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Precise asymptotics for products of sums and U-statistics

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Abstract. Let $\{X, X_i, i \ge 1\}$ be a sequence of independent and identically distributed positive random variables with $E(X) = \mu > 0$, $\mathrm{Var}(X) < \infty$. Put $S_n = \sum_{i=1}^n X_i$ and let $g(x)$ be a positive and differentiable function defined on $(0, +\infty)$ satisfying some mild conditions. We prove that, for any $s > 1$,

$$\lim_{\varepsilon \to 0} \varepsilon^{1/s} \sum_{n=1}^{\infty} g'(n) P\left\{\left|\log\left(\prod_{j=1}^{n} \frac{S_j}{j\mu}\right)\right| \ge \varepsilon \sqrt{n} g^s(n)\right\} = E|N|^{1/s},$$

where N is a standard normal random variable. This result was also extended to product of U-statistics.

1 Introduction

Let $\{X, X_i, i \ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.), $S_n = \sum_{i=1}^n X_i$, $n \ge 1$. Many authors discussed the precise rate and limit value of $\sum_{n=1}^{\infty} \varphi(n, \varepsilon) P\{|S_n| \ge \varepsilon g(n)\}$ as $\varepsilon \downarrow a$, $a \ge 0$, where $\varphi(n, \varepsilon)$ and $g(n)$ are positive functions defined on $[0, \infty)$. We call $\varphi(\cdot, \cdot)$ and $g(\cdot)$ weight function and boundary function, respectively. The first result was due to Heyde (1975), who proved that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P\{|S_n| \ge \varepsilon n\} = EX^2,$$

if $EX = 0$ and $EX^2 < \infty$. Later, Gut and Spăataru (2000a, 2000b) studied this type of asymptotic result in the Baum–Katz (1965) and Davis (1968) law of large numbers and in the law of the iterated logarithm and called it precise asymptotics. The following precise asymptotics in the Davis (1968) law of large numbers are cited from Gut and Spătaru (2000a). During the entire text, N denotes a standard normal random variable.

Theorem A. *Suppose that $EX = 0$ and $EX^2 = \sigma^2 < \infty$, then for $1 \le p < \delta < 2$ we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2(\delta-p)/(2-p)} \sum_{n \ge 1} n^{\delta/p-2} P\{|S_n| \ge \varepsilon n^{1/p}\} = \frac{p\sigma^{2(\delta-p)/(2-p)}}{\delta - p} E|N|^{2(\delta-p)/(2-p)}.$$

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Theorem B. *Suppose that* $EX = 0$ *and* $EX^2 = \sigma^2 < \infty$, *then for* $0 \le \delta \le 1$ *we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2\delta+2} \sum_{n\ge 1} n^{(\log n)^2/n} P\{|S_n| \ge \varepsilon \sqrt{n \log n}\} = \frac{\sigma^{2\delta+2}}{\delta+1} E|N|^{(2\delta+2)}.$$

Recently, Cheng and Wang (2005) extended Theorems A and B to the more general case when the underling distribution is in the domain of attraction of a stable law with index in (0, 2]. They also discussed the general weight functions and boundary functions. Based on the result of Gut (2002), Wang and Yang (2003) studied more general weight functions and boundary functions of this type of precise asymptotics for record times (cf. Gut (2002) for the definition).

In this paper, the aim is to extend the above theorems to product of partial sums and U-statistics, and to get the corresponding results for more general weight functions and boundary functions. As direct conclusions, several similar results to Theorems A and B were obtained. Before providing the main results, let us review the history of the product of partial sums and U-statistics.

While considering limiting properties of sums of records (cf. Arnold and Villasenor (1998) for the definition), Arnold and Villasenr (1998) obtained the following version of the central limit theorem (CLT) for a sequence $\{X, X_i, i \ge 1\}$ of i.i.d. exponential random variables with the mean equal to 1:

$$\left(\frac{\prod_{k=1}^n S_k}{n!}\right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}N}. \tag{1.1}$$

Later Rempala and Wesoaowski (2002) extended such a CLT to general i.i.d. positive r.v. and obtained the following result.

