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STOCHASTIC GEOMETRIC WAVE EQUATIONS WITH VALUES IN COMPACT RIEMANNIAN HOMOGENEOUS SPACES

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Let M be a compact Riemannian homogeneous space (e.g., a Euclidean sphere). We prove existence of a global weak solution of the stochastic wave equation $\mathbf{D}_t \partial_t u = \sum_{k=1}^{d} \mathbf{D}_{x_k} \partial_{x_k} u + f_u(Du) + g_u(Du)\dot{W}$ in any dimension $d \geq 1$, where f and g are continuous multilinear maps, and W is a spatially homogeneous Wiener process on $\mathbb{R}^d$ with finite spectral measure. A nonstandard method of constructing weak solutions of SPDEs, that does not rely on martingale representation theorem, is employed.

1. Introduction. Wave equations subject to random perturbations and/or forcing have been a subject of deep and extensive studies in the last forty years. One of the reasons for this is that they find applications in physics, relativistic quantum mechanics or oceanography; see, for instance, Cabaña [9], Carmona and Nualart [10, 11], Chow [15], Dalang [18–20], Marcus and Mizel [36], Maslowski and Seidler [37], Millet and Morien [38], Ondreját [43, 46], Peszat and Zabczyk [51, 52], Peszat [50], Millet and Sanz-Sole [39] and references therein. All these research papers are concerned with equations whose solutions take values in Euclidean spaces. However, many theories and models in modern physics, such as harmonic gauges in general relativity, nonlinear σ-models in particle systems, electro-vacuum Einstein equations or Yang–Mills field theory, that require the solutions to take values are a Riemannian manifold; see, for instance, Ginibre and Velo [26] and Shatah and Struwe [57]. Stochastic wave equations with values in Riemannian manifolds were first studied by the authors of the present paper in [6]; see also [7], where the existence and the uniqueness of global strong solutions was proven for equations defined on the one-dimensional Minkowski space $\mathbb{R}^{1+1}$ and arbitrary target Riemannian manifold. In the present paper, we strive to obtain a global existence result for equations on general Minkowski space $\mathbb{R}^{1+d}$, $d \in \mathbb{N}$, however, for the price that the target space is a particular Riemannian manifold—a compact homogeneous space, for example, a sphere.

Let us first briefly compare our results (obtained in this paper as well as in the earlier one [6]) with those for the deterministic equations. For more details

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on the latter, we refer the reader to nice surveys on geometric wave equations by Shatah and Struwe [57] and Tataru [58]. Existence and uniqueness of global solutions is known for the wave equations for an arbitrary target manifold provided that the Minkowski space of the equation is either $\mathbb{R}^{1+1}$ or $\mathbb{R}^{1+2}$; see Ladyzhenskaya and Shubov [35], Ginibre and Velo [26], Gu [27], Shatah [56], Zhou [60], Christodoulou and Tahvildar-Zadeh [16] and Müller and Struwe [41]. In the former case, depending on the regularity of the initial conditions, the global solutions are known to exist in the weak [60], respectively, the strong, sense; see [26, 27] and [56]. In the latter case the existence of global weak solutions has been established in [16] and [41]. In the more interesting and difficult case of $\mathbb{R}^{1+d}$ with $d \geq 3$, counterexamples have been constructed (see, e.g., [13, 56] and [57]), showing that smooth solutions may explode in a finite time and that weak solutions can be nonunique. Notwithstanding, existence of global solutions can be proven for particular target manifold; for example, for compact Riemannian homogeneous spaces, see Freire [23]. The aim of the current paper is to consider a stochastic counterpart of Freire's result. In other words, we will prove existence of global solutions for the wave equation with values a compact Riemannian homogeneous space even if it is subject to particular (however quite general) random perturbations.

Toward this end, we assume that M is a *compact Riemannian homogeneous space* (see Sections 2 and 5 for more details), and we consider the initial value problem for the following stochastic wave equation:

$$\begin{aligned}\mathbf{D}_t \partial_t u = \sum_{k=1}^{d} \mathbf{D}_{x_k} \partial_{x_k} u + f(u, \partial_t u, \partial_{x_1} u, \dots, \partial_{x_d} u) \\ + g(u, \partial_t u, \partial_{x_1} u, \dots, \partial_{x_d} u) \dot{W}\end{aligned} \tag{1.1}$$

with a random initial data $(u_0, v_0) \in TM$. Here $\mathbf{D}$ is the connection on the pullback bundle $u^{-1}TM$ induced by the Riemannian connection on M; see, for example, [6] and [57]. In a simpler way (see [6]),

$$[\mathbf{D}_t \partial_t \gamma](t) = \nabla_{\partial_t \gamma(t)} (\partial_t \gamma)(t), \qquad t \in I, \tag{1.2}$$

is the *acceleration* of the curve $\gamma : I \to M$, $I \subset \mathbb{R}$, at $t \in I$. Note, however, that deep understanding of the covariant derivative $\mathbf{D}$ is not necessary for reading this paper. We will denote by $T^k M$, for $k \in \mathbb{N}$, the vector bundle over M whose fibre at $p \in M$ is equal to $(T_p M)^k$, the k-fold cartesian product of $T_p M$. The nonlinear term f (and analogously g) in equation (1.1) will be assumed to be of the following form (see Section 4):

$$\begin{aligned} f : T^{d+1} M \ni (p, v_0, \dots, v_k) \\ \mapsto f_0(p) v_0 + \sum_{k=1}^{d} f_k(p) v_k + f_{d+1}(p) \in TM,\end{aligned} \tag{1.3}$$

where f_{d+1} and g_{d+1} are continuous vector fields on M, $f_0, g_0 : M \to \mathbb{R}$ are continuous functions and $f_k, g_k : TM \to TM$, $k = 1, \dots, d$, are continuous vector bundles homomorphisms; see Definition 4.1. Finally, we assume that W is a spatially homogeneous Wiener process.

Equation (1.1) is written in a formal way, but we showed in [6] that there are various equivalent rigorous definitions of a solution to (1.1). In the present paper we are going to use the one in which, in view of the Nash isometric embedding theorem [42], M is assumed to be isometrically embedded into a certain euclidean space $\mathbb{R}^n$ (and so we can identify M with its image). Hence, M is assumed to be a submanifold in $\mathbb{R}^n$, and in this case, we study, instead of (1.1), the following classical second order SPDE:

$$\partial_{tt} u = \Delta u + \mathbf{S}_u(\partial_t u, \partial_t u) - \sum_{k=1}^{d} \mathbf{S}_u(u_{x_k}, u_{x_k}) + f_u(Du) + g_u(Du)\dot{W}, \tag{1.4}$$

where $\mathbf{S}$ is the second fundamental form of the submanifold $M \subseteq \mathbb{R}^n$.

We could summarize our proof in the following way. We introduce an approximation of problem (1.4) via penalization. We find a sufficiently large space in which the laws of the approximated sequence are tight. This space has also to be small enough so that, after using the Skorokhod embedding theorem, the convergence in that space is strong enough for the sequences of approximated solutions, as well as some auxiliary processes, to be convergent. Finally, we use the symmetry of the target manifold to identify the limit with a solution to problem (1.4). Let us also point out that our proof of the main theorem is based on a method (recently introduced by the authors) of constructing weak solutions of SPDEs, that does not rely on any kind of martingale representation theorem.

2. Notation and conventions. We will denote by $B_R(a)$, for $a \in \mathbb{R}^d$ and $R > 0$, the open ball in $\mathbb{R}^d$ with center at a, and we put $B_R = B_R(0)$. If X is a normed vector space, then by B_R^X we will denote the ball in X centered at 0 of radius R. Now we will list notations used throughout the whole paper.

- $\mathbb{N} = \{0, 1, \dots\}$ denotes the set of natural numbers, $\mathbb{R}_+ = [0, \infty)$, Leb denotes the Lebesgue measure, $L^p = L^p(\mathbb{R}^d; \mathbb{R}^n)$, $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^n)$, L^p_{loc} is a metrizable topological vector space equipped with a natural countable family of seminorms $(p_j)_{j\in\mathbb{N}}$ defined by

$$p_j(u) := \|u\|_{L^p(B_j)}, \qquad u \in L^p_{\mathrm{loc}}, j \in \mathbb{N}. \tag{2.1}$$

- $W^{k,p}_{\mathrm{loc}} = W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^n)$, for $p \in [1, \infty]$ and $k \in \mathbb{N}$, is the space of all elements $u \in L^p_{\mathrm{loc}}$ whose weak derivatives up to order k belong to L^p_{loc}. $W^{k,p}_{\mathrm{loc}}$ is a metrizable topological vector space equipped with a natural countable family of seminorms $(p_j)_{j\in\mathbb{N}}$,

$$p_j(u) := \|u\|_{W^{k,p}(B_j)}, \qquad u \in W^{k,p}_{\mathrm{loc}}, j \in \mathbb{N}. \tag{2.2}$$

The spaces $W^{k,2}$ and $W^{k,2}_{\mathrm{loc}}$ are denoted by H^k and H^k_{loc}, respectively.
- $\mathscr{H}_O = H^1(O;\mathbb{R}^n) \oplus L^2(O;\mathbb{R}^n)$ if O is an open subset of $\mathbb{R}^d$ and, for $R > 0$, $\mathscr{H}_R = H_{B_R}$.
- $\mathscr{H} = H^1(\mathbb{R}^d;\mathbb{R}^n) \oplus L^2(\mathbb{R}^d;\mathbb{R}^n)$ and $\mathscr{H}_{\mathrm{loc}} = H^1_{\mathrm{loc}}(\mathbb{R}^d) \oplus L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$.
- $\mathscr{D} = \mathscr{D}(\mathbb{R}^d;\mathbb{R}^n)$ is the class of all compactly supported C^∞-class functions $\varphi:\mathbb{R}^d \to \mathbb{R}^n$.
- Whenever $\mathcal{Y}$ denotes a certain class of functions defined on $\mathbb{R}^d$, by $\mathcal{Y}_{\mathrm{comp}}$ we will denote the space of those elements of $\mathcal{Y}$ whose support is a compact subset of $\mathbb{R}^d$. For instance, $L^2_{\mathrm{comp}}(\mathbb{R}^d)$ and $H^k_{\mathrm{comp}}(\mathbb{R}^d)$.
- If Z is a topological space equipped with a countable system of pseudometrics $(\rho_m)_{m\in\mathbb{N}}$, then, without further reference, we will assume that the topology of Z is metrized by a metric

$$\rho(a,b) = \sum_{m=0}^{\infty} \frac{1}{2^m} \min\{1, \rho_m(a,b)\}, \qquad a,b \in Z. \tag{2.3}$$

- By $\mathscr{T}_2(X,Y)$ we will denote the class of Hilbert–Schmidt operators from a separable Hilbert space X to Y. By $\mathscr{L}(X,Y)$ we will denote the space of all linear continuous operators from a topological vector space X to Y; see [54], Chapter I. Both these spaces will be equipped with the strong σ-algebra, that is, the σ-algebra generated by the family of maps $\mathscr{T}_2(X,Y) \ni B \mapsto Bx \in Y$ or $\mathscr{L}(X,Y) \ni B \mapsto Bx \in Y$, $x \in X$.
- If (X,ρ) is a metric space, then we denote by $C(\mathbb{R}_+;X)$ the space of continuous functions $f:\mathbb{R}_+ \to X$. The space $C(\mathbb{R}_+;X)$ in endowed with the metric ρ_C, defined by the following formula:

$$\rho_C(f,g) = \sum_{m=1}^{\infty} 2^{-m} \min\Big\{1, \sup_{t\in[0,m]} \rho(f(t),g(t))\Big\}, \qquad f,g \in C(\mathbb{R}_+,X). \tag{2.4}$$

- If X is a locally convex space, then by $C_w(\mathbb{R}_+;X)$ we denote the space of all weakly continuous functions $f:\mathbb{R}_+ \to X$, endowed with the locally convex topology, generated by the a family $\|\cdot\|_{m,\varphi}$ of pseudonorms, defined by

$$\|f\|_{m,\varphi} = \sup_{t\in[0,m]} |\varphi(f(t))|, \qquad m \in \mathbb{N}, \varphi \in X^*. \tag{2.5}$$

- By π_R we will denote various restriction maps to the ball B_R, for example, $\pi_R : L^2_{\mathrm{loc}} \ni v \mapsto v|_{B_R} \in L^2(B_R)$ or $\pi_R : \mathscr{H}_{\mathrm{loc}} \ni z \mapsto z|_{B_R} \in \mathscr{H}_R$. In the danger of ambiguity we will make this precise.
- ζ is a smooth symmetric density on $\mathbb{R}^d$ supported in the unit ball. If $m \in \mathbb{R}_+$, then we put $\zeta_m(\cdot) = m^d\zeta(m\cdot)$. The sequence $(\zeta_m)_{m=1}^\infty$ is called *an approximation of identity*.
- By $\mathscr{S}$ (see, e.g., [54]) we will denote the Schwartz space of $\mathbb{R}$-valued rapidly decreasing C^∞-class functions on $\mathbb{R}^d$. By $\mathscr{S}'$ we will denote the space of tempered distributions on $\mathbb{R}^d$, the dual of the space $\mathscr{S}$. The Fourier transform, in

the cases of $\mathscr{S}$, $\mathscr{S}'$ as well as L^2, we will denote by $\widehat{\ }$. For example, $\widehat{\varphi} \in \mathscr{S}$ will denote the Fourier transform of a function $\varphi \in \mathscr{S}$.
- Given a positive measure μ on $\mathbb{R}^d$, we will denote by $L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu)$ the subspace of $L^2(\mathbb{R}^d, \mu; \mathbb{C})$ consisting of all ψ such that $\psi = \psi_{(\mathrm{s})}$, where $\psi_{(\mathrm{s})}(\cdot) = \overline{\psi(-\cdot)}$.

As mentioned in the Introduction, throughout the whole paper we will assume that M is a compact Riemannian homogeneous space; see Section 5 for details. We put:

- $\mathscr{H}_{\mathrm{loc}}(M) = \{(u, v) \in \mathscr{H}_{\mathrm{loc}} : v(x) \in T_{u(x)}M \text{ for a.e. } x \in M\}$. The strong, respectively, weak, topologies on $\mathscr{H}_{\mathrm{loc}}(M)$, are by definition the traces of the strong (resp., weak) topologies on $\mathscr{H}_{\mathrm{loc}}$. In particular, a function $u : [0, \infty) \to \mathscr{H}_{\mathrm{loc}}(M)$ is weakly continuous if and only if u is weakly continuous viewed as a $\mathscr{H}_{\mathrm{loc}}$-valued function.

3. The Wiener process. Given a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ is a filtration, an $\mathscr{S}'$-valued process $W = (W_t)_{t\geq 0}$ is called a spatially homogeneous Wiener process with a spectral measure μ which, throughout the paper we always assume to be positive, symmetric and to satisfy $\mu(\mathbb{R}^d) < \infty$, if and only if the following three conditions are satisfied:

- $W\varphi := (W_t\varphi)_{t\geq 0}$ is a real $\mathbb{F}$-Wiener process, for every $\varphi \in \mathscr{S}$;
- $W_t(a\varphi + \psi) = aW_t(\varphi) + W_t(\psi)$ almost surely for all $a \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $\varphi, \psi \in \mathscr{S}$;
- $\mathbb{E}\{W_t\varphi_1 W_t\varphi_2\} = t\langle\widehat{\varphi}_1, \widehat{\varphi}_2\rangle_{L^2(\mu)}$ for all $t \geq 0$ and $\varphi_1, \varphi_2 \in \mathscr{S}$, where $L^2(\mu) = L^2(\mathbb{R}^d, \mu; \mathbb{C})$.

REMARK 3.1. The reader is referred to the works by Peszat and Zabczyk [51, 52] and Brzeźniak and Peszat [8] for further details on spatially homogeneous Wiener processes.

Let us denote by $H_\mu \subseteq \mathscr{S}'$ the reproducing kernel Hilbert space of the $\mathscr{S}'$-valued random vector $W(1)$; see, for example, [17]. Then W is an H_μ-cylindrical Wiener process. Moreover, see [8] and [51]; then the following result identifying the space H_μ is known.

PROPOSITION 3.2.

$$H_\mu = \{\widehat{\psi\mu} : \psi \in L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu)\},$$

$$\langle\widehat{\psi\mu}, \widehat{\varphi\mu}\rangle_{H_\mu} = \int_{\mathbb{R}^d} \psi(x)\overline{\varphi(x)}\, d\mu(x), \qquad \psi, \varphi \in L^2_{(\mathrm{s})}(\mathbb{R}^d, \mu).$$

See [43] for a proof of the following lemma that states that under some assumptions, H_μ is a function space and that multiplication operators are Hilbert–Schmidt from H_μ to L^2.

