \left[e^{\int_0^{u+s} C(\alpha, T, h)\lambda(v) \, dv} \right]^{1/2} \, du \, ds \\
 & \leq \frac{e^{C(\alpha, T, h)C_\epsilon}}{T^2} \int_{t-T}^t \int_0^T \mathbb{E}^{P^\varnothing} \left[e^{\epsilon C(\alpha, T, h)N[0, u+s]h(0)} \right]^{1/2} \, du \, ds \\
 & \leq e^{C(\alpha, T, h)C_\epsilon} \mathbb{E}^{P^\varnothing} \left[e^{\epsilon C(\alpha, T, h)N[0, t+T]h(0)} \right]^{1/2}, \tag{3.17}
 \end{aligned}$$

where $C(\alpha, T, h) = \exp(2(e^{\|f\|_{L^\infty}} + 1)\alpha T h(0)) - 1$. Thus,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[e^{(e^{\|f\|_{L^\infty}} + 1)/T} \alpha \int_{t-T}^t \int_0^T \int_0^{u+s} h(u+s-v) \, dN_v \, du \, ds \right] = 0. \tag{3.18}$$

Similarly, we can estimate the first term.

For the third term, by Jensen's inequality, we have

$$\begin{aligned}
 & \mathbb{E}^{P^\varnothing} \left[e^{(e^{\|f\|_{L^\infty}} + 1)/T} \int_0^t \int_0^T \alpha \sum_{\tau \in (\omega_t)^-} h(u+s-\tau) \, du \, ds \right] \\
 & \leq \frac{1}{T} \int_0^T \mathbb{E}^{P^\varnothing} \left[e^{\alpha(\exp(\|f\|_{L^\infty}) + 1)} \int_0^t \sum_{\tau \in (\omega_t)^-} h(u+s-\tau) \, ds \right] \, du \\
 & \leq \mathbb{E}^{P^\varnothing} \left[e^{\alpha(\exp(\|f\|_{L^\infty}) + 1)} \int_0^t \sum_{\tau \in (\omega_t)^-} h(s-\tau) \, ds \right] \\
 & = \mathbb{E}^{P^\varnothing} \left[e^{\alpha(\exp(\|f\|_{L^\infty}) + 1)} \int_0^t \int_0^t \sum_{k=0}^\infty h(s+kt+t-u) \, dN_u \, ds \right]. \tag{3.19}
 \end{aligned}$$

Since we assume that $h(\cdot)$ is decreasing, $\int_{kt}^{(k+1)t} h(s) ds \geq th((k+1)t)$. Thus

$$\sum_{k=0}^{\infty} h(s+kt+t-u) \leq h(s+t-u) + \frac{1}{t} \int_{s+t-u}^{\infty} h(v) dv. \tag{3.20}$$

Let $C(\alpha, f) = \alpha(\exp(\|f\|_{L^\infty}) + 1)$ and $H(t) = \int_t^\infty h(s) ds$. Then,

$$\begin{aligned} & \mathbb{E}^{P^\varnothing} \left[e^{\alpha(\exp(\|f\|_{L^\infty})+1) \int_0^t \int_0^t \sum_{k=0}^{\infty} h(s+kt+t-u) dN_u ds} \right] \\ & \leq \mathbb{E}^{P^\varnothing} \left[e^{C(\alpha, f) \int_0^t \int_0^t (1/t) H(s+t-u) dN_u ds + C(\alpha, f) \int_0^t \int_0^t h(s+t-u) ds} dN_u \right] \\ & = \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t [(C(\alpha, f)/t) \int_0^t H(s+t-u) ds] dN_u + \int_0^t \int_0^t C(\alpha, f) h(s+t-u) ds} dN_u \right]. \end{aligned} \tag{3.21}$$

Notice that

$$\mathbb{E}^{P^\varnothing} \left[e^{\int_0^t [(C(\alpha, f)/t) \int_0^t H(s+t-u) ds] dN_u} \right] \leq \mathbb{E}^{P^\varnothing} \left[e^{[(C(\alpha, f)/t) \int_0^t H(s) ds] N_t} \right], \tag{3.22}$$

where $\frac{C(\alpha, f)}{t} \int_0^t H(s) ds \rightarrow 0$ as $t \rightarrow \infty$, which implies that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t [(C(\alpha, f)/t) \int_0^t H(s+u) ds] dN_u} \right] = 0. \tag{3.23}$$

Moreover,

$$\begin{aligned} & \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t \int_0^t C(\alpha, f) h(s+t-u) ds} dN_u \right] \\ & \leq \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t (e^{2 \int_0^t C(\alpha, f) h(s+t-u) ds} - 1) \lambda(u) du} \right]^{1/2} \\ & \leq e^{(1/2) C_\epsilon \int_0^t (e^{2 \int_0^t C(\alpha, f) h(s+t-u) ds} - 1) du} \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t (e^{\int_0^t 2C(\alpha, f) h(s+t-u) ds} - 1) \epsilon \sum_{\tau < u} h(u-\tau) du} \right]^{1/2} \\ & \leq e^{(1/2) C_\epsilon \int_0^t (e^{2C(\alpha, f) H(t-u)} - 1) du} \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t (e^{2C(\alpha, f) \|h\|_{L^1}} - 1) \epsilon \sum_{\tau < u} h(u-\tau) du} \right]^{1/2} \\ & \leq e^{(1/2) C_\epsilon \int_0^t (e^{2C(\alpha, f) H(u)} - 1) du} \mathbb{E}^{P^\varnothing} \left[e^{\epsilon (e^{2C(\alpha, f) \|h\|_{L^1}} - 1) \|h\|_{L^1} N_t} \right]^{1/2}. \end{aligned} \tag{3.24}$$