Theorem C. *Let* $\{X, X_i, i \ge 1\}$ *be i.i.d. positive random variables with* $E(X_1) = \mu > 0$ *and* $\mathrm{Var}(X_1) = \sigma^2 < \infty$. *Denote by* $S_n = \sum_{i=1}^n X_i$ *and* $\gamma = \sigma/\mu$ *the coefficient of variation. Then*

$$\left(\frac{\prod_{k=1}^n S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{d} e^{\sqrt{2}N}. \tag{1.2}$$

This result was extended by Qi (2003), Lu and Qi (2004) to a general case when the underling distribution is in the domain of attraction of a stable law with index in Arnold and Villasenor (1998) and Baum and Katz (1965). For more general result, we refer to Rempala and Wesolowski (2005), Zhang and Huang (2007), Matula and Stępien (2008, 2009).

However, this type of CLT also can be extended to U-statistics. Corresponding to a symmetric kernel function h (cf. Serfling (1980) for the definition), the U-statistic is defined as follows

$$U_n = \frac{1}{\binom{n}{m}} \sum_{1\le i_1 < \cdots < i_m \le n} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}), \tag{1.3}$$

where the summation is over all distinct combination of m elements from $\{1, \ldots, n\}$, the (X_i) are i.i.d. positive random variables.

Denote $h_1(x_1) = Eh(x_1, X_2, \ldots, X_m)$ and $\varsigma_1 = \mathrm{Var}(h_1(X_1))$. It is a known result (see, e.g., Serfling (1980), p. 192) that if $Eh^2(X_1, X_2, \ldots, X_m) < \infty$ and $\varsigma_1 > 0$, then

$$n^{1/2}(U_n - \mu') \xrightarrow{d} N,$$

where $\mu' = E(h(X_1, X_2, \ldots, X_m))$.

Rempala and Wesoaowski (2002) extended Theorem C to U-statistics and obtained the following theorem.

Theorem D. *Let U_n be a statistics given by* (1.3) *and denote by $\gamma' = \sqrt{\varsigma_1}/\mu'$ the coefficient of variation. Assume that $P(h(X_1, \ldots, X_m) > 0) = 1$, $\varsigma_1 > 0$. Then*

$$\left(\prod_{k=m}^{n} \frac{U_k}{\mu'}\right)^{1/(m\gamma'\sqrt{n})} \xrightarrow{d} e^{\sqrt{2}N}.$$

In this paper, based on Theorems C and D, the precise asymptotics for products of sums and U-statistics will be considered. The paper is organized as follows. Section 2 displays the two main results, Sections 3 and 4 prove the two main results, respectively.

2 Main results

The main results read as follows.

Theorem 2.1. *Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. positive random variables with $E(X) = \mu > 0$, $\mathrm{Var}(X) = \sigma^2 < \infty$. Put $S_n = \sum_{i=1}^{n} X_i$ and let $g(x)$ be a positive and differentiable function defined on $(0, +\infty)$. Suppose that the following conditions are satisfied*:

(A1) *$g(x) \uparrow \infty$, as $x \to \infty$;*
(A2) *$g'(x)$ is monotone on $[1, +\infty)$;*
(A3) *if is $g'(x)$ is monotone nondecreasing, we assume that $\lim_{x\to\infty} \frac{g'(x+1)}{g'(x)} = 1$. Then for any $s > 1$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/s} \sum_{n=1}^{\infty} g'(n) P\left\{\left|\log\left(\prod_{j=1}^{n} \frac{S_j}{j\mu}\right)\right| \geq \varepsilon\sqrt{n} g^s(n)\right\} = E|N|^{1/s}. \tag{2.1}$$

Remark 2.1. The conditions on $g(x)$ were first introduced by Cheng and Wang (2004). More general conditions on weighted function and boundary function, one can find in Wang and Yang (2003).

