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Weak quenched limiting distributions for transient one-dimensional random walk in a random environment

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Abstract. We consider a one-dimensional, transient random walk in a random i.i.d. environment. The asymptotic behaviour of such random walk depends to a large extent on a crucial parameter $\kappa > 0$ that determines the fluctuations of the process. When $0 < \kappa < 2$, the averaged distributions of the hitting times of the random walk converge to a κ-stable distribution. However, it was shown recently that in this case there does not exist a quenched limiting distribution of the hitting times. That is, it is not true that for almost every fixed environment, the distributions of the hitting times (centered and scaled in any manner) converge to a non-degenerate distribution. We show, however, that the quenched distributions do have a limit in the weak sense. That is, the quenched distributions of the hitting times – viewed as a random probability measure on $\mathbb{R}$ – converge in distribution to a random probability measure, which has interesting stability properties. Our results generalize both the averaged limiting distribution and the non-existence of quenched limiting distributions.

Résumé. Nous considérons une marche aléatoire unidimensionnelle dans un environnement i.i.d. Le comportement asymptotique d'une telle marche aléatoire dépend largement d'un paramètre crucial κ qui détermine les fluctuations du processus. Si $0 < \kappa < 2$, alors les distributions moyennées des temps d'atteinte de la marche aléatoire convergent vers une loi κ-stable. Cependant, il a été récemment prouvé que dans ce cas là, il n'existe pas de distribution limite des temps d'atteinte à environnement fixé. C'est-à-dire, il n'est pas vrai que presque tout environnement fixé, les distributions des temps d'atteinte (centrés et normalisés de quelque manière que ce soit) convergent vers une distribution non dégénérée. Nous montrons néanmoins que les distributions à environnement fixé ont une limite au sens faible. Plus précisément, les distributions à environnement fixé des temps d'atteinte – vues comme des mesures de probabilité aléatoires sur $\mathbb{R}$ – convergent en distribution vers une mesure de probabilité aléatoire qui a d'intéressantes propriétés de stabilité. Nos résultats généralisent à la fois la limite des distributions moyennisées et la non existence de distributions limites à environnement fixé.

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1. Introduction

A random walk in a random environment (RWRE) is a Markov chain with transition probabilities that are chosen randomly ahead of time. The collection of transition probabilities are referred to as the *environment* for the random walk.

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We will be concerned with nearest-neighbor RWRE on $\mathbb{Z}$, in which case the space of environments may be identified with $\Omega = [0,1]^{\mathbb{Z}}$, endowed with the cylindrical σ-field. Environments $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega$ are chosen according to a probability measure P on Ω.

Given an environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega$ and an initial location $x \in \mathbb{Z}$, we let $\{X_n\}_{n\geq 0}$ be the Markov chain with law P_ω^x defined by $P_\omega^x(X_0 = x) = 1$, and

$$P_\omega^x(X_{n+1} = z | X_n = y) = \begin{cases} \omega_y, & z = y+1, \\ 1-\omega_y, & z = y-1, \\ 0, & \text{otherwise.} \end{cases}$$

Since the environment ω is random, $P_\omega^x(\cdot)$ is a random probability measure and is called the *quenched* law. By averaging over all environments we obtain the *averaged* law

$$\mathbb{P}^x(\cdot) = \int_\Omega P_\omega^x(\cdot) P(\mathrm{d}\omega).$$

Since we will usually be concerned with RWRE starting at $x = 0$, we will denote P_ω^0 and $\mathbb{P}^0$ by P_ω and $\mathbb{P}$, respectively. Expectations with respect to P, P_ω and $\mathbb{P}$ will be denoted by E_P, E_ω and $\mathbb{E}$, respectively. Throughout the paper we will use $\mathbf{P}$ to denote a generic probability law, separate from the RWRE, with corresponding expectations $\mathbf{E}$.

We will make the following assumptions on the distribution P on environments

Assumption 1. *The environments are i.i.d. That is,* $\{\omega_x\}_{x\in\mathbb{Z}}$ *is an i.i.d. sequence of random variables under the measure* P.

Assumption 2. *The expectation* $E_P[\log\rho_0]$ *is well defined and* $E_P[\log\rho_0] < 0$. *Here* $\rho_i = \rho_i(\omega) = \frac{1-\omega_i}{\omega_i}$, *for all* $i \in \mathbb{Z}$.

In Solomon's seminal paper on RWRE [18], he showed that Assumptions 1 and 2 imply that the RWRE is transient to $+\infty$. That is, $\mathbb{P}(\lim_{n\to\infty} X_n = +\infty) = 1$. Moreover, Solomon also proved a law of large numbers with an explicit formula for the limiting velocity $\mathrm{v}_P = \lim_{n\to\infty} X_n/n$. Interestingly, $\mathrm{v}_P > 0$ if and only if $E_P[\rho_0] < 1$, and thus one can easily construct examples of RWRE that are transient with "zero speed."

Soon after Solomon's original paper, Kesten, Kozlov and Spitzer [11] analyzed the limiting distributions of transient RWRE under the following additional assumption.

Assumption 3. *The distribution of* $\log\rho_0$ *is non-lattice under* P, *and there exists a* $\kappa > 0$ *such that* $E_P[\rho_0^\kappa] = 1$ *and* $E_P[\rho_0^\kappa \log\rho_0] < \infty$.

Kesten, Kozlov and Spitzer obtained limiting distributions for the random walk X_n by first analyzing the limiting distributions of the hitting times

$$T_x := \inf\{n \geq 0\colon\ X_n = x\}.$$

Let $\Phi(x)$ be the distribution function of the standard normal distribution, and let $L_{\kappa,b}(x)$ be the distribution function of a totally skewed to the right stable istribution of index $\kappa \in (0,2)$ with scaling parameter $b > 0$ and zero shift; see [16].

Theorem 1.1 (Kesten, Kozlov and Spitzer [11]). *Suppose that Assumptions* 1–3 *hold, and let* $x \in \mathbb{R}$.

(1) *If* $\kappa \in (0,1)$, *then there exists a constant* $b > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/\kappa}} \leq x\right) = L_{\kappa,b}(x).$$

(2) *If* $\kappa = 1$, *then there exist constants* $A, b > 0$ *and a sequence* $D(n) \sim A\log n$ *so that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - nD(n)}{n} \leq x\right) = L_{1,b}(x).$$

(3) *If $\kappa \in (1,2)$, then there exists a constant $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{n^{1/\kappa}} \le x\right) = L_{\kappa,b}(x).$$

(4) *If $\kappa = 2$, then there exists a constant $\sigma > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{\sigma\sqrt{n\log n}} \le x\right) = \Phi(x).$$

(5) *If $\kappa > 2$, then there exists a constant $\sigma > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - n/\mathrm{v}_P}{\sigma\sqrt{n}} \le x\right) = \Phi(x).$$

Theorem 1.1 is then used in [11] in the natural way to obtain averaged limiting distributions for the random walk itself, but for the sake of space we do not state the precise statement here. It should be noted that a formula for the scaling parameter $b > 0$ appearing above when $\kappa < 2$ has been obtained recently in [7,8].

It was not until more recently that the limiting distributions of the hitting time and the random walk were studied under the quenched distribution. In the case when $\kappa > 2$, Alili proved a quenched central limit theorem for the hitting times of the form

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma_1\sqrt{n}} \le x\right) = \Phi(x), \quad \forall x \in \mathbb{R},\ P\text{-a.s.}, \tag{1}$$

where $\sigma_1^2 = E_P[\mathrm{Var}_\omega T_1] < \infty$ [1]. The environment-dependent centering term $E_\omega T_n$ makes it difficult to use (1) to obtain a quenched central limit theorem for the random walk, but this difficulty was overcome independently by Goldsheid [10] and Peterson [12] to obtain a quenched central limit theorem for the random walk (also with an environment-dependent centering).

When $\kappa < 2$ the situation is quite different. Even though one could reasonably expect that, similarly to (1), a limiting stable distribution of index κ existed (possibly with environment-dependent centering or scaling), this has turn out not be the case. In fact, it was shown in [13,14] that quenched limiting distributions do not exist when $\kappa < 2$. For P-a.e. environment ω, there exist two (random) subsequences $n_k = n_k(\omega)$ and $m_k = m_k(\omega)$ so that the limiting distributions of T_{n_k} and $T_{n'_k}$ under the measure P_ω are Gaussian and shifted exponential, respectively. That is,

$$\lim_{k\to\infty} P_\omega\left(\frac{T_{n_k} - E_\omega T_{n_k}}{\sqrt{\mathrm{Var}_\omega T_{n_k}}} \le x\right) = \Phi(x), \quad \forall x \in \mathbb{R},$$

and

$$\lim_{k\to\infty} P_\omega\left(\frac{T_{m_k} - E_\omega T_{m_k}}{\sqrt{\mathrm{Var}_\omega T_{m_k}}} \le x\right) = \begin{cases} 1 - \mathrm{e}^{-x-1}, & x > -1, \\ 0, & x \le -1, \end{cases} \quad \forall x \in \mathbb{R}.$$

These subsequences were then used to show the non-existence of quenched limiting distributions for the random walk as well [13,14].

These results of [13,14] are less than completely satisfying because one would like to be able to say something about the quenched distribution after a large number of steps. Also, the existence of subsequential limiting distributions that are Gaussian and shifted exponential begs the question of whether and what other types of distributions are possible to obtain through subsequences. The proof of the non-existence of quenched limiting distributions in [13] implies, for large n, the magnitude of the hitting time T_n is determined, to a large extent, by the amount of time it takes the random walk to pass a few "large traps" in the interval $[0, n]$. Moreover, as was shown in [13], Corollary 4.5, the time to cross a "large trap" is approximately an exponential random variable with parameter depending on the "size" of the trap. Therefore, one would hope that the quenched distribution of T_n could be described in terms of some random (depending on ω) weighted sum of exponential random variables. Our main results confirm this by showing that the quenched distribution – viewed as a random probability measure on $\mathbb{R}$ – converges in distribution on the space of probability measures to the law of a certain random infinite weighted sum of exponential random variables.

Before stating our main result, we introduce some notation. Let $\mathcal{M}_1$ be the space of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field. Recall that $\mathcal{M}_1$ is a complete, separable metric space when equipped with the Prohorov metric

$$\rho(\pi,\mu) = \inf\{\varepsilon > 0\colon \pi(A) \le \mu(A^\varepsilon) + \varepsilon, \mu(A) \le \pi(A^\varepsilon) + \varepsilon\ \forall A \in \mathcal{B}(\mathbb{R})\}, \quad \pi,\mu \in \mathcal{M}_1, \tag{2}$$

where $A^\varepsilon := \{x \in \mathbb{R}\colon |x-y| < \varepsilon \text{ for some } y \in A\}$ is the ε-neighbourhood of A. By a random probability measure we mean a $\mathcal{M}_1$-valued random variable, and we denote convergence in distribution of a sequence of random probability measures by $\mu_n \Longrightarrow \mu$; see [2]. This notation does carry the danger of being confused with the weak convergence of probability measures on $\mathbb{R}$, but we prefer it to the more proper, but awkward, notation $\mathcal{L}_{\mu_n} \Longrightarrow \mathcal{L}_\mu$ with $\mathcal{L}_\mu$ being the law of a random measure μ.

Next, let $\mathcal{M}_p$ be the space of Radon point processes on $(0,\infty]$; these are the point processes assigning a finite mass to all sets $(x,\infty]$ with $x > 0$. We equip $\mathcal{M}_p$ with the standard topology of vague convergence. This topology can be metrized to make $\mathcal{M}_p$ a complete separable metric space; see [15], Proposition 3.17. For point processes in $\mathcal{M}_p$ we denote vague convergence by $\zeta_n \xrightarrow{v} \zeta$. An $\mathcal{M}_p$-valued random variable will be called a random point process, and, as above, we will use the somewhat improper notation $\zeta_n \Longrightarrow \zeta$ to denote convergence in distribution of random point processes.

We define a mapping $\bar{H}\colon \mathcal{M}_p \to \mathcal{M}_1$ in the following manner. Let $\zeta = \sum_{i\ge1} \delta_{x_i}$, where (x_i) is an arbitrary enumeration of the points of $\zeta \in \mathcal{M}_p$. We let $\bar{H}(\zeta)$ to be the probability measure defined by

$$\bar{H}(\zeta)(\cdot) = \begin{cases} \mathbf{P}\left(\sum_{i\ge1} x_i(\tau_i - 1) \in \cdot\right), & \sum_{i\ge1} x_i^2 < \infty, \\ \delta_0(\cdot), & \text{otherwise}, \end{cases} \tag{3}$$

where, under a probability measure $\mathbf{P}$, (τ_i) is a sequence of i.i.d. mean 1 exponential random variables. Note that the condition $\sum_{i\ge1} x_i^2 < \infty$ guarantees that the sum inside the probability converges $\mathbf{P}$-a.s. It is clear that the mapping $\bar{H}$ is well defined in the sense that $\bar{H}(\zeta)$ does not depend on the enumeration of the points of ζ. We defer the proof of the following lemma to Appendix A.

Lemma 1.2. *The map $\bar{H}$ is measurable.*

We are now ready to state our first main result, describing the weak quenched limiting distribution for the hitting times centered by the quenched mean.

Theorem 1.3. *Let Assumptions 1–3 hold, and for any $\omega \in \Omega$ let $\mu_{n,\omega} \in \mathcal{M}_1$ be defined by*

$$\bar{\mu}_{n,\omega}(\cdot) = P_\omega\left(\frac{T_n - E_\omega T_n}{n^{1/\kappa}} \in \cdot\right). \tag{4}$$

Then there exists a $\lambda > 0$ such that $\bar{\mu}_{n,\omega} \Longrightarrow \bar{H}(N_{\lambda,\kappa})$ where $N_{\lambda,\kappa}$ is a non-homogeneous Poisson point process on $(0,\infty)$ with intensity $\lambda x^{-\kappa-1}$.