Notice that it holds for any $\epsilon > 0$ and $\frac{1}{t} \int_0^t (e^{2C(\alpha, f) H(u)} - 1) du \rightarrow 0$ as $t \rightarrow \infty$, which implies that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[e^{\int_0^t \int_0^t C(\alpha, f) h(s+t-u) ds} dN_u \right] = 0. \tag{3.25}$$

Putting all these together and applying Hölder's inequality several times, we get, for any $q > 0$, $T > 0$ and $F \in \mathcal{C}_T$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\varnothing} \left[\exp \left\{ q \left| \frac{1}{T} \int_0^t F(\theta_s \omega_t) ds - \frac{1}{T} \int_0^t F(\theta_s \omega) ds \right| \right\} \right] = 0. \tag{3.26}$$

□

Lemma 16. *We have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sup_{F \in \mathcal{C}_T} \int F(\omega) Q(d\omega) \geq H(Q). \tag{3.27}$$

Proof. Assume $H(Q) < \infty$. For any $\epsilon > 0$, there exists some f_ϵ such that

$$\mathbb{E}^Q \left[\int_0^1 f_\epsilon dN_s - \int_0^1 (e^{f_\epsilon} - 1) \lambda ds \right] \geq H(Q) - \epsilon. \tag{3.28}$$

We can find a sequence $f_T \in \mathcal{B}(\mathcal{F}_s^{-(T-1)}) \cap C(\Omega \times \mathbb{R}) \rightarrow f_\epsilon$ as $T \rightarrow \infty$ and by Fatou's lemma,

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \frac{1}{T} \sup_{F \in \mathcal{C}_T} \int_{\Omega} F(\omega) Q(d\omega) \\ & \geq \liminf_{T \rightarrow \infty} \mathbb{E}^Q \left[\int_0^1 f_T dN_s - \int_0^1 (e^{f_T} - 1) \lambda ds \right] \geq H(Q) - \epsilon. \end{aligned} \tag{3.29}$$

If $H(Q) = \infty$, then, for any $M > 0$, there exists some f_M such that

$$\mathbb{E}^Q \left[\int_0^1 f_M dN_s - \int_0^1 (e^{f_M} - 1) \lambda ds \right] \geq M. \tag{3.30}$$

Repeat the same argument as in the case that $H(Q) < \infty$. □

Lemma 17. For any compact set A ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in A) \leq - \inf_{Q \in A} H(Q). \tag{3.31}$$

Proof. Notice that

$$\mathbb{E}^{P^\varnothing} [e^{N[0,t]}] \leq \mathbb{E}^{P^\varnothing} [e^{(e^2-1) \int_0^t \lambda(s) ds}]^{1/2} \leq \mathbb{E}^{P^\varnothing} [e^{(e^2-1)\epsilon h(0)N[0,t]+C_\epsilon(e^2-1)}]^{1/2}. \tag{3.32}$$

By choosing $\epsilon > 0$ small enough, we have $\mathbb{E}^{P^\varnothing} [e^{N[0,t]}] \leq e^{Ct}$ for some constant $C > 0$. Therefore

$$\limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P^\varnothing(N[0, t] > \ell t) = -\infty, \tag{3.33}$$

which implies (by comparing $\int_{\Omega} N[0, 1] dR_{t,\omega}$ and $N[0, t]/t$ and the superexponential estimates in Lemma 20)

$$\limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P^\varnothing \left(\int_{\Omega} N[0, 1] dR_{t,\omega} > \ell \right) = -\infty. \tag{3.34}$$

Therefore, we only need to consider the compact set A such that for any $Q \in A$, $\mathbb{E}^Q[N[0, 1]] < \infty$.

Now for any A compact consisting of Q with $\mathbb{E}^Q[N[0, 1]] < \infty$ and for any $F \in \mathcal{C}_T$ and for any $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality, Chebychev's inequality and Lemma 13,

$$\begin{aligned} & P^\varnothing(R_{t,\omega} \in A) \\ & \leq \mathbb{E}^{P^\varnothing} [e^{(1/(pT)) \int_0^t F(\theta_s \omega_t) ds}] \cdot \exp \left\{ -\frac{t}{pT} \inf_{Q \in A} \int_{\Omega} F(\omega) Q(d\omega) \right\} \\ & \leq \mathbb{E}^{P^\varnothing} [e^{(1/T) \int_0^t F(\theta_s \omega) ds}]^{1/p} \mathbb{E}^{P^\varnothing} [e^{(q/(pT)) | \int_0^t F(\theta_s \omega_t) ds - \int_0^t F(\theta_s \omega) ds |}]^{1/q} \\ & \quad \cdot \exp \left\{ -\frac{t}{pT} \inf_{Q \in A} \int_{\Omega} F(\omega) Q(d\omega) \right\} \\ & \leq \mathbb{E}^{P^\varnothing} [e^{(q/(pT)) | \int_0^t F(\theta_s \omega_t) ds - \int_0^t F(\theta_s \omega) ds |}]^{1/q} \cdot \exp \left\{ -\frac{t}{pT} \inf_{Q \in A} \int_{\Omega} F(\omega) Q(d\omega) \right\}. \end{aligned} \tag{3.35}$$