LEMMA 3.3. *Assume that $\mu(\mathbb{R}^d) < \infty$. Then the reproducing kernel Hilbert space H_μ is continuously embedded in the space $C_b(\mathbb{R}^d)$, and for any $g \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ and a Borel set $D \subseteq \mathbb{R}^d$, the multiplication operator $m_g = \{H_\mu \ni \xi \mapsto g \cdot \xi \in L^2(D)\}$ is Hilbert–Schmidt. Moreover, there exists a universal constant c_μ such that*

$$\|m_g\|_{\mathscr{T}_2(H_\mu, L^2(D))} \le c_\mu \|g\|_{L^2(D)}. \tag{3.1}$$

4. The main result. Roughly speaking our main result states that for each reasonable initial data equation (1.4) has a weak solution both in the PDE and in the Stochastic Analysis senses. By a weak solution to equation (1.4) in the PDE sense, we mean a process that satisfies a variational form identity with a certain class of test functions. By a weak solution in the Stochastic Analysis sense to equation (1.4), we mean a stochastic basis, a spatially homogeneous Wiener process (defined on that stochastic basis) and a continuous adapted process z such that (1.4) is satisfied; see the formulation of Theorem 4.4 below. We recall that $\mathbf{S}$ is the second fundamental tensor/form of the isometric embedding $M \subseteq \mathbb{R}^n$.

DEFINITION 4.1. A continuous map $\lambda : TM \to TM$ is a vector bundles homomorphisms if and only if, for every $p \in M$, the map $\lambda_p : T_pM \to T_pM$ is linear.

In our two previous papers [6, 7] we found two equivalent definitions of a solution to the stochastic geometric wave equation (1.1): intrinsic and extrinsic. Contrary to those papers, in the present article, we only deal with the extrinsic solutions (as they refer to the ambient space $\mathbb{R}^n$). Hence, since we do not introduce (neither use) an alternative notion of an intrinsic solution, we will not use the adjective "extrinsic." We will discuss these issues in a subsequent publication.

ASSUMPTION 4.2. We assume that f_0, g_0 are continuous functions on M, $f_1, \dots, f_d$, $g_1, \dots, g_d$ are continuous vector bundles homomorphisms and f_{d+1}, g_{d+1} are continuous vector fields on M. For $b \in \{f, g\}$, we set

$$\begin{aligned} & b(p, \xi_0, \dots, \xi_d) = b_0(p)\xi_0 + \sum_{k=1}^{d} b_k(p)\xi_k + b_{d+1}(p), \qquad p \in M, \\ & \qquad (\xi_i)_{i=0}^{d} \in [T_pM]^{d+1}. \end{aligned} \tag{4.1}$$

DEFINITION 4.3. Suppose that Θ is a Borel probability measure on $\mathscr{H}_{\mathrm{loc}}(M)$. A system $\mathfrak{U} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, z)$ consisting of (1) a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, (2) a spatially homogeneous Wiener process W and (3) an adapted, weakly-continuous $\mathscr{H}_{\mathrm{loc}}(M)$-valued process $z = (u, v)$ is called a weak solution to equation (1.1) if and only if for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the following equalities holds $\mathbb{P}$-a.s., for all $t \geq 0$:

$$\langle u(t), \varphi\rangle = \langle u(0), \varphi\rangle + \int_0^t \langle v(s), \varphi\rangle \, ds, \tag{4.2}$$

$$\begin{aligned} \langle v(t), \varphi\rangle = \langle v(0), \varphi\rangle &+ \int_0^t \langle \mathbf{S}_{u(s)}(v(s), v(s)), \varphi\rangle \\ &+ \int_0^t \langle f(z(s), \nabla u(s)), \varphi\rangle \, ds + \int_0^t \langle u(s), \Delta\varphi\rangle \, ds \\ &- \sum_{k=1}^d \int_0^t \langle \mathbf{S}_{u(s)}(\partial_{x_k} u(s), \partial_{x_k} u(s)), \varphi\rangle \\ &+ \int_0^t \langle g(z(s), \nabla u(s)) \, dW, \varphi\rangle, \end{aligned} \tag{4.3}$$

where we assume that all integrals above are convergent, and we use notation (4.1).

We will say that the system $\mathfrak{U}$ is a weak solution to the problem (1.1) with the initial data Θ, if and only if it is a weak solution to equation (1.1), and

$$\text{the law of } z(0) \text{ is equal to } \Theta. \tag{4.4}$$

THEOREM 4.4. *Assume that μ is a positive, symmetric Borel measure on $\mathbb{R}^d$ such that $\mu(\mathbb{R}^d) < \infty$. Assume that M is a compact Riemannian homogeneous space. Assume that Θ is a Borel probability measure on $\mathscr{H}_{\mathrm{loc}}(M)$ and that the coefficients f and g satisfy Assumption 4.2. Then there exists a weak solution to problem (1.1) with the initial data Θ.*

REMARK 4.5. We do not claim uniqueness of a solution in Theorem 4.4; cf. Freire [23] where uniqueness of solutions is not known in the deterministic case either.

REMARK 4.6. Note that the solution from Theorem 4.4 satisfies only $u(t, \omega, \cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$, $t \geq 0$, $\omega \in \Omega$. Hence, for $d \geq 2$, the function $u(t, \omega, \cdot)$ need not be continuous in general.

REMARK 4.7. In the above theorem we assume that f_0 and g_0 are real functions and not general vector bundles homomorphisms. We do not know whether our result is true under these more general assumptions.

Theorem 4.4 states the mere existence of a solution. The next result tells us that, among all possible solutions, there certainly exists one that satisfies the "local energy estimates."

In order to make this precise, we define the following family of energy functions: $\mathbf{e}_{x,T}(t,\cdot,\cdot)$, where $x \in \mathbb{R}^n$, $T > 0$ and $t \in [0,T]$,

$$\mathbf{e}_{x,T}(t,u,v) = \int_{B(x,T-t)} \left\{ \frac{1}{2}|u(y)|^2 + \frac{1}{2}|\nabla u(y)|^2 + \frac{1}{2}|v(y)|^2 + \mathbf{s}^2 \right\} dy, \tag{4.5}$$
$$(u,v) \in \mathscr{H}_{\mathrm{loc}}.$$

In the above the constant $\mathbf{s}^2$ is defined by

$$\mathbf{s}^2 = \max\{\|f_{d+1}\|_{L^\infty(M)}, \|f_{d+1}\|^2_{L^\infty(M)} + \|g_{d+1}\|^2_{L^\infty(M)}\}. \tag{4.6}$$

THEOREM 4.8. *Assume that μ, M, Θ, f and g satisfy the assumptions of Theorem 4.4. Then there exists a weak solution $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, z, W)$ of (1.1) with initial data Θ such that*

$$\mathbb{E}\Big\{\mathbf{1}_A(z(0)) \sup_{s\in[0,t]} L(\mathbf{e}_{x,T}(s,z(s)))\Big\} \le 4e^{Ct}\mathbb{E}\{\mathbf{1}_A(z(0))L(\mathbf{e}_{x,T}(0,z(0)))\} \tag{4.7}$$

holds for every $T \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $t \in [0,T]$, $A \in \mathscr{B}(\mathscr{H}_{\mathrm{loc}})$ and every nonnegative nondecreasing function $L \in C[0,\infty) \cap C^2(0,\infty)$ satisfying (for some $c \in \mathbb{R}_+$)

$$tL'(t) + \max\{0, t^2 L''(t)\} \le cL(t), \qquad t > 0. \tag{4.8}$$

The constant C in (4.7) depends only on c, c_μ and on the $L^\infty(M)$-norms of $(f_i, g_i)_{i\in\{0,\dots,d+1\}}$.

5. The target manifold M. Let M be a compact Riemannian manifold, and let the following hypotheses be satisfied:

(M1) There exists a metric-preserving diffeomorphism of M to a submanifold in $\mathbb{R}^n$ for some $n \in \mathbb{N}$ (and from now on we will identify M with its image).

(M2) There exists a C^∞-class function $F : \mathbb{R}^n \to [0,\infty)$ such that $M = \{x : F(x) = 0\}$, and F is constant outside some large ball in $\mathbb{R}^n$.

(M3) There exists a finite sequence $(A^i)_{i=1}^N$ of skew symmetric linear operators on $\mathbb{R}^n$ such that for each $i \in \{1,\dots,N\}$,

$$\langle \nabla F(x), A^i x\rangle = 0 \qquad \text{for every } x \in \mathbb{R}^n, \tag{5.1}$$

$$A^i p \in T_p M \qquad \text{for every } p \in M. \tag{5.2}$$

(M4) There exists a family $(h_{ij})_{1\le i,j\le N}$ of C^∞-class $\mathbb{R}$-valued functions on M such that

$$\xi = \sum_{i=1}^N \sum_{j=1}^N h_{ij}(p)\langle \xi, A^i p\rangle_{\mathbb{R}^n} A^j p, \qquad p \in M, \xi \in T_p M. \tag{5.3}$$

REMARK 5.1. Let us assume that M is a compact Riemannian manifolfd with a compact Lie group G acting transitively by isometries on M; that is, there exists a smooth map $\pi : G \times M \ni (g, p) \mapsto gp \in M$ such that, with e being the unit element in G, for all $g_0, g_1 \in G$ and $p \in M$, $\pi(e, p) = p$ and $\pi(g_0 g_1, p) = g_0 \pi(g_1, p)$, there exists (equivalently, for all) $p_0 \in M$ such that $\{\pi(g, p_0) : g \in G\} = M$, and for every $g \in G$, the map $\pi_g : M \ni p \mapsto \pi(g, p) \in M$ is an isometry. We will show that M satisfies conditions (M1)–(M4). By [33], Theorem 2.20 and Corollary 2.23 (see also [49], Theorem 3.1), for every $p \in M$ the set $G_p = \{g \in G : \pi(g, p) = p\}$ is a closed Lie subgroup of G, and the map $\pi^p : G \ni g \mapsto gp \in M$ is a locally trivial fiber bundle over M with fibre G_p. In particular, for every $p \in M$, the map π^p is a submersion. Moreover, by the Moore–Schlafly theorem [40], there exists an isometric embedding $\Phi : M \hookrightarrow \mathbb{R}^n$ and an orthogonal representation, that is, a smooth Lie group homomorphism, $\rho : G \to SO(n)$, where $SO(n)$ is the orthogonal group, such that $\Phi(gp) = \rho(g)\Phi(p)$, for all $p \in M, g \in G$. Each matrix $A \in SO(n)$ we identify with a linear operator on $\mathbb{R}^n$ (with respect to the canonical ONB of $\mathbb{R}^n$). Let $\{X_i : i \in I\}$ be a basis in $T_e G$, and let us denote, for each $i \in I$, $A_i = d_e\rho(X_i) \in so(n)$. Then, for each $i \in I$, A_i is identified with a skew-self-adjoint linear map in $\mathbb{R}^n$. Let us also put $N = \Phi(M)$. Then, since as observed earlier π^p is a submersion for each $p \in M$, we infer that

$$\operatorname{linspan}\{A_i x : i\} = T_x N \qquad \text{for every } x \in N. \tag{5.4}$$

Let us choose a smooth function $h : \mathbb{R}^n \to \mathbb{R}_+$ such that $N = h^{-1}(\{0\})$ and $h - 1$ has compact support. Let us denote by ν_G a probability measure on G that is invariant with respect to the right multiplication. Then a function $F : \mathbb{R}^n \ni x \mapsto \int_G h(\rho(g)x)\nu_G(dg)$ has the following properties: (i) F is of C^∞-class; (ii) the function $F - 1$ has compact support; (iii) $N = F^{-1}(\{0\})$; (iv) the function F is ρ-invariant, that is, $F(\rho(g)x) = F(x)$ for all $g \in G$ and $x \in \mathbb{R}^n$. Hence, for every $i \in I$, if $\gamma : [0, 1] \to G$ is a smooth curve such that $\gamma(0) = e$ and $\dot\gamma(0) = X_i$, then by property (iv) above and the chain rule, for every $x \in \mathbb{R}^n$, $0 = \frac{d}{dt} F(\rho(\gamma(t))x)|_{t=0} = d_x F(A_i x)$. This proves the first one of the two additional properties of the function F: (v) for every $i \in I$ and $x \in \mathbb{R}^n$, $\langle \nabla F(x), A_i x\rangle = 0$, (vi) for every $i \in I$ and each $x \in N$, $A_i x \in T_x N$ and for each $x \in N$, the set $\{A_i x : i \in I\}$ spans the tangent space $T_x N$. To prove the first part of (vi) it is sufficient to observe that if $x \in N$, i is fixed, and γ is as earlier, then $\rho(\gamma(t))x \in N$ for every $t \in [0, 1]$, and so $A_i x = \frac{d}{dt}\rho(\gamma(t))x|_{t=0} \in T_x N$. The second part of (vi) is simply (5.4).

In view of (5.4) for each $x \in N$ we can find $i_1, \ldots, i_{\dim M}$ and a neighborhood U_x of x in N such that $A_{i_1} y, \ldots, A_{i_{\dim M}} y$ is a basis of $T_y N$ for each $y \in U_x$. By the Gram–Schmidt orthogonalization procedure, we can find C^∞-class functions $\alpha_{jk} : U_x \to \mathbb{R}$, $j, k = 1, \ldots, \dim M$, such that for each $y \in U_x$, the vectors $Z_j(y) = \sum_{k=1}^{\dim M} \alpha_{jk}(y) A_{i_k} y$, $1, \ldots, \dim M$ form an ONB of $T_y N$. Hence, if

$h_{i_k i_l} := \sum_{j=1}^{\dim M} \alpha_{jk}\alpha_{jl}$ and $h_{kl} := 0$ for all other indices, then

$$\xi = \sum_{j=1}^{\dim M} \langle \xi, Z_j(y)\rangle Z_j(y) = \sum_{\alpha}\sum_{\beta} h_{\alpha\beta}(y)\langle \xi, A_\alpha y\rangle A_\beta y, \qquad \xi \in T_yN,\ y \in U_x. \tag{5.5}$$

This local equality can be extended to a global one by employing the partition of unity argument. Hence we have shown that M satisfies all four assumptions (M1)–(M4).

REMARK 5.2. Let us note here that condition (5.2) is a consequence of condition (5.1) if the normal space $(T_pM)^\perp$ is one-dimensional (which is not assumed here), for example, if $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$.

REMARK 5.3. Let us denote by $\tilde{h}_{ij}$ a smooth compactly supported extension of the function h_{ij} to $\mathbb{R}^n$. For $k \in \{1, \ldots, N\}$ let us define a map $Y^k : \mathbb{R}^n \ni x \mapsto \sum_{j=1}^N \tilde{h}_{kj}(x)A^j x \in \mathbb{R}^n$. For each $k \in \{1, \ldots, N\}$ and for all $x \in \mathbb{R}^n$, $\tilde{Y}^k(x)$ is a skew symmetric linear operator in $\mathbb{R}^n$. Let us also denote by Y^k the restriction of $\tilde{Y}^k$, that is,

$$Y^k(p) = \sum_{j=1}^N h_{kj}(p)A^j p, \qquad p \in M. \tag{5.6}$$

In view of assumption (M3), for each $p \in M$, $Y^k(p) \in T_pM$, and hence Y^k, can be viewed as a vector field on M. Moreover, identity (5.3) from assumption (M4) can be equivalently expressed in terms of the vector fields Y^k, with $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}^n}$, as follows:

$$\xi = \sum_{k=1}^N \langle \xi, A^k p\rangle Y^k p, \qquad p \in M, \xi \in T_pM. \tag{5.7}$$

Identity (5.7) is a close reminiscence of formula (7) in [30], Lemma 2.

The following lemma will prove most useful in the *identification* part of the proof of the existence of a solution. Let us recall that by $\mathbf{S}$ we denote the second fundamental form of the submanifold $M \subset \mathbb{R}^n$.

LEMMA 5.4. *For every $(p, \xi) \in TM$, we have*

$$\mathbf{S}_p(\xi, \xi) = \sum_{k=1}^N \langle \xi, A^k p\rangle d_pY^k(\xi), \tag{5.8}$$

where $d_pY^k(\xi) := d_p\tilde{Y}^k(\xi)$, and $d_p\tilde{Y}^k$ is the Frèchet derivative of the map $\tilde{Y}^k$ at p.