In Theorem 2.1, let $g(x) = x^{(\delta-p)/(2p)}$, $s = \frac{2-p}{\delta-p}$, where $0 < p < \delta < 2$, we have

Corollary 2.1. *For* $0 < p < \delta < 2$, *then*

$$\lim_{\varepsilon\to 0}\varepsilon^{(\delta-p)/(2-p)}\sum_{n=1}^{\infty}n^{(\delta-3p)/(2p)}P\left\{\left|\log\left(\prod_{j=1}^{n}\frac{S_j}{j\mu}\right)\right|\geq \varepsilon n^{1/p}\right\}=E|N|^{(\delta-p)/(2-p)}.$$

In Theorem 2.1, let $g(x)=(\log x)^{\delta+1}$, $s=\frac{1}{2(\delta+1)}$, where $-1<\delta<-\frac{1}{2}$, we have

Corollary 2.2. *For* $-1<\delta<-\frac{1}{2}$, *then*

$$\lim_{\varepsilon\to 0}\varepsilon^{2\delta+2}\sum_{n=1}^{\infty}\frac{(\log n)^{\delta}}{n}P\left\{\left|\log\left(\prod_{j=1}^{n}\frac{S_j}{j\mu}\right)\right|\geq \varepsilon\sqrt{n\log n}\right\}=\frac{E|N|^{2\delta+2}}{\delta+1}.$$

In Theorem 2.1, let $g(x)=(\log\log x)^{\delta+1}$, $s=\frac{1}{2(\delta+1)}$, where $-1<\delta<-\frac{1}{2}$, we have

Corollary 2.3. *For* $-1<\delta<-\frac{1}{2}$, *then*

$$\lim_{\varepsilon\to 0}\varepsilon^{2\delta+2}\sum_{n=1}^{\infty}\frac{(\log\log n)^{\delta}}{n\log n}P\left\{\left|\log\left(\prod_{j=1}^{n}\frac{S_j}{j\mu}\right)\right|\geq \varepsilon\sqrt{n\log\log n}\right\}=\frac{E|N|^{2\delta+2}}{\delta+1}.$$

The result of Theorem 2.1 can be extended to U-statistics as follows.

Theorem 2.2. *Let* U_n *be a statistics given by* (1.3). *Assume that* $P(h(X_1,\ldots,X_m)>0)=1$, $\varsigma_1>0$. *Let* $g(x)$ *be a positive and differentiable function defined on* $(0,+\infty)$. *Suppose that the conditions* (A1), (A2) *and* (A3) *are satisfied. Then for any* $s>1$, *we have*

$$\lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n=1}^{\infty}g'(n)P\left\{\left|\log\left(\prod_{j=m}^{n}\frac{U_j}{\mu'}\right)\right|\geq \varepsilon m\sqrt{2n}g^{s}(n)\right\}=E|N|^{1/s}. \tag{2.2}$$

3 Proof of Theorem 2.1

In this section, the following notations will be used.

Let $b_{k,n}=\sum_{i=k}^{n}1/i$ and $s_{k,n}=(\sum_{i=1}^{k}b_{i,n}^{2})^{1/2}$ for $k\leq n$ with $b_{k,n}=0$ if $k>n$. Furthermore, let $Z_i=\frac{X_i-\mu}{\sigma}$, $i=1,2,\ldots$. Define a triangular array $Y_{1,n},Y_{2,n},\ldots,Y_{n,n}$ as $Y_{k,n}=b_{k,n}Z_k$ and set $S_{k,n}=\sum_{i=1}^{k}Y_{i,n}$ for $1\leq k\leq n$. From Gonechigdanzan and Rempala (2006), we have

$$s_{n,n}^{2}=2n-b_{1,n},\ S_{n,n}/s_{n,n}\xrightarrow{d}N,\qquad \text{as } n\to\infty. \tag{3.1}$$