Remark 1.4. *The Gaussian and centered exponential distributions that were shown in [13] to be subsequential quenched limiting distributions of the hitting times are both, clearly, in the support of the random limiting probability measure obtained in Theorem 1.3. Indeed, letting $\zeta_k = k\delta_{k^{-1/2}} \in \mathcal{M}_p$ we see that $\bar{H}(\zeta_1)$ is a centered exponential distribution, and the central limit theorem implies that $\lim_{k\to\infty} \bar{H}(\zeta_k)$ is a standard Gaussian distribution.*

Remark 1.5. *One can represent the non-homogeneous Poisson process $N_{\lambda,\kappa}$ as*

$$N_{\lambda,\kappa} = \sum_{j=1}^\infty \delta_{(\lambda/\kappa)^{1/\kappa} \Gamma_j^{-1/\kappa}},$$

where $(\Gamma_j)_{j\geq 1}$ is the increasing sequence of the points of the unit rate homogeneous Poisson process on $(0,\infty)$. *In particular, the points of $N_{\lambda,\kappa}$ are square summable with probability* 1 *if $\kappa < 2$ (and square summable with probability* 0 *if $\kappa \geq 2$). Furthermore, the random limiting distribution in Theorem* 1.3 *can be written in the form*

$$\bar{H}(N_{\lambda,\kappa})(\cdot) = \mathbf{P}\left((\lambda/\kappa)^{1/\kappa}\sum_{j=1}^{\infty}\Gamma_j^{-1/\kappa}(\tau_j - 1) \in \cdot\right), \tag{5}$$

and we recall that the probability in (5) *is taken with respect to the exponential random variables (τ_j), while keeping the standard Poisson arrivals (Γ_j) fixed.*

The random probability measure $L = H(N_{\lambda,\kappa})$ above has a curious stability property in $\mathcal{M}_1$: if $L_1, \ldots, L_n$ are i.i.d. copies of L, then

$$L_1 * \cdots * L_n(\cdot) \stackrel{\text{law}}{=} L\left(\cdot/n^{1/\kappa}\right) \tag{6}$$

for $n = 1, 2, \ldots$. To see why this is true, represent each L_i as in (5), *but using an independent sequence of Poisson arrivals for each $i = 1, \ldots, n$. Then the n-fold convolution $L_1 * \cdots * L_n$ has the same representation, but the sequence of the standard Poisson arrivals has to be replaced by a superposition of n such independent sequences. Since a superposition of independent Poisson processes is, once again, a Poisson process and the mean measures add up, we conclude that*

$$L_1 * \cdots * L_n(\cdot) \stackrel{\text{law}}{=} \mathbf{P}\left((\lambda/\kappa)^{1/\kappa}\sum_{j=1}^{\infty}\tilde{\Gamma}_j^{-1/\kappa}(\tau_j - 1) \in \cdot\right),$$

where $(\tilde{\Gamma}_j)_j$ is the increasing sequence of the points of a homogeneous Poisson random measure on $(0,\infty)$ *with intensity n. Since the sequence $(\Gamma_j/n)_j$ also forms a Poisson random measure with intensity n,* (6) *follows.*

Since we know that when $\kappa < 2$ there is no centering and scaling that results in convergence to a deterministic distribution, we have some flexibility in choosing what centering and scaling to work with. For example, if we use the averaged centering and scaling in Theorem 1.1, then a slightly different random probability distribution will appear in the limit. Before stating this result we need to introduce some more notation. Define mappings $H, H_\varepsilon : \mathcal{M}_p \to \mathcal{M}_1$, $\varepsilon > 0$, as follows. For $\zeta = \sum_{i\geq 1}\delta_{x_i}$, $H(\zeta)$ and $H_\varepsilon(\zeta)$ are the probability measures defined by

$$H(\zeta)(\cdot) = \begin{cases} \mathbf{P}\left(\sum_{i\geq 1} x_i\tau_i \in \cdot\right) & \text{if } \sum_{i\geq 1} x_i < \infty, \\ \delta_0, & \sum_{i\geq 1} x_i = \infty, \end{cases} \tag{7}$$

and

$$H_\varepsilon(\zeta)(\cdot) = \mathbf{P}\left(\sum_{i\geq 1} x_i\tau_i \mathbf{1}_{\{x_i > \varepsilon\}} \in \cdot\right). \tag{8}$$

As was the case in the definition of $\bar{H}$ in (3), the definition of $H(\zeta)$ does not depend on a particular enumeration of the points of ζ. Furthermore, an obvious modification of the proof of Lemma 1.2 shows that the map H is measurable. The maps H_ε are even (almost) continuous, as will be seen in Section 7.

Theorem 1.6. *Let Assumptions* 1–3 *hold. For $\lambda, \kappa > 0$ let $N_{\lambda,\kappa}$ be a non-homogeneous Poisson point process on* $(0,\infty)$ *with intensity $\lambda x^{-\kappa-1}$. Then for every $\kappa \in (0,2)$ there is a $\lambda > 0$ such that the following statements hold.*

(1) *If $\kappa \in (0,1)$, then*

$$\mu_{n,\omega}(\cdot) = P_\omega\left(\frac{T_n}{n^{1/\kappa}} \in \cdot\right) \Longrightarrow H(N_{\lambda,\kappa}).$$

(2) *If* $\kappa = 1$, *then*

$$\mu_{n,\omega}(\cdot) = P_\omega\left(\frac{T_n - nD(n)}{n} \in \cdot\right) \Longrightarrow \lim_{\varepsilon\to 0^+}\left[H_\varepsilon(N_{\lambda,1}) * \delta_{-c_{\lambda,1}(\varepsilon)}\right],$$

where $c_{\lambda,1}(\varepsilon) = \int_\varepsilon^1 \lambda x^{-1}\,\mathrm{d}x = \lambda\log(1/\varepsilon)$, *and* $D(n)$ *is a sequence such that* $D(n) \sim A\log n$ *for some* $A > 0$.

(3) *If* $\kappa \in (1, 2)$, *then*

$$\mu_{n,\omega}(\cdot) = P_\omega\left(\frac{T_n - n/\mathrm{v}_P}{n^{1/\kappa}} \in \cdot\right) \Longrightarrow \lim_{\varepsilon\to 0^+}\left[H_\varepsilon(N_{\lambda,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)}\right],$$

where $c_{\lambda,\kappa}(\varepsilon) = \int_\varepsilon^\infty \lambda x^{-\kappa}\,\mathrm{d}x = \frac{\lambda}{\kappa-1}\varepsilon^{-(\kappa-1)}$.

Remark 1.7. *The limits as* $\varepsilon \to 0^+$ *in the cases* $1 \le \kappa < 2$ *in Theorem* 1.6 *are weak limits in* $\mathcal{M}_1$. *The fact that these limits exist is standard*; *see e.g.* [16]. *As we show in Section* 7, *fixing a Poisson process* $N_{\lambda,\kappa}$ *on some probability space* (*for example*, *as in Remark* 1.5), *even convergence with probability* 1 *holds*.

The limiting random probability measures obtained in the different parts of Theorem 1.6 *also have stability properties in* $\mathcal{M}_1$, *similar to the stability property of* $\bar{H}(N_{\lambda,\kappa})$ *described in Remark* 1.5. *Specifically, if* $L_1, L_2, \ldots, L_n$ *are i.i.d. copies of the limiting random probability measure* L *in Theorem* 1.6, *then the stability relation for the convolution operation* (6) *still holds if* $\kappa \neq 1$. *In the case* $\kappa = 1$, *the corresponding stability relation is*

$$L_1 * \cdots * L_n(\cdot) \stackrel{\text{law}}{=} L(\cdot/n - \lambda\log n). \tag{9}$$

The proof is similar to the argument used in Remark 1.5. *We omit the details*.

The statement (and proof) of the weak quenched limits with the quenched centering (Theorem 1.3) is much simpler than the corresponding result with the averaged centering (Theorem 1.6). However, in transferring a limiting distribution from the hitting times T_n to the location of the random walk X_n it is easier to use the averaged centering.

Corollary 1.8. *Let Assumptions* 1–3 *hold for some* $\kappa \in (0, 2)$, *and let* $\lambda > 0$ *be given by Theorem* 1.6.

(1) *If* $\kappa \in (0, 1)$, *then for any* $x \in \mathbb{R}$,

$$P_\omega\left(\frac{X_n}{n^\kappa} < x\right) \Longrightarrow H(N_{\lambda,\kappa})\big(x^{-1/\kappa}, \infty\big).$$

(2) *If* $\kappa = 1$, *then there exists a sequence* $\delta(n) \sim n/(A\log n)$ (*with* $A > 0$ *as in the conclusion of Theorem* 1.6) *such that for any* $x \in \mathbb{R}$,

$$P_\omega\left(\frac{X_n - \delta(n)}{n/(\log n)^2} < x\right) \Longrightarrow \lim_{\varepsilon\to 0^+}\big(H_\varepsilon(N_{\lambda,1}) * \delta_{-c_{\lambda,1}(\varepsilon)}\big)\big(-A^2x, \infty\big).$$

(3) *If* $\kappa \in (1, 2)$, *then for any* $x \in \mathbb{R}$,

$$P_\omega\left(\frac{X_n - n\mathrm{v}_P}{n^{1/\kappa}} < x\right) \Longrightarrow \lim_{\varepsilon\to 0^+}\big(H_\varepsilon(N_{\lambda,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)}\big)\big(-x\mathrm{v}_P^{-1-1/\kappa}, \infty\big).$$

Remark 1.9. *The type of convergence in Corollary* 1.8 *is weaker than that in Theorems* 1.3 *and* 1.6. *Instead of proving that the quenched distribution of* X_n (*centered and scaled*) *converges in distribution on the space* $\mathcal{M}_1$, *we only prove that certain projections of the quenched law converge in distribution as real valued random variables. We suspect that, with some extra work, the techniques of this paper could be used to prove a limiting distribution for the full quenched distribution of* X_n, *but we will leave that for a future paper. Some results in this direction have previously been obtained in* [6].

Remark 1.10. *Theorem* 1.6 *and Corollary* 1.8 *generalize the stable limiting distributions under the averaged law* [11]. *For instance, when* $\kappa \in (0,1)$,

$$\mathbb{P}\left(\frac{T_n}{n^{1/\kappa}} \le x\right) = E_P\left[P_\omega\left(\frac{T_n}{n^{1/\kappa}} \le x\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbf{E}\big[H(N_{\lambda,\kappa})(-\infty, x]\big],$$

and it is easy to see that $\mathbf{E}[H(N_{\lambda,\kappa})(-\infty,x]] = L_{\kappa,b}(x)$ *for some* $b>0$.

The structure of the paper is as follows. In Section 2 we introduce some notation and review some basic facts that we will need. Then, in Section 3 we outline a general method for transferring a limiting distribution result for one sequence of random probability measures to another sequence of random probability measures by constructing a coupling between the two sequences. The method developed in Section 3 is then implemented several times in Section 4 to reduce the study of the quenched distribution of the hitting times T_n to the quenched distribution of a certain environment-dependent mixture of exponential random variables. Then, these environment-dependent mixing coefficients are shown in Section 5 to be related to a non-homogeneous Poisson point process $N_{\lambda,\kappa}$. In Section 6 we complete the proof of Theorem 1.3 by proving a weak quenched limiting distribution for this mixture of exponentials. The proof of Theorem 1.6 is similar to the proof of Theorem 1.3, and in Section 7 we indicate how to complete the parts of the proof that are different. Finally, in Section 8 we give the proof of the Corollary 1.8.

Before turning to the proofs, we make one remark on the writing style. Throughout the paper, we will use c, C and C' to denote generic constants that may change from line to line. Specific constants that remain fixed throughout the paper are denoted C_0, C_1, etc.

Remark 1.11. *Soon after this work had been completed and posted on the arXiv, two other papers* [5,8] *appeared giving independent proofs of some of the main results of this paper. A few brief remarks are in order on the differences between these papers. Neither of the above papers state weak quenched limits with the averaged centering as in Theorem* 1.6 (*although this should follow easily from Corollary* 1 *in* [5]) *nor do they discuss the quenched distribution of* X_n *as in Corollary* 1.8. *In* [5], *instead of studying the hitting times* T_n *directly the authors study the amount of time spent in the interval* $[0,n)$ *– which can easily be seen to have the same weak quenched limiting distributions as the hitting times. In* [8], *the authors prove Theorem* 1.3 *under the stronger Wasserstein* W^1 *metric on the space* $\mathcal{M}_1$. *However, an analysis of the proof in the current paper* (*especially the coupling technique introduced in Section* 3) *reveals that it should be easily adaptable to the Wasserstein metric as well.*

Remark 1.12. *After the initial submission of this paper, we also became aware of* [17] *which provides a systematic study of stable random probability distributions – that is, random probability distributions with stability properties like* (6) *or* (9). *Moreover, in* [17] *the authors study a simpler model of random motion in a random environment and obtain weak quenched limiting distributions for the hitting times similar to Theorems* 1.3 *and* 1.6.

2. Background

In this section we introduce some notation that will be used throughout the rest of the paper. For RWRE on $\mathbb{Z}$, many quenched probabilities and expectations are explicitly solvable in terms of the environment. It is in order to express these formulas compactly that we need this additional notation. Recall that $\rho_x = (1-\omega_x)/\omega_x$, $x\in\mathbb{Z}$. Then, for $i\le j$ we let

$$\Pi_{i,j} = \prod_{x=i}^{j} \rho_x, \qquad R_{i,j} = \sum_{k=i}^{j} \Pi_{i,k} \quad \text{and} \quad W_{i,j} = \sum_{k=i}^{j} \Pi_{k,j}. \tag{10}$$

Denote

$$R_i = \lim_{j\to\infty} R_{i,j} = \sum_{k=i}^{\infty} \Pi_{i,k} \quad \text{and} \quad W_j = \lim_{i\to-\infty} W_{i,j} = \sum_{k=-\infty}^{j} \Pi_{k,j}. \tag{11}$$

Note that Assumption 2 implies that R_i and W_j are finite with probability 1 for all $i, j \in \mathbb{Z}$. The following formulas are extremely useful (see [19] for a reference)

$$P_\omega^x(T_i > T_j) = \frac{R_{i,x-1}}{R_{i,j-1}} \quad \text{and} \quad P_\omega^x(T_i < T_j) = \frac{\Pi_{i,x-1} R_{x,j-1}}{R_{i,j-1}}, \quad i < x < j, \tag{12}$$

$$E_\omega^i T_{i+1} = 1 + 2W_i, \quad i \in \mathbb{Z}. \tag{13}$$

As in [13,14], we define the "ladder locations" ν_i of the environment by

$$\nu_0 = 0 \quad \text{and} \quad \nu_i = \inf\{n > \nu_{i-1}\colon \Pi_{\nu_{i-1},n-1} < 1\}, \quad i \ge 1. \tag{14}$$

Since the environment is i.i.d., the sections of the environment $\{\omega_x\colon \nu_{i-1} \le x < \nu_i\}$ between successive ladder locations are also i.i.d. However, the environment directly to the left of $\nu_0 = 0$ is different from the environment to the left of ν_i for $i > 1$. Thus, as in [13,14] it is convenient to define a new probability law on environments by

$$Q(\cdot) = P(\cdot | \Pi_{i,-1} < 1, \text{ all } i \le -1); \tag{15}$$

by Assumption 2 the condition is an event of positive probability.

Two facts about the distribution Q will be important to keep in mind throughout the remainder of the paper.

- Under the measure Q the environments stationary under shifts by the ladder locations ν_i.
- Since, under P, the environment is i.i.d., the measure Q coincides with the measure P on $\sigma(\omega_x\colon x \ge 0)$.

Often for convenience we will denote ν_1 by ν. It was shown in [14], Lemma 2.1, that the distribution of ν (which is the same under P and Q) has exponential tails. That is, there exist constants $C, C' > 0$ such that

$$P(\nu > x) = Q(\nu > x) \le C' \mathrm{e}^{-Cx}, \quad x \ge 0. \tag{16}$$

In particular this implies that $\lim_{n\to\infty} \nu_n / n = \bar{\nu} := E_Q \nu = E_P \nu$, both P and Q-a.s.