PROOF. Let us denote in this proof by $i : M \hookrightarrow \mathbb{R}^n$ the natural embedding of M into $\mathbb{R}^n$. Let us take $p \in M$ and $\xi \in T_pM$. Let $I \subset \mathbb{R}$ be an open interval such that $0 \in I$. Let $\gamma : I \to M$ be a curve such that $\gamma(0) = p$ and $\dot\gamma(0) = (i \circ \gamma)^{\cdot}(0) = \xi$. Then by identity (5.7) we get

$$(i \circ \gamma)^{\cdot}(t) = \sum_{k=1}^{N} \langle (i \circ \gamma)^{\cdot}, A^k \gamma(t) \rangle Y^k(\gamma(t)), \qquad t \in I. \tag{5.9}$$

By taking the standard $\mathbb{R}^n$-valued derivative of (5.9), we get

$$\begin{aligned}(i \circ \gamma)^{\cdot\cdot}(t) &= \sum_{k=1}^{N} \langle (i \circ \gamma)^{\cdot\cdot}(t), A^k \gamma(t) \rangle Y^k(\gamma(t)) \\ &\quad + \sum_{k=1}^{N} \langle (i \circ \gamma)^{\cdot}(t), A^k (i \circ \gamma)^{\cdot}(t) \rangle Y^k(\gamma(t)) \\ &\quad + \sum_{k=1}^{N} \langle (i \circ \gamma)^{\cdot\cdot}(t), A^k \gamma(t) \rangle \big(d_{\gamma(t)} Y^k\big)\big((i \circ \gamma)^{\cdot}(t)\big), \qquad t \in I.\end{aligned} \tag{5.10}$$

On the other hand, by formula ([7], (2.5)) (see also [48], Corollary 4.8), we have

$$(i \circ \gamma)^{\cdot\cdot}(t) = \nabla_{\dot\gamma(t)} \dot\gamma(t) + \mathbf{S}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)), \qquad t \in I. \tag{5.11}$$

In other words, for $t \in I$, the tangential part of $(i \circ \gamma)^{\cdot\cdot}(t)$ is equal to $\nabla_{\dot\gamma(t)} \dot\gamma(t)$ while the normal part of $\ddot\gamma(t)$ is equal to $\mathbf{S}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))$. Since $A^k \gamma(t) \in T_{\gamma(t)}M$ by part (5.2) of assumption (M3), in view of identity (5.7), we infer that

$$\begin{aligned}\sum_{k=1}^{N} \langle (i \circ \gamma)^{\cdot\cdot}(t), A^k \gamma(t) \rangle Y^k(\gamma(t)) &= \sum_{k=1}^{N} \langle \nabla_{\dot\gamma(t)} \dot\gamma(t), A^k \gamma(t) \rangle Y^k(\gamma(t)) \\ &= \nabla_{\dot\gamma(t)} \dot\gamma(t), \qquad t \in I.\end{aligned} \tag{5.12}$$

Since A^k are skew-symmetric, the middle term on the RHS of (5.11) is equal to 0, and thus in view of (5.11), (5.12) and (5.10), we infer that

$$\begin{aligned}&\mathbf{S}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) \\ &\quad = \sum_{k=1}^{N} \langle (i \circ \gamma)^{\cdot}(t), A^k \gamma(t) \rangle \big(d_{\gamma(t)} Y^k\big)\big((i \circ \gamma)^{\cdot}(t)\big), \qquad t \in I.\end{aligned} \tag{5.13}$$

Putting $t = 0$ in equality (5.13) we get identity (5.8). The proof is now complete. $\square$

6. Outline of the proof of the main theorem. The main idea of the proof of Theorem 4.4 can be seen from the following result. The proof of this result follows by applying the Itô formula from [5] and using the material discussed in Section 5. The proof of the converse part can be reproduced from the proof of Lemma 9.10.

PROPOSITION 6.1. *Assume that M is a compact Riemannian homogeneous space and that the coefficients f and g satisfy Assumption 4.2. Suppose that a system*

$$(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, (u, v)) \tag{6.1}$$

is a weak solution of (1.1). Assume that $A : \mathbb{R}^d \to \mathbb{R}^d$ is a skew-symmetric linear operator satisfying condition (5.2). Define a process $\mathbf{M}$ by the following formula:

$$\mathbf{M}(t) := \langle v(t), Au(t)\rangle_{\mathbb{R}^n}, \qquad t \geq 0. \tag{6.2}$$

Then for every function $\varphi \in H^1_{\mathrm{comp}}$, the following equality holds almost surely:

$$\begin{aligned}\langle \varphi, \mathbf{M}(t)\rangle = \langle \varphi, \mathbf{M}(0)\rangle &- \sum_{k=1}^{d} \left\langle \partial_{x_k}\varphi, \int_0^t \langle \partial_{x_k} u(s), u(s)\rangle_{\mathbb{R}^n}\, ds \right\rangle \\ &+ \left\langle \varphi, \int_0^t \langle f(u(s), v(s), \nabla u(s)), Au(s)\rangle\, ds \right\rangle \\ &+ \left\langle \varphi, \int_0^t \langle g(u(s), v(s), \nabla u(s)), Au(s)\rangle\, dW(s) \right\rangle, \qquad t \geq 0.\end{aligned} \tag{6.3}$$

Conversely, assume that a system (6.1) satisfies all the conditions of Definition 4.3 of a weak solution to equation (1.1) but (4.3). Suppose that a finite sequence $(A^i)_{i=1}^N$ of skew symmetric linear operators in $\mathbb{R}^n$ satisfies conditions (5.1), (5.2) and (M4). For each $i \in \{1, \dots, N\}$ define a process $\mathbf{M}^i$ by formula (6.2) with $A = A^i$. Suppose that for every function $\varphi \in H^1_{\mathrm{comp}}$ each $\mathbf{M}^i$ satisfies equality (6.3) with $A = A^i$. Then the process (u, v) satisfies the equality (4.3).

The first step of the proof of Theorem 4.4 consists of introducing a penalized and regularized stochastic wave equation (7.5)–(7.6), that is,

$$\begin{aligned}\partial_{tt} U^m = \Delta U^m - m\nabla F(U^m) &+ f^m\big(U^m, \nabla_{(t,x)} U^m\big) \\ &+ g^m\big(U^m, \nabla_{(t,x)} U^m\big)\, dW^m\end{aligned} \tag{6.4}$$

$$\text{law of } (U^m(0), \partial_t U^m(0)) = \Theta. \tag{6.5}$$

Had we assumed that the coefficients f and g were sufficiently regular, we would have simply put $f^m = f$ and $g^m = g$ above. The existence of a unique global solution $Z^m = (U^m, V^m)$ to the problem (6.4)–(6.5) is more or less standard. In Section 8, by using uniform energy estimates, we will show that the sequence

$(Z^m)_{m\in\mathbb{N}}$ is tight on an appropriately chosen Fréchet space, and by employing Jakubowski's generalization [31] of the Skorokhod embedding theorem, we construct a version z^m of Z^m such that it converges in law to a certain process z. In order to prove that z is a weak solution of (1.4), we construct processes M_k^i defined by a formula analogous to formula (6.1); see formula (9.4). We prove that the sequence $\mathbf{M}_k^i$ is convergent and denote the limit by $\mathbf{M}^i$. Moreover, we show that z takes values in the tangent bundle TM. The proof is concluded by constructing an appropriate Wiener process (see Lemma 9.9) and showing, by employing the argument needed to prove the converse part of Proposition 6.1, that the process z is indeed a weak solution of (1.4). We remark that our method of constructing weak solutions to stochastic PDEs does not employ any martingale representation theorem (and we are not aware of such results in the Fréchet spaces anyway).

7. Preparation for the proof of the main theorem.

7.1. *Approximation of coefficients.* Let $(\zeta_m)_{m=1}^{\infty}$ be the approximation of identity introduced in Section 2. Let us assume that J is a continuous vector field on M, h a continuous real function on M and λ a continuous vector bundle homomorphisms from TM to TM. Let $\pi:\mathbb{R}^n \to \mathscr{L}(\mathbb{R}^n,\mathbb{R}^n)$ be a smooth compactly supported function such that for every $p\in M$, $\pi(p)$ is the orthogonal projection from $\mathbb{R}^n$ onto T_pM. The vector field J, the function h and the $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$-valued function $\lambda\circ\pi|_M$ (all defined on M) can be extended to continuous compactly supported functions, all denoted again by the same symbols, $J:\mathbb{R}^n\to\mathbb{R}^n$, $h:\mathbb{R}^n\to\mathbb{R}$ and $\lambda:\mathbb{R}^n\to\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n)$. By a standard approximation argument (invoking the convolution with functions ζ_m) we can find sequences of C^∞-class functions $J^m:\mathbb{R}^n\to\mathbb{R}^n$, $h^m:\mathbb{R}^n\to\mathbb{R}$, $\lambda^m:\mathbb{R}^n\to\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n)$ supported in a compact set in $\mathbb{R}^n$ such that $J^m\to J$, $h^m\to h$ and $\lambda^m\to\lambda$ uniformly on $\mathbb{R}^n$.

When we specify the above to our given data, continuous vector fields f_{d+1}, g_{d+1} on M, continuous functions f_0, g_0 on M and continuous vector bundle homeomorphisms $f_1,\dots,f_d$, $g_1,\dots,g_d$ on TM, we can construct the following sequences of approximating smooth functions, with $i\in\{1,\dots,d\}$:

$$
\begin{aligned}
&f_0^m, g_0^m:\mathbb{R}^n\to\mathbb{R}, \qquad f_i^m, g_i^m:\mathbb{R}^n\to\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n),\\
&f_{d+1}^m, g_{d+1}^m:\mathbb{R}^n\to\mathbb{R}^n, \qquad m\in\mathbb{N},
\end{aligned}
\tag{7.1}
$$

such that for some $R_0>0$ and every $i\in\{0,\dots,d+1\}$,

$$
\bigcup_{m\in\mathbb{N}}[\operatorname{supp}(f_i^m)\cup\operatorname{supp}(g_i^m)]\subset B(0,R_0)\subset\mathbb{R}^n,
\tag{7.2}
$$

$$
\begin{aligned}
&|f_{d+1}^m|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\le|f_{d+1}|_{L^\infty(M,\mathbb{R}^n)},\\
&|g_{d+1}^m|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\le|g_{d+1}|_{L^\infty(M,\mathbb{R}^n)}, \qquad m\in\mathbb{N},
\end{aligned}
\tag{7.3}
$$

$$
f_i^m\to f_i \quad\text{and}\quad g_i^m\to g_i \qquad \text{as } m\to\infty, \text{ uniformly on } \mathbb{R}^n.
\tag{7.4}
$$

7.2. *Solutions to an approximated problem.* Let the Borel probability measure Θ on $\mathscr{H}_{\mathrm{loc}}(M)$ be as in Theorem 4.4. It follows from known results (see, e.g., [47]) that for each $m \in \mathbb{N}$ there exists a weak solution of the following problem:

$$\begin{aligned}\partial_{tt} U^m &= \Delta U^m - m\nabla F(U^m) + f^m\big(U^m, \nabla_{(t,x)} U^m\big) \\ &\quad + g^m\big(U^m, \nabla_{(t,x)} U^m\big)\, dW^m\end{aligned} \tag{7.5}$$

$$\text{law of } (U^m(0), \partial_t U^m(0)) = \Theta, \tag{7.6}$$

where the coefficients f^m (resp., g^m) are defined by (1.3), with f_i being replaced by f_i^m (resp., by g_i^m). In other words for every $m \in \mathbb{N}$, there exists:

(i) a complete stochastic basis $(\Omega^m, \mathscr{F}^m, \mathbb{F}^m, \mathbb{P}^m)$, where $\mathbb{F}^m = (\mathscr{F}_t^m)_{t\geq 0}$;

(ii) a spatially homogeneous $\mathbb{F}^m$-Wiener process W^m with spectral measure μ; and

(iii) an $\mathbb{F}^m$-adapted $\mathscr{H}_{\mathrm{loc}}$-valued weakly continuous process $Z^m = (U^m, V^m)$ such that Θ is equal to the law of $Z^m(0)$, and for every $t \geq 0$ and $\varphi \in \mathscr{D}(\mathbb{R}^d, \mathbb{R}^n)$ the following equalities hold almost surely:

$$\langle U^m(t), \varphi\rangle_{\mathbb{R}^n} = \langle U^m(0), \varphi\rangle_{\mathbb{R}^n} + \int_0^t \langle V^m(s), \varphi\rangle_{\mathbb{R}^n}\, ds, \tag{7.7}$$

$$\begin{aligned}\langle V^m(t), \varphi\rangle_{\mathbb{R}^n} &= \langle V^m(0), \varphi\rangle_{\mathbb{R}^n} \\ &\quad + \int_0^t \langle -m\nabla F(U^m(s)) + f^m(Z^m(s), \nabla U^m(s)), \varphi\rangle_{\mathbb{R}^n}\, ds \\ &\quad + \int_0^t \langle U^m(s), \Delta\varphi\rangle_{\mathbb{R}^n}\, ds \\ &\quad + \int_0^t \langle g^m(Z^m(s), \nabla U^m(s))\, dW_s^m, \varphi\rangle_{\mathbb{R}^n}.\end{aligned} \tag{7.8}$$

The processes Z^m need not take values in the tangent bundle TM, and since the diffusion nonlinearity is not Lipschitz, it only exists in the weak probabilistic sense.

REMARK 7.1. Let us point out that for each $m \in \mathbb{N}$, $Z^m(0)$ is $\mathscr{F}_0^m$-measurable $\mathscr{H}_{\mathrm{loc}}(M)$-valued random variables whose law is equal to Θ. In particular, our initial data satisfy $U_0^m(\omega) \in M$ and $V_0^m(\omega) \in T_{U_0^m(\omega)}M$ a.e. for every $\omega \in \Omega$.

8. Tightness of the approximations. Lemma 8.1 below constitutes the first step toward proving Theorem 4.8. In its formulation we use the following generalized family of energy functions $\mathbf{e}_{x,T,mF}$, where $x \in \mathbb{R}^n$, $T > 0$, $m \in \mathbb{N}$ and the constant $\mathbf{s}^2$ was defined in (4.6) [compare with (4.5)],

$$\begin{aligned}\mathbf{e}_{x,T,mF}(t,u,v) &= \int_{B(x,T-t)} \left\{\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{2}|v|^2 + mF(u) + \mathbf{s}^2\right\} dy, \\ &\qquad t \in [0,T], (u,v) \in \mathscr{H}_{\mathrm{loc}}.\end{aligned} \tag{8.1}$$

LEMMA 8.1. *There exists a weak solution* $(\Omega^m, \mathscr{F}^m, (\mathscr{F}_t^m), \mathbb{P}^m, Z^m = (U^m, V^m), W^m)$ *to* (7.5)–(7.6) *such that*

$$\mathbb{E}^m\Big[1_A(Z^m(0)) \sup_{s\in[0,t]} L(\mathbf{e}_{x,T,mF}(s, Z^m(s)))\Big] \le 4e^{Ct}\mathbb{E}^m[1_A(Z^m(0))L(\mathbf{e}_{x,T,mF}(0, Z^m(0)))] \tag{8.2}$$

holds for every $T \ge 0$, $t \in [0,T]$, $A \in \mathscr{B}(\mathscr{H}_{\mathrm{loc}})$, $m \in \mathbb{N}$ *whenever* $L \in C[0,\infty) \cap C^2(0,\infty)$ *is a nondecreasing function such that, for some* $c > 0$,

$$tL'(t) + \max\{0, t^2L''(t)\} \le cL(t), \qquad t > 0. \tag{8.3}$$

The constant C *depends on* c, c_μ *and on* $\|f^j\|_{L^\infty}$, $\|g^j\|_{L^\infty}$, $j = 0, 1, \dots, d+1$.

PROOF. This is a direct application of Theorems 5.1 and 5.2 in [47]. We use bound (7.3) according to which the $\|f^m_{d+1}\|_{L^\infty}$ norm is bounded by $\|f_{d+1}\|_{L^\infty(M)}$. $\square$

In the following lemma we use the notions introduced in Appendices B and C.