Setting $T_n = \frac{1}{\gamma}\log(\prod_{j=1}^n \frac{S_j}{\mu j})$ and notice that $\log(1+x) = x + \frac{\theta}{2}x^2$ as $|x| < 1$, where $\theta \in (-1, 0)$, then

$$\begin{aligned} T_n &= \frac{1}{\gamma}\sum_{j=1}^n \log\frac{S_j}{\mu j} \\ &= \frac{1}{\gamma}\sum_{j=1}^n \left(\frac{S_j}{\mu j} - 1\right) + \frac{1}{\gamma}\sum_{j=1}^n \frac{\theta_j}{2}\left(\frac{S_j}{\mu j} - 1\right)^2 \\ &= S_{n,n} + \frac{1}{\gamma}\sum_{j=1}^n \frac{\theta_j}{2}\left(\frac{S_j}{\mu j} - 1\right)^2 \\ &=: S_{n,n} + R_n. \end{aligned}$$

In order to prove the theorem, we need to know the convergence rate of the remainder term R_n. As usual, $a_n \ll b_n$ means $\limsup_{n\to\infty} |a_n/b_n| < +\infty$.

Lemma 3.1. *Under the conditions of Theorem 2.1, we have $E|R_n| \ll \log n$.*

Proof.

$$\begin{aligned} E|R_n| &= E\left|\frac{1}{\gamma}\sum_{j=1}^n \frac{\theta_j}{2}\left(\frac{S_j}{\mu j} - 1\right)^2\right| \\ &\ll \sum_{j=1}^n E\left(\frac{S_j}{\mu j} - 1\right)^2 \qquad (3.2) \\ &\ll \sum_{j=1}^n \frac{1}{j} \ll \log n. \end{aligned}$$

□

The proof of Theorem 2.1 is via the following four propositions. Let C denote positive constants whose values may vary from place to place.

Proposition 3.1. *For any $s > 1$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/s} \sum_{n=1}^\infty g'(n) P\{|N| \geq \varepsilon g^s(n)\} = E|N|^{1/s}. \qquad (3.3)$$

Proof. Let $y = \varepsilon g^s(x)$, for any constant $C \in (0, \infty)$ we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{1/s} \int_C^\infty g'(x) P\{|N| \geq \varepsilon g^s(x)\}\,dx \\ &\quad= \lim_{\varepsilon\to 0} \frac{1}{s}\int_{\varepsilon g^s(C)}^\infty y^{1/s-1} P\{|N| \geq y\}\,dy \qquad (3.4) \end{aligned}$$

$$= \frac{1}{s}\int_0^\infty y^{1/s-1} P\{|N| \geq y\}\,dy$$
$$= E|N|^{1/s}.$$

Using (A2) and (A3), we can prove that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{1/s}\int_1^\infty g'(x)P\{|N|\geq \varepsilon g^s(x)\}\,dx \\ &\quad \leq \lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n=1}^\infty g'(n)P\{|N|\geq \varepsilon g^s(n)\} \\ &\quad \leq \lim_{\varepsilon\to 0}\varepsilon^{1/s}\int_0^\infty g'(x)P\{|N|\geq \varepsilon g^s(x)\}\,dx. \end{aligned} \tag{3.5}$$

In fact, if $g'(x)$ is monotone nonincreasing, (3.5) holds obviously; if $g'(x)$ is monotone nondecreasing, by $\lim_{x\to\infty}\frac{g(x+1)}{g(x)} = 1$, for any $\delta > 0$ there exists a positive integer N_0 such that $x > N_0$, we have $g'(x) < (1+\delta)g'(x-1)$, therefore

$$\begin{aligned} &\lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n=1}^\infty g'(n)P\{|N|\geq \varepsilon g^s(n)\} \\ &\quad = \lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n=N_0+1}^\infty g'(n)P\{|N|\geq \varepsilon g^s(n)\} \\ &\quad \leq (1+\delta)\lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n=N_0+1}^\infty g'(n-1)P\{|N|\geq \varepsilon g^s(n)\} \\ &\quad \leq (1+\delta)\lim_{\varepsilon\to 0}\varepsilon^{1/s}\int_{N_0}^\infty g'(x)P\{|N|\geq \varepsilon g^s(x)\}\,dx. \end{aligned}$$

Let $\delta \downarrow 0$, we get the right inequality in (3.5), similarly, the left one in (3.5) is valid. And (3.3) follows by (3.4) and (3.5). □

In the following propositions, we will denote $a(\varepsilon) = g^{-1}(M\varepsilon^{-1/s})$, $M > 0$, and $g^{-1}(x)$ is the inverse function of $g(x)$.