In contrast, it was shown in [14], Theorem 1.4, that, under Assumption 3, the distribution of the first hitting time $E_\omega T_\nu$ has power tails under the measure Q. That is, there exists a constant C_0 such that

$$Q(E_\omega T_\nu > x) \sim C_0 x^{-\kappa}, \quad x \to \infty. \tag{17}$$

3. A general method for transferring weak quenched limits

Our strategy for proving weak quenched limits for the hitting times will be to first prove a weak quenched limiting distribution for a related sequence of random variables. Then by exhibiting a coupling between the two sequences of random variables we will be able to conclude that the hitting times have the same weak quenched limiting distribution. The second of these steps is accomplished through the following lemma. It applies to random probability measures on $\mathbb{R}^2$, which are simply random variables taking values in $\mathcal{M}_1(\mathbb{R}^2)$. The latter space is the space of all probability measures on $\mathbb{R}^2$ which can be turned into a complete, separable metric space in the same way as it was done to the space $\mathcal{M}_1$ in Section 1. The two maps assigning each probability measure in $\mathcal{M}_1(\mathbb{R}^2)$ its two marginal probability measures are automatically continuous.

Lemma 3.1. *Let $\theta_n, n = 1, 2, \ldots,$ be a sequence of random probability measures on $\mathbb{R}^2$ defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let γ_n and γ_n' be the two marginals of θ_n, $n = 1, 2, \ldots$. Suppose that for every $\delta > 0$*

$$\lim_{n\to\infty} \mathbf{P}\big(\theta_n\big(\big\{(x, y)\colon |x - y| \ge \delta\big\}\big) > \delta\big) = 0. \tag{18}$$

If $\gamma_n \Longrightarrow \gamma$ for some $\gamma \in \mathcal{M}_1$, then $\gamma_n' \Longrightarrow \gamma$ as well.

Remark 3.2. *Generally the space Ω will be the space of environments and $\mathbf{P}$ will be the measure Q on environments defined in (15). However, in one application (Lemma 4.2 below) we will use slightly different spaces and measures and so we need to state Lemma 3.1 in this more general form.*

Proof of Lemma 3.1. The definition of the Prohorov metric ρ in (2) implies that, if $\theta_n(\{(x,y)\colon |x-y|\geq\delta\})\leq\delta$, then $\rho(\gamma_n,\gamma_n')\leq\delta$. Therefore, the assumption (18) implies that $\rho(\gamma_n,\gamma_n')\to 0$ in probability. Now the statement of the lemma follows from Theorem 3.1 in [2]. $\square$

The following is an immediate corollary.

Corollary 3.3. *Under the setup of Lemma 3.1, assume that*

$$\mathbf{E}_{\theta_n}|X-Y|\to 0, \quad \textit{in } \mathbf{P}\textit{-probability} \tag{19}$$

(here X and Y are the coordinate variables in $\mathbb{R}^2$ and $\mathbf{E}_{\theta_n}$ is expectation with respect to the measure θ_n). If $\gamma_n \Longrightarrow \gamma$ for some $\gamma\in\mathcal{M}_1$, then $\gamma_n'\Longrightarrow\gamma$ as well.

Proof. The claim follows immediately from Lemma 3.1 and Markov's inequality via

$$\mathbf{P}\bigl(\theta_n\bigl(|X-Y|\geq\delta\bigr)\geq\delta\bigr)\leq\mathbf{P}\bigl(\mathbf{E}_{\theta_n}|X-Y|\geq\delta^2\bigr). \qquad \square$$

Remark 3.4. *By the Cauchy–Scwarz inequality, a sufficient condition for (19) is*

$$\mathbf{E}_{\theta_n}(X-Y)\to 0 \quad \textit{and} \quad \mathbf{Var}_{\theta_n}(X-Y)\to 0, \quad \textit{in } \mathbf{P}\textit{-probability}. \tag{20}$$

4. A series of reductions

In this section we repeatedly apply Lemma 3.1 and Corollary 3.3 to reduce the problem of finding weak quenched limits of the hitting times T_n to the problem of finding weak quenched limits of a simpler sequence of random variables that is a random mixture of exponential distributions.

First of all, instead of studying the quenched distributions of the hitting times, it will be more convenient to study the hitting times along the random sequence of the ladder locations ν_n. Since by (16), the distance between consecutive ladder locations has exponential tails, and $\nu_n/n\to\bar{\nu}=E_P\nu_1$ the quenched distribution of T_n should be close to the quenched distribution of $T_{\nu_{\bar{\alpha}n}}$ with $\bar{\alpha}=1/\bar{\nu}$ (for ease of notation we will write $\nu_{\bar{\alpha}n}$ instead of $\nu_{\lfloor\bar{\alpha}n\rfloor}$). Based on this, we will reduce our problem to proving a quenched weak limit theorem for $T_{\nu_n}=\sum_{i=1}^n(T_{\nu_i}-T_{\nu_{i-1}})$. Secondly, as mentioned in the Introduction, the proof of the non-existence of quenched limiting distributions for hitting times in [13] hinged on two observations. The first of these says that, for large n, the magnitude of T_{ν_n} is mainly determined by the increments $T_{\nu_i}-T_{\nu_{i-1}}$ for those $i=1,\dots,n$ for which there is a large "trap" between the ladder locations ν_{i-1} and ν_i. The second observation is that, when there is a large "trap" between ν_{i-1} and ν_i, the time to cross from ν_{i-1} to ν_i is, approximately, an exponential random variable with a large mean. That is, $T_{\nu_i}-T_{\nu_{i-1}}$ may be approximated by $\beta_i\tau_i$ where

$$\beta_i=\beta_i(\omega)=E_\omega^{\nu_{i-1}}T_{\nu_i}=E_\omega(T_{\nu_i}-T_{\nu_{i-1}}), \tag{21}$$

and τ_i is a mean 1 exponential random variable that is independent of everything else.

When analyzing the hitting times of the ladder locations T_{ν_n} the measure Q is more convenient to use than the measure P since, under Q, the environment is stationary under shifts of the environment by the ladder locations. In particular, $\{\beta_i\}_{i\geq 1}$ is a stationary sequence under Q. The main result of this section is the following proposition.

Proposition 4.1. *For $\omega\in\Omega$, suppose that P_ω is expanded so that there exists a sequence τ_i which, under P_ω, is an i.i.d. sequence of mean 1 exponential random variables. Let $\bar{\sigma}_{n,\omega}\in\mathcal{M}_1$ be defined by*

$$\bar{\sigma}_{n,\omega}(\cdot)=P_\omega\left(\frac{1}{n^{1/\kappa}}\sum_{i=1}^n\beta_i(\tau_i-1)\in\cdot\right), \tag{22}$$

where $\beta_i=\beta_i(\omega)$ is given by (21). If $\bar{\sigma}_{n,\omega}\stackrel{Q}{\Longrightarrow}\bar{H}(N_{\lambda,\kappa})$ then $\bar{\mu}_{n,\omega}\stackrel{P}{\Longrightarrow}\bar{H}(N_{\lambda/\bar{\nu},\kappa})$, where $\bar{\mu}_{n,\omega}$ is defined in (4).

Lemma 3.1 says that weak imits for one sequence of $\mathcal{M}_1$-valued random variables can be transferred to another sequence of $\mathcal{M}_1$-valued random variables if these random probability measures can be coupled in a nice way. We pursue this idea and prove Proposition 4.1 by establishing the series of lemmas below. All of these results will be proved using Lemma 3.1 and Corollary 3.3.

Lemma 4.2. *If* $\bar{\mu}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})$ *then* $\bar{\mu}_{n,\omega} \overset{P}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})$.

Lemma 4.3. *For* $\omega \in \Omega$, *let* $\bar{\phi}_{n,\omega} \in \mathcal{M}_1$ *be defined by*

$$\bar{\phi}_{n,\omega}(\cdot) = P_\omega\left(\frac{T_{\nu_n} - E_\omega T_{\nu_n}}{n^{1/\kappa}} \in \cdot\right) = P_\omega\left(\frac{1}{n^{1/\kappa}}\sum_{i=1}^{n}(T_{\nu_i} - T_{\nu_{i-1}} - \beta_i) \in \cdot\right).$$

If $\bar{\phi}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})$ *then* $\bar{\mu}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda/\bar{\nu},\kappa})$.

Lemma 4.4. *If* $\bar{\sigma}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})$ *then* $\bar{\phi}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})$.

Proof of Lemma 4.2. Recall that P and Q are identical on $\sigma(\omega_x\colon x \geq 0)$. We start with a coupling of P and Q that that produces two environments that agree on the non-negative integers. Let ω be an environment with distribution P and let $\tilde{\omega}$ be an independent environment with distribution Q. Then, construct the environment ω' by letting

$$\omega'_x = \begin{cases} \tilde{\omega}_x, & x \leq -1, \\ \omega_x, & x \geq 0. \end{cases}$$

Then ω' has distribution Q and is identical to ω on the non-negative integers. Let $\mathbf{P}$ be the joint distribution of (ω, ω') in the above coupling. Given a pair of environments (ω, ω'), we will construct coupled random walks $\{X_n\}$ and $\{X'_n\}$ with hitting times $\{T_n\}$ and $\{T'_n\}$, respectively, so that the marginal distributions of $\{X_n\}$ and $\{X'_n\}$ are P_ω and $P_{\omega'}$ respectively. Let $P_{\omega,\omega'}$ denote the joint distribution of $\{X_n\}$ and $\{X'_n\}$ with expectations denoted by $E_{\omega,\omega'}$, and consider random probability measures on $\mathbb{R}^2$ defined by

$$\theta_n(\cdot) = P_{\omega,\omega'}\left[\left(\frac{T_n - E_{\omega,\omega'}T_n}{n^{1/\kappa}}, \frac{T'_n - E_{\omega,\omega'}T'_n}{n^{1/\kappa}}\right) \in \cdot\right].$$

We wish to construct the coupled random walks so that

$$\lim_{n\to\infty} n^{-1/\kappa} E_{\omega,\omega'}\left|(T_n - E_{\omega,\omega'}T_n) - \left(T'_n - E_{\omega,\omega'}T'_n\right)\right| = 0, \quad \mathbf{P}\text{-a.s.} \tag{23}$$

This will be more than enough to satisfy conditions (19) of Corollary 3.3, and the conclusion of Lemma 4.2 will follow.

We now show how to construct coupled random walks $\{X_n\}$ and $\{X'_n\}$. Since the environments ω and ω' agree on the non-negative integers, our coupling will cause the two walks to move in the same manner at all locations $x \geq 0$. Precisely, on their respective ith visits to site $x \geq 0$, they will both either move to the right or both move to the left. To do this, let $\bar{\xi} = \{\xi_{x,i}\}_{x\in\mathbb{Z},i\geq 1}$ be a collection of i.i.d. standard uniform random variables that is independent of everything else. Then, given (ω, ω') and $\bar{\xi}$, construct the random walks as follows:

$$X_0 = 0 \quad \text{and} \quad X_{n+1} = \begin{cases} X_n + 1 & \text{if } X_n = x, \#\{k \leq n\colon X_k = x\} = i \text{ and } \xi_{x,i} \leq \omega_x, \\ X_n - 1 & \text{if } X_n = x, \#\{k \leq n\colon X_k = x\} = i \text{ and } \xi_{x,i} > \omega_x, \end{cases}$$

and

$$X'_0 = 0 \quad \text{and} \quad X'_{n+1} = \begin{cases} X'_n + 1 & \text{if } X'_n = x, \#\left\{k \leq n\colon X'_k = x\right\} = i \text{ and } \xi_{x,i} \leq \omega'_x, \\ X'_n - 1 & \text{if } X'_n = x, \#\left\{k \leq n\colon X'_k = x\right\} = i \text{ and } \xi_{x,i} > \omega'_x. \end{cases}$$

Having constructed our coupling, we now turn to the proof of (23). It is enough in fact to show that

$$\sup_n E_{\omega,\omega'}\big|T_n - T'_n\big| < \infty \quad \text{and} \quad \sup_n \big|E_{\omega,\omega'}T_n - E_{\omega,\omega'}T'_n\big| < \infty, \quad \mathbf{P}\text{-a.s.} \tag{24}$$

To show the second inequality in (24), we use the explicit formula (13) for the quenched expectations of hitting times, so that

$$E_\omega T_n = n + 2\sum_{i=0}^{n} W_i = n + 2\sum_{i=0}^{n}(W_{0,i} + \Pi_{0,i}W_{-1}) = n + 2\sum_{i=0}^{n} W_{0,i} + 2W_{-1}R_{0,n}.$$

Similarly (with the obvious notation for corresponding random variables corresponding to ω'),

$$E_{\omega'}T'_n = n + 2\sum_{i=0}^{n} W'_{0,i} + 2W'_{-1}R'_{0,n-1} = n + 2\sum_{i=0}^{n} W_{0,i} + 2W'_{-1}R_{0,n},$$

where the second equality is valid because $\omega_x = \omega'_x$ for all $x \ge 0$. Thus,

$$\sup_n \big|E_{\omega,\omega'}T_n - E_{\omega,\omega'}T'_n\big| = \sup_n 2R_{0,n}\big|W_{-1} - W'_{-1}\big| = 2R_0\big|W_{-1} - W'_{-1}\big| < \infty, \quad \mathbf{P}\text{-a.s.}$$

Turning to the first inequality in (24), let

$$L_n := \sum_{k=0}^{T_n} \mathbf{1}_{\{X_k<0\}}, \qquad L'_n := \sum_{k=0}^{T'_n} \mathbf{1}_{\{X_k<0\}},$$

be the number of visits by by the walks $\{X_n\}$ and $\{X'_n\}$, correspondingly, to the negative integers, by the time they reach site $x = n$. The coupling of T_n and T'_n constructed above is such that $|T_n - T'_n| = |L_n - L'_n|$. Therefore,

$$E_{\omega,\omega'}\big|T_n - T'_n\big| = E_{\omega,\omega'}\big|L_n - L'_n\big| \le E_\omega L_n + E_{\omega'}L'_n.$$

Letting $L = \lim_{n\to\infty} L_n$ and $L' = \lim_{n\to\infty} L'_n$ denote the total amount of time spent in the negative integers by the random walks $\{X_n\}$ and $\{X'_n\}$, respectively, we need only to show that $E_\omega L + E_{\omega'}L' < \infty$, $\mathbf{P}$-a.s. To this end, note that $L = \sum_{i=1}^{G} U_i$ where G is the number of times the random walk $\{X_n\}$ steps from 0 to -1 and the U_i is the amount of time it takes to reach 0 after the ith visit to -1. Note that G is a geometric random variable starting from 0 with success parameter $P_\omega(T_{-1} = \infty) > 0$, and that the U_i are independent (and independent of G) with common distribution equal to that of the time it takes a random walk in environment ω to reach 0 when starting at -1. Thus, by first conditioning on G, we obtain that

$$E_\omega L = E_\omega\big[G\big(E_\omega^{-1}T_0\big)\big] = \big(E_\omega^{-1}T_0\big)\frac{P_\omega(T_{-1} < \infty)}{P_\omega(T_{-1} = \infty)}.$$