LEMMA 8.2. *Assume that* $r < \min\{2, \frac{d}{d-1}\}$. *Then*:

(1) *the sequence* $\{U^m\}$ *is tight on* $C_w(\mathbb{R}_+; H^1_{\mathrm{loc}})$;
(2) *the sequence* $\{V^m\}$ *is tight on* $\mathbb{L} = L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$;
(3) *and, for every* $i \in \{1, \dots, N\}$, *the sequence* $\langle V^m, A^iU^m\rangle_{\mathbb{R}^d}$ *is tight on* $C_w(\mathbb{R}_+; L^r_{\mathrm{loc}})$.

PROOF OF CLAIM (1). Let us now take and fix $\varepsilon > 0$. In view of Corollary C.1 it is enough to find a sequence $\{a_k\}_{k\in\mathbb{N}}$ such that

$$\mathbb{P}^m\left(\bigcup_{k=1}^{\infty}\{\|U^m\|_{L^\infty((0,k);H^1(B_k))} + \|U^m\|_{C^1([0,k];L^2(B_k))} > a_k\}\right) \le \varepsilon, \qquad m \in \mathbb{N}. \tag{8.4}$$

If we denote, for $\delta > 0$ and $k, m \in \mathbb{N}$, $Q_{m,k,\delta} = \{\|Z^m(0)\|_{\mathscr{H}_{2k}} \le \delta\}$, then by definition (8.1) of the function $\mathbf{e}$ we can find a constant $c > 0$ such that

$$\mathbb{E}^m[1_{Q_{m,k,\delta}}[\|U^m\|_{L^\infty((0,k);H^1(B_k))} + \|U^m\|_{C^1([0,k];L^2(B_k))}]] \le c\mathbb{E}^m\Big[1_{Q_{m,k,\delta}} \sup_{s\in[0,k]} L(\mathbf{e}_{0,2k,m}(s, Z^m(s)))\Big] \tag{8.5}$$

for all $\delta > 0$ and $k, m \in \mathbb{N}$.

On the other hand, since the sequence $\{\mathbf{s}_m\}$ in (8.1) is bounded, by applying Lemma 8.1 with function $L(\cdot)=\sqrt{2\cdot}$, we infer that for $\delta>0$ and $k\in\mathbb{N}$, we can find $C_{k,\delta}>0$ such that

$$c\mathbb{E}^m\Big[1_{Q_{m,k,\delta}}\sup_{s\in[0,k]}L(\mathbf{e}_{0,2k,m}(s,Z^m(s)))\Big]\le C_{k,\delta}\qquad\text{for every } m\in\mathbb{N}. \tag{8.6}$$

If we put $a_k=C_{k,\delta_k}\varepsilon^{-1}2^{k+1}$, for $k\in\mathbb{N}$, then by (8.5) and (8.6) in view of the Chebyshev inequality we infer that

$$\begin{aligned}&\mathbb{P}^m\big[1_{Q_{m,k,\delta_k}}\{\|U^m\|_{L^\infty((0,k);H^1(B_k))}+\|U^m\|_{C^1([0,k];L^2(B_k))}>a_k\}\big]\\&\quad\le\varepsilon 2^{-k-1}.\end{aligned} \tag{8.7}$$

Since the measure Θ is Radon, for $k\in\mathbb{N}$, we can find $\delta_k>0$ such that $\Theta(\{z\in\mathscr{H}_{\mathrm{loc}}:\|z\|_{\mathscr{H}_{2k}}\ge\delta_k\})<\frac{\varepsilon}{2^{k+1}}$. Hence, since the law of Z^m under $\mathbb{P}^m$ is equal to Θ we infer that

$$\mathbb{P}^m(Q_{m,k,\delta_k})>1-\frac{\varepsilon}{2^{k+1}},\qquad k\in\mathbb{N}. \tag{8.8}$$

Summing up, (8.4) follows from inequalities (8.7) and (8.8). $\square$

PROOF OF CLAIM (2). As far as the sequence $\{V^m\}$ is concerned, let us observe that by Lemma 8.1, in the same way as in inequalities (8.5) and (8.6), we get also the following one

$$\begin{aligned}&\mathbb{E}^m\big[1_{Q_{m,k,\delta}}\|V^m\|_{L^\infty((0,k);L^2(B_k))}\big]\\&\quad\le\mathbb{E}^m\Big[1_{Q_{m,k,\delta}}\sup_{s\in[0,k]}L(\mathbf{e}_{0,2k,m}(s,Z^m(s)))\Big]\le C_{k,\delta}.\end{aligned}$$

Hence for each $k\in\mathbb{N}$, $\mathbb{P}^m\{\|V^m\|_{L^\infty((0,k);L^2(B_k))}>a_k\}\le\varepsilon 2^{-k}$ and arguing as above, but now using Corollary B.2, we infer that the sequence $\{V^m\}$ is tight on $\mathbb{L}=L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2_{\mathrm{loc}})$. $\square$

PROOF OF CLAIM (3). Since the assumptions of the Itô lemma from [5] [with $q=2$, $k=n$, $w=v$ and $Y(y)=A^iy$] are satisfied, by the properties of function F and operators A^i listed in Section 5, we infer that for every $t\ge0$ and $\varphi\in\mathscr{D}(\mathbb{R}^d)$, the following equality:

$$\begin{aligned}&b(V^m(t),A^iU^m(t),\varphi)\\&\quad=b(V^m(0),A^iU^m(0),\varphi)-\sum_{k=1}^d\int_0^t b(\partial_{x_k}U^m(s),A^iU^m(s),\partial_{x_k}\varphi)\,ds\\&\qquad+\int_0^t b(f^m(Z^m(s),\nabla U^m(s)),A^iU^m(s),\varphi)\,ds\\&\qquad+\int_0^t b(g^m(Z^m(s),\nabla U^m(s))\,dW^m,A^iU^m(s),\varphi)\end{aligned}$$

holds almost surely. Hence, in view of Appendix C, for every $R > 0$, the equality

$$\begin{aligned}\langle V^m(t), A^i U^m(t)\rangle_{\mathbb{R}^n} \\ &= \langle V^m(0), A^i U^m(0)\rangle_{\mathbb{R}^n} + \sum_{k=1}^d \partial_{x_k}\left[\int_0^t \langle \partial_{x_k} U^m(s), A^i U^m(s)\rangle_{\mathbb{R}^n}\, ds\right] \\ &\quad + \int_0^t \langle f^m(Z^m(s), \nabla U^m(s)), A^i U^m(s)\rangle_{\mathbb{R}^n}\, ds \\ &\quad + \int_0^t \langle g^m(Z^m(s), \nabla U^m(s)), A^i U^m(s)\rangle_{\mathbb{R}^n}\, dW^m\end{aligned} \tag{8.9}$$

holds in $\mathbb{W}_R^{-1,r}$ for every $t \ge 0$, almost surely. Indeed, by the Gagliardo–Nirenberg inequality (G–NI) and the Hölder inequality, we get

$$\begin{aligned}\|ab\|_{L^r(\mathbb{R}^d)} &\le \|a\|_{L^2(\mathbb{R}^d)}\|b\|_{L^{2r/(2-r)}(\mathbb{R}^d)} \\ &\le c\|a\|_{L^2(\mathbb{R}^d)}\|b\|_{H^1(\mathbb{R}^d)}, \qquad a \in L^2(\mathbb{R}^d), b \in H^1(\mathbb{R}^d).\end{aligned} \tag{8.10}$$

Therefore, the first deterministic integral in (8.9) converges in L^r_{loc}. But the the map $\partial_{x_k} : L^r_{\mathrm{loc}} \to \mathbb{W}^{-1,r}_{\mathrm{loc}}$ is continuous, and so the first term in (8.9) is a well defined $\mathbb{W}_R^{-1,r}$-valued random variable for each $R > 0$.

Since the functions f_i^m, g_i^m have compact support, we can find $T > 0, c > 0$ such that

$$\begin{aligned}|\langle f^m(y, w), A^i y\rangle| + |\langle g^m(y, w), A^i y\rangle| \\ \le c\mathbf{1}_{[-T,T]}(|y|)(1 + |w|), \qquad (y, w) \in \mathbb{R}^n \times [\mathbb{R}^n]^{d+1}.\end{aligned} \tag{8.11}$$

Therefore, the stochastic and the second deterministic integrals are convergent in L^2_{loc}. Since $r < \frac{d}{d-1}$, by the G–NI

$$L^2_{\mathrm{loc}} \hookrightarrow \mathbb{W}_R^{-1,r}, \qquad R > 0. \tag{8.12}$$

Thus the stochastic and the second deterministic integrals are convergent in $\mathbb{W}_R^{-1,r}$, $R > 0$.

Next let us choose $p > 4$ and $\gamma > 0$ such that $\gamma + \frac{2}{p} < \frac{1}{2}$. Let us denote

$$I^{(2)}(t) = \int_0^t \langle \partial_{x_l} U^m(s), A^i U^m(s)\rangle_{\mathbb{R}^n}\, ds, \qquad t \ge 0. \tag{8.13}$$

$$I^{(3)}(t) = \int_0^t \langle f^m(Z^m(s), \nabla U^m(s)), A^i U^m(s)\rangle_{\mathbb{R}^n}\, ds, \qquad t \ge 0. \tag{8.14}$$

$$I^{(4)}(t) = \int_0^t \langle g^m(Z^m(s), \nabla U^m(s)), A^i U^m(s)\rangle_{\mathbb{R}^n}\, dW^m, \qquad t \ge 0. \tag{8.15}$$

Then, if we denote $Q_{m,k,\delta} = \{\|Z^m(0)\|_{\mathscr{H}_{2k}} \le \delta\}$, by (8.12), the Garsia–Rumsey–Roedemich lemma [25] and the Burkholder inequality, we have

$$
\begin{aligned}
&\mathbb{E}^m\big[1_{Q_{m,k,\delta}}\|I^{(4)}\|^p_{C^\gamma([0,k];\mathbb{W}_k^{-1,r})}\big] \\
&\quad\le c_{k,r}\mathbb{E}^m\|1_{Q_{m,k,\delta}}I^{(4)}\|^p_{C^\gamma([0,k],L^2(B_k))} \\
&\quad\le \tilde{c}_{k,r,p}\mathbb{E}^m\bigg[1_{Q_{m,k,\delta}}\int_0^k \|\langle g^m(Z^m(s),\nabla U^m(s)), \\
&\qquad\qquad A^i U^m(s)\rangle_{\mathbb{R}^n}\|^p_{\mathscr{T}_2(H_\mu,L^2(B_k))}\,ds\bigg].
\end{aligned}
\tag{8.16}
$$

Applying Lemma 3.3, inequality (8.11) and Lemma 8.1 we infer that for a generic constant $C>0$, $\|\langle g^m(Z,\nabla U), A^iU\rangle_{\mathbb{R}^n}\|^p_{\mathscr{T}_2(H_\mu,L^2(B_k))} \le C\|\langle g^m(Z,\nabla U), A^iU\rangle_{\mathbb{R}^n}\|^p_{L^2(B_k)} \le C(1+\|Z^m(s)\|^p_{\mathscr{H}_k}) \le C[1+\mathbf{e}^{p/2}_{0,2k,m}(s,Z^m(s))]$. Hence the RHS of inequality (8.16) is bounded by

$$
\begin{aligned}
&\mathbb{E}^m\big[1_{Q_{m,k,\delta}}\|I^{(4)}\|^p_{C^\gamma([0,k];\mathbb{W}_k^{-1,r})}\big] \\
&\quad\le c^0_{k,r,p}\mathbb{E}^m\bigg[1_{Q_{m,k,\delta}}\int_0^k [1+\mathbf{e}^{p/2}_{0,2k,m}(s,Z^m(s))]\,ds\bigg] \\
&\quad\le C_{k,r,p,\delta}.
\end{aligned}
$$

Analogously, by the Hölder inequality and Lemma 8.1,

$$
\begin{aligned}
&\mathbb{E}^m\big[1_{Q_{m,k,\delta}}\|I^{(3)}\|^p_{C^\gamma([0,k];\mathbb{W}_k^{-1,r})}\big] \\
&\quad\le c_{k,r}\mathbb{E}^m\|1_{Q_{m,k,\delta}}I^{(3)}\|^p_{C^\gamma([0,k],L^2(B_k))} \\
&\quad\le c^0_{k,r,p}\mathbb{E}^m\bigg[1_{Q_{m,k,\delta}}\int_0^k [1+\mathbf{e}^{p/2}_{0,2k,m}(s,Z^m(s))]\,ds\bigg] \\
&\quad\le C_{k,r,p,\delta}.
\end{aligned}
\tag{8.17}
$$

Concerning the process $I^{(2)}$, an analogous argument yields

$$
\begin{aligned}
\|\partial_{x_l}I^{(2)}\|^p_{C^\gamma([0,k],\mathbb{W}_k^{-1,r})} &\le c_{k,p}\int_0^k \|\langle\partial_{x_l}U^m(s), A^iU^m(s)\rangle_{\mathbb{R}^n}\|^p_{L^r(B_k)}\,ds \\
&\le \tilde{c}_{k,r,p}\int_0^k \mathbf{e}^p_{0,2k,m}(s,Z^m(s))\,ds.
\end{aligned}
$$

Hence, by Lemma 8.1, we infer that

$$
\mathbb{E}^m\big[1_{Q_{m,k,\delta}}\|\partial_{x_l}I^{(2)}_m\|^p_{C^\gamma([0,k],\mathbb{W}_k^{-1,r})}\big] \le C_{k,p,\delta}.
\tag{8.18}
$$

Moreover, since $\mathbb{W}_k^{-1,r} \hookrightarrow L^r(B_k)$ continuously, by (8.10) we have

$$\mathbb{E}^m\Big[1_{Q_{m,k,\delta}} \sup_{s\in[0,k]} \|\langle V^m(s), A^i U^m(s)\rangle_{\mathbb{R}^n}\|^p_{\mathbb{W}_k^{-1,r}}\Big]$$
$$\leq \tilde{c}_r \mathbb{E}^m\Big[1_{Q_{m,k,\delta}} \sup_{s\in[0,k]} \mathbf{e}^p_{0,2k,m}(s, Z^m(s))\Big] \leq c_{k,r,p,\delta}.$$

Summing up, we proved that there exists a constant $C_{k,r,p,\delta} > 0$ such that

$$\begin{aligned}\mathbb{E}^m\{1_{Q_{m,k,\delta}}[\|\langle V^m, A^i U^m\rangle_{\mathbb{R}^n}\|_{L^\infty([0,k];L^r(B_k))\cap C^\gamma([0,k],\mathbb{W}_k^{-1,r})}]^p\}\\ \leq C_{k,r,p,\delta}.\end{aligned} \tag{8.19}$$

Let $\varepsilon > 0$ be fixed. Define a sequence $(a_k)_{k=1}^\infty$ of positive real numbers by $C_{k,r,p,\delta_k} = a_k^p \varepsilon 2^{-k-1}$, $k \in \mathbb{N}$. Then by (8.19) and (8.8), for each $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}^m\{&\|\langle V^m, A^i U^m\rangle_{\mathbb{R}^n}\|_{L^\infty([0,k];L^r(B_k))}\\ &+ \|\langle V^m, A^i U^m\rangle_{\mathbb{R}^n}\|_{C^\gamma([0,k],\mathbb{W}_k^{-1,r})} > a_k\}\\ &\leq \varepsilon 2^{-k}.\end{aligned} \tag{8.20}$$

Hence, by Proposition C.1 the sequence $\langle V^m, A^i U^m\rangle_{\mathbb{R}^n}$ is tight on $C_w(\mathbb{R}_+; L^r_{\text{loc}})$. $\square$

9. Proof of the main result. Let us consider the approximating sequence of processes $(Z^m)_{m\in\mathbb{N}}$, where $Z^m = (U^m, V^m)$ for $m \in \mathbb{N}$, introduced Lemma 8.1. Let us also consider the following representation of Wiener processes W^m:

$$W_t^m = \sum_i \beta_i^m(t) e_i, \qquad t \geq 0, \tag{9.1}$$

where $\beta = (\beta^1, \beta^2, \dots)$ are independent real standard Wiener processes, and $\{e_i : i \in \mathbb{N}\}$ is an orthonormal basis in H_μ; see Proposition 3.2.