Proposition 3.2. *For any $M > 0$, we have*

$$\lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n\leq a(\varepsilon)} g'(n)\big|P\big\{|T_n|\geq \varepsilon\sqrt{2n}g^s(n)\big\} - P\{|N|\geq \varepsilon g^s(n)\}\big| = 0. \tag{3.6}$$

Proof. Let

$$\Delta_n = \sup_x\big|P\big(|T_n|\geq \sqrt{2n}x\big) - P(|N|\geq x)\big|.$$

By (1.2), we have

$$\frac{T_n}{\sqrt{2n}} \xrightarrow{d} N,$$

then

$$\frac{|T_n|}{\sqrt{2n}} \xrightarrow{d} |N|.$$

By the continuity of $\Psi(x) = P(|N| \geq x)$, we have $\lim_{n\to\infty} \Delta_n = 0$. Note that

$$\sum_{n \leq a(\varepsilon)} g'(n) \approx \int_1^{a(\varepsilon)} g'(x)\,dx \approx g(a(\varepsilon)) = M\varepsilon^{-1/s},$$

where $A(x) \approx B(x)$ means that there exist positive constants $C_1 < C_2$ such that $C_1 A(x) \leq B(x) \leq C_2 A(x)$. Thus, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{1/s} \sum_{n \leq a(\varepsilon)} g'(n) \big| P\{|T_n| \geq \varepsilon\sqrt{2n} g^s(n)\} - P\{|N| \geq \varepsilon g^s(n)\}\big| \\
&\quad \leq \lim_{\varepsilon\to 0} \varepsilon^{1/s} \sum_{n \leq a(\varepsilon)} g'(n)\Delta_n = 0.
\end{aligned}$$

□

Proposition 3.3. *For any $s > 1$, we have*

$$\lim_{M\to\infty} \lim_{\varepsilon\downarrow 0} \varepsilon^{1/s} \sum_{n > a(\varepsilon)} g'(n) P\{|N| \geq \varepsilon g^s(n)\} = 0. \tag{3.7}$$

Proof. At first, by a similar argument as in that of Proposition 3.1, we have the following relations between the integral and the series.

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{1/s} \sum_{n \geq a(\varepsilon)+1} g'(n) P\{|N| \geq \varepsilon g^s(n)\} \\
&\quad \leq \lim_{\varepsilon\to 0} C\varepsilon^{1/s} \int_{a(\varepsilon)}^{\infty} g'(x) P\{|N| \geq \varepsilon g^s(x)\}\,dx.
\end{aligned} \tag{3.8}$$

By (3.8) and Markov inequality, we have

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \varepsilon^{1/s} \sum_{n > a(\varepsilon)} g'(n) P\{|N| \geq \varepsilon g^s(n)\} &\leq \lim_{\varepsilon\downarrow 0} C\varepsilon^{1/s} \int_{a(\varepsilon)}^{\infty} g'(x) P\{|N| \geq \varepsilon g^s(x)\}\,dx \\
&\leq \lim_{\varepsilon\downarrow 0} C\varepsilon^{1/s} \int_{a(\varepsilon)}^{\infty} g'(x) \varepsilon^{-2} g^{-2s}(x)\,dx \\
&= \lim_{\varepsilon\downarrow 0} C\varepsilon^{1/s-2} g^{-2s+1}(a(\varepsilon)) \\
&= CM^{-2s+1}.
\end{aligned}$$

The result follows by letting $M \to \infty$. □

Proposition 3.4. *For any $s > 1$, we have*

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|T_n|\ge\varepsilon\sqrt{2n}g^s(n)\}=0. \tag{3.9}$$