Similarly,

$$E_{\omega'}L' = \big(E_{\omega'}^{-1}T_0\big)\frac{P_{\omega'}(T_{-1} < \infty)}{P_{\omega'}(T_{-1} = \infty)}.$$

This completes the proof since $E_\omega^{-1}T_0$ and $E_{\omega'}^{-1}T_0$ are finite, $\mathbf{P}$-a.s. by (13). □

Proof of Lemma 4.3. For $\omega \in \Omega$, let $\hat\phi_{n,\omega} \in \mathcal{M}_1$ be defined by

$$\hat\phi_{n,\omega}(A) = P_\omega\bigg(\frac{T_{\nu_{\bar\alpha n}} - E_\omega T_{\nu_{\bar\alpha n}}}{n^{1/\kappa}} \in A\bigg) = \bar\phi_{\lfloor\bar\alpha n\rfloor,\omega}\bigg(\frac{n^{1/\kappa}}{\lfloor\bar\alpha n\rfloor^{1/\kappa}}A\bigg).$$

Since $n^{1/\kappa}/\lfloor\bar{\alpha}n\rfloor^{1/\kappa} \to \bar{\alpha}^{-1/\kappa} = \bar{\nu}^{1/\kappa}$ as $n \to \infty$, it follows (for example, by Lemma 3.1) that

$$\bar{\phi}_{n,\omega}(\cdot) \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})(\cdot) \quad \text{implies that} \quad \hat{\phi}_{n,\omega}(\cdot) \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})\big(\bar{\nu}^{1/\kappa}\cdot\big).$$

Now, it follows from (5) that $\bar{H}(N_{\lambda,\kappa})(\bar{\nu}^{1/\kappa}\cdot) \overset{\text{law}}{=} \bar{H}(N_{\lambda/\bar{\nu},\kappa})(\cdot)$. Therefore, the claim of the lemma will follow once we check that

$$\hat{\phi}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa}) \quad \text{implies that} \quad \bar{\mu}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa}). \tag{25}$$

To show (25) we will verify condition (20) of the remark following Corollary 3.3. Since both $\hat{\phi}_{n,\omega}$ and $\bar{\mu}_{n,\omega}$ are mean zero distributions on $\mathbb{R}$, it is enough to show that

$$\lim_{n\to\infty} Q\big(n^{-2/\kappa}\operatorname{Var}_\omega(T_n - T_{\nu_{\bar{\alpha}n}}) > \delta\big) = 0, \quad \forall\delta > 0. \tag{26}$$

To this end, note that if $\nu_{\bar{\alpha}n} \le n \le \nu_k$ then $\operatorname{Var}_\omega(T_n - T_{\nu_{\bar{\alpha}n}}) = \sum_{x=\nu_{\bar{\alpha}n}+1}^{n}\operatorname{Var}_\omega(T_x - T_{x-1}) \le \operatorname{Var}_\omega(T_{\nu_k} - T_{\nu_{\bar{\alpha}n}})$. A similar inequality holds if $\nu_k \le n \le \nu_{\bar{\alpha}n}$. Using this, we obtain that for any $\varepsilon > 0$

$$\begin{aligned} Q\big(\operatorname{Var}_\omega(T_n - T_{\nu_{\bar{\alpha}n}}) > \delta n^{2/\kappa}\big) &\le Q\big(|n - \nu_{\bar{\alpha}n}| > \varepsilon n\big) + Q\big(\operatorname{Var}_\omega(T_{\nu_{[\bar{\alpha}n]+[\varepsilon n]}} - T_{\nu_{\bar{\alpha}n}}) > \delta n^{2/\kappa}\big) \\ &\quad + Q\big(\operatorname{Var}_\omega(T_{\nu_{\bar{\alpha}n}} - T_{\nu_{[\bar{\alpha}n]-[\varepsilon n]}}) > \delta n^{2/\kappa}\big) \\ &= Q\big(|n - \nu_{\bar{\alpha}n}| > \varepsilon n\big) + 2Q\big(\operatorname{Var}_\omega(T_{\nu_{\varepsilon n}}) > \delta n^{2/\kappa}\big), \end{aligned} \tag{27}$$

where the last equality is due to the fact that, under the measure Q, the environment is stationary under shifts of the ladder locations. The first term in (27) vanishes since $\nu_{\bar{\alpha}n}/n \to 1$, Q-a.s., by the law of large numbers. For the second term in (27), recall that $n^{-2/\kappa}\operatorname{Var}_\omega T_{\nu_n}$ has a κ-stable limiting distribution under Q [13], Theorem 1.3. Thus, there exists a $b > 0$ such that

$$\lim_{n\to\infty} Q\big(\operatorname{Var}_\omega(T_{\nu_{\varepsilon n}}) > \delta n^{2/\kappa}\big) = 1 - L_{\kappa,b}\big(\delta\varepsilon^{-2/\kappa}\big).$$

Since the right-hand side can be made arbitrarily small by taking $\varepsilon \to 0$, we have finished the proof of (26) and, thus, also of the lemma. □

Proof of Lemma 4.4. The proof of the lemma consists of showing that we can couple the standard exponential random variables of Proposition 4.1 with the random walk $\{X_n\}$ in such a way that condition (20) of the remark following Corollary 3.3 holds. Since the relevant random probability measures have zero means, we only need to ensure that

$$\lim_{n\to\infty} Q\left(n^{-2/\kappa}\operatorname{Var}_\omega\left(T_{\nu_n} - E_\omega T_{\nu_n} - \sum_{i=1}^{n}\beta_i(\tau_i - 1)\right) > \delta\right) = 0, \quad \forall\delta > 0. \tag{28}$$

We will perform the coupling in such a way that the sequence of pairs $(T_{\nu_i} - T_{\nu_{i-1}}, \tau_i)$ is independent under the quenched law P_ω. Since $E_\omega T_{\nu_n} = \sum_{i=1}^{n}\beta_i$, this will imply that

$$\operatorname{Var}_\omega\left(T_{\nu_n} - E_\omega T_{\nu_n} - \sum_{i=1}^{n}\beta_i(\tau_i - 1)\right) = \sum_{i=1}^{n}\operatorname{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}} - \beta_i\tau_i).$$

As in [14], for any i define

$$M_i = \max\{\Pi_{\nu_{i-1},j}\colon \nu_{i-1} \le j < \nu_i\}. \tag{29}$$

The utility of the sequence M_i is that it is roughly comparable to β_i and $\sqrt{\operatorname{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}})}$, but M_i is an i.i.d. sequence of random variables (see [14], Eqs (15) and (63), for precise statements regarding these comparisons). In

[14], Lemma 5.5, it was shown that for any $0 < \varepsilon < 1$,

$$\lim_{n\to\infty} Q\left(\frac{1}{n^{2/k}}\sum_{i=1}^{n} \mathrm{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}})\mathbf{1}_{\{M_i \le n^{(1-\varepsilon)/\kappa}\}} > \delta\right) = 0, \quad \forall \delta > 0.$$

A similar argument (see also the proof of [14], Lemma 3.1) implies that

$$\lim_{n\to\infty} Q\left(\frac{1}{n^{2/k}}\sum_{i=1}^{n} \beta_i^2 \mathbf{1}_{\{M_i \le n^{(1-\varepsilon)/\kappa}\}} > \delta\right) = 0, \quad \forall \delta > 0.$$

Then, since $\mathrm{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}} - \beta_i\tau_i) \le 2\,\mathrm{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}}) + 2\beta_i^2$, in order to guarantee (28) it is enough to perform a coupling in such a way that for some $0 < \varepsilon < 1$,

$$\lim_{n\to\infty} Q\left(\frac{1}{n^{2/k}}\sum_{i=1}^{n} \mathrm{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}} - \beta_i\tau_i)\mathbf{1}_{\{M_i > n^{(1-\varepsilon)/\kappa}\}} > \delta\right) = 0, \quad \forall \delta > 0. \tag{30}$$

Recall that we separately couple each exponential random variable τ_i with the corresponding crossing time $T_{\nu_i} - T_{\nu_{i-1}}$. For simplicity of notation we will describe this coupling when $i = 1$, and we will denote ν_1, β_1 and τ_1 by ν, β and τ, respectively.

First, note that T_ν can be constructed by doing repeated excursions from the origin. Let $T_0^+ = \inf\{n > 0\colon X_n = 0\}$ be the first return time to the origin, and let $\{F^{(j)}\}_{j\ge 1}$ be an i.i.d. sequence of random variables all having the distribution of T_0^+ under $P_\omega(\cdot|T_0^+ < T_\nu)$. Also, let let S be independent of the $\{F^{(j)}\}$ and have the same distribution as T_ν under $P_\omega(\cdot|T_\nu < T_0^+)$. Finally, let N be independent of S and the $\{F^{(j)}\}$ and have a geometric distribution starting from 0 with success parameter $p_\omega = P_\omega(T_\nu < T_0^+)$. Then we can construct T_ν by letting

$$T_\nu = S + \sum_{j=1}^{N} F^{(j)}. \tag{31}$$

Note that

$$\beta = E_\omega T_\nu = E_\omega S + \frac{1-p_\omega}{p_\omega}\bigl(E_\omega F^{(1)}\bigr). \tag{32}$$

Given this construction of T_ν, the most natural way to couple T_ν with τ is to provide a coupling between τ and N. We set

$$N = \lfloor c_\omega \tau\rfloor, \quad \text{where } c_\omega = \frac{-1}{\log(1-p_\omega)}, \tag{33}$$

so that N is exactly a geometric random variable with parameter p_ω.

For this coupling, we obtain the following bound on $\mathrm{Var}_\omega(T_\nu - \beta\tau)$.

Lemma 4.5. *Let T_ν and $\beta\tau$ be coupled using* (31) *and* (33). *Then*,

$$\mathrm{Var}_\omega(T_\nu - \beta\tau) \le (E_\omega S)^2 + \frac{(E_\omega F^{(1)})^2}{3} + \mathrm{Var}_\omega(T_\nu) - \bigl(E_\omega F^{(1)}\bigr)^2 \mathrm{Var}_\omega(N). \tag{34}$$

Proof. First of all, note that

$$\begin{aligned}\mathrm{Var}_\omega(T_\nu - \beta\tau) &= \mathrm{Var}_\omega\left(S + \sum_{j=1}^{N} F^{(j)} - \beta\tau\right) = \mathrm{Var}_\omega(S) + \mathrm{Var}_\omega\left(\sum_{j=1}^{N} F^{(j)} - \beta\tau\right)\\ &= \mathrm{Var}_\omega(S) + \mathrm{Var}_\omega\bigl(F^{(1)}\bigr)\bigl(E_\omega\lfloor c_\omega\tau\rfloor\bigr) + \mathrm{Var}_\omega\bigl(\lfloor c_\omega\tau\rfloor\bigl(E_\omega F^{(1)}\bigr) - \beta\tau\bigr).\end{aligned} \tag{35}$$

Since $\lfloor c_\omega \tau \rfloor$ is independent of $c_\omega \tau - \lfloor c_\omega \tau \rfloor$, we can use the identity for β in (32) to write, with the help of a bit of algebra,

$$\begin{aligned}
\mathrm{Var}_\omega\big(\lfloor c_\omega \tau \rfloor \big(E_\omega F^{(1)}\big) - \beta\tau\big) &= \big(E_\omega F^{(1)}\big)^2 \mathrm{Var}_\omega\big(\lfloor c_\omega \tau \rfloor\big) + \beta^2 - 2\big(E_\omega F^{(1)}\big)\beta \,\mathrm{Cov}\big(\lfloor c_\omega \tau \rfloor, \tau\big) \\
&= \big(\big(E_\omega F^{(1)}\big)^2 - 2\big(E_\omega F^{(1)}\big)\beta/c_\omega\big) \mathrm{Var}_\omega\big(\lfloor c_\omega \tau \rfloor\big) + \beta^2 \\
&= (E_\omega S)^2 + 2(E_\omega S)\big(E_\omega F^{(1)}\big)\frac{1-p_\omega}{p_\omega^2}\big(p_\omega + \log(1-p_\omega)\big) \\
&\quad + \big(E_\omega F^{(1)}\big)^2 \frac{1-p_\omega}{p_\omega^2}\left(2 - p_\omega + 2\frac{1-p_\omega}{p_\omega}\log(1-p_\omega)\right).
\end{aligned}$$

Using a Taylor series expansion of $\log(1-p)$ for $|p| < 1$, one can show that for any $p \in [0, 1)$,

$$p + \log(1-p) = -\sum_{k=2}^{\infty} \frac{p^k}{k} \le 0$$

and

$$\frac{1-p}{p^2}\left(2 - p + 2\frac{1-p}{p}\log(1-p)\right) = 1/3 - \sum_{k=1}^{\infty} \frac{4p^k}{(k+1)(k+2)(k+3)} \le \frac{1}{3}.$$

Therefore,

$$\mathrm{Var}_\omega\big(\lfloor c_\omega \tau \rfloor \big(E_\omega F^{(j)}\big) - \beta\tau\big) \le (E_\omega S)^2 + \frac{(E_\omega F^{(1)})^2}{3}.$$

Recalling (35), we obtain that

$$\mathrm{Var}_\omega(T_\nu - \beta\tau) \le \mathrm{Var}_\omega(S) + (E_\omega S)^2 + \frac{(E_\omega F^{(1)})^2}{3} + \mathrm{Var}_\omega\big(F^{(1)}\big)\big(E_\omega \lfloor c_\omega \tau \rfloor\big).$$

Since (31) implies that

$$\mathrm{Var}_\omega(T_\nu) = \mathrm{Var}_\omega(S) + \mathrm{Var}_\omega\left(\sum_{i=1}^{N} F^{(i)}\right) = \mathrm{Var}_\omega(S) + \big(E_\omega F^{(1)}\big)^2 \mathrm{Var}_\omega(N) + \mathrm{Var}_\omega\big(F^{(1)}\big)(E_\omega N),$$

the bound (34) follows. □

The utility of the upper bound in Lemma 4.5 is that $E_\omega F^{(1)}$ and $E_\omega S$ are relatively small when M_1 is large.

Lemma 4.6. *For* $0 < \varepsilon < 1$,

$$Q\big(E_\omega S > n^{6\varepsilon/\kappa}, M_1 > n^{(1-\varepsilon)/\kappa}\big) = \mathrm{o}\big(n^{-1}\big) \tag{36}$$

and

$$Q\big(E_\omega F^{(1)} > n^{6\varepsilon/\kappa}, M_1 > n^{(1-\varepsilon)/\kappa}\big) = \mathrm{o}\big(n^{-1}\big). \tag{37}$$

The bound (36) on the tail decay of $E_\omega S$ was proved in [14], Corollary 4.2. The proof of (37) is similar and involves straightforward but rather tedious computations using explicit formulas for quenched expectations and variances of hitting times conditioned on exiting an interval on a certain side. We defer the proof to Appendix B.