Assume that $r < \min\{2, \frac{d}{d-1}\}$ is fixed. Then Lemma 8.2, Corollary B.3, Proposition C.2 and Corollary A.2 yield[2] that there exists:

- a probability space $(\Omega, \mathscr{F}, \mathbb{P})$,
- a subsequence m_k,
- the following sequences of Borel measurable functions:

(9.2)

$(u^k)_{k\in\mathbb{N}}$	with values in $C(\mathbb{R}_+, H^1_{\text{loc}})$
$(v^k)_{k\in\mathbb{N}}$	with values in $C(\mathbb{R}_+, L^2_{\text{loc}})$
$(w^k)_{k\in\mathbb{N}}$	with values in $C(\mathbb{R}_+, \mathbb{R}^{\mathbb{N}})$

[2]Let us recall that we used there to denote by $\mathbb{L}$ the space $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2_{\text{loc}})$.

- the following Borel random variables:

(9.3)

v_0	with values in L^2_{loc}
u	with values in $C_w(\mathbb{R}_+; H^1_{\mathrm{loc}})$
$\bar{v}$	with values in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$
w	with values in $C(\mathbb{R}_+, \mathbb{R}^{\mathbb{N}})$
$M^i, i = 1, \dots, N$	with values in $C_w(\mathbb{R}_+; L^r_{\mathrm{loc}})$

such that, with the notation $z^k = (u^k, v^k)$, $k \in \mathbb{N}$ and

$$M^i_k := \langle v^k, A^i u^k \rangle_{\mathbb{R}^n}, \qquad i = 1, \dots, N, k \in \mathbb{N}, \tag{9.4}$$

the following conditions are satisfied:

(R1) for every $k \in \mathbb{N}$, the law of (Z^{m_k}, β^{m_k}) coincides with the law of (z^k, w^k) on $\mathscr{B}(C(\mathbb{R}_+, \mathscr{H}_{\mathrm{loc}}) \times C(\mathbb{R}_+, \mathbb{R}^{\mathbb{N}}))$;

(R2) pointwise on Ω the following convergences hold:

$$\begin{aligned} u^k &\to u && \text{in } C_w(\mathbb{R}_+; H^1_{\mathrm{loc}}), \\ v^k &\to \bar{v} && \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}}), \\ v^k(0) &\to v_0 && \text{in } L^2_{\mathrm{loc}}, \\ M^i_k &\to M^i && \text{in } C_w(\mathbb{R}_+; L^r_{\mathrm{loc}}), \\ w^k &\to w && \text{in } C(\mathbb{R}_+, \mathbb{R}^{\mathbb{N}}); \end{aligned} \tag{9.5}$$

(R3) the law of $(u(0), v_0)$ is equal to Θ.

In particular, the conclusions of Lemma 8.1 hold for this new system of processes. This is summarized in the proposition below.

PROPOSITION 9.1. *If ρ is the constant from Lemma 8.1, then inequality (8.2) holds. Thus, for any nondecreasing function $L \in C[0, \infty) \cap C^2(0, \infty)$ satisfying condition (8.3), we have*

$$\begin{aligned} &\mathbb{E}\Big[\mathbf{1}_A(z^k(0)) \sup_{s\in[0,t]} L(\mathbf{e}_{x,T,m_k}(s, z^k(s)))\Big] \\ &\qquad \le 4e^{\rho t}\mathbb{E}[\mathbf{1}_A(z^k(0)) L(\mathbf{e}_{x,T,m_k}(0, z^k(0)))] \end{aligned} \tag{9.6}$$

for every $k \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathbb{R}^d$, $A \in \mathscr{B}(\mathscr{H}_{\mathrm{loc}})$.

Before we continue, let us observe that the compactness of the embedding $H^1_{\mathrm{loc}} \hookrightarrow L^2_{\mathrm{loc}}$ and properties (9.3) and (9.5) imply the following auxiliary result.

PROPOSITION 9.2. *In the above framework, all the trajectories of the process u belong to $C(\mathbb{R}_+, L^2_{\mathrm{loc}})$ and for every $t \in \mathbb{R}_+$, $u^k(t) \to u(t)$ in L^2_{loc}.*

We also introduce a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ of σ-algebras on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ defined by $\mathscr{F}_t = \sigma\{\sigma\{v_0, u(s), w(s) : s \in [0,t]\} \cup \{N : \mathbb{P}(N) = 0\}\}$, $t \geq 0$.

Our first result states, roughly speaking, that the limiting process u takes values in the set M. To be precise, we have the following.

PROPOSITION 9.3. *There exists a set $Q_u \in \mathscr{F}$ such that $\mathbb{P}(Q_u) = 1$ and, for every $\omega \in Q_u$ and $t \geq 0$, $u(t,\omega) \in M$ almost everywhere on $\mathbb{R}^d$.*

PROOF. Let us fix $T > 0$ and $\delta > 0$. In view of the definition (8.1) of the function $\mathbf{e}_{0,T,m}$, inequality (9.6) yields that for some finite constant $C_{T,\delta}$,

$$\mathbb{E}\left[1_{B_\delta^{\mathscr{H}_T}}(z_0^k)\int_{B_{T-t}} m_k F(u^k(t))\,dx\right] \leq C_{T,\delta}, \qquad t \in [0,T]. \tag{9.7}$$

Since $\|z^k(0)\|_{\mathscr{H}_T} \to \|(u(0), v_0)\|_{\mathscr{H}_T}$ and by Proposition 9.2 for every $t \in [0, T]$, $u^k(t) \to u(t)$ in $L^2(B_{T-t})$, by applying the Fatou lemma we infer that

$$\begin{aligned}\mathbb{E}&\left[1_{B_\delta^{\mathscr{H}_T}}(z_0)\int_{B_{T-t}} F(u(t))\,dx\right]\\ &\leq \liminf_{k\to\infty}\mathbb{E}\left[1_{B_\delta^{\mathscr{H}_T}}(z_0^k)\int_{B_{T-t}} F(u^k(t))\,dx\right].\end{aligned} \tag{9.8}$$

On the other hand, since $m_k \nearrow \infty$, by (9.7) we get $\liminf_{k\to\infty}\mathbb{E}[1_{B_\delta^{\mathscr{H}_T}}(z_0^k) \times \int_{B_{T-t}} F(u^k(t))\,dx] = 0$ and so $\mathbb{E}[1_{B_\delta^{\mathscr{H}_T}}(z_0)\int_{B_{T-t}} F(u(t))\,dx] = 0$. Taking the limits as $\delta \nearrow \infty$ and $T \nearrow \infty$, we get that for any $t \geq 0$, $\mathbb{E}\int_{\mathbb{R}^d} F(u(t))\,dx = 0$. Since $F \geq 0$ and $M = F^{-1}(\{0\})$ we infer that for each $t \in \mathbb{R}_+$, $u(t,x) \in M$ for Leb a.a. $x \in \mathbb{R}^d$, $\mathbb{P}$-almost surely. Hence there exists a set Ω_u of full measure such that

$$\mathrm{Leb}\big(\{x \in \mathbb{R}^d : u(q,\omega,x) \notin M\}\big) = 0 \qquad \text{for every } q \in \mathbb{Q}_+ \text{ and every } \omega \in \Omega_u.$$

Let us take $t \in \mathbb{R}_+$, $\omega \in \Omega_*$. Obviously we can find a sequence $(q_n)_{n=1}^\infty \subset \mathbb{Q}_+$ such that $q_n \to t$. Hence, by Proposition 9.2, there exists a subsequence $(n_k)_{k=1}^\infty$ such that for Leb-almost every x, $u(q_{n_k},\omega,x) \to u(t,\omega,x)$ as $k \to \infty$. Hence by closedness of the set M we infer that also $u(t,\omega,x) \in M$ for Leb-almost every x. The proof is complete. □

The last result suggests the following definition.

DEFINITION 9.4. Set

$$\mathbf{u}(t,\omega) = \begin{cases} u(t,\omega), & \text{for } t \geq 0 \text{ and } \omega \in Q_u,\\ \mathbf{p}, & \text{for } t \geq 0 \text{ and } \omega \in \Omega \setminus Q_u,\end{cases} \tag{9.9}$$

where $\mathbf{p}(x) = p$, $x \in \mathbb{R}^d$ for some fixed (but otherwise arbitrary) point $p \in M$.

Let $\bar{v}$ be the $\mathbb{L}$-valued random variable as in (9.3) and (9.5). In view of Proposition B.4 there exits a a measurable L^2_{loc}-valued process v such that for every $\omega \in \Omega$, the function $v(\cdot, \omega)$ is a representative of $\bar{v}(\omega)$.

LEMMA 9.5. *There exists an $\mathbb{F}$-progressively measurable L^2_{loc}-valued process $\mathbf{V}$ such that* Leb $\otimes \mathbb{P}$-*a.e.*, $\mathbf{V} = v$ *and*, $\mathbb{P}$-*almost surely*,

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \mathbf{V}(s)\, ds \text{ in } L^2_{\mathrm{loc}} \qquad \text{for all } t \geq 0. \tag{9.10}$$

Moreover $\mathbf{V}(t, \omega) \in T_{\mathbf{u}(t,\omega)}M$, Leb-*a.e. for every* $(t, \omega) \in \mathbb{R}_+ \times \Omega$. *Finally, there exists an $\mathscr{F}_0$-measurable L^2_{loc}-valued random variable $\mathbf{v}_0$ such that*

$$\mathbf{v}_0 = v_0, \qquad \mathbb{P} \text{ almost surely} \tag{9.11}$$

and, for every $\omega \in \Omega$,

$$\mathbf{v}_0(\omega) \in T_{\mathbf{u}(0,\omega)}M, \qquad \text{Leb-}a.e. \tag{9.12}$$

PROOF. Let us fix $t > 0$. Since the map

$$C(\mathbb{R}_+, H^1_{\mathrm{loc}}) \times C(\mathbb{R}_+, L^2_{\mathrm{loc}}) \ni (u, v) \mapsto u(t) - u(0) - \int_0^t v(s)\, ds \in L^2_{\mathrm{loc}}$$

is continuous, by identity (7.7) and property (R1) on page 1957, we infer that $\mathbb{P}$-almost surely

$$u^k(t) = u^k(0) + \int_0^t v^k(s)\, ds, \qquad t \geq 0.$$

Here we used the following simple rule. If R is a Borel map such that $R(X) = 0$ a.s., X and Y have equal laws, then $R(Y) = 0$ a.s. Hence, if $\varphi \in L^2_{\mathrm{comp}}$ then $\mathbb{P}$-almost surely

$$\begin{aligned} &\langle \varphi, u(t) \rangle - \langle \varphi, u(0) \rangle - \int_0^t \langle \varphi, v(s) \rangle\, ds \\ &\quad = \lim_{k \to \infty} \left[\langle \varphi, u^k(t) \rangle - \langle \varphi, u^k(0) \rangle - \int_0^t \langle \varphi, v^k(s) \rangle\, ds \right] = 0. \end{aligned} \tag{9.13}$$

Let us define an L^2_{loc}-valued process q by

$$q(t, \omega) = \begin{cases} \lim_{j \to \infty} j\left[u(t, \omega) - u\left(\max\left\{t - \frac{1}{j}, 0\right\}, \omega\right)\right], \\ \qquad \text{if the } L^2_{\mathrm{loc}}\text{-limit exists}, \\ 0, \qquad \text{otherwise}. \end{cases}$$

Then q is an L^2_{loc}-valued $\mathbb{F}$-progressively measurable and by (9.13), $q = v$ Leb $\otimes \bar{\mathbb{P}}$-almost everywhere. In particular, there exists a $\mathbb{P}$-conegligible set $N \subset \Omega$ such that $q(\cdot, \omega) = v(\cdot, \omega)$ a.e. for every $\omega \in N$. Hence all the paths of the process

$1_N q$ belong to the space $L^\infty_{\rm loc}(\mathbb{R}_+; L^2_{\rm loc})$. Hence (9.10) follows provided we define the process $\mathbf{V}$ to be equal $1_N q$.

Concerning the 2nd and the 3rd issue, let us observe that by [28], page 108, there exists a smooth compactly supported function $H:\mathbb{R}^n \to \mathbb{R}^n$ such that, for every $p \in M$, $H(p) = p$ and, with $H'(p) = d_p H \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is the Fréchet derivative of H at p,

$$H'(p)\xi = \xi \quad \Longleftrightarrow \quad \xi \in T_p M. \tag{9.14}$$

Since by (9.4), $\mathbb{P}$-almost surely, the following identity is satisfied in $L^2_{\rm loc}$, for every $t \geq 0$:

$$\int_0^t H'(\mathbf{u}(s))v(s)\,ds = H(\mathbf{u}(t)) - H(\mathbf{u}(0)) = \mathbf{u}(t) - \mathbf{u}(0) = \int_0^t v(s)\,ds,$$

we may conclude that $v = H'(\mathbf{u})v$ for Leb $\otimes\, \mathbb{P}$-almost every (t, ω). Hence, $\mathbf{V} = H'(\mathbf{u})(\mathbf{V})$ on a $\mathbb{F}$-progressively measurable and Leb $\otimes\, \mathbb{P}$-conegligible set. This, in view of (9.14), implies inclusion (9.12).

Finally, in order to prove (9.3), let us observe that since the following map $L^2_{\rm loc} \times L^2_{\rm loc} \to L^2_{\rm loc} : (u, v) \mapsto H'(u)v - v$ is continuous, $H'(u^k(0))v^k(0) = v^k(0)$, for every $k \in \mathbb{N}$, almost surely. Therefore, almost surely, $H'(\mathbf{u}(0))v_0 = v_0$. $\square$

Before we formulate the next result let us define process $\mathbf{v}$ by

$$\mathbf{v}(t,\omega) := \sum_{i,j=1}^N h_{ij}(\mathbf{u}(t,\omega))\mathbf{M}^i(t,\omega)A^j\mathbf{u}(t,\omega), \qquad \omega \in \Omega, t \geq 0. \tag{9.15}$$

LEMMA 9.6. *There exists a $\mathbb{P}$-conegligible set $Q \in \mathscr{F}$ such that if the process $\mathbf{M}^i$ is defined by $\mathbf{M}^i = 1_Q M^i$, $i \in \{1, \dots, N\}$, then the following properties are satisfied*:

(i) *For every $i \in \{1, \dots, N\}$ the process $\mathbf{M}^i$ is an $L^2_{\rm loc}$-valued $\mathbb{F}$-adapted and weakly continuous.*

(ii) *The following three identities hold for every $\omega \in Q$*:

$$\mathbf{M}^i(t,\omega) = \langle \mathbf{V}(t,\omega), A^i\mathbf{u}(t,\omega)\rangle_{\mathbb{R}^n} \qquad \textit{for a.e. } t \geq 0, \tag{9.16}$$

$$\mathbf{v}_0(\omega) = \sum_{i,j=1}^N h_{ij}(\mathbf{u}(0,\omega))\mathbf{M}^i(0,\omega)A^j\mathbf{u}(0,\omega), \tag{9.17}$$

$$\mathbf{V}(t,\omega) = \mathbf{v}(t,\omega) \qquad \textit{for a.e. } t \geq 0, \tag{9.18}$$

$$\mathbf{v}(t,\omega) \in T_{\mathbf{u}(t,\omega)M}, \qquad t \geq 0. \tag{9.19}$$

Moreover, with $\mathbf{z} = (\mathbf{u}, \mathbf{v})$, *for every* $\omega \in Q$, *for almost every* $t \geq 0$,

$$\lim_{k\to\infty} \langle f^{m_k}(z^k(t,\omega)), A^i u^k(t,\omega)\rangle_{\mathbb{R}^n} = \langle f(\mathbf{z}(t,\omega), \nabla\mathbf{u}(t,\omega)), A^i \mathbf{u}(t,\omega)\rangle_{\mathbb{R}^n}, \tag{9.20}$$

$$\lim_{k\to\infty} \langle g^{m_k}(z^k(t,\omega)), A^i u^k(t,\omega)\rangle_{\mathbb{R}^n} = \langle g(\mathbf{z}(t,\omega), \nabla\mathbf{u}(t,\omega)), A^i \mathbf{u}(t,\omega)\rangle_{\mathbb{R}^n}, \tag{9.21}$$

where the limits are with respect to the weak topology on L^2_{loc}.

PROOF. Since by (9.5) $v^k \to \bar{v}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$ on Ω, we infer that for every $R > 0$ and $\omega \in \Omega$ the sequence $\|v^k(\omega)\|_{L^\infty((0,R),L^2(B_R))}$, $k \in \mathbb{N}$ is bounded. Since, by (9.2), v^k is a continuous L^2_{loc}-valued process, $\|v^k(\omega)\|_{L^\infty((0,R),L^2(B_R))} = \|v^k(\omega)\|_{C([0,R],L^2(B_R))}$, and hence also the following sequence

$$\big(\|v^k(\omega)\|_{C([0,R],L^2(B_R))}\big)_{k\in\mathbb{N}}$$

is bounded.