By Lemma 15, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P^\varnothing(R_{t,\omega} \in A) \leq -\frac{1}{p} \inf_{Q \in A} \frac{1}{T} \int_{\Omega} F(\omega) Q(d\omega). \tag{3.36}$$

Since it holds for any $p > 1$, we get

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P^{\mathcal{Q}}(R_{t,\omega} \in A) \leq - \inf_{Q \in A} \frac{1}{T} \int_{\Omega} F(\omega) Q(d\omega). \tag{3.37}$$

For any compact A , given $Q \in A$ and $\epsilon > 0$, by Lemma 16, there exists $T_Q > 0$ and $F_Q \in \mathcal{C}_{T_Q}$ such that $\frac{1}{T_Q} \int_{\Omega} F_Q(\omega) Q(d\omega) \geq \inf_{A \in Q} H(Q) - \frac{1}{2}\epsilon$. Since the linear integral is a continuous functional of Q (see the proof of Lemma 6), there exists a neighborhood G_Q of Q such that $\frac{1}{T_Q} \int_{\Omega} F_Q(\omega) Q(d\omega) \geq \inf_{A \in Q} H(Q) - \epsilon$ for all $Q \in G_Q$. Since A is compact, there exists $G_{Q_1}, \dots, G_{Q_\ell}$ such that $A \subset \bigcup_{j=1}^{\ell} G_{Q_j}$. Hence

$$\inf_{1 \leq j \leq \ell} \sup_{T > 0} \sup_{F \in \mathcal{C}_T} \inf_{Q \in G_j} \frac{1}{T} \int_{\Omega} F(\omega) Q(d\omega) \geq \inf_{Q \in A} H(Q) - \epsilon. \tag{3.38}$$

Note that for any A, B ,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in A \cup B) \\ & \leq \max \left\{ \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in A), \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in B) \right\}. \end{aligned} \tag{3.39}$$

Thus, for $A \subset \bigcup_{j=1}^{\ell} G_j$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in A) \leq - \inf_{1 \leq j \leq \ell} \sup_{T > 0} \sup_{F \in \mathcal{C}_T} \inf_{Q \in G_j} \frac{1}{T} \int_{\Omega} F(\omega) Q(d\omega). \tag{3.40}$$

Hence $\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in A) \leq - \inf_{Q \in A} H(Q)$ for any compact A . □

Theorem 18 (Upper bound). *For any closed set C ,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in C) \leq - \inf_{Q \in C} H(Q). \tag{3.41}$$

Proof. For any closed set C and compact \mathcal{A}^n which is defined in Lemma 25, we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in C) \\ & \leq \max \left\{ \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in C \cap \mathcal{A}^n), \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in (\mathcal{A}^n)^c) \right\}. \end{aligned} \tag{3.42}$$

Since $C \cap \mathcal{A}^n$ is compact, by Lemma 17,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in C \cap \mathcal{A}^n) \leq - \inf_{Q \in C \cap \mathcal{A}^n} H(Q) \leq - \inf_{Q \in C} H(Q). \tag{3.43}$$

Furthermore, by Lemma 24, we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in (\mathcal{A}^n)^c) \\ & = \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(R_{t,\omega} \in \bigcup_{j=n}^{\infty} \mathcal{A}_{1/j,j,j}^c \right) \end{aligned}$$

$$\begin{aligned} &\leq \max_{j \geq n} \max \left\{ \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \int_0^t \chi_{N[0,1] \geq j}(\theta_s \omega_t) \, ds \geq \varepsilon(j) \right), \right. \\ &\quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \int_0^t \chi_{N[0,1/j] \geq 2}(\theta_s \omega_t) \, ds \geq (1/j)g(1/j) \right), \\ &\quad \left. \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \int_0^t N[0, 1] \chi_{N[0,1] \geq \ell}(\theta_s \omega_t) \, ds \geq m(\ell) \right) \right\} \end{aligned} \tag{3.44}$$

$$\rightarrow -\infty \tag{3.45}$$

as $n \rightarrow \infty$. Hence, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P(R_{t,\omega} \in C) \leq - \inf_{Q \in C} H(Q). \tag{3.46}$$

□

4. Superexponential estimates

In order to get the full large deviation principle, we need the upper bound inequality valid for any closed set instead of for any compact set, which requires some superexponential estimates.

Lemma 19. *For any $q > 0$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\otimes} \left[e^{q \int_0^t h(t-s) \, dN_s} \right] = 0. \tag{4.1}$$

Proof.

$$\begin{aligned} \mathbb{E}^{P^\otimes} \left[e^{q \int_0^t h(t-s) \, dN_s} \right] &\leq \mathbb{E}^{P^\otimes} \left[e^{\int_0^t (e^{2qh(t-s)} - 1) \lambda(\sum_{0 < \tau < s} h(s-\tau)) \, ds} \right]^{1/2} \\ &\leq \mathbb{E}^{P^\otimes} \left[e^{(C_\epsilon + h(0)\epsilon N_t) \int_0^t (e^{2qh(t-s)} - 1) \, ds} \right]^{1/2}. \end{aligned} \tag{4.2}$$