We now proceed to finish the proof of Lemma 4.4 by extending the coupling of T_ν with τ to all crossing times and showing that the resulting coupling satisfies (30). As was done for T_ν in (31) we may decompose $T_{\nu_i} - T_{\nu_{i-1}}$ so that, with the obvious notation,

$$T_{\nu_i} - T_{\nu_{i-1}} = S_i + \sum_{j=1}^{N_i} F_i^{(j)}.$$

Lemma 4.5 tells us that

$$\begin{aligned}\sum_{i=1}^n &\operatorname{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}} - \beta_i\tau_i)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}} \\ &\le \sum_{i=1}^n \biggl((E_\omega S_i)^2 + \frac{(E_\omega F_i^{(1)})^2}{3} + \operatorname{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}}) - \bigl(E_\omega F_i^{(1)}\bigr)^2 \operatorname{Var}_\omega(N_i)\biggr)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}}.\end{aligned}$$

An immediate consequence of Lemma 4.6 is that for any $0<\varepsilon<1$, on an event of probability converging to one, all the $E_\omega S_i$ and $E_\omega F_i^{(1)}$ with $i \le n$ are less than $n^{6\varepsilon/\kappa}$ when $M_1 > n^{(1-\varepsilon)/\kappa}$. Thus, by choosing $0 < 12\varepsilon/\kappa < 2/\kappa - 1$ we obtain that

$$\lim_{n\to\infty} Q\Biggl(\frac{1}{n^{2/\kappa}}\sum_{i=1}^n\biggl((E_\omega S_i)^2 + \frac{(E_\omega F_i^{(1)})^2}{3}\biggr)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}} > \delta\Biggr) = 0, \quad \forall\delta>0.$$

Therefore, to prove (30) it is enough to show

$$\lim_{n\to\infty} Q\Biggl(\frac{1}{n^{2/\kappa}}\sum_{i=1}^n\bigl(\operatorname{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}}) - \bigl(E_\omega F_i^{(1)}\bigr)^2\operatorname{Var}_\omega N_i\bigr)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}} > \delta\Biggr) = 0, \quad \forall\delta>0. \tag{38}$$

In [14], it was shown that, when M_1 is large, $\beta_1^2 = (E_\omega T_\nu)^2$ is comparable to $\operatorname{Var}_\omega T_{\nu_1}$. In fact, as was shown in the proof of Corollary 5.6 in [14],

$$\lim_{n\to\infty} Q\Biggl(n^{-2/\kappa}\Biggl|\sum_{i=1}^n\bigl(\operatorname{Var}_\omega(T_{\nu_i} - T_{\nu_{i-1}}) - \beta_i^2\bigr)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}}\Biggr| > \delta\Biggr) = 0, \quad \forall\delta>0.$$

Therefore it only remains to show that

$$\lim_{n\to\infty} Q\Biggl(n^{-2/\kappa}\sum_{i=1}^n\bigl(\beta_i^2 - \bigl(E_\omega F_i^{(1)}\bigr)^2\operatorname{Var}_\omega(N_i)\bigr)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}} > \delta\Biggr) = 0, \quad \forall\delta>0.$$

Note that by (32)

$$\begin{aligned}\beta^2 - \bigl(E_\omega F^{(1)}\bigr)^2\operatorname{Var}_\omega(N) &= (E_\omega S)^2 + 2(E_\omega S)\bigl(E_\omega F^{(1)}\bigr)(E_\omega N) - \bigl(E_\omega F^{(1)}\bigr)^2\bigl(E_\omega N^2\bigr) \\ &\le (E_\omega S)^2 + 2(E_\omega S)(E_\omega T_\nu).\end{aligned}$$

On the event where $E_\omega S_i \le n^{6\varepsilon/\kappa}$ for all $i \le n$ with $M_i > n^{(1-\varepsilon)/\kappa}$ we have

$$\begin{aligned}\sum_{i=1}^n\bigl(\beta_i^2 - \bigl(E_\omega F_i^{(1)}\bigr)^2\operatorname{Var}_\omega(N_i)\bigr)\mathbf{1}_{\{M_i>n^{(1-\varepsilon)/\kappa}\}} &\le n^{1+12\varepsilon/\kappa} + 2n^{6\varepsilon/\kappa}\sum_{i=1}^n E_\omega^{\nu_{i-1}} T_{\nu_i} \\ &= n^{1+12\varepsilon/\kappa} + 2n^{6\varepsilon/\kappa} E_\omega T_{\nu_n}.\end{aligned}$$

Again, applying Lemma 4.6 with $0 < 12\varepsilon/\kappa < 2/\kappa - 1$, we see that for any $\delta > 0$,

$$\limsup_{n\to\infty} Q\left(n^{-2/\kappa} \sum_{i=1}^{n} \bigl(\beta_i^2 - \bigl(E_\omega F_i^{(1)}\bigr)^2 \operatorname{Var}_\omega N_i\bigr)\mathbf{1}_{\{M_i > n^{(1-\varepsilon)/\kappa}\}} > \delta\right)$$
$$\le \limsup_{n\to\infty} Q\left(n^{-2/\kappa+6\varepsilon/\kappa} E_\omega T_{\nu_n} > \frac{\delta}{2}\right),$$

and so the proof will be complete once we show that $n^{-2/\kappa+\varepsilon} E_\omega T_{\nu_n} = n^{-2/\kappa+\varepsilon} \sum_{i=1}^n \beta_i$ converges in probability to 0 for $\varepsilon > 0$ small enough.

If $\kappa < 1$, then since $n^{-1/\kappa} E_\omega T_{\nu_n}$ converges in distribution [14], Theorem 1.1, choosing $\varepsilon < 1/\kappa$ works. If $\kappa > 1$ then since $E_\omega T_{\nu_n} = \sum_{i=1}^n \beta_i$ and the β_i are stationary and integrable under Q (see (17)), the ergodic theorem implies that $n^{-1} E_\omega T_{\nu_n}$ converges and, hence, choosing $\varepsilon < 2/\kappa - 1$ works. Finally, when $\kappa = 1$ it follows from (17) that for any $0 < p < 1$, $E_Q(\sum_{i=1}^n \beta_i)^p \le a_p n$ for some $a_p \in (0, \infty)$, so choosing $\varepsilon < 1$ works. $\square$

We conclude this section by noting that with a few minor modifications of the proof of Proposition 4.1 we can obtain the following analog in the case of the averaged centering.

Proposition 4.7. *For $\omega \in \Omega$, suppose that P_ω is expanded so that there exists a sequence τ_i which, under P_ω, is an i.i.d. sequence of mean 1 exponential random variables. Let $\sigma_{n,\omega} \in \mathcal{M}_1$ be defined by*

$$\sigma_{n,\omega}(\cdot) = \begin{cases} P_\omega\bigl(\frac{1}{n^{1/\kappa}} \sum_{i=1}^n \beta_i \tau_i \in \cdot\bigr), & \kappa < 1, \\ P_\omega\bigl(\frac{1}{n} \sum_{i=1}^n \bigl(\beta_i \tau_i - D'(n)\bigr) \in \cdot\bigr), & \kappa = 1, \\ P_\omega\bigl(\frac{1}{n^{1/\kappa}} \sum_{i=1}^n (\beta_i \tau_i - \bar\beta) \in \cdot\bigr), & \kappa \in (1,2), \end{cases} \tag{39}$$

where $D'(n) = E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \bar\nu n\}}] \sim C_0 \log(n)$ and $\bar\beta = E_Q[\beta_1] = E_Q[E_\omega T_\nu]$. Let $c_{\lambda,\kappa}(\varepsilon)$ be as in Theorem 1.6, and set $\tilde c_{\lambda,1}(\varepsilon) = \int_\varepsilon^{\bar\nu} \lambda x^{-1}\,\mathrm{d}x = c_{\lambda,1}(\varepsilon) + \lambda \log(\bar\nu)$. If

$$\sigma_{n,\omega} \overset{Q}{\Longrightarrow} \begin{cases} H(N_{\lambda,\kappa}), & \kappa < 1, \\ \lim_{\varepsilon\to 0^+} H_\varepsilon(N_{\lambda,1}) * \delta_{-\tilde c_{\lambda,1}(\varepsilon)}, & \kappa = 1, \\ \lim_{\varepsilon\to 0^+} H_\varepsilon(N_{\lambda,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)}, & \kappa \in (1,2), \end{cases}$$

then

$$\mu_{n,\omega} \overset{P}{\Longrightarrow} \begin{cases} H(N_{\lambda/\bar\nu,\kappa}), & \kappa < 1, \\ \lim_{\varepsilon\to 0^+} H_\varepsilon(N_{\lambda/\bar\nu,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)}, & \kappa \in [1,2), \end{cases}$$

where $\mu_{n,\omega}$ is as in Theorem 1.6.

Remark 4.8. *In the case $\kappa = 1$, the relation between the sequences $D(n)$ and $D'(n)$ can be given by*

$$D(n) = \frac{\lfloor n/\bar\nu \rfloor}{n} D'\bigl(\lfloor n/\bar\nu \rfloor\bigr) = \frac{\lfloor n/\bar\nu \rfloor}{n} E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \bar\nu \lfloor n/\bar\nu \rfloor\}}].$$

5. Analysis of the crossing times

By Propositions 4.1 and 4.7, our work is reduced to studying the distribution of a random mixture of exponential random variables, where the random coefficients are the average crossing times $\beta_i = E_\omega^{\nu_{i-1}} T_{\nu_i}$ in (21). The following proposition, which is the main result of this section, establishes a Poisson limit of point processes arising from the random coefficients β_i.

Proposition 5.1. *For $n \geq 1$ let $N_{n,\omega}$ be a point process defined by*

$$N_{n,\omega} = \sum_{i=1}^{n} \delta_{\beta_i / n^{1/\kappa}}. \tag{40}$$

Then, under the measure Q, $N_{n,\omega}$ converges weakly in the space $\mathcal{M}_p$ to a non-homogeneous Poisson point process with intensity $\lambda x^{-\kappa-1}$, where $\lambda = C_0\kappa$ and C_0 is the constant in (17). That is, $N_{n,\omega} \overset{Q}{\Longrightarrow} N_{\lambda,\kappa}$.

Proof. For a point process $\zeta = \sum_{i\geq 1} \delta_{x_i} \in \mathcal{M}_p$ and a function $f:(0,\infty] \to \mathbb{R}_+$, define the Laplace functional $\zeta(f) = \sum_{i\geq 1} f(x_i)$. Since the weak convergence in the space $\mathcal{M}_p$ is equivalent to convergence of the Laplace functionals evaluated at all continuous functions with compact support of the type $[\delta,\infty]$ for some $\delta > 0$ (see Proposition 3.19 in [15]), the statement of the proposition will follow once we check that for any such f

$$\lim_{n\to\infty} E_Q\left[\mathrm{e}^{-N_{n,\omega}(f)}\right] = \exp\left\{-\int_0^\infty \left(1 - \mathrm{e}^{-f(x)}\right)\lambda x^{-\kappa-1}\,\mathrm{d}x\right\}. \tag{41}$$

Remark 5.2. *An inspection of the argument of Propositions* 3.16 *and* 3.19 *in* [15] *reveals that the convergence in* (41) *for all continuous functions with compact support as above will follow once it is checked for such functions that are, in addition, Lipschitz continuous on* $(0,\infty)$.

Recall from (17) that $Q(\beta_1 > x) \sim C_0 x^{-\kappa}$. Thus, if the (β_i) were i.i.d., the conclusion of the proposition would follow immediately; see e.g. Proposition 3.21 in [15]. Since the sequence (β_i) is only stationary under Q, our strategy is to show that the dependence between the (β_i) is weak enough so that the point process $N_{n,\omega}$ converges weakly to the same limit as if the (β_i) were i.i.d.

Recalling the notation in (10) and (11) and the formula for quenched expectations of hitting times in (13), we may write

$$\begin{aligned}
\beta_i = E_\omega^{\nu_{i-1}} T_{\nu_i} &= \nu_i - \nu_{i-1} + 2\sum_{j=\nu_{i-1}}^{\nu_i - 1} W_j \\
&= \nu_i - \nu_{i-1} + 2\sum_{j=\nu_{i-1}}^{\nu_i - 1} W_{\nu_{i-1},j} + 2W_{\nu_{i-1}-1} R_{\nu_{i-1},\nu_i - 1}.
\end{aligned}$$

Thus, $\beta_i = A_i Z_i + Y_i$, where

$$A_i = W_{\nu_{i-1}-1}, \qquad Z_i = 2R_{\nu_{i-1},\nu_i-1} \quad \text{and} \quad Y_i = \nu_i - \nu_{i-1} + 2\sum_{j=\nu_{i-1}}^{\nu_i-1} W_{\nu_{i-1},j}.$$

Note that Y_i and Z_i only depend on the environment from ν_{i-1} to $\nu_i - 1$, and therefore $\{(Y_i, Z_i)\}_{i\geq 1}$ is an i.i.d. sequence of random variables with the same distribution as

$$(Y_1, Z_1) = \left(\nu + 2\sum_{j=0}^{\nu-1} W_{0,j}, 2R_{0,\nu-1}\right).$$

Also, note that the sequence $\{A_i\}_{i\geq 1}$ is stationary under the measure Q. From this decomposition of β_i we can see that the reason (β_i) is not an i.i.d. sequence is that the sequence (A_i) is not i.i.d. The random variables (A_i) all have the same distribution under Q as $A_1 = W_{-1}$. Furthermore, W_{-1} has exponential tails under Q. That is, there exist constants $C, C' > 0$ such that

$$Q(W_{-1} > x) \leq C'\mathrm{e}^{-Cx}; \tag{42}$$

see Lemma 4.2.2 in [12]. In addition, W_{-1} can be very well approximated by $W_{-j,-1}$ for large j. That is, there exist constants $C_1, C_2, C_3 > 0$ such that for every $j = 1, 2, \ldots,$

$$Q\bigl(W_{-1} - W_{-j,-1} > \mathrm{e}^{-C_1 j}\bigr) \le C_2 \mathrm{e}^{-C_3 j}. \tag{43}$$

To see this, defining the ladder locations ν_{-k} to the left of the origin in the natural way (see [14]), observe that for any $c > 0$,

$$\begin{aligned} Q\bigl(W_{-1} - W_{\nu_{-k},-1} > \mathrm{e}^{-ck}\bigr) &\le \mathrm{e}^{ck} E_Q[W_{-1} - W_{\nu_{-k},-1}] \\ &= \mathrm{e}^{ck} E_Q[\Pi_{\nu_{-k},-1} W_{\nu_{-k}-1}] = \mathrm{e}^{ck} E_Q[\Pi_{0,\nu-1}]^k E_Q[W_1]. \end{aligned}$$

Since $E_Q[\Pi_{0,\nu-1}] < 1$ by the definition of the ladder locations, choosing c small enough gives us an exponential bound $Q(W_{-1} - W_{\nu_{-k},-1} > \mathrm{e}^{-ck}) \le C' \mathrm{e}^{-Ck}$, $k = 1, 2, \ldots,$ for some positive C, C'. The bound (43) now follows by writing, for $a > 0$,

$$Q\bigl(W_{-1} - W_{-j,-1} > \mathrm{e}^{-cj}\bigr) \le Q\bigl(W_{-1} - W_{-\nu_{aj},-1} > \mathrm{e}^{-cj}\bigr) + Q(\nu_{aj} > j),$$

and noticing that, by (16), for $a > 0$ small enough, the latter probability is exponentially small as a function of j.