Let us now fix $t \geq 0$ and $\omega \in \Omega$. Since then for every $R > 0$ the sequence $\|v^k(t,\omega)\|_{L^2(B_R)}$ is bounded, by employing the diagonalization procedure, we can find an element $\theta(t,\omega) \in L^2_{\mathrm{loc}}$ and a subsequence $(k_j)_{j=1}^\infty$, depending on t and ω, such that

$$v^{k_j}(t,\omega) \to \theta(t,\omega) \qquad \text{weakly in } L^2_{\mathrm{loc}}. \tag{9.22}$$

Since by Proposition 9.2 $u^{k_j}(t,\omega) \to u(t,\omega)$ strongly in L^2_{loc}, we infer that $\langle A^i u^{k_j}(t,\omega), v^{k_j}(t,\omega)\rangle_{\mathbb{R}^n}$ converges to $\langle A^i u(t,\omega), \theta(t,\omega)\rangle_{\mathbb{R}^n}$ in the sense of distributions and hence for any $\varphi \in \mathscr{D}$

$$\begin{aligned}\langle M^i(t,\omega), \varphi\rangle &= \lim_{j\to\infty} \langle M^i_{k_j}(t,\omega), \varphi\rangle \\ &= \lim_{j\to\infty} \langle\langle A^i u^{k_j}(t,\omega), v^{k_j}(t,\omega)\rangle_{\mathbb{R}^n}, \varphi\rangle \\ &= \langle\langle A^i u(t,\omega), \theta(t,\omega)\rangle_{\mathbb{R}^n}, \varphi\rangle,\end{aligned} \tag{9.23}$$

where the 1st identity above follows from $(9.5)_4$, the 2nd follows from (9.4) the 3rd follows from (9.22) and Proposition 9.2. Summarizing, we proved that

$$M^i(t,\omega) = \langle A^i u(t,\omega), \theta(t,\omega)\rangle_{\mathbb{R}^n}, \qquad (t,\omega) \in \mathbb{R}_+ \times \Omega. \tag{9.24}$$

Let $Q := Q_u$ be the event introduced in Proposition 9.3 and let $\mathbf{u}$ be the process introduced in Definition 9.4. Let us assume that $\omega \in Q$. Then, since $\mathbf{u}$ is an M-valued process and hence uniformly bounded, it follows from (9.24) and (9.22) that for every $R > 0$, we have

$$\sup_{t\in[0,R]} \|M^i(t,\omega)\|_{L^2(B_R)} \leq c \sup_{k\geq 0} \|v^k(\omega)\|_{C([0,R];L^2(B_R))} < \infty. \tag{9.25}$$

So we conclude that the process $\mathbf{M}_i := 1_{Q_u} M^i$ takes values in the space L^2_{loc}. Consequently, in view of Lemma D.4, as $M^i \in C_w(\mathbb{R}_+; L^r_{\mathrm{loc}})$, and by (9.25), $\mathbf{M}_i$ has weakly continuous paths in L^2_{loc}. In this way the proof of one part of claim (i) is complete. Later on we will deal with the adaptiveness of the process $1_{Q_u} M^i$.

In the next part of the proof we shall deal with (9.16). For this aim let us observe that for any $\varphi \in \mathscr{D}$ and $(t,\omega) \in \mathbb{R}_+ \times \Omega$, by $(9.5)_{1,2}$ and Proposition B.4, we infer that

$$\begin{aligned}\int_0^t \langle \varphi, M^i(s)\rangle\, ds &= \lim_{k\to\infty} \int_0^t \langle \varphi, \langle v^k(s), A^i u^k(s)\rangle_{\mathbb{R}^n}\rangle\, ds \\ &= \lim_{k\to\infty} \int_0^t \langle \varphi, \langle v^k(s), A^i u(s)\rangle_{\mathbb{R}^n}\rangle\, ds \\ &= \int_0^t \langle \varphi, \langle v(s), A^i u(s)\rangle_{\mathbb{R}^n}\rangle\, ds.\end{aligned}$$

Hence we infer that for every $\omega \in \Omega$

$$\langle v(t,\omega), A^i u(t,\omega)\rangle_{\mathbb{R}^n} = M^i(t,\omega) \qquad \text{for almost every } t \geq 0 \tag{9.26}$$

that proves (9.16).

Concerning the proofs of (9.20) and (9.21), let us observe that we only need to prove the former one, as the proof of the latter is identical. Moreover, in view of formulae (1.3) and (9.26) we need to deal with the following three limits, weakly in L^2_{loc}, on $\mathbb{R}_+ \times \Omega$:

$$\lim_{k\to\infty} \langle f_0^{m_k}(u^k) v^k, A^i u^k\rangle_{\mathbb{R}^n} = f_0(u) M^i, \tag{9.27}$$

$$\lim_{k\to\infty} \langle f_l^{m_k}(u^k) \partial_{x_l} u^k, A^i u^k\rangle_{\mathbb{R}^n} = \langle f_l(u)\partial_{x_l} u, A^i u\rangle_{\mathbb{R}^n}, \tag{9.28}$$

$$\lim_{k\to\infty} \langle f_{d+1}^{m_k}(u^k), A^i u^k\rangle_{\mathbb{R}^n} = \langle f_{d+1}(u), A^i u\rangle_{\mathbb{R}^n}. \tag{9.29}$$

The last of these three follows easily from Proposition 9.2 [according to which for every $t \in \mathbb{R}_+$ and every $R > 0$ $u^k(t) \to u(t)$ in $L^2(B_R)$] and the convergence (7.4). The proofs of the middle ones are more complex but can be done in a similar (but simpler) way than the proof of the first (which we present below).

To prove (9.27) let us choose $R > 0$ such that (7.2) holds, in particular $\bigcup_{m\in\mathbb{N}} \operatorname{supp}(f_0^m) \subset B(0,R)$. Since $f_0^m \to f_0$ uniformly $f_0^m(u) M^i \in L^2_{\mathrm{loc}}$ by (9.25), by $(9.5)_4$ and

$$|f_0^{m_k}(u^k)\langle v^k, A^i u^k\rangle_{\mathbb{R}^n}| \leq c_R \mathbf{1}_{B_R}(u^k) |v^k|_{\mathbb{R}^n},$$

and by the Lebesgue dominated convergence theorem, we infer that for every $\varphi \in L^2_{\mathrm{comp}}$

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \varphi f_0^{m_k}(u^k) M_k^i\, dx = \int_{\mathbb{R}^d} \varphi f_0(u) M^i\, dx \qquad \text{on } \mathbb{R}_+ \times \Omega, \tag{9.30}$$

which proves (9.27). As mentioned earlier, this proves (9.20).

To prove that the process $\mathbf{M}^i$ is adapted, let us first notice that, by (9.26) and Lemma 9.5, for almost every $t \geq 0$, $M^i(t) = \langle \mathbf{V}(t), A^i u(t)\rangle_{\mathbb{R}^n}$ almost surely, hence $M^i(t): \Omega \to L^r_{\text{loc}}$ is $\mathscr{F}_t$-measurable for almost every $t \geq 0$. Also, since $\langle v_0, A^i u(0)\rangle = M^i(0)$ on Ω, the random variable $M^i(0): \Omega \to L^r_{\text{loc}}$ is $\mathscr{F}_0$-measurable. Now, if $\varphi \in L^r_{\text{comp}}$, then $\langle \varphi, M^i\rangle$ is continuous; hence M^i is $\mathbb{F}$-adapted in L^r_{loc}.

Finally we will prove the 2nd and 3rd identities in claim (ii). For this aim let H be the function introduced in the proof of Lemma 9.5. Then by (9.15) for every $q \in \mathbb{Q}_+$,

$$H'(\mathbf{u}(q,\omega))\mathbf{v}(q,\omega) = \mathbf{v}(q,\omega) \qquad \text{almost surely.}$$

Since both the left-hand side and the right-hand side of the last equality are weakly continuous in L^2_{loc}, the proof of both identities (9.17) and (9.18) is complete. In conclusion, the proof of Lemma 9.6 is complete. □

The proof of the following lemma will be given jointly with the proof of Lemma 9.8.

LEMMA 9.7. *The processes $(w_l)_{l=1}^{\infty}$ are i.i.d. real $\mathbb{F}$-Wiener processes.*

To formulate the next result let us define a distribution-valued process P^i by formula

$$\begin{aligned} P^i(t) = \mathbf{M}^i(t) - \mathbf{M}^i(0) - \sum_{j=1}^{d} \partial_{x_j}\left[\int_0^t \langle \partial_{x_j}\mathbf{u}(\tau), A^i\mathbf{u}(\tau)\rangle_{\mathbb{R}^n}\, d\tau\right] \\ - \int_0^t \langle f(\mathbf{u}(\tau), \mathbf{V}(\tau), \nabla\mathbf{u}(\tau)), A^i\mathbf{u}(\tau)\rangle_{\mathbb{R}^n}\, d\tau, \qquad t \geq 0. \end{aligned} \tag{9.31}$$

Since the integrals in (9.31) are convergent in L^2_{loc}, the process P^i takes values in $\mathbb{W}^{-1,2}_R$ for every $R > 0$; see Appendix C. Moreover, we have the following result.

LEMMA 9.8. *For any $\varphi \in W^{1,r}_{\text{comp}}$ the process $\langle \varphi, P^i\rangle$ is an $\mathbb{F}$-martingale, and its quadratic and cross variations satisfy, respectively, the following*:

$$\langle\langle \varphi, P^i\rangle\rangle = \int_0^{\cdot} \|[\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}]^*\varphi\|^2_{H_\mu}\, ds, \tag{9.32}$$

$$\langle\langle \varphi, P^i\rangle, w_l\rangle = \int_0^{\cdot} \langle[\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i, u(s)\rangle_{\mathbb{R}^n}]^*\varphi, e_l\rangle_{H_\mu}\, ds, \tag{9.33}$$

where $g^*\varphi$ denotes the only element in H_μ such that $\langle g\xi, \varphi\rangle = \langle \xi, g^*\varphi\rangle_{H_\mu}$, $\forall \xi \in H_\mu$.

PROOF. Let us fix a real number $p \in [2, \infty)$ and $R > 0$, and a function $\varphi \in W^{1,r}_{\mathrm{comp}}$ such that $\operatorname{supp} \varphi \subset B_R \subset \mathbb{R}^d$. Then, by employing the argument used earlier in the paragraph between (8.9) and (8.11), the maps ${}^1_R N^i_k : C(\mathbb{R}_+, \mathscr{H}_{\mathrm{loc}}) \to C(\mathbb{R}_+, \mathbb{W}^{-1,r}_R)$, ${}^2_\varphi N^i_k : C(\mathbb{R}_+, \mathscr{H}_{\mathrm{loc}}) \to C(\mathbb{R}_+)$ and ${}^{\;3}_{\varphi,l} N^i_k : C(\mathbb{R}_+, \mathscr{H}_{\mathrm{loc}}) \to C(\mathbb{R}_+)$ defined by, for $t \in \mathbb{R}_+$,

$$
\begin{aligned}
[{}^1_R N^i_k(u,v)](t) = {} & \langle v(t), A^i u(t)\rangle_{\mathbb{R}^n} - \langle v(0), A^i u(0)\rangle_{\mathbb{R}^n} \\
& - \sum_{j=1}^d \partial_{x_j}\left[\int_0^t \langle \partial_{x_j} u(s), A^i u(s)\rangle_{\mathbb{R}^n}\, ds\right] \\
& - \int_0^t \langle f^{m_k}(z(s), \nabla u(s)), A^i u(s)\rangle_{\mathbb{R}^n}\, ds, \\
[{}^2_\varphi N^i_k(u,v)](t) = {} & \int_0^t \|[\langle g^{m_k}(z(s), \nabla u(s)), A^i u(s)\rangle_{\mathbb{R}^n}]^* \varphi\|^2_{H_\mu}\, ds, \\
[{}^{\;3}_{\varphi,l} N^i_k(u,v)](t) = {} & \int_0^t \langle [\langle g^{m_k}(z(s), \nabla u(s)), A^i u(s)\rangle_{\mathbb{R}^n}]^* \varphi, e_l\rangle_{H_\mu}\, ds
\end{aligned}
$$

are continuous. Thus, if we set $z^k = (u^k, v^k)$, the laws of the random variables $({}^1_R N^i_k(z^k), {}^2_\varphi N^i_k(z^k), {}^{\;3}_{\varphi,l} N^i_k(z^k), z^k, w^k)$ and $({}^1_R N^i_k(Z^{m_k}), {}^2_\varphi N^i_k(Z^{m_k}), {}^{\;3}_{\varphi,l} N^i_k(Z^{m_k}), Z^{m_k}, \beta^{m_k})$ are equal on

$$
\begin{aligned}
& \mathscr{B}(C(\mathbb{R}_+; \mathbb{W}^{-1,r}_R)) \otimes \mathscr{B}(C(\mathbb{R}_+)) \otimes \mathscr{B}(C(\mathbb{R}_+)) \\
& \quad \otimes \mathscr{B}(C(\mathbb{R}_+; \mathscr{H}_{\mathrm{loc}})) \otimes \mathscr{B}(C(\mathbb{R}_+; \mathbb{R}^{\mathbb{N}})).
\end{aligned}
$$

Since by identity (8.9),

$$
{}^1_R N^i_k(Z^{m_k})(t) = \int_0^t \langle g^{m_k}(Z^{m_k}(s), \nabla U^{m_k}(s)), A^i U^{m_k}(s)\rangle_{\mathbb{R}^n}\, dW^{m_k}\, ds
$$

for $t \in \mathbb{R}_+$ in $\mathbb{W}^{-1,r}_R$, by the Burkholder–Gundy–Davis inequality and Lemmata 3.3 and 8.1, we infer that for any $\delta > 0$,

$$
\begin{aligned}
& \mathbb{E} 1_{B^{\mathscr{H}_{2R}}_\delta}(z^k(0)) \Big[\sup_{t\in[0,R]} |\langle \varphi, {}^1_R N^i_k(z^k)(t)\rangle|^p \\
& \qquad\qquad + \sup_{t\in[0,R]} |{}^2_\varphi N^i_k(z^k)(t)|^{p/2} + \sup_{t\in[0,R]} |{}^{\;3}_{\varphi,l} N^i_k(z^k)(t)|^p\Big] \\
& \quad = \mathbb{E}^{m_k} 1_{B^{\mathscr{H}_{2R}}_\delta}(Z^{m_k}(0)) \\
& \qquad \times \Big[\sup_{t\in[0,R]} |\langle \varphi, {}^1_R N^i_k(Z^{m_k})(t)\rangle|^p \\
& \qquad\quad + \sup_{t\in[0,R]} |{}^2_\varphi N^i_k(Z^{m_k})(t)|^{p/2} + \sup_{t\in[0,R]} |{}^{\;3}_{\varphi,l} N^i_k(Z^{m_k})(t)|^p\Big] \\
& \quad \le c_{p,R,\delta} \|\varphi\|^p_{L^2(B_R)}.
\end{aligned}
\tag{9.34}
$$

Moreover, by property (R1) on page 1957 we have

$$\mathbb{E}^{m_k}|\beta_l^{m_k}(r)|^p = \mathbb{E}|w_l^k(r)|^p = c_{p,r}, \qquad r \geq 0. \tag{9.35}$$

Let us consider times $s, t \in \mathbb{R}_+$ such that $s < t$. We can always assume that $t < R$. Let us choose numbers $s_1, \dots, s_K$ such that $0 \leq s_1 \leq \cdots \leq s_K \leq s$. Let $h : \mathbb{R}^K \times \mathbb{R}^{K \times K} \times [C(\mathbb{R}_+; \mathbb{R}^{\mathbb{N}})]^K \to [0, 1]$ be a continuous function and $\varphi_1, \dots, \varphi_K \in L^2_{\mathrm{comp}}$. Let us denote

$$\begin{aligned}
\tilde{a}_k &= h(\langle \varphi_{i_1}, V^{m_k}(0)\rangle_{i_1 \leq K}, \langle \varphi_{i_2}, U^{m_k}(s_{i_3})\rangle_{i_2, i_3 \leq K}, (\beta^{m_k}(s_{i_4}))_{i_4 \leq K}),\\
a_k &= h(\langle \varphi_{i_1}, v^k(0)\rangle_{i_1 \leq K}, \langle \varphi_{i_2}, u^k(s_{i_3})\rangle_{i_2, i_3 \leq K}, (w^k(s_{i_4}))_{i_4 \leq K}),\\
a &= h(\langle \varphi_{i_1}, v_0\rangle_{i_1 \leq K}, \langle \varphi_{i_2}, \mathbf{u}(s_{i_3})\rangle_{i_2, i_3 \leq K}, (w(s_{i_4}))_{i_4 \leq K}),\\
\tilde{q}_k &= \tilde{a}_k \mathbf{1}_{B_\delta^{\mathscr{H}_{2R}}}(Z^{m_k}(0)), \qquad q_k = a_k \mathbf{1}_{B_\delta^{\mathscr{H}_{2R}}}(z^k(0)),\\
q &= a \mathbf{1}_{B_\delta^{\mathscr{H}_{2R}}}((\mathbf{u}(0), \mathbf{v}_0)).
\end{aligned}$$