Note that $\int_0^t (e^{2qh(t-s)} - 1) \, ds = \int_0^t (e^{2qh(s)} - 1) \, ds \in L_1$ since $h \in L^1$. Therefore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\otimes} \left[e^{q \int_0^t h(t-s) \, dN_s} \right] \leq c(\epsilon), \tag{4.3}$$

where $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Since it holds for any ϵ , we get the desired result. □

Lemma 20. *For any $q > 0$ and $T > 0$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^{P^\otimes} \left[e^{qN[t,t+T]} \right] = 0. \tag{4.4}$$

Therefore, for any $\epsilon > 0$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P^\otimes(N[t, t + T] \geq \epsilon t) = -\infty. \tag{4.5}$$

Proof. By Hölder’s inequality,

$$\begin{aligned} \mathbb{E}^{P^\otimes} \left[e^{qN[t,t+T]} \right] &\leq \mathbb{E}^{P^\otimes} \left[e^{\int_t^{t+T} \lambda(\sum_{0 < \tau < s} h(s-\tau)) \, ds} \right]^{1/2} \\ &\leq e^{(1/2)(e^{2q} - 1)C_\epsilon T} \cdot \mathbb{E}^{P^\otimes} \left[e^{\epsilon(e^{2q} - 1)h(0)N[t,t+T] + \epsilon(e^{2q} - 1) \int_0^t h(t-s) \, dN_s} \right]^{1/2} \\ &\leq e^{(1/2)(e^{2q} - 1)C_\epsilon T} \cdot \mathbb{E}^{P^\otimes} \left[e^{2\epsilon(e^{2q} - 1)h(0)N[t,t+T]} \right]^{1/4} \cdot \mathbb{E}^{P^\otimes} \left[e^{2\epsilon(e^{2q} - 1) \int_0^t h(t-s) \, dN_s} \right]^{1/4}. \end{aligned} \tag{4.6}$$

Choose $\epsilon < q[2(e^{2q} - 1)h(0)]^{-1}$. We have

$$\mathbb{E}^{P^\otimes} [e^{qN[t,t+T]}]^{3/4} \leq e^{(1/2)(e^{2q}-1)C_\epsilon T} \cdot \mathbb{E}^{P^\otimes} [e^{2\epsilon(e^{2q}-1)\int_0^t h(t-s)dN_s}]^{1/4}. \tag{4.7}$$

By Lemma 19, we get the desired result. □

Lemma 21. *We have the following superexponential estimates.*

(i) For any $\epsilon > 0$,

$$\limsup_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{\delta t} \int_0^t \chi_{N[0,\delta] \geq 2}(\theta_s, \omega) ds \geq \epsilon\right) = -\infty. \tag{4.8}$$

(ii) For any $\epsilon > 0$,

$$\limsup_{M \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{t} \int_0^t \chi_{N[0,1] \geq M}(\theta_s, \omega) ds \geq \epsilon\right) = -\infty. \tag{4.9}$$

(iii) For any $\epsilon > 0$,

$$\limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{t} \int_0^t N[0, 1] \chi_{N[0,1] \geq \ell}(\theta_s, \omega) ds \geq \epsilon\right) = -\infty. \tag{4.10}$$

Proof. (i) Let us define

$$N_{\ell'}[0, t] = \int_0^t \chi_{\lambda(s) < \ell'} dN_s, \quad \hat{N}_{\ell'}[0, t] = \int_0^t \chi_{\lambda(s) \geq \ell'} dN_s. \tag{4.11}$$

Then $N[0, t] = N_{\ell'}[0, t] + \hat{N}_{\ell'}[0, t]$ and $N_{\ell'}[0, t]$ has compensator $\int_0^t \lambda(s) \chi_{\lambda(s) < \ell'} ds$ and $\hat{N}_{\ell'}[0, t]$ has compensator $\int_0^t \lambda(s) \chi_{\lambda(s) \geq \ell'} ds$. Notice that

$$\chi_{N[0,\delta] \geq 2} \leq \chi_{N_{\ell'}[0,\delta] \geq 2} + \chi_{\hat{N}_{\ell'}[0,\delta] \geq 1}. \tag{4.12}$$

It is clear that $N_{\ell'}$ is dominated by the usual Poisson process with rate ℓ' . By Lemma 22,

$$\limsup_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{\delta t} \int_0^t \chi_{N_{\ell'}[0,\delta] \geq 2}(\theta_s, \omega) ds \geq \frac{\epsilon}{2}\right) = -\infty. \tag{4.13}$$

On the other hand,

$$\begin{aligned} \frac{1}{\delta} \int_0^t \chi_{\hat{N}_{\ell'}[0,\delta] \geq 1}(\theta_s, \omega) ds &= \frac{1}{\delta} \int_0^t \chi_{\hat{N}_{\ell'}[s,s+\delta] \geq 1}(\omega) ds \\ &\leq \frac{1}{\delta} \int_0^t \hat{N}_{\ell'}[s, s + \delta] ds \\ &= \frac{1}{\delta} \int_s^{t+\delta} \hat{N}_{\ell'}[0, s] ds - \frac{1}{\delta} \int_0^t \hat{N}_{\ell'}[0, s] ds \\ &\leq \hat{N}_{\ell'}[0, t] + N[t, t + \delta]. \end{aligned} \tag{4.14}$$