Proof. Note that $T_n = S_{n,n} + R_n$, we have

$$P\{|T_n|\ge\varepsilon\sqrt{2n}g^s(n)\}\le P\left\{|S_{n,n}|\ge\frac{\varepsilon}{2}\sqrt{2n}g^s(n)\right\}+P\left\{|R_n|\ge\frac{\varepsilon}{2}\sqrt{2n}g^s(n)\right\}.$$

Therefore in order to show (3.9), it suffices to show that

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|S_{n,n}|\ge\varepsilon\sqrt{2n}g^s(n)\}=0 \tag{3.10}$$

and

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|R_n|\ge\varepsilon\sqrt{2n}g^s(n)\}=0. \tag{3.11}$$

By (3.8), (3.1) and Markov inequality, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|S_{n,n}|\ge\varepsilon\sqrt{2n}g^s(n)\}\\
&\quad\le\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)\frac{2n-b_1}{\varepsilon^2 2ng^{2s}(n)}\\
&\quad\le\lim_{\varepsilon\downarrow 0}C\varepsilon^{1/s}\int_{a(\varepsilon)}^{\infty} g'(x)\varepsilon^{-2}g^{-2s}(x)\,dx\\
&\quad=\lim_{\varepsilon\downarrow 0}C\varepsilon^{1/s-2}g^{-2s+1}(a(\varepsilon))\\
&\quad=CM^{-2s+1},
\end{aligned}$$

and (3.10) follows by letting $M\to\infty$. Similarly, by (3.8), (3.2) and Markov inequality, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|R_n|\ge\varepsilon\sqrt{2n}g^s(n)\}\\
&\quad\le\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)\frac{\log n}{\varepsilon\sqrt{2n}g^s(n)}\\
&\quad\le\lim_{\varepsilon\downarrow 0}C\varepsilon^{1/s}\int_{a(\varepsilon)}^{\infty} g'(x)\varepsilon^{-1}x^{-1/2}g^{-s}(x)\log x\,dx\\
&\quad\le\lim_{\varepsilon\downarrow 0}C\varepsilon^{1/s-1}g^{-s+1}(a(\varepsilon))\\
&\quad=CM^{-s+1},
\end{aligned}$$

and (3.11) follows by letting $M\to\infty$. □

4 Proof of Theorem 2.2

Define the projection of the U-statistic U_n as

$$\widehat{U}_n = \frac{m}{n}\sum_{i=1}^{n} h_1(X_i) - (m-1)\mu'.$$

Note that it is exactly a sum of an i.i.d. random variables. Put $J_n = U_n - \widehat{U}_n$. It is known (Serfling (1980), p. 189) that if $Eh^2(X_1,\dots,X_m) < \infty$,

$$EJ_n^2 = O(n^{-2}) \qquad \text{as } n \to \infty. \tag{4.1}$$

In this section, we will use the notations defined in the beginning of Section 3. We only change the definition of Z_i to $Z_i = \frac{h_1(X_i)-\mu'}{\sqrt{\varsigma_1}}, i = 1, 2, \dots$. Similar to Section 3, we also can obtain (3.1).

Now, setting $W_n = \frac{1}{\gamma'}\sum_{j=m}^{n} \log \frac{U_j}{\mu'}$. Similarly,