Keeping the exponential bounds (42) and (43) in mind, we modify the sequence of the crossing times in order to reduce the dependence. For $n \ge 1$ we set $A_i^{(n)} = W_{\nu_{i-1} - \lfloor\sqrt{n}\rfloor, \nu_{i-1}-1}$ and $\beta_i^{(n)} = A_i^{(n)} Z_i + Y_i$, $i = 1, 2, \ldots.$ Notice that $\beta_i^{(n)}$ and $\beta_j^{(n)}$ are independent if $|i - j| > \sqrt{n}$. Next, we give a comparison of $\beta_i^{(n)}$ with β_i that will allow us to analyze the tail behaviour of the random variables $(\beta_i^{(n)})$.

Lemma 5.3. *There exist constants*, $C, C' > 0$ *such that*

$$Q\bigl(\beta_1 - \beta_1^{(n)} > \mathrm{e}^{-n^{1/4}}\bigr) \le C \mathrm{e}^{-C'\sqrt{n}}, \quad n = 1, 2, \ldots.$$

Proof. From the decompositions of β_i and $\beta_i^{(n)}$ we obtain that $\beta_i - \beta_i^{(n)} = (A_i - A_i^{(n)}) Z_i$. Note that $Z_1 = 2R_{0,\nu-1} \le 2R_0$. By (17) there exists a constant C such that $Q(Z_1 > x) \le C x^{-\kappa}$ for all $x > 0$. Therefore, for any $x > 0$

$$\begin{aligned} Q\bigl(\beta_1 - \beta_1^{(n)} > x\bigr) &\le Q\bigl(A_1 - A_1^{(n)} > \mathrm{e}^{-C_1\sqrt{n}}\bigr) + Q\bigl(Z_1 > \mathrm{e}^{C_1\sqrt{n}} x\bigr) \\ &\le C_2 \mathrm{e}^{-C_3\sqrt{n}} + C \mathrm{e}^{-C_1\kappa\sqrt{n}} x^{-\kappa}. \end{aligned}$$

Choosing $x = \mathrm{e}^{-n^{1/4}}$ completes the proof. □

Based on the truncated crossing times $(\beta_i^{(n)})$ we define a sequence of point processes by

$$N_{n,\omega}^{(n)} = \sum_{i \ge 1} \delta_{\beta_i^{(n)}/n^{1/\kappa}}, \quad n = 1, 2, \ldots.$$

Lemma 5.4. $N_{n,\omega}^{(n)} \overset{Q}{\Longrightarrow} N_{\lambda,\kappa}$ *as* $n \to \infty$ *for* $\lambda = C_0$, *the constant in* (17).

Proof. Let $f : (0, \infty] \to \mathbb{R}_+$ be a continuous functon vanishing for all $0 < x < \delta$ for some $\delta > 0$, and Lipshitz on the interval (δ, ∞). We will prove the following analogue of (41):

$$\lim_{n\to\infty} E_Q\bigl[\mathrm{e}^{-N_{n,\omega}^{(n)}(f)}\bigr] = \exp\biggl\{-\int_0^\infty \bigl(1 - \mathrm{e}^{-f(x)}\bigr)\lambda x^{-\kappa-1}\,\mathrm{d}x\biggr\}. \tag{44}$$

According to Remark 5.2, this will give us the claim of the lemma.

For $0 < \tau < 1$ we define a sequence of random random variables

$$K_n(\tau) = \mathrm{card}\bigl\{i = 1, \ldots, n\text{: both } \beta_i^{(n)} > \delta n^{1/\kappa} \text{ and } \beta_j^{(n)} > \delta n^{1/\kappa} \text{ for some } i + 1 \le j \le i + \tau n,\ j \le n\bigr\}.$$

We claim that

$$\lim_{\tau\to 0}\limsup_{n\to\infty} Q\bigl(K_n(\tau)>0\bigr)=0. \tag{45}$$

To see this, let $0<\varepsilon<1$, and consider a sequence of events

$$B_n(\varepsilon)=\bigl\{\text{for some } i=1,\dots,n,\ \beta_i^{(n)}>\delta n^{1/\kappa} \text{ but } \max(Y_i,Z_i)\le \varepsilon n^{1/\kappa}\bigr\}.$$

Since by (17) there exists a constant C such that $Q(\max(Y_1,Z_1)>x)\le Cx^{-\kappa}$ for all $x>0$, while by (42) the random variable A_1 has an exponentially fast decaying tail, we see that

$$\begin{aligned}
Q\bigl(B_n(\varepsilon)\bigr) &\le nQ\bigl(\max(Y_1,Z_1)\le \varepsilon n^{1/\kappa}, \beta_1^{(n)}>\delta n^{1/\kappa}\bigr)\\
&\le nQ\bigl(\max(Y_1,Z_1)\le \varepsilon n^{1/\kappa}, (A_1+1)\max(Y_1,Z_1)>\delta n^{1/\kappa}\bigr)\\
&= O\bigl(nQ\bigl(\max(Y_1,Z_1)>\delta n^{1/\kappa}\bigr)E_Q\bigl((A_1+1)^\kappa \mathbf{1}(A_1+1>\delta/\varepsilon)\bigr)\bigr)\\
&= O\bigl(\delta^{-\kappa}E_Q\bigl((A_1+1)^\kappa \mathbf{1}(A_1+1>\delta/\varepsilon)\bigr)\bigr)
\end{aligned}$$

as in, for example, Breiman's lemma ([3]). Therefore,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} Q\bigl(B_n(\varepsilon)\bigr)=0. \tag{46}$$

For $\tau,\varepsilon>0$

$$\begin{aligned}
Q\bigl(K_n(\tau)>0\bigr) &\le Q\bigl(B_n(\varepsilon)\bigr)+Q\bigl(\text{for some } i=1,\dots,n, \text{ some } i+1\le j\le i+\tau n,\\
&\qquad \max(Y_i,Z_i)>\varepsilon n^{1/\kappa} \text{and} \max(Y_j,Z_j)>\varepsilon n^{1/\kappa}\bigr)\\
&\le Q\bigl(B_n(\varepsilon)\bigr)+\tau n^2\bigl(Q\bigl(\max(Y_1,Z_1)>\varepsilon n^{1/\kappa}\bigr)\bigr)^2\\
&\le Q\bigl(B_n(\varepsilon)\bigr)+C^2\varepsilon^{-2\kappa}\tau.
\end{aligned}$$

We conclude that

$$\lim_{\tau\to 0}\limsup_{n\to\infty} Q\bigl(K_n(\tau)>0\bigr)\le \limsup_{n\to\infty} Q\bigl(B_n(\varepsilon)\bigr),$$

and so (45) follows from (46).

Fix, for a moment, $\varepsilon>0$ and take $\tau>0$ such that for some n_0 we have $Q(K_n(\tau)>0)\le\varepsilon$ for all $n\ge n_0$; this is possible by (45). Consider the random sets

$$D_n=\bigl\{i=1,\dots,n\colon\ \beta_i^{(n)}>\delta n^{1/\kappa}\bigr\}.$$

Since $f(x)=0$ if $x\le\delta$, we can write

$$\begin{aligned}
E_Q\bigl[\mathrm{e}^{-N_{n,\omega}^{(n)}(f)}\bigr] &= E_Q\exp\Bigl\{-\sum_{i\in D_n} f\bigl(\beta_i^{(n)}/n^{1/\kappa}\bigr)\Bigr\}\\
&= E_Q\Bigl[\exp\Bigl\{-\sum_{i\in D_n} f\bigl(\beta_i^{(n)}/n^{1/\kappa}\bigr)\Bigr\}\mathbf{1}\bigl(K_n(\tau)=0\bigr)\Bigr]\\
&\quad + E_Q\Bigl[\exp\Bigl\{-\sum_{i\in D_n} f\bigl(\beta_i^{(n)}/n^{1/\kappa}\bigr)\Bigr\}\mathbf{1}\bigl(K_n(\tau)>0\bigr)\Bigr]\\
&:= H_n^{(1)}+H_n^{(2)}.
\end{aligned} \tag{47}$$

By the choice of τ,

$$\limsup_{n\to\infty} H_n^{(2)} \le \limsup_{n\to\infty} Q\bigl(K_n(\tau) > 0\bigr) \le \varepsilon. \tag{48}$$

Moreover, given the event $\{K_n(\tau) = 0\}$, the points in the random set D_n are separated, for large n, by more than $\sqrt{n}$ and, hence, given also the random set D_n, the random variables $\beta_i^{(n)}$, $i \in D_n$, are independent, each one with the corresponding conditional distribution. That is,

$$H_n^{(1)} = Q\bigl(K_n(\tau)=0\bigr)E_Q\bigl\{\bigl[E_Q\bigl(\exp\bigl\{-f\bigl(\beta_1^{(n)}/n^{1/\kappa}\bigr)\bigr\}|\beta_1^{(n)} > \delta n^{1/\kappa}\bigr)\bigr]^{\operatorname{card} D_n}\big|K_n(\tau)=0\bigr\}.$$

The power law (17) and Lemma 5.3 show the weak convergence to the Pareto distribution

$$Q\bigl(\beta_1^{(n)}/n^{1/\kappa} > t|\beta_1^{(n)} > \delta n^{1/\kappa}\bigr) \to (t/\delta)^{-\kappa}$$

for $t \ge \delta$, and so by the bounded convergence theorem,

$$E_Q\bigl(\exp\bigl\{-f\bigl(\beta_1^{(n)}/n^{1/\kappa}\bigr)\bigr\}|\beta_1^{(n)} > \delta n^{1/\kappa}\bigr) \to \int_1^\infty \mathrm{e}^{-f(\delta t)}\kappa t^{-(\kappa+1)}\,\mathrm{d}t.$$

Now the claim (44) follows from (47), (48) and the following limiting statement: for the constant C_0 in (17),

$$\begin{aligned}\exp\bigl\{-C_0(1-\alpha)\delta^{-\kappa}\bigr\} &\le \lim_{\tau\to0}\liminf_{n\to\infty} E_Q\bigl(\alpha^{\operatorname{card} D_n}|K_n(\tau)=0\bigr)\\ &= \lim_{\tau\to0}\limsup_{n\to\infty} E_Q\bigl(\alpha^{\operatorname{card} D_n}|K_n(\tau)=0\bigr) \le \exp\bigl\{-C_0(1-\alpha)\delta^{-\kappa}\bigr\}\end{aligned} \tag{49}$$

for all $0 < \alpha < 1$. In order to complete the proof of the lemma it, therefore, remains to prove (49).

We split the set $\{1, 2, \ldots, n\}$ into a union of the following sets. Let

$$\begin{aligned}
&I_{1.n} = \bigl\{1, \ldots, \bigl[n^{3/4}\bigr]\bigr\}, \qquad J_{1.n} = \bigl\{\bigl[n^{3/4}\bigr]+1, \ldots, \bigl[n^{3/4}\bigr]+\bigl[n^{2/3}\bigr]\bigr\},\\
&I_{2.n} = \bigl\{\bigl[n^{3/4}\bigr]+\bigl[n^{2/3}\bigr]+1, \ldots, 2\bigl[n^{3/4}\bigr]+\bigl[n^{2/3}\bigr]\bigr\},\\
&J_{2.n} = \bigl\{2\bigl[n^{3/4}\bigr]+\bigl[n^{2/3}\bigr]+1, \ldots, 2\bigl[n^{3/4}\bigr]+2\bigl[n^{2/3}\bigr]\bigr\},
\end{aligned}$$

etc. (the last interval can be a bit shorter than the rest). Clearly, the cardinality m_n of the union of all intervals $J_{k.n}$ satisfies $m_n/n \to 0$ as $n \to \infty$. We write $D_n = D_n^{(I)} \cup D_n^{(J)}$, where $D_n^{(I)}$ (resp. $D_n^{(J)}$) contains all the points of D_n that are in one of the intervals $I_{k.n}$ (resp. $J_{k.n}$). Observe that the intervals $I_{k.n}$ are separated by more that $\sqrt{n}$, so for i and j in two different of this type, $\beta_i^{(n)}$ and $\beta_j^{(n)}$ are independent. We have

$$\begin{aligned}E_Q\bigl(\alpha^{\operatorname{card} D_n}\mathbf{1}\bigl(K_n(\tau)=0\bigr)\bigr) &\le E_Q\bigl(\alpha^{\operatorname{card} D_n^{(I)}}\bigr)\\ &= \bigl(E_Q\alpha^{\operatorname{Card}(D_n\cap I_{1.n})}\bigr)^{[n/([n^{3/4}]+[n^{2/3}])]}.\end{aligned}$$

Repeating the argument leading to (45) (that shows that $\beta_i^{(n)}$ and $\beta_j^{(n)}$ can both exceed $\delta n^{1/\kappa}$ for $0 < |i-j| \le n^{3/4}$ only on an event of a vanishing probability) tells us that

$$\begin{aligned}
&Q\bigl(\operatorname{Card}(D_n \cap I_{1.n}) = 1\bigr) \sim n^{3/4}Q\bigl(\beta_1^{(n)} > \delta n^{1/\kappa}\bigr) \sim n^{3/4}C_0\delta^{-\kappa}n^{-1} = C_0\delta^{-\kappa}n^{-1/4},\\
&Q\bigl(\operatorname{Card}(D_n \cap I_{1.n}) > 1\bigr) = \mathrm{o}\bigl(n^{-1/4}\bigr).
\end{aligned}$$

Therefore,

$$E_Q\alpha^{\operatorname{Card}(D_n\cap I_{1.n})} = 1-(1-\alpha)C_0\delta^{-\kappa}n^{-1/4} + \mathrm{o}\bigl(n^{-1/4}\bigr),$$

implying that

$$\limsup_{n\to\infty} E_Q\big(\alpha^{\operatorname{card} D_n} \mid K_n(\tau)=0\big) \le \frac{1}{Q(K_n(\tau)=0)} \exp\big\{-C_0(1-\alpha)\delta^{-\kappa}\big\},$$

and the upper limit part in (49) follows from (45).