By Lemma 20, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{t} N[t, t + \delta] \geq \frac{\epsilon}{4}\right) = -\infty, \tag{4.15}$$

for any $\delta > 0$. Hence

$$\limsup_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{t} N[t, t + \delta] \geq \frac{\epsilon}{4}\right) = -\infty. \tag{4.16}$$

Finally, for some positive $h(\ell')$ to be chosen later,

$$\begin{aligned} P\left(\frac{1}{t} \hat{N}_{\ell'}[0, t] \geq \frac{\epsilon}{4}\right) &\leq \mathbb{E}[e^{h(\ell') \hat{N}_{\ell'}[0, t]}] e^{-th(\ell')\epsilon/4} \\ &\leq \mathbb{E}[e^{(e^{2h(\ell')} - 1) \int_0^t \lambda(s) \chi_{\lambda(s) \geq \ell'} ds}]^{1/2} e^{-th(\ell')\epsilon/4}. \end{aligned} \tag{4.17}$$

Let $f(z) = \frac{z}{\lambda(z)}$. Then $f(z) \rightarrow \infty$ as $z \rightarrow \infty$. Let $Z_s = \sum_{\tau \in \omega[0, s]} h(s - \tau)$. Then, by the definition of $\lambda(s)$ and abusing the notation a little bit, we see that $\lambda(s) = \lambda(Z_s)$. Since $\lambda(\cdot)$ is increasing, its inverse function λ^{-1} exists and $\lambda^{-1}(\ell') \rightarrow \infty$ as $\ell' \rightarrow \infty$. We have

$$\begin{aligned} \mathbb{E}[e^{(e^{2h(\ell')} - 1) \int_0^t \lambda(s) \chi_{\lambda(s) \geq \ell'} ds}]^{1/2} &\leq \mathbb{E}[e^{(e^{2h(\ell')} - 1) \int_0^t \lambda(Z_s) \chi_{Z_s \geq \lambda^{-1}(\ell')} ds}]^{1/2} \\ &\leq \mathbb{E}[e^{(e^{2h(\ell')} - 1) \int_0^t \lambda(Z_s) (f(Z_s) / (\inf_{z \geq \ell'} f(\lambda^{-1}(z)))) ds}]^{1/2}. \end{aligned} \tag{4.18}$$

It is clear that $\lim_{\ell' \rightarrow \infty} \inf_{z \geq \ell'} f(\lambda^{-1}(z)) = \infty$. Choose

$$h(\ell') = \frac{1}{2} \log \left[\inf_{z \geq \ell'} f(\lambda^{-1}(z)) + 1 \right]. \tag{4.19}$$

Then $h(\ell') \rightarrow \infty$ as $\ell' \rightarrow \infty$ and

$$\begin{aligned} \mathbb{E}[e^{(e^{2h(\ell')} - 1) \int_0^t \lambda(Z_s) (f(Z_s) / (\inf_{z \geq \ell'} f(\lambda^{-1}(z)))) ds}]^{1/2} &= \mathbb{E}[e^{\int_0^t Z_s ds}]^{1/2} \\ &= \mathbb{E}[e^{\int_0^t \sum_{\tau \in \omega[0, s]} h(s - \tau) ds}]^{1/2} \\ &\leq \mathbb{E}[e^{\|h\|_{L^1} N_t}]^{1/2}. \end{aligned} \tag{4.20}$$

Hence,

$$\limsup_{\ell' \rightarrow \infty} \limsup_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{t} \hat{N}_{\ell'}[0, t] \geq \frac{\epsilon}{4}\right) = -\infty. \tag{4.21}$$

(ii) It is easy to see that (iii) implies (ii).

(iii) Observe first that

$$\begin{aligned} N[s, s + 1] \chi_{N[s, s + 1] \geq \ell} &\leq N_{\ell'}[s, s + 1] \chi_{N_{\ell'}[s, s + 1] \geq \ell/2} + \hat{N}_{\ell'}[s, s + 1] \chi_{\hat{N}_{\ell'}[s, s + 1] \geq \ell/2} \\ &\quad + \frac{\ell}{2} \chi_{N_{\ell'}[s, s + 1] \geq \ell/2} + \frac{\ell}{2} \chi_{\hat{N}_{\ell'}[s, s + 1] \geq \ell/2}. \end{aligned} \tag{4.22}$$

For the first term, notice that $N_{\ell'}$ is dominated by a usual Poisson process with rate ℓ' . Thus, by Lemma 23,

$$\limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P\left(\frac{1}{t} \int_0^t N_{\ell'}[s, s + 1] \chi_{N_{\ell'}[s, s + 1] \geq \ell/2}(\omega) ds \geq \frac{\epsilon}{4}\right) = -\infty. \tag{4.23}$$

For the second term, $\hat{N}_{\ell'}[s, s + 1] \chi_{\hat{N}_{\ell'}[s, s + 1] \geq \ell/2} \leq \hat{N}_{\ell'}[s, s + 1]$ and

$$\int_0^t \hat{N}_{\ell'}[s, s + 1] ds \leq \hat{N}_{\ell'}[0, t] + N[t, t + 1]. \tag{4.24}$$

By Lemma 20,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} N[t, t + 1] \geq \frac{\epsilon}{8} \right) = -\infty, \tag{4.25}$$

and by the same argument as in (i),

$$\limsup_{\ell' \rightarrow \infty} \limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \hat{N}_{\ell'}[0, t] \geq \frac{\epsilon}{8} \right) = -\infty. \tag{4.26}$$