$$\begin{aligned}
W_n &= \frac{1}{\gamma'}\sum_{j=m}^{n} \log \frac{U_j}{\mu'} \\
&= \frac{1}{\gamma'}\sum_{j=m}^{n}\left(\frac{U_j}{\mu'} - 1\right) + \frac{1}{\gamma'}\sum_{j=m}^{n}\frac{\theta_j}{2}\left(\frac{U_j}{\mu'} - 1\right)^2 \\
&= \frac{1}{\gamma'}\sum_{j=m}^{n}\left(\frac{\widehat{U}_j}{\mu'} - 1\right) + \frac{1}{\gamma'}\sum_{j=m}^{n}\frac{J_j}{\mu'} + \frac{1}{\gamma'}\sum_{j=m}^{n}\frac{\theta_j}{2}\left(\frac{U_j}{\mu'} - 1\right)^2 \\
&= \frac{1}{\gamma'}\sum_{j=1}^{n}\left(\frac{\sum_{k=1}^{j} h_1(X_k)}{\mu' j} - 1\right) - \frac{1}{\gamma'}\sum_{j=1}^{m-1}\left(\frac{\sum_{k=1}^{j} h_1(X_k)}{\mu' j} - 1\right) \\
&\quad + \frac{1}{\gamma'}\sum_{j=m}^{n}\frac{J_j}{\mu'} + \frac{1}{\gamma'}\sum_{j=m}^{n}\frac{\theta_j}{2}\left(\frac{U_j}{\mu'} - 1\right)^2 \\
&=: S_{n,n} - Q_{n,1} + Q_{n,2} + Q_{n,3}.
\end{aligned}$$

For $Q_{n,1}$,

$$E|Q_{n,1}|^2 = O(1). \tag{4.2}$$

For $Q_{n,2}$, By (4.1), we have

$$E|Q_{n,2}| \ll \sum_{j=m}^{n} E|J_j| \le \sum_{j=m}^{n} (E|J_j|^2)^{1/2} \ll \sum_{j=m}^{n} (j^{-2})^{1/2} \le \log n. \tag{4.3}$$

For $Q_{n,3}$, by Lemma A in Serfling (1980), p. 185, we have

$$E|Q_{n,3}| \ll \sum_{j=m}^{n} E(U_j - \mu')^2 \ll \sum_{j=m}^{n} \frac{1}{j} \ll \log n. \tag{4.4}$$

The proof of Theorem 2.2 is via Propositions 3.1, 3.3 and the following two propositions.

Proposition 4.1. *For any* $M > 0$, *we have*

$$\lim_{\varepsilon\to 0}\varepsilon^{1/s}\sum_{n\le a(\varepsilon)} g'(n)\big|P\{|W_n|\ge \varepsilon m\sqrt{2n}g^s(n)\} - P\{|N|\ge \varepsilon g^s(n)\}\big| = 0. \tag{4.5}$$

Proof. Similar to Proposition 3.2. □

Proposition 4.2. *For any* $s > 1$, *we have*

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|W_n|\ge \varepsilon m\sqrt{2n}g^s(n)\} = 0. \tag{4.6}$$

Proof. Note that $W_n = S_{n,n} - Q_{n,1} + Q_{n,2} + Q_{n,3}$, we have

$$\begin{aligned}
&P\{|W_n|\ge \varepsilon m\sqrt{2n}g^s(n)\}\\
&\quad\le P\left\{S_{n,n}\ge \frac{\varepsilon}{4}m\sqrt{2n}g^s(n)\right\} + P\left\{Q_{n,1}\ge \frac{\varepsilon}{4}m\sqrt{2n}g^s(n)\right\}\\
&\qquad + P\left\{Q_{n,2}\ge \frac{\varepsilon}{4}m\sqrt{2n}g^s(n)\right\} + P\left\{Q_{n,3}\ge \frac{\varepsilon}{4}m\sqrt{2n}g^s(n)\right\}.
\end{aligned}$$

Therefore in order to show (4.6), it suffices to show that

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|S_{n,n}|\ge \varepsilon m\sqrt{2n}g^s(n)\} = 0, \tag{4.7}$$

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|Q_{n,1}|\ge \varepsilon m\sqrt{2n}g^s(n)\} = 0, \tag{4.8}$$

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|Q_{n,2}|\ge \varepsilon m\sqrt{2n}g^s(n)\} = 0 \tag{4.9}$$

and

$$\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\varepsilon^{1/s}\sum_{n>a(\varepsilon)} g'(n)P\{|Q_{n,3}|\ge \varepsilon m\sqrt{2n}g^s(n)\} = 0. \tag{4.10}$$

Then, along the same lines as those of the proofs of Proposition 3.4, these results can be obtained by using (3.1), (4.2), (4.3) and (4.4). □

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