Then $\mathbb{E}^{m_k}\tilde{a}_k(\beta_l^{m_k}(t) - \beta_l^{m_k}(s)) = \mathbb{E}a_k(w_l^k(t) - w_l^k(s)) = 0$ and similarly,

$$\begin{aligned}
&\mathbb{E}^{m_k}\tilde{q}_k[\langle \varphi, \tfrac{1}{R}N_k^i(Z^{m_k})(t)\rangle - \langle \varphi, \tfrac{1}{R}N_k^i(Z^{m_k})(s)\rangle] \\
&\qquad = \mathbb{E}q_k[\langle \varphi, \tfrac{1}{R}N_k^i(z^k)(t)\rangle - \langle \varphi, \tfrac{1}{R}N_k^i(z^k)(s)\rangle] = 0,\\
&\mathbb{E}^{m_k}\tilde{q}_k[\langle \varphi, \tfrac{1}{R}N_k^i(Z^{m_k})(t)\rangle^2 - {}^{2}_{\varphi}N_k^i(Z^{m_k})(t)\\
&\qquad\quad - \langle \varphi, \tfrac{1}{R}N_k^i(Z^{m_k})(s)\rangle^2 + {}^{2}_{\varphi}N_k^i(Z^{m_k})(s)] = 0,\\
&\mathbb{E}^{m_k}\tilde{a}_k[\beta_l^{m_k}(t)\beta_j^{m_k}(t) - t\delta_{lj} - \beta_l^{m_k}(s)\beta_j^{m_k}(s) + s\delta_{lj}] = 0,\\
&\mathbb{E}^{m_k}\tilde{q}_k[\langle \varphi, \tfrac{1}{R}N_k^i(Z^{m_k})(t)\rangle\beta_l^{m_k}(t) - {}^{3}_{\varphi,l}N_k^i(Z^{m_k})(t)]
\end{aligned} \tag{9.36}$$

$$\qquad - \mathbb{E}^{m_k}\tilde{q}_k[\langle \varphi, \tfrac{1}{R}N_k^i(Z^{m_k})(s)\rangle\beta_l^{m_k}(s) - {}^{3}_{\varphi,l}N_k^i(Z^{m_k})(s)] = 0. \tag{9.37}$$

Next, since by Lemma 3.3 $\|(g^{m_k})^*\varphi\|^2_{H_\mu} = \sum_l \langle g^{m_k}, \varphi e_l\rangle^2$ and

$$\sum_l \|\varphi e_l\|^2_{L^2(B_R)} \leq c\|\varphi\|^2_{L^2(B_R)}$$

by applying the compactness of the embedding of $H^1(B_R)$ in $L^{2r/(r-2)}(B_R)$, Lemma 9.6, property (R2) from page 1957 and the Lebesgue dominated convergence theorem, we infer that the following three limits in $C(\mathbb{R}_+)$ exist almost surely:

$$\begin{aligned}
&\lim_{k\to\infty}\langle \varphi, \tfrac{1}{R}N_k^i(z^k)\rangle = \langle \varphi, P^i\rangle,\\
&\qquad \lim_{k\to\infty} N_\varphi^{2i,k}(z^k) = \int_0^\cdot \|[\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}]^*\varphi\|^2_{H_\mu}\, ds\\
&\qquad\qquad\qquad\qquad\quad =: P_\varphi^{2i},
\end{aligned}$$

$$\lim_{k\to\infty} {}_{\varphi,l}^{3}N_k^i(z^k) = \int_0^{\cdot} \langle [\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), A^i \mathbf{u}(s)\rangle_{\mathbb{R}^n}]^* \varphi, e_l\rangle_{H_\mu}\, ds$$
$$=: P_{\varphi,l}^{3i}.$$

So, in view of the uniform boundedness (9.34) and (9.35) in $L^p(\Omega)$ for p large enough, the integrals (9.36)–(9.37) converge as $k \nearrow \infty$ provided that $\mathbb{P}\{\|(\mathbf{u}(0), \mathbf{v}_0)\|_{\mathscr{H}_{2R}} = \delta\} = 0$, and so we obtain

$$\mathbb{E}q\langle\varphi, P^i(t)\rangle = \mathbb{E}q\langle\varphi, P^i(s)\rangle,$$
$$\mathbb{E}q[\langle\varphi, P^i(t)\rangle^2 - P_\varphi^{2i}(t)] = \mathbb{E}q[\langle\varphi, P^i(s)\rangle^2 - P_\varphi^{2i}(s)],$$
$$\mathbb{E}a[w_l(t)w_j(t) - t\delta_{lj}] = \mathbb{E}a[w_l(s)w_j(s) - s\delta_{lj}],$$
$$\mathbb{E}aw_l(t) = \mathbb{E}aw_l(s),$$
$$\mathbb{E}q[\langle\varphi, P^i(t)\rangle w_l(t) - P_{\varphi,l}^{3i}(t)] = \mathbb{E}q[\langle\varphi, P^i(s)\rangle w_l(s) - P_{\varphi,l}^{3i}(s)].$$

Hence in view of Corollary E.1, with $a_\delta = 1_{B_\delta^{\mathscr{H}_{2R}}}((\mathbf{u}(0), \mathbf{v}_0))$,

$$\mathbb{E}[a_\delta\langle\varphi, P^i(t)\rangle|\mathscr{F}_s] = a_\delta\langle\varphi, P^i(s)\rangle,$$
$$\mathbb{E}\{a_\delta[\langle\varphi, P^i(t)\rangle^2 - P_\varphi^{2i}(t)]|\mathscr{F}_s\} = a_\delta[\langle\varphi, P^i(s)\rangle^2 - P_\varphi^{2i}(s)],$$
$$\mathbb{E}[w_l(t)w_j(t) - t\delta_{lj}|\mathscr{F}_s] = w_l(s)w_j(s) - s\delta_{lj},$$
$$\mathbb{E}[w_l(t)|\mathscr{F}_s] = w_l(s),$$
$$\mathbb{E}\{a_\delta[\langle\varphi, P^i(t)\rangle w_l(t) - P_{\varphi,l}^{3i}(t)]|\mathscr{F}_s\} = a_\delta[\langle\varphi, P^i(s)\rangle w_l(s) - P_{\varphi,l}^{3i}(s)].$$

Therefore $w_1, w_2, \dots$ are i.i.d. $\mathbb{F}$-Wiener processes, $a_\delta\langle\varphi, P^i\rangle$ is an $\mathbb{F}$-martingale on $[0, T]$ and the quadratic and the cross variations satisfy $\langle a_\delta\langle\varphi, P^i\rangle\rangle = a_\delta P_\varphi^{2i}$ and $\langle a_\delta\langle\varphi, P^i\rangle, w_l\rangle = a_\delta P_{\varphi,l}^{3i}$ on $[0, R]$. In order to finish the proof we introduce the following $\mathbb{F}$-stopping times:

$$\tau_l = \inf\biggl\{t \in [0, R] : \sup_{s\in[0,t]} |\langle\varphi, P^i\rangle(t)|$$
$$+ \int_0^t \|[\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}]^*\varphi\|_{H_\mu}^2\, ds \ge l\biggr\}.$$

By letting $\delta \nearrow \infty$ we deduce that $(\tau_l)_{l=1}^\infty$ localizes $\langle\varphi, P^i\rangle$, P_φ^{2i} and $P_{\varphi,l}^{3i}$ on $[0, R]$. The result now follows by letting $R \nearrow \infty$. □

PROPOSITION 9.9. *Let $(e_l)_{l=1}^\infty$ be an ONB of the RKHS H_μ, and let us set*

$$W\psi = \sum_{l=1}^{\infty} w_l e_l(\psi), \qquad \psi \in \mathscr{S}(\mathbb{R}^d). \tag{9.38}$$

Then W is a spatially homogeneous $\mathbb{F}$-Wiener process with spectral measure μ, and for every function $\varphi \in H^1_{\mathrm{comp}}$ the following equality holds almost surely:

$$
\begin{aligned}
\langle \varphi, \mathbf{M}^i(t)\rangle = \langle \varphi, \mathbf{M}^i(0)\rangle \\
&- \sum_{k=1}^d \Big\langle \partial_{x_k}\varphi, \int_0^t \langle \partial_{x_k}\mathbf{u}(s), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}\, ds\Big\rangle \\
&+ \Big\langle \varphi, \int_0^t \langle f(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle\, ds\Big\rangle \\
&+ \Big\langle \varphi, \int_0^t \langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle\, dW(s)\Big\rangle, \\
&\qquad t \geq 0.
\end{aligned}
\tag{9.39}
$$

PROOF. By Lemma 9.7 we infer that $W\varphi$ is an $\mathbb{F}$-Wiener process and $\mathbb{E}|W_t\varphi|^2 = t\sum_l |e_l(\varphi)|^2 = t\|\widehat{\varphi}\|^2_{L^2(\mu)}$. Hence W is a spatially homogeneous $\mathbb{F}$-Wiener process with spectral measure μ.

Let now P^i be the process defined by formula (9.31). Let $\varphi \in W^{1,r}_{\mathrm{comp}}$. In order to prove equality (9.39) it is enough to show that

$$
\Big\langle \langle \varphi, P^i\rangle - \int \langle\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}\, dW, \varphi\rangle\Big\rangle = 0.
\tag{9.40}
$$

The following sequence of equalities concludes the proof of (9.40):

$$
\begin{aligned}
&\Big\langle \langle \varphi, P^i\rangle, \int \langle\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}\, dW, \varphi\rangle\Big\rangle \\
&\quad = \sum_l \Big\langle \langle \varphi, P^i\rangle, \int \langle\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n} e_l, \varphi\rangle\, dw_l\Big\rangle \\
&\quad = \sum_l \int \langle\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n} e_l, \varphi\rangle\, d\langle\langle \varphi, P^i\rangle, w_l\rangle \\
&\quad = \sum_l \int \langle\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n} e_l, \varphi\rangle^2\, ds \\
&\quad = \int \|[\langle g(\mathbf{u}(s)\mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}]^*\varphi\|^2_{H_\mu}\, ds \\
&\quad = \langle\langle \varphi, P^i\rangle\rangle = \Big\langle \int \langle\langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i\mathbf{u}(s)\rangle_{\mathbb{R}^n}\, dW, \varphi\rangle\Big\rangle. \qquad \square
\end{aligned}
$$

LEMMA 9.10. *The L^2_{loc}-valued process $\mathbf{v}$ introduced in (9.15) is $\mathbb{F}$-adapted and weakly continuous. Moreover, $\mathbf{v}(t) \in T_{\mathbf{u}(t)}M$ for every $t \geq 0$ almost surely and* *for every* $\varphi \in \mathscr{D}(\mathbb{R}^d)$

$$
\begin{aligned}
\langle \mathbf{v}(t), \varphi\rangle = \langle \mathbf{v}(0), \varphi\rangle &+ \int_0^t \langle \mathbf{u}(s), \Delta\varphi\rangle\, ds \\
&+ \int_0^t \langle \mathbf{S}_{\mathbf{u}(s)}(\mathbf{v}(s), \mathbf{v}(s)), \varphi\rangle \\
&- \sum_{k=1}^d \int_0^t \langle \mathbf{S}_{\mathbf{u}(s)}(\partial_{x_k}\mathbf{u}(s), \partial_{x_k}\mathbf{u}(s)), \varphi\rangle \\
&+ \int_0^t \langle f(\mathbf{u}(s), \mathbf{v}(s), \nabla\mathbf{u}(s)), \varphi\rangle\, ds \\
&+ \int_0^t \langle g(\mathbf{u}(s), \mathbf{v}(s), \nabla\mathbf{u}(s))\, dW, \varphi\rangle
\end{aligned}
\tag{9.41}
$$

almost surely for every $t \geq 0$.

PROOF. Obviously the process $\mathbf{v}$ is L^2_{loc}-valued. The $\mathbb{F}$-adaptiveness and the weak continuity of $\mathbf{v}$ follow from the definition (9.15) and Lemma 9.6.

In order to prove the equality (9.41) let us take a test function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and functions h_{ij} as in assumption (M4). Then we consider vector fields Y^i, $i = 1, \dots, n$, defined by formula (5.6), that is, $Y^i(x) = \sum_{j=1}^N h_{ij}(x) A^j x$. Let $\mathbf{M}^i$ be the process introduced in Lemma 9.6 and which satisfies the identity (9.41). Then by applying the Itô lemma from [5] to the processes $\mathbf{u}$ and $\mathbf{M}^i$ and the vector field Y^i, we get the following equality:

$$
\begin{aligned}
&\sum_{i=1}^N \langle \mathbf{M}^i(t) Y^i(\mathbf{u}(t)), \varphi\rangle \\
&\quad = \sum_{i=1}^N \langle \mathbf{M}^i(0) Y^i(\mathbf{u}(0)), \varphi\rangle + \sum_{i=1}^N \int_0^t \langle \mathbf{M}^i(s) \big(d_{\mathbf{u}(s)} Y^i\big)(\mathbf{V}(s)), \varphi\rangle\, ds \\
&\qquad - \sum_{i=1}^N \sum_{l=1}^d \int_0^t \langle \langle \partial_{x_l}\mathbf{u}(s), A^i \mathbf{u}(s)\rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \partial_{x_l}\varphi\rangle\, ds \\
&\qquad - \sum_{i=1}^N \sum_{l=1}^d \int_0^t \big\langle \langle \partial_{x_l}\mathbf{u}(s), A^i \mathbf{u}(s)\rangle_{\mathbb{R}^n} \big(d_{\mathbf{u}(s)} Y^i\big)(\partial_{x_l}\mathbf{u}(s)), \varphi\big\rangle\, ds \\
&\qquad + \sum_{i=1}^N \int_0^t \langle \langle f(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i \mathbf{u}(s)\rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \varphi\rangle\, ds \\
&\qquad + \sum_{i=1}^N \int_0^t \langle \langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla\mathbf{u}(s)), A^i \mathbf{u}(s)\rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s))\, dW, \varphi\rangle
\end{aligned}
\tag{9.42}
$$

$\mathbb{P}$-almost surely for every $t \geq 0$.