For the third term, notice that

$$\int_0^t \frac{\ell}{2} \chi_{N_{\ell'}[s, s+1] \geq \ell/2} ds \leq \int_0^t N_{\ell'}[s, s + 1] \chi_{N_{\ell'}[s, s+1] \geq \ell/2}(\omega) ds. \tag{4.27}$$

We can get the same superexponential estimate as before. Finally, for the fourth term,

$$\int_0^t \frac{\ell}{2} \chi_{\hat{N}_{\ell'}[s, s+1] \geq \ell/2} ds \leq \int_0^t \hat{N}_{\ell'}[s, s + 1](\omega) ds. \tag{4.28}$$

We can get the same superexponential estimate as before. □

Lemma 22. Assume N_t is a usual Poisson process with constant rate λ . Then, for any $\epsilon > 0$

$$\limsup_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\frac{1}{\delta t} \int_0^t \chi_{N[s, s+\delta] \geq 2}(\omega) ds \geq \epsilon \right) = -\infty. \tag{4.29}$$

Proof. Let $f(\delta, \omega) = \frac{1}{h(\delta)} \chi_{N[0, \delta] \geq 2}(\omega)$, where $h(\delta)$ is to be chosen later. By Jensen’s inequality and stationarity and independence of increments of Poisson process, we have

$$\begin{aligned} \mathbb{E} \left[e^{\int_0^t (1/\delta) f(\delta, \theta_s, \omega) ds} \right] &\leq \mathbb{E} \left[e^{(1/\delta) \int_0^\delta \sum_{j=0}^{\lfloor t/\delta \rfloor} f(\delta, \theta_{s+j\delta\omega}) ds} \right] \\ &\leq \mathbb{E} \left[\frac{1}{\delta} \int_0^\delta e^{\sum_{j=0}^{\lfloor t/\delta \rfloor} f(\delta, \theta_{s+j\delta\omega})} ds \right] \\ &= \mathbb{E} \left[e^{\sum_{j=0}^{\lfloor t/\delta \rfloor} f(\delta, \theta_{j\delta\omega})} \right] \\ &= \mathbb{E} \left[e^{f(\delta, \omega)} \right]^{\lfloor t/\delta \rfloor + 1} \\ &= \left\{ e^{1/h(\delta)} (1 - e^{-\lambda\delta} - \lambda\delta e^{-\lambda\delta}) + e^{-\lambda\delta} + \lambda\delta e^{-\lambda\delta} \right\}^{\lfloor t/\delta \rfloor + 1} \\ &\leq (M' e^{1/h(\delta)} \lambda^2 \delta^2 + 1)^{\lfloor t/\delta \rfloor + 1}, \end{aligned} \tag{4.30}$$

for some $M' > 0$. Choose $h(\delta) = \frac{1}{\log(1/\delta)}$. Thus,

$$\mathbb{E} \left[e^{\int_0^t (1/\delta) f(\delta, \theta_s, \omega) ds} \right] \leq (M' \delta + 1)^{\lfloor t/\delta \rfloor + 1} \leq e^{Mt}, \tag{4.31}$$

for some $M > 0$. Therefore, by Chebychev’s inequality,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P} \left(\frac{1}{\delta h(\delta) t} \int_0^t \chi_{N[s, s+\delta] \geq 2}(\omega) ds \geq \frac{\epsilon}{h(\delta)} \right) \leq M - \frac{\epsilon}{h(\delta)}, \tag{4.32}$$

which holds for any $\delta > 0$. Letting $\delta \rightarrow 0$, we get the desired result. □

Lemma 23. Assume N_t is a usual Poisson process with constant rate λ . Then, for any $\epsilon > 0$,

$$\limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \int_0^t N[0, 1] \chi_{N[0, 1] \geq \ell}(\theta_s \omega) \, ds \geq \epsilon \right) = -\infty. \tag{4.33}$$

Proof. Let $h(\ell)$ be some function of ℓ to be chosen later. Following the same argument as in the proof of Lemma 22, we have

$$\begin{aligned} \mathbb{P} \left(h(\ell) \int_0^t N[0, 1] \chi_{N[0, 1] \geq \ell}(\theta_s \omega) \, ds \geq \epsilon h(\ell) t \right) &\leq \mathbb{E} \left[e^{h(\ell) \int_0^t N[0, 1] \chi_{N[0, 1] \geq \ell}(\theta_s \omega) \, ds} \right] e^{-\epsilon h(\ell) t} \\ &\leq \mathbb{E} \left[e^{h(\ell) N[0, 1] \chi_{N[0, 1] \geq \ell}} \right]^{[t]+1} e^{-\epsilon h(\ell) t} \\ &= \left\{ \mathbb{P}(N[0, 1] < \ell) + \sum_{k=\ell}^{\infty} e^{h(\ell)k} e^{-\lambda \frac{\lambda^k}{k!}} \right\}^{[t]+1} e^{-\epsilon h(\ell) t} \\ &\leq \left\{ 1 + C_1 \sum_{k=\ell}^{\infty} e^{h(\ell)k + \log(\lambda)k - \log(k)k} \right\}^{[t]+1} e^{-\epsilon h(\ell) t} \\ &\leq \left\{ 1 + C_2 e^{h(\ell)\ell + \log(\lambda)\ell - \log(\ell)\ell} \right\}^{[t]+1} e^{-\epsilon h(\ell) t}. \end{aligned} \tag{4.34}$$