Similarly,

$$\begin{aligned} E_Q\big(\alpha^{\operatorname{card} D_n}\mathbf{1}\big(K_n(\tau)=0\big)\big) &\ge E_Q\big(\alpha^{\operatorname{card} D_n^{(I)}}\mathbf{1}\big(K_n(\tau)=0, D_n^{(J)}=0\big)\big) \\ &\ge E_Q\big(\alpha^{\operatorname{card} D_n^{(I)}}\big) - Q\big(K_n(\tau)>0\big) - Q\big(D_n^{(J)}>0\big). \end{aligned}$$

The last term vanishes in the limit since $m_n/n \to 0$. Therefore,

$$\liminf_{n\to\infty} E_Q\big(\alpha^{\operatorname{card} D_n} \mid K_n(\tau)=0\big) \ge \exp\big\{-C_0(1-\alpha)\delta^{-\kappa}\big\} - Q\big(K_n(\tau)>0\big),$$

and the lower limit part in (49) follows from (45) as well. □

Now we are ready to finish the proof of Proposition 5.1, which we accomplish by checking (41) for non-negative continuous functions f on $(0,\infty]$ with compact support that are Lipschitz continuous on $(0,\infty)$. For any such function f,

$$\begin{aligned} E\big[\mathrm{e}^{-N_{n,\omega}(f)}\big] &= E\left[\exp\left\{-\sum_{i=1}^n f\big(\beta_i/n^{1/\kappa}\big)\right\}\right] \\ &= E\left[\mathrm{e}^{-N_{n,\omega}^{(n)}(f)} \exp\left\{-\sum_{i=1}^n \big(f\big(\beta_i/n^{1/\kappa}\big) - f\big(\beta_i^{(n)}/n^{1/\kappa}\big)\big)\right\}\right]. \end{aligned}$$

Now, let

$$\Omega_n := \big\{\omega\in\Omega\colon\ \beta_i - \beta_i^{(n)} \le \mathrm{e}^{-n^{1/4}}, \forall i=1,2,\dots,n\big\}.$$

Lemma 5.3 implies that $Q(\Omega_n^c)\to 0$ as $n\to\infty$. Since f is Lipschitz with some constant c, on the event Ω_n we have

$$\begin{aligned} \left|\sum_{i=1}^n \big(f\big(\beta_i/n^{1/\kappa}\big) - f\big(\beta_i^{(n)}/n^{1/\kappa}\big)\big)\right| &\le \frac{c}{n^{1/\kappa}}\sum_{i=1}^n \big|\beta_i - \beta_i^{(n)}\big| \\ &\le c n^{1-1/\kappa}\mathrm{e}^{-n^{1/4}}, \end{aligned}$$

and so by Lemma 5.4

$$\lim_{n\to\infty} E\big[\mathrm{e}^{-N_{n,\omega}(f)}\big] = \lim_{n\to\infty} E\big[\mathrm{e}^{-N_{n,\omega}^{(n)}(f)}\mathbf{1}_{\Omega_n}\big] = \mathbf{E}\big[\mathrm{e}^{-N_{\lambda,\kappa}(f)}\big],$$

proving (41). □

In addition to the already established convergence of the point processes $(N_{n,\omega})$, in the sequel we will also need the following tail bound on the sums of the average crossing times β_i that are not extremely large.

Lemma 5.5. *Let $\kappa\in[1,2)$. Then for any $\delta>0$,*

$$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty} Q\left(\frac{1}{n^{1/\kappa}}\left|\sum_{i=1}^n \big(\beta_i\mathbf{1}_{\{\beta_i\le\varepsilon n^{1/\kappa}\}} - E_Q[\beta_1\mathbf{1}_{\{\beta_1\le\varepsilon n^{1/\kappa}\}}]\big)\right| \ge \delta\right) = 0.$$

Proof. Clearly, $\beta_i \mathbf{1}_{\{\beta_i \le \varepsilon n^{1/\kappa}\}} = \beta_i \wedge \varepsilon n^{1/\kappa} - \varepsilon n^{1/\kappa} \mathbf{1}_{\{\beta_i > \varepsilon n^{1/\kappa}\}}$. Therefore,

$$Q\left(\frac{1}{n^{1/\kappa}}\left|\sum_{i=1}^n \bigl(\beta_i \mathbf{1}_{\{\beta_i \le \varepsilon n^{1/\kappa}\}} - E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \varepsilon n^{1/\kappa}\}}]\bigr)\right| \ge \delta\right)$$

$$\le Q\left(\frac{1}{n^{1/\kappa}}\left|\sum_{i=1}^n \bigl(\beta_i \wedge \varepsilon n^{1/\kappa} - E_Q\bigl[\beta_1 \wedge \varepsilon n^{1/\kappa}\bigr]\bigr)\right| \ge \delta/2\right) \tag{50}$$

$$+ Q\left(\varepsilon\left|\sum_{i=1}^n \mathbf{1}_{\{\beta_i > \varepsilon n^{1/\kappa}\}} - nQ\bigl(\beta_1 > \varepsilon n^{1/\kappa}\bigr)\right| \ge \delta/2\right). \tag{51}$$

We will first handle the term in (51). For $\varepsilon > 0$, let $G_\varepsilon : \mathcal{M}_p \to \mathbb{Z}_+$ be defined by $G_\varepsilon(\zeta) = \sum_{i\ge1} \mathbf{1}_{\{x_i > \varepsilon\}}$ when $\zeta = \sum_{i\ge1} \delta_{x_i}$. Then, since G_ε is continuous on the set $\mathcal{M}_p^{(\varepsilon)} = \{\zeta(\{\varepsilon\}) = 0\}$, we conclude by Proposition 5.1 and the continuous mapping theorem that $\sum_{i=1}^n \mathbf{1}_{\{\beta_i > \varepsilon n^{1/\kappa}\}} = G_\varepsilon(N_{n,\omega}) \Longrightarrow G_\varepsilon(N_{\lambda,\kappa})$. Further, it follows from (17) that $nQ(\beta_1 > \varepsilon n^{1/\kappa}) \to C_0 \varepsilon^{-\kappa} = \mathbf{E}[G_\varepsilon(N_{\lambda,\kappa})]$ as $n \to \infty$. Now, since $G_\varepsilon(N_{\lambda,\kappa})$ has Poisson distribution with mean $\lambda\varepsilon^{-\kappa}/\kappa$, we see that

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty} Q\left(\varepsilon\left|\sum_{i=1}^n \mathbf{1}_{\{\beta_i > \varepsilon n^{1/\kappa}\}} - nQ\bigl(\beta_1 > \varepsilon n^{1/\kappa}\bigr)\right| \ge \delta/2\right) \le \lim_{\varepsilon\to0} \mathbf{P}\left(\bigl|G_\varepsilon(N_{\lambda,\kappa}) - \mathbf{E}\bigl[G_\varepsilon(N_{\lambda,\kappa})\bigr]\bigr| \ge \frac{\delta}{2\varepsilon}\right)$$

$$\le \lim_{\varepsilon\to0} \frac{4\varepsilon^2}{\delta^2} \mathbf{Var}\bigl(G_\varepsilon(N_{\lambda,\kappa})\bigr) = \lim_{\varepsilon\to0} \frac{4\varepsilon^{2-\kappa}\lambda}{\delta^2\kappa} = 0.$$

Next, we estimate the probability in (50). By Chebychev's inequality and the fact that the β_i are stationary under Q, this probability is bounded above by

$$\frac{4}{\delta^2 n^{2/\kappa}} \operatorname{Var}_Q\left(\sum_{i=1}^n \beta_i \wedge \varepsilon n^{1/\kappa}\right)$$

$$= \frac{4}{\delta^2 n^{2/\kappa}} n \operatorname{Var}_Q\bigl(\beta_1 \wedge \varepsilon n^{1/\kappa}\bigr) + \frac{8}{\delta^2 n^{2/\kappa}} \sum_{k=1}^n (n-k) \operatorname{Cov}_Q\bigl(\beta_1 \wedge \varepsilon n^{1/\kappa}, \beta_{k+1} \wedge \varepsilon n^{1/\kappa}\bigr). \tag{52}$$

Now, the tail decay (17) of β_1 and Karamata's theorem (see p. 17 in [15]) imply that

$$\limsup_{n\to\infty} n^{-(2/\kappa-1)} \operatorname{Var}_Q\bigl(\beta_1 \wedge \varepsilon n^{1/\kappa}\bigr) \le \lim_{n\to\infty} n^{-(2/\kappa-1)} E_Q\bigl[\beta_1^2 \wedge \varepsilon^2 n^{2/\kappa}\bigr] = \frac{2C_0}{2-\kappa} \varepsilon^{2-\kappa}.$$

Since $\kappa < 2$ this vanishes as $\varepsilon \to 0$ and so to finish the proof of the lemma it is enough to show that

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty} \frac{1}{n^{2/\kappa}} \sum_{k=1}^n (n-k) \operatorname{Cov}_Q\bigl(\beta_1 \wedge \varepsilon n^{1/\kappa}, \beta_{k+1} \wedge \varepsilon n^{1/\kappa}\bigr) = 0. \tag{53}$$

To bound the covariance terms, we use (13) to write

$$\begin{aligned} \beta_{k+1} &= \sum_{j=\nu_k}^{\nu_{k+1}-1} (1 + 2W_j) \\ &= \nu_{k+1} - \nu_k + 2 \sum_{j=\nu_k}^{\nu_{k+1}-1} W_{\nu_1,j} + 2W_{\nu_1-1} \Pi_{\nu_1,\nu_k-1} R_{\nu_k,\nu_{k+1}-1} \\ &=: \tilde{\beta}_{k+1} + 2W_{\nu_1-1} \Pi_{\nu_1,\nu_k-1} R_{\nu_k,\nu_{k+1}-1}. \end{aligned}$$

Note that $\tilde{\beta}_{k+1}$ is independent of β_1, so that for some constant C'

$$\begin{aligned}
\mathrm{Cov}_Q\big(\beta_1 \wedge \varepsilon n^{1/\kappa}, \beta_{k+1} \wedge \varepsilon n^{1/\kappa}\big) &= \mathrm{Cov}_Q\big(\beta_1 \wedge \varepsilon n^{1/\kappa}, \beta_{k+1} \wedge \varepsilon n^{1/\kappa} - \tilde{\beta}_{k+1} \wedge \varepsilon n^{1/\kappa}\big) \\
&\le \sqrt{\mathrm{Var}_Q\big(\beta_1 \wedge \varepsilon n^{1/\kappa}\big)}\sqrt{\mathrm{Var}_Q\big(\beta_{k+1} \wedge \varepsilon n^{1/\kappa} - \tilde{\beta}_{k+1} \wedge \varepsilon n^{1/\kappa}\big)} \\
&\le C' \varepsilon^{1-\kappa/2} n^{1/\kappa - 1/2} \sqrt{E_Q\big[(\beta_{k+1} - \tilde{\beta}_{k+1})^2 \mathbf{1}_{\{\tilde{\beta}_{k+1} \le \varepsilon n^{1/\kappa}\}}\big]}
\end{aligned} \tag{54}$$

for n large enough. An examination of the formula for $\tilde{\beta}_{k+1}$ shows that $R_{\nu_k, \nu_{k+1}-1} \le \tilde{\beta}_{k+1}$. Therefore,

$$\begin{aligned}
E_Q\big[(\beta_{k+1} - \tilde{\beta}_{k+1})^2 \mathbf{1}_{\{\tilde{\beta}_{k+1} \le \varepsilon n^{1/\kappa}\}}\big] &= 4 E_Q\big[W_{\nu_1 - 1}^2 \Pi_{\nu_1, \nu_k - 1}^2 R_{\nu_k, \nu_{k+1}-1}^2 \mathbf{1}_{\{\tilde{\beta}_{k+1} \le \varepsilon n^{1/\kappa}\}}\big] \\
&\le 4 E_Q\big[W_{\nu_1 - 1}^2\big] E_Q\big[\Pi_{\nu_1, \nu_k - 1}^2\big] E_Q\big[R_{\nu_k, \nu_{k+1}-1}^2 \mathbf{1}_{\{R_{\nu_k, \nu_{k+1}-1} \le \varepsilon n^{1/\kappa}\}}\big] \\
&= 4 E_Q\big[W_{-1}^2\big] E_Q\big[\Pi_{0, \nu-1}^2\big]^{k-1} E_Q\big[R_{0, \nu-1}^2 \mathbf{1}_{\{R_{0,\nu-1} \le \varepsilon n^{1/\kappa}\}}\big],
\end{aligned} \tag{55}$$

where in the last step we used the invariance of the distribution Q under shifts by the ladder locations ν_i. Further, $E_Q[W_{-1}^2] < \infty$ by (42), and $E_Q[\Pi_{0,\nu-1}] < 1$ by the definition of the ladder locations. Also, since $R_{0,\nu-1} \le \beta_1$, $E_Q[R_{0,\nu-1}^2 \mathbf{1}_{\{R_{0,\nu-1} \le \varepsilon n^{1/\kappa}\}}] \le C' \varepsilon^{2-\kappa} n^{2/\kappa - 1}$ for large n. Combining this with (54) and (55) we see that for some $0 < \rho < 1$,

$$\mathrm{Cov}_Q\big(\beta_1 \wedge \varepsilon n^{1/\kappa}, \beta_{k+1} \wedge \varepsilon n^{1/\kappa}\big) \le \big(C'\big)^2 \varepsilon^{2-\kappa} n^{2/\kappa - 1} \rho^k,$$

and this bound on the covariance is sufficient to prove (53). This finishes the proof of the lemma. □

We conclude this section by giving a corollary of Lemma 5.5 that is of independent interest. In [14] it was shown that, if $0 < \kappa < 1$, then $n^{-1/\kappa} E_\omega T_{\nu_n} = n^{-1/\kappa} \sum_{i=1}^n \beta_i$ converges in distribution to a κ-stable random variable. The following corollary shows that $E_\omega T_{\nu_n}$ has a stable limit law when $\kappa \in [1, 2)$ as well.

Corollary 5.6. *If $\kappa = 1$, then there exists a $b > 0$ and a sequence $D''(n) = E[\beta_1 \mathbf{1}_{\{\beta_1 \le n\}}] \sim C_0 \log n$ such that*

$$\lim_{n\to\infty} Q\left(\frac{E_\omega T_{\nu_n} - n D''(n)}{n} \le x\right) = L_{1,b}(x), \quad \forall x \in \mathbb{R}.$$

If $\kappa \in (1, 2)$, then

$$\lim_{n\to\infty} Q\left(\frac{E_\omega T_{\nu_n} - n E_Q[E_\omega T_{\nu_1}]}{n^{1/\kappa}} \le x\right) = L_{\kappa,b}(x), \quad \forall x \in \mathbb{R}.$$

In both cases $b^\kappa = \lambda/\kappa$.