Next, by identity (9.15) in Lemma 9.6, we have, for each $t \geq 0$ and $\omega \in \Omega$, $\langle \mathbf{v}(t,\omega), \varphi \rangle = \sum_{i=1}^{N} \langle \mathbf{M}^i(t) Y^i(\mathbf{u}(t)), \varphi \rangle$ and by identity (5.7) we have, for each $s \geq 0$ and $\omega \in \Omega$,

$$\sum_{i=1}^{N} \sum_{l=1}^{d} \langle \langle \partial_{x_l} \mathbf{u}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \partial_{x_l} \varphi \rangle = \sum_{l=1}^{d} \langle \partial_{x_l} \mathbf{u}(s), \partial_{x_l} \varphi \rangle. \tag{9.43}$$

Furthermore, from Lemma 5.4 we infer that for each $s \geq 0$ and $\omega \in \Omega$,

$$\begin{aligned} &\sum_{i=1}^{N} \sum_{l=1}^{d} \big\langle \langle \partial_{x_l} \mathbf{u}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} \big(d_{\mathbf{u}(s)} Y^i\big)(\partial_{x_l} \mathbf{u}(s)), \varphi \big\rangle \\ &\qquad = \sum_{l=1}^{d} \big\langle \mathbf{S}_{\mathbf{u}(s)}(\partial_{x_l} \mathbf{u}(s), \partial_{x_l} \mathbf{u}(s)), \varphi \big\rangle. \end{aligned} \tag{9.44}$$

Similarly, by the identities (5.8) and (9.16) in Lemmata 5.4 and 9.6, we infer that for a.e. $s \geq 0$,

$$\begin{aligned} \sum_{i=1}^{N} \big\langle \mathbf{M}^i(s) \big(d_{\mathbf{u}(s)} Y^i\big)(\mathbf{V}(s)), \varphi \big\rangle &= \sum_{i=1}^{N} \big\langle \langle \mathbf{V}(s), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} \big(d_{\mathbf{u}(s)} Y^i\big)(\mathbf{V}(s)), \varphi \big\rangle \\ &= \big\langle \mathbf{S}_{\mathbf{u}(s)}(\mathbf{V}(s), \mathbf{V}(s)), \varphi \big\rangle \\ &= \big\langle \mathbf{S}_{\mathbf{u}(s)}(\mathbf{v}(s), \mathbf{v}(s)), \varphi \big\rangle \end{aligned} \tag{9.45}$$

holds a.s. Moreover, by a similar argument based on (5.7) we can deal with the integrands of the last two terms on the RHS of (9.42). Indeed by (5.6) we get

$$\begin{aligned} &\sum_{i=1}^{N} \langle \langle f(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \varphi \rangle \\ &\qquad = \langle f(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), \varphi \rangle, \end{aligned} \tag{9.46}$$

$$\begin{aligned} &\sum_{i=1}^{N} \langle \langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), A^i \mathbf{u}(s) \rangle_{\mathbb{R}^n} Y^i(\mathbf{u}(s)), \varphi \rangle \\ &\qquad = \langle g(\mathbf{u}(s), \mathbf{V}(s), \nabla \mathbf{u}(s)), \varphi \rangle. \end{aligned} \tag{9.47}$$

Summing up, we infer from equality (9.42), and the other equalities which follow it, that for every $t \geq 0$ almost surely

$$\begin{aligned} \langle \mathbf{v}(t), \varphi \rangle = \langle \mathbf{v}(0), \varphi \rangle &- \int_0^t \sum_{l=1}^{d} \langle \partial_{x_l} \mathbf{u}(s), \partial_{x_l} \varphi \rangle \, ds \\ &- \int_0^t \sum_{l=1}^{d} \big\langle \mathbf{S}_{\mathbf{u}(s)}(\partial_{x_l} \mathbf{u}(s), \partial_{x_l} \mathbf{u}(s)), \varphi \big\rangle \, ds \end{aligned}$$

$$+\int_0^t \langle f(\mathbf{u}(s), \mathbf{v}(s), \nabla \mathbf{u}(s)), \varphi\rangle$$
$$+\int_0^t \langle g(\mathbf{u}(s), \mathbf{v}(s), \nabla \mathbf{u}(s))\, dW(s), \varphi\rangle.$$

This concludes the proof of Lemma 9.10. □

To complete the proof of the existence of a solution, that is, the proof of Theorem 4.4, let us observe that the above equality is nothing else but (4.3). Moreover, (4.2) follows from (9.10) and (9.15). This proves that if the process $\mathbf{z} := (\mathbf{u}, \mathbf{v})$ then $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P}, W, z)$ a weak solution to equation (1.1).

APPENDIX A: THE JAKUBOWSKI'S VERSION OF THE SKOROKHOD REPRESENTATION THEOREM

THEOREM A.1. *Let X be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : X \to \mathbb{R}$ that separate points of X. Let us denote by $\mathscr{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*:

(j1) *every compact subset of X is metrizable*;
(j2) *every Borel subset of a σ-compact set in X belongs to $\mathscr{S}$*;
(j3) *every probability measure supported by a σ-compact set in X has a unique Radon extension to the Borel σ-algebra on X*;
(j4) *if (μ_m) is a tight sequence of probability measures on $(X, \mathscr{S})$, then there exists a subsequence (m_k), a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with X-valued Borel measurable random variables X_k, X such that μ_{m_k} is the law of X_k and X_k converge almost surely to X. Moreover, the law of X is a Radon measure.*

PROOF. See [31]. □

COROLLARY A.2. *Under the assumptions of Theorem A.1, if Z is a Polish space and $b : Z \to X$ is a continuous injection, then $b[B]$ is a Borel set whenever B is Borel in Z.*

PROOF. See Corollary A.2 in [47]. □

APPENDIX B: THE SPACE $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$

Let $\mathbb{L} = L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2_{\mathrm{loc}})$ be the space of equivalence classes $[f]$ of all measurable functions $f : \mathbb{R}_+ \to L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ such that $\|f\|_{L^2(B_n)} \in L^\infty(0, n)$ for every $n \in \mathbb{N}$. The space $\mathbb{L}$ is equipped with the locally convex topology generated by functionals

$$f \mapsto \int_0^n \int_{B_n} \langle g(t, x), f(t, x)\rangle_{\mathbb{R}^n}\, dx\, dt, \tag{B.1}$$

where $n \in \mathbb{N}$ and $g \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^d))$.

Let us also define a space

$$Y_m = L^1((0, m), L^2(B_m)). \tag{B.2}$$

Let us recall that $L^\infty((0, m), L^2(B_m)) = Y_m^*$. Consider the following natural restriction maps:

$$\pi_m : L^2(\mathbb{R}^d) \ni g \mapsto g|_{B_m} \in L^2(B_m), \tag{B.3}$$

$$l_m : \mathbb{L} \ni f \mapsto (\pi_m \circ f)|_{[0,m]} \in (Y_m^*, w^*). \tag{B.4}$$

The following results describe some properties of the space $\mathbb{L}$.

LEMMA B.1. *A map $l = (l_m(f))_{m\in\mathbb{N}} : \mathbb{L} \to \prod_{m\in\mathbb{N}}(Y_m^*, w^*)$ is a homeomorphism onto a closed subset of $\prod_{m\in\mathbb{N}}(Y_m^*, w^*)$.*

PROOF. The proof is straightforward. □

COROLLARY B.2. *Given any sequence $(a_m)_{m=1}^\infty$ of positive numbers, the set*

$$\{f \in \mathbb{L} : \|f\|_{L^\infty((0,m),L^2(B_m))} \le a_m, m \in \mathbb{N}\} \tag{B.5}$$

is compact in $\mathbb{L}$.

PROOF. The proof follows immediately from Lemma B.1 and the Banach–Alaoglu theorem since a product of compacts is a compact by the Tychonov theorem. □

COROLLARY B.3. *The Skorokhod representation Theorem A.1 holds for every tight sequence of probability measures defined on $(\mathbb{L}, \sigma(\mathbb{L}^*))$, where the σ-algebra $\sigma(\mathbb{L}^*)$ is the σ-algebra on $\mathbb{L}$ generated by $\mathbb{L}^*$.*

PROOF. Since each Y_m is a separable Banach space, there exists a sequence $(j_{m,k})_{k=1}^\infty$, such that each $j_{m,k} : (Y_m^*, w^*) \to \mathbb{R}$ is a continuous function, and $(j_{m,k})_{k=1}^\infty$ separate points of Y_m^*. Consequently, such a separating sequence of continuous functions exists for product space $\prod(Y_m^*, w^*)$, and, by Lemma B.1, for the $\mathbb{L}$ as well. Existence of a separating sequence of continuous functions is sufficient for the Skorokhod representation theorem to hold by the Jakubowski theorem [31]. □

PROPOSITION B.4. *Let $\bar{\xi}$ be an $\mathbb{L}$-valued random variable. Then there exists a measurable L^2_{loc}-valued process ξ such that for every $\omega \in \Omega$,*

$$[\xi(\cdot, \omega)] = \bar{\xi}(\omega). \tag{B.6}$$

PROOF. Let $(\varphi_n)_{n=1}^{\infty}$ be an approximation of identity on $\mathbb{R}$. Let us fix $t \geq 0$ and $n \in \mathbb{N}^*$. Then the linear operator

$$I_n(t) : \mathbb{L} \ni f \mapsto \int_0^{\infty} \varphi_n(t-s) f(s)\, ds \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \tag{B.7}$$

is well defined, and for all $\psi \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^* = L^2_{\mathrm{comp}}(\mathbb{R}^d)$ and $t \geq 0$, the function $\psi \circ I_n(t) : \mathbb{L} \to \mathbb{R}$ is continuous. Hence in view of Corollary E.1 the map $I_n(t)$ is Borel measurable. We put

$$I(t) : \mathbb{L} \ni f \mapsto \begin{cases} \lim_{n\to\infty} I_n(t)(t), & \text{provided the limit in } L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ exists}, \\ 0, & \text{otherwise}. \end{cases} \tag{B.8}$$

Then (by employing the Lusin theorem [53] in case (ii)) we infer that given $f \in \mathbb{L}$:

(i) the map $\mathbb{R}_+ \ni t \mapsto I_n(t) f \in L^2_{\mathrm{loc}}$ is continuous, and
(ii) $\lim_{n\to\infty} I_n(t) f$ exists in L^2_{loc} for almost every $t \in \mathbb{R}_+$ and $[I(\cdot) f] = f$.

If we next define L^2_{loc}-valued stochastic processes ξ_n, for $n \in \mathbb{N}^*$, and ξ by $\xi_n(t,\omega) = I_n(t)(\bar{\xi}(\omega))$ and $\xi(t,\omega) = I(t)(\bar{\xi}(\omega))$ for $(t,\omega) \in \mathbb{R}_+ \times \Omega$, then by (i) above we infer that ξ_n is continuous and so measurable. Hence the process ξ is also measurable, and by (ii) above, given $\omega \in \Omega$, the function $\{\mathbb{R}_+ \ni t \mapsto \xi(t,\omega)\}$ is a representative of $\bar{\xi}(\omega)$. The proof is complete. $\square$

APPENDIX C: THE SPACE $C_w(\mathbb{R}_+; W^{k,p}_{\mathrm{loc}})$, $k \geq 0$, $1 < p < \infty$

Let us introduce the spaces, for $l \geq 0$, $R > 0$ and $p, p' \in (1,\infty)$ satisfying $\frac{1}{p'} + \frac{1}{p} = 1$, $W^{l,p}_R = \{f \in W^{l,p}(\mathbb{R}^d; \mathbb{R}^n) : f = 0 \text{ on } \mathbb{R}^d \setminus B_R\}$, $\mathbb{W}^{l,p}_R = W^{l,p}(B_R; \mathbb{R}^n)$ and $\mathbb{W}^{-l,p}_R = (W^{l,p'}_R)^*$. Let us recall that by $(W^{k,p}(B_R), \mathrm{w})$ we mean the space $W^{k,p}(B_R)$ endowed with the weak topology. We now formulate the first of the two main results in this Appendix. The proofs of them can be found in [47]; see Corollary B.2 and Proposition B.3.

COROLLARY C.1. *Assume that $\gamma > 0$, $1 < r, p < \infty$, $-\infty < l \leq k$ satisfy $\frac{1}{p} - \frac{k}{d} \leq \frac{1}{r} - \frac{l}{d}$. Then for any sequence $a = (a_m)_{m=1}^{\infty}$ of positive numbers, the set $K(a) := \{f \in C_{\mathrm{w}}(\mathbb{R}_+; W^{k,p}_{\mathrm{loc}}) : \|f\|_{L^\infty([0,m], W^{k,p}(B_m))} + \|f\|_{C^\gamma([0,m], \mathbb{W}^{l,r}_m)} \leq a_m, m \in \mathbb{N}\}$ is a metrizable compact subset of $C_{\mathrm{w}}(\mathbb{R}_+; W^{k,p}_{\mathrm{loc}})$.*

PROPOSITION C.2. *The Skorokhod representation Theorem A.1 holds for every tight sequence of probability measures defined on the σ-algebra generated by the following family of maps*:

$$\{C_{\mathrm{w}}(\mathbb{R}_+; W^{k,p}_{\mathrm{loc}}) \ni f \mapsto \langle \varphi, f(t) \rangle \in \mathbb{R}\} : \varphi \in \mathscr{D}(\mathbb{R}^d, \mathbb{R}^n), \qquad t \in [0,\infty).$$

APPENDIX D: TWO ANALYTIC LEMMATA

LEMMA D.1. *Suppose that $(D, \mathcal{S}, \mu)$ is a finite measure space with $\mu \geq 0$. Assume that a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ is of linear growth. Then the following functional*:

$$F(u) := \int_D f(u(x))\mu(dx), \qquad u \in L^2(D, \mu), \tag{D.1}$$

is well defined. Moreover, if f is of C^1 class such that f' is of linear growth, then F is also of C^1 class and

$$d_u F(v) = F'(u)(v) = \int_D f'(u(x))v(x)\mu(dx), \qquad u, v \in L^2(D, \mu). \tag{D.2}$$

PROOF. The measurability and the linear growth of f, together with the assumption that $\mu(D) < \infty$ [what implies that $L^2(D, \mu) \subset L^1(D, \mu)$], imply that F is well defined.

Let us fix for each $u \in L^2(D, \mu)$. The linear growth of f' and the Cauchy–Schwarz inequality imply that the functional Φ defined by the RHS of the inequality D.2 is a bounded linear functional on $L^2(D, \mu)$. It is standard to show that F is Fréchet differentiable at u and that $d_F = \Phi$. $\square$

LEMMA D.2. *Suppose that an $L^2(D, \mu)$-valued sequence v^k converges weakly in $L^2(D, \mu)$ to $v \in L^2(D, \mu)$. Suppose also that an $L^2(D, \mu)$-valued sequence u^k converges strongly in $L^2(D, \mu)$ to $u \in L^2(D, \mu)$. Then for any $\varphi \in L^\infty(D, \mu)$*,

$$\int_D u^k(x)v^k(x)\varphi(x)\,dx \to \int_D u(x)v(x)\varphi(x)\,dx. \tag{D.3}$$

PROOF. We have the following inequality:

$$\begin{aligned}
&\left|\int_D u^k(x)v^k(x)\varphi(x)\,dx - \int_D u(x)v(x)\varphi(x)\,dx\right| \\
&\quad \leq \left|\int_D \bigl(v^k(x) - v(x)\bigr)u(x)\varphi(x)\,dx\right| \\
&\quad \leq \left(\int_D |u^k(x) - u(x)|^2\,dx\right)^{1/2}\left(\int_D |v^k(x)|^2\,dx\right)^{1/2}|\varphi|_{L^\infty}.
\end{aligned} \tag{D.4}$$

Since on the one hand, by the first assumption the sequence, $(\int_D |v^k(x)|^2\,dx)^{1/2} = |v^k|_{L^2}$ is bounded, and $\int_D (v^k(x) - v(x))u(x)\varphi(x)\,dx$ converges to 0, and on the other hand, by the second assumption $\int_D |u^k(x) - u(x)|^2\,dx$ converges to 0, the result follows by applying (D.4). $\square$

The same proof as above applies to the following result.

LEMMA D.3. *Suppose that an $L^2_{\mathrm{loc}}(\mathbb{R}^d)$-valued sequence v^k converges weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ to $v \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose also that an $L^2_{\mathrm{loc}}(\mathbb{R}^d)$-valued sequence u^k converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ to $u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Then for any function $\varphi \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$,*

$$\int_D u^k(x)v^k(x)\varphi(x)\,dx \to \int_D u(x)v(x)\varphi(x)\,dx. \tag{D.5}$$

PROOF. The set D in the proof of Lemma D.2 has to be chosen so that $\operatorname{supp}\varphi \subset D$. □

LEMMA D.4. *Suppose that an $L^2(D,\mu)$-valued bounded sequence v^k converges weakly in $L^1(D,\mu)$ to $v \in L^2(D,\mu)$. Then $v^k \to v$ weakly in $L^2(D,\mu)$.*

APPENDIX E: A MEASURABILITY LEMMA

Let X be a separable Fréchet space (with a sequence of seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$, let X_k be separable Hilbert spaces and $i_k : X \to X_k$ linear maps such that $\|i_k(x)\|_{X_k} = \|x\|_k$, $k \geq 1$. Let $(\varphi_{k,j})_{j\in\mathbb{N}} \subset X_k^*$ separate points of X_k. Then (see Appendix C in [47]) the maps $(\varphi_{k,j} \circ i_k)_{k,j\in\mathbb{N}}$ generate the Borel σ-field on X. This implies the following important result.

COROLLARY E.1. *There exists a countable system $(\varphi_k)_{k\in\mathbb{N}} \subset \mathscr{D}$ such that for every $m \geq 0$ the maps $H^m_{\mathrm{loc}} \ni h \mapsto \langle h, \varphi_k\rangle_{L^2} \in \mathbb{R}$, $k \in \mathbb{N}$ generate the Borel σ-algebra on H^m_{loc}.*

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