Choosing $h(\ell) = (\log(\ell))^{1/2}$ will do the work. □

The following Lemma 24 provides us the superexponential estimates that we need. These superexponential estimates have basically been done in Lemma 21. The difference is that in the statement in Lemma 21, we used ω and in Lemma 24 it is changed to ω_t which is what we needed. Lemma 24 has three statements. Part (i) says if you start with a sequence of simple point processes, the limiting point process may not be simple, but that has probability that is superexponentially small. Part (ii) is the usual superexponential we would expect if $\mathcal{M}_S(\Omega)$ were equipped with weak topology. But since we are using a strengthened weak topology with the convergence of first moment as well, we will also need Part (iii).

Lemma 24. We have the following superexponential estimates.

(i) For some $g(\delta) \rightarrow 0$ as $\delta \rightarrow 0$,

$$\limsup_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{\delta t} \int_0^t \chi_{N[0, \delta] \geq 2}(\theta_s \omega_t) \, ds \geq g(\delta) \right) = -\infty. \tag{4.35}$$

(ii) For some $\varepsilon(M) \rightarrow 0$ as $M \rightarrow \infty$,

$$\limsup_{M \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \int_0^t \chi_{N[0, 1] \geq M}(\theta_s \omega_t) \, ds \geq \varepsilon(M) \right) = -\infty. \tag{4.36}$$

(iii) For some $m(\ell) \rightarrow 0$ as $\ell \rightarrow \infty$,

$$\limsup_{\ell \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \int_0^t N[0, 1] \chi_{N[0, 1] \geq \ell}(\theta_s \omega_t) \, ds \geq m(\ell) \right) = -\infty. \tag{4.37}$$

Proof. We can replace the ϵ in the statement of Lemma 21 by $g(\delta)$, $\varepsilon(M)$ and $m(\ell)$ by a standard analysis argument. We can also replace the ω in Lemma 21 by ω_t here since

$$\left| \int_0^t \chi_{N[0, \delta] \geq 2}(\theta_s \omega_t) \, ds - \int_0^t \chi_{N[0, \delta] \geq 2}(\theta_s \omega) \, ds \right| \leq 2\delta, \tag{4.38}$$

$$\left| \int_0^t \chi_{N[0, 1] \geq M}(\theta_s \omega_t) \, ds - \int_0^t \chi_{N[0, 1] \geq M}(\theta_s \omega) \, ds \right| \leq 2, \tag{4.39}$$

and

$$\begin{aligned}
 & \left| \int_0^t N[0, 1] \chi_{N[0, 1] \geq \ell}(\theta_s \omega_t) \, ds - \int_0^t N[0, 1] \chi_{N[0, 1] \geq \ell}(\theta_s \omega) \, ds \right| \\
 & \leq \int_{t-1}^t N[s, s+1](\omega) \, ds + \int_{t-1}^t N[s, s+1](\omega_t) \, ds \\
 & \leq N[t-1, t+1](\omega) + N[t-1, t+1](\omega_t) \\
 & = N[t-1, t+1](\omega) + N[t-1, t](\omega) + N[0, 1](\omega).
 \end{aligned} \tag{4.40}$$

By Lemma 20, we have the superexponential estimate, for any $\epsilon > 0$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P \left(\frac{1}{t} \{ N[t-1, t+1](\omega) + N[t-1, t](\omega) + N[0, 1](\omega) \} \geq \epsilon \right) = -\infty. \tag{4.41}$$

□

Lemma 25. For any $\delta, M > 0, \ell > 0$, define

$$\begin{aligned}
 \mathcal{A}_\delta &= \{ Q \in \mathcal{M}_S(\Omega) : Q(N[0, \delta] \geq 2) \leq \delta g(\delta) \}, \\
 \mathcal{A}_M &= \{ Q \in \mathcal{M}_S(\Omega) : Q(N[0, 1] \geq M) \leq \epsilon(M) \}, \\
 \mathcal{A}_\ell &= \left\{ Q \in \mathcal{M}_S(\Omega) : \int_{N[0, 1] \geq \ell} N[0, 1] \, dQ \leq m(\ell) \right\},
 \end{aligned} \tag{4.42}$$

where $\epsilon(M) \rightarrow 0$ as $M \rightarrow \infty$, $m(\ell) \rightarrow 0$ as $\ell \rightarrow \infty$ and $g(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Let $\mathcal{A}_{\delta, M, \ell} = \mathcal{A}_\delta \cap \mathcal{A}_M \cap \mathcal{A}_\ell$ and

$$\mathcal{A}^n = \bigcap_{j=n}^\infty \mathcal{A}_{1/j, j, j}. \tag{4.43}$$

Then, \mathcal{A}^n is compact.

Proof. Observe that for $\beta > 0$,

$$K_\beta = \bigcap_{k=1}^\infty \{ \omega : \{ N[-k, -(k-1)](\omega) \leq \beta \ell_k \} \cap \{ N[k-1, k](\omega) \leq \beta \ell_k \} \} \tag{4.44}$$

are relatively compact sets in Ω . Let $\overline{K_\beta}$ be the closure of K_β , which is then compact.

For any $Q \in \mathcal{A}^n$, $Q(N[0, 1] \geq M) \leq \epsilon(M)$ for any $M \geq n$. We can choose β big enough and an increasing sequence ℓ_k such that $\beta \ell_1 \geq n$ and $\infty > \sum_{k=1}^\infty \epsilon(\beta \ell_k) \rightarrow 0$ as $\beta \rightarrow \infty$, uniformly for $Q \in \mathcal{A}^n$,

$$\begin{aligned}
 Q(\overline{K_\beta}^c) &\leq Q(K_\beta^c) \\
 &= Q \left(\bigcup_{k=1}^\infty \{ N[-k, -(k-1)](\omega) > \beta \ell_k \} \cap \{ N[k-1, k](\omega) > \beta \ell_k \} \right) \\
 &\leq \sum_{k=1}^\infty \{ Q(N[-(k-1), -k] > \beta \ell) + Q(N[k-1, k] > \beta \ell_k) \} \\
 &= 2 \sum_{k=1}^\infty Q(N[0, 1] > \beta \ell_k) \\
 &\leq 2 \sum_{k=1}^\infty \epsilon(\beta \ell_k) \rightarrow 0
 \end{aligned} \tag{4.45}$$

as $\beta \rightarrow \infty$. Therefore, \mathcal{A}^n is tight in the weak topology and by Prokhorov theorem \mathcal{A}^n is precompact in the weak topology. In other words, for any sequence in \mathcal{A}^n , there exists a subsequence, say Q_n such that $Q_n \rightarrow Q$ weakly as $n \rightarrow \infty$ for some Q . By the definition of \mathcal{A}^n , Q_n are uniformly integrable, which implies that $\int N[0, 1] dQ_n \rightarrow \int N[0, 1] dQ$ as $n \rightarrow \infty$. It is also easy to see that \mathcal{A}^n is closed by checking that each $\mathcal{A}_{1/j, j, j}$ is closed. That implies that $Q \in \mathcal{A}^n$. Finally, we need to check that Q is a simple point process. Let $I_{j, \delta} = [(j-1)\delta, j\delta]$. We have for any $Q \in \mathcal{A}^n$,

$$\begin{aligned} Q(\exists t: N[t-, t] \geq 2) &= Q\left(\bigcup_{k=1}^{\infty} \{\exists t \in [-k, k]: N[t-, t] \geq 2\}\right) \\ &= Q\left(\bigcup_{k=1}^{\infty} \bigcap_{\delta > 0} \bigcup_{j=-[k/\delta]+1}^{[k/\delta]} \{\omega: \#\{\omega \cup I_{j, \delta}\} \geq 2\}\right) \\ &\leq \sum_{k=1}^{\infty} \inf_{\delta=1/m, m \geq n} \sum_{j=-[k/\delta]+1}^{[k/\delta]} Q(\#\{\omega \cup I_{j, \delta}\} \geq 2) \\ &\leq \sum_{k=1}^{\infty} \inf_{\delta=1/m, m \geq n} \{2[k/\delta]\delta g(\delta)\} \\ &= 0. \end{aligned} \tag{4.46}$$

Hence, \mathcal{A}^n is precompact in our topology. Since \mathcal{A}^n is closed, it is compact. \square

5. Concluding remarks

In this paper, we obtained a process-level large deviation principle for a wide class of simple point processes, i.e. nonlinear Hawkes process. Indeed, the methods and ideas should apply to some other simple point processes as well and we should expect to get the same expression for the rate function $H(Q)$. For $H(Q) < \infty$, it should satisfy

$$H(Q) = \int_{\Omega} \int_0^1 \lambda(\omega, s) - \hat{\lambda}(\omega, s) + \log\left(\frac{\hat{\lambda}(\omega, s)}{\lambda(\omega, s)}\right) \hat{\lambda}(\omega, s) ds Q(d\omega), \tag{5.1}$$

where $\lambda(\omega, s)$ is the intensity of the underlying simple point process. Now, it would be interesting to ask for what conditions for a simple point process would guarantee the process-level large deviation principle that we obtained in our paper? First, we have to assume that $\lambda(\omega, t)$ is predictable and progressively measurable. Second, from the proof of the upper bound in our paper, the key assumption we used about nonlinear Hawkes process is that $\lim_{z \rightarrow \infty} \frac{\lambda(z)}{z} = 0$. That is crucial to guarantee the superexponential estimates we need for the upper bound. If for a simple point process, we have $\lambda(\omega, t) \leq F(N(t, \omega))$ for some sublinear function $F(\cdot)$, we would expect the superexponential estimates still works for the upper bound. Third, it is not enough to have $\lambda(\omega, t) \leq F(N(t, \omega))$ for sublinear $F(\cdot)$ to get the full large deviation principle. The reason is that in the proof of lower bound, in particular, in Lemma 9, we need to use the fact that if $\lambda(\omega, t)$ has memory, the memory will decay to zero eventually over time. For nonlinear Hawkes process, this is guaranteed by the assumption that $\int_0^{\infty} h(t) dt < \infty$, which is crucial in the proof of Lemma 9. Indeed for any simple point process P , if you want to define P^{ω^-} , the probability measure conditional on the past history ω^- , to make sense of it, you have to have some regularities to ensure that the memory of the history will decay to zero eventually over time. From this respective, nonlinear Hawkes process form a rich and ideal class for which the process-level large deviation principle holds.

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