Proof. This is a direct application of Proposition 5.1 and Lemma 5.5 to Theorem 3.1 in [4]. □

6. Weak quenched limits of hitting times – Quenched centering

Having done the necessary preperatory work in Sections 4 and 5 we are now ready to prove Theorem 1.3. Recall, that by Proposition 4.1 it is enough to show that $\bar{\sigma}_{n,\omega} \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa})$ for some $\lambda > 0$, where $\bar{\sigma}_{n,\omega} = \bar{H}(N_{n,\omega})$ is given in (22), while $\bar{H}$ and $N_{n,\omega}$ are defined by (3) and (40), respectively. Since $N_{n,\omega} \overset{Q}{\Longrightarrow} N_{\lambda,\kappa}$ by Proposition 5.1, if the mapping $\bar{H} : \mathcal{M}_p \to \mathcal{M}_1$ were continuous the statement of Theorem 1.3 would follow by the continuous mapping theorem. Unfortunately, $\bar{H}$ is not a continuous mapping. To overcome this, we employ a truncation technique.

For $\varepsilon > 0$ define the a mapping $\bar{H}_\varepsilon : \mathcal{M}_p \to \mathcal{M}_1$ by modifying the definition (22) as follows:

$$\bar{H}_\varepsilon(\zeta)(\cdot) = \mathbf{P}\left(\sum_{i\ge 1} x_i(\tau_i - 1)\mathbf{1}_{\{x_i > \varepsilon\}} \in \cdot\right), \quad \text{when } \zeta = \sum_{i\ge 1} \delta_{x_i}. \tag{56}$$

It turns out that this mapping is continuous on the relevant subset of $\mathcal{M}_p$.

Lemma 6.1. *$\bar{H}_\varepsilon$ is continuous on the set $\mathcal{M}_p^{(\varepsilon)} := \{\zeta \in \mathcal{M}_p \colon \zeta(\{\varepsilon\}) = 0\}$.*

Proof. Let $\zeta_n \xrightarrow{v} \zeta \in \mathcal{M}_p^{(\varepsilon)}$. Then, by [15], Proposition 3.13, there exists an integer M and a labelling of the points of ζ and ζ_n (for n sufficiently large) such that

$$\zeta\big(\cdot \cap (\varepsilon,\infty)\big) = \sum_{i=1}^{M} \delta_{x_i} \quad \text{and} \quad \zeta_n\big(\cdot \cap (\varepsilon,\infty)\big) = \sum_{i=1}^{M} \delta_{x_i^{(n)}},$$

with $(x_1^{(n)}, x_2^{(n)}, \dots, x_M^{(n)}) \to (x_1, x_2, \dots, x_M)$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \bar{H}_\varepsilon(\zeta_n)(\cdot) = \lim_{n\to\infty} \mathbf{P}\left(\sum_{i=1}^{M} x_i^{(n)}(\tau_i - 1) \in \cdot\right) = \mathbf{P}\left(\sum_{i=1}^{M} x_i(\tau_i - 1) \in \cdot\right) = \bar{H}_\varepsilon(\zeta)(\cdot)$$

in the space $\mathcal{M}_1$. □

Proof of Theorem 1.3. Since $\mathbf{P}(N_{\lambda,\kappa} \notin \mathcal{M}_p^{(\varepsilon)}) = 0$, Proposition 5.1, Lemma 6.1 and the continuous mapping theorem [2], Theorem 2.7, imply that for every $\varepsilon > 0$,

$$\bar{H}_\varepsilon(N_{n,\omega}) \overset{Q}{\Longrightarrow} \bar{H}_\varepsilon(N_{\lambda,\kappa}), \quad \text{as } n \to \infty. \tag{57}$$

Next, we claim that

$$\lim_{\varepsilon\to 0^+} \bar{H}_\varepsilon(N_{\lambda,\kappa}) = \bar{H}(N_{\lambda,\kappa}), \quad \mathbf{P}\text{-a.s.} \tag{58}$$

and

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} Q\big(\rho\big(\bar{H}_\varepsilon(N_{n,\omega}), \bar{H}(N_{n,\omega})\big) \ge \delta\big) = 0, \quad \forall \delta > 0. \tag{59}$$

By [2], Theorem 3.2, this will show that

$$\bar{\sigma}_{n,\omega} = \bar{H}(N_{n,\omega}) \overset{Q}{\Longrightarrow} \bar{H}(N_{\lambda,\kappa}),$$

which, by Proposition 4.1, is enough for the the conclusion of Theorem 1.3. Thus, it only remains to prove (58) and (59). Since the claim (58) follows from the continuity of the map $\bar{H}_2$ in the proof of Lemma 1.2 in Appendix A, we prove (59).

Recall that for any two random variables X and Y defined on the same probability space, with respective laws $\mathcal{L}_X$ and $\mathcal{L}_Y$, $\rho(\mathcal{L}_X, \mathcal{L}_Y) \le (E|X - Y|^2)^{1/3}$. Therefore,

$$\rho\big(\bar{H}_\varepsilon(N_{n,\omega}), \bar{H}(N_{n,\omega})\big) \le \left(\frac{1}{n^{2/\kappa}} \sum_{i=1}^{n} \beta_i^2 \mathbf{1}_{\{\beta_i/n^{1/\kappa} \le \varepsilon\}}\right)^{1/3}$$

and so by the Markov inequality, (17) and Karamata's theorem,

$$\begin{aligned}\limsup_{n\to\infty} Q\big(\rho\big(\bar{H}_\varepsilon(N_{n,\omega}), \bar{H}(N_{n,\omega})\big) \ge \delta\big) &\le \limsup_{n\to\infty} Q\left(\frac{1}{n^{2/\kappa}} \sum_{i=1}^n \beta_i^2 \mathbf{1}_{\{\beta_i/n^{1/\kappa} \le \varepsilon\}} \ge \delta^3\right) \\ &\le \limsup_{n\to\infty} \frac{n^{1-2/\kappa}}{\delta^3} E_Q\big[\beta_1^2 \mathbf{1}_{\{\beta_1 \le \varepsilon n^{1/\kappa}\}}\big] \\ &= \frac{C_0 \kappa \varepsilon^{2-\kappa}}{(2-\kappa)\delta^3}.\end{aligned}$$

Since $\kappa < 2$ the right-hand side tends to 0 as $\varepsilon \to 0$. This completes the proof of (59) and thus also the proof of the Theorem 1.3. □

7. Weak quenched limiting distributions – Averaged centering

In this section we prove weak convergence with the averaged centering stated in Theorem 1.6. The argument is similar in most respects to the proof of Theorem 1.3 in the previous section, so we will concentrate now on those parts of the argument that are different. Recall that by Proposition 4.7 we only need to establish a weak quenched limit for

$$\sigma_{n,\omega} = \begin{cases} H(N_{n,\omega}), & \kappa \in (0,1), \\ H(N_{n,\omega}) * \delta_{-D'(n)}, & \kappa = 1, \\ H(N_{n,\omega}) * \delta_{-\bar{\beta} n^{1-1/\kappa}}, & \kappa \in (1,2), \end{cases} \tag{60}$$

where $H : \mathcal{M}_p \to \mathcal{M}_1$ is given by (7). We will use Proposition 5.1 and, once again, we have to use a truncated version of the mapping H. We will use the mapping H_ε defined in (8). The following lemma, whose proof is identical to that of Lemma 6.1, shows that H_ε is also continuous on the relevant subset of $\mathcal{M}_p$.

Lemma 7.1. *H_ε is continuous on $\mathcal{M}_p^{(\varepsilon)} = \{\zeta \in \mathcal{M}_p: \zeta(\{\varepsilon\}) = 0\}$.*

An immediate consequence of Lemma 7.1 and Proposition 5.1 is

$$H_\varepsilon(N_{n,\omega}) \overset{Q}{\Longrightarrow} H_\varepsilon(N_{\lambda,\kappa}). \tag{61}$$

We divide the remainder of the proof of Theorem 1.6 into two cases: $\kappa \in (0,1)$ and $\kappa \in [1,2)$.

Case I: $\kappa \in (0,1)$. The case $\kappa \in (0,1)$ is almost identical to the proof of Theorem 1.3. Due to (61), it is enough to show that

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(N_{\lambda,\kappa}) = H(N_{\lambda,\kappa}), \quad \mathbf{P}\text{-a.s.} \tag{62}$$

and

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} Q\big(\rho\big(H_\varepsilon(N_{n,\omega}), H(N_{n,\omega})\big) \ge \delta\big) = 0, \quad \forall \delta > 0. \tag{63}$$

The proof of (62) is similar to that of (58). The main difference between the proof of (63) and that of (59) is that now we are using the fact that for any two random variables X and Y defined on the same probability space, with respective laws $\mathcal{L}_X$ and $\mathcal{L}_Y$, $\rho(\mathcal{L}_X, \mathcal{L}_Y) \le (E|X-Y|)^{1/2}$, after which one uses once again (17) and Karamata's theorem.

Case II: $\kappa \in [1,2)$. The difference in this case is that centering is needed. Let

$$c_n(\varepsilon) = \begin{cases} E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \in (\varepsilon n, \bar{\nu} n]\}}] & \text{if } \kappa = 1, \\ n^{1-1/\kappa} E_Q[\beta_1 \mathbf{1}_{\{\beta_1 > \varepsilon n^{1/\kappa}\}}] & \text{if } \kappa \in (1,2). \end{cases}$$

Recalling the definitions from the statement of Proposition 4.7, we see that the tail decay of β_1 implies that

$$\lim_{n\to\infty} c_n(\varepsilon) = \begin{cases} \tilde{c}_{\lambda,1}(\varepsilon) & \text{if } \kappa = 1, \\ c_{\lambda,\kappa}(\varepsilon) & \text{if } \kappa \in (1,2), \end{cases} \qquad \text{where } \lambda = \kappa C_0.$$

Combining this with (61) we obtain that

$$H_\varepsilon(N_{n,\omega}) * \delta_{-c_n(\varepsilon)} \Longrightarrow \begin{cases} H_\varepsilon(N_{\lambda,1}) * \delta_{-\tilde{c}_{\lambda,1}(\varepsilon)} & \text{if } \kappa = 1, \\ H_\varepsilon(N_{\lambda,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)} & \text{if } \kappa \in (1,2). \end{cases} \tag{64}$$

We use, once again, [2], Theorem 3.2. By (64), in the case $\kappa \in (1,2)$, weak convergence of the measures $\sigma_{n,\omega}$ in (60) will follow once we show that

$$H_\varepsilon(N_{\lambda,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)} \quad \text{converges } \mathbf{P}\text{-a.s. as } \varepsilon \to 0^+, \tag{65}$$

and

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} Q\big(\rho\big(H(N_{n,\omega}) * \delta_{-\bar{\beta} n^{1-1/\kappa}}, H_\varepsilon(N_{n,\omega}) * \delta_{-c_n(\varepsilon)}\big) \geq \delta\big) = 0, \quad \forall \delta > 0. \tag{66}$$

The argument in the case $\kappa = 1$ is exactly the same if one replaces every instance of $\bar{\beta} n^{1-1/\kappa}$ and $c_{\lambda,\kappa}(\varepsilon)$ with $D'(n)$ and $\tilde{c}_{\lambda,1}(\varepsilon)$, respectively. Thus we will only give the proof in the case $\kappa \in (1,2)$.

To prove (65), let $\xi_1 > \xi_2 > \cdots$ be the points of $N_{\lambda,\kappa}$. By Theorem 3.12.2 in [16], the shifted truncated sums

$$\sum_{i\geq 1} \xi_i \tau_i \mathbf{1}_{\{\xi_i > \varepsilon\}} - c_{\lambda,\kappa}(\varepsilon)$$

converge a.s. as $\varepsilon \to 0^+$. The convergence above is true for almost every realization of the joint sequence $(\xi_i, \tau_i)_{i\geq 1}$, but by Fubini's theorem the same remains true for a.e. realization of the Poisson process $N_{\lambda,\kappa}$. Since a.s. convergence implies weak convergence, we obtain (65).

Turning now to the proof of (66), we use the same upper bound on the Prohorov's distance as in the proof of Theorem 1.3. Since $\bar{\beta} n^{1-1/\kappa} - c_n(\varepsilon) = n^{1-1/\kappa} E_Q[\beta_1 \mathbf{1}_{\{\beta_1/n^{1/\kappa} \leq \varepsilon\}}]$, we have

$$\begin{aligned} &\rho\big(H(N_{n,\omega}) * \delta_{-\bar{\beta} n^{1-1/\kappa}}, H_\varepsilon(N_{n,\omega}) * \delta_{-c_n(\varepsilon)}\big) \\ &\quad \leq \left(\frac{2}{n^{2/\kappa}} \left(\sum_{i=1}^n \big\{\beta_i \mathbf{1}_{\{\beta_i/n^{1/\kappa} \leq \varepsilon\}} - E_Q[\beta_1 \mathbf{1}_{\{\beta_1/n^{1/\kappa} \leq \varepsilon\}}]\big\} \right)^2 \right)^{1/3} \\ &\qquad + \left(\frac{2}{n^{2/\kappa}} \sum_{i=1}^n \beta_i^2 \mathbf{1}_{\{\beta_i/n^{1/\kappa} \leq \varepsilon\}} \right)^{1/3}. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} Q\big(\rho\big(H(N_{n,\omega}) * \delta_{-\bar{\beta} n^{1-1/\kappa}}, H_\varepsilon(N_{n,\omega}) * \delta_{-c_n(\varepsilon)}\big) \geq \delta\big)$$

$$\leq \limsup_{n\to\infty} Q\left(\frac{2}{n^{2/\kappa}} \left(\sum_{i=1}^n \big\{\beta_i \mathbf{1}_{\{\beta_i/n^{1/\kappa} \leq \varepsilon\}} - E_Q[\beta_1 \mathbf{1}_{\{\beta_1/n^{1/\kappa} \leq \varepsilon\}}]\big\} \right)^2 \geq \frac{\delta^3}{8} \right) \tag{67}$$

$$+ \limsup_{n\to\infty} Q\left(\frac{2}{n^{2/\kappa}} \sum_{i=1}^n \beta_i^2 \mathbf{1}_{\{\beta_i/n^{1/\kappa} \leq \varepsilon\}} \geq \frac{\delta^3}{8} \right). \tag{68}$$

Lemma 5.5 implies that (67) vanishes as $\varepsilon \to 0$, and (as in the proof of Theorem 1.3) Markov's inequality, (17) and Karamata's theorem imply that (68) vanishes as $\varepsilon \to 0$ as well. This completes the proof of a limiting distribution for $\sigma_{n,\omega}$, and the proof of Theorem 1.6 follows by an application of Proposition 4.7.

8. Converting from time to space

In this section we show that the weak quenched limit theorem for the hitting times T_n in Theorem 1.6 implies the weak quenched limit theorem for the random walk X_n in Corollary 1.8.

For any $t \geq 0$, let

$$X_t^* = \max\{X_k\colon\ k \leq t\} = \max\{n \in \mathbb{Z}\colon\ T_n \leq t\}$$

be the farthest the random walk has traversed to the right by time t. The usefulness of X_t^* stems from the identity of the events

$$\left\{X_t^* < x\right\} = \{T_x > t\} \quad \text{and} \quad \left\{X_t^* \geq x\right\} = \{T_x \leq t\}. \tag{69}$$

The following lemma implies that X_n typically is very close to X_n^*.

Lemma 8.1. *Let Assumptions 1 and 2 hold. Then*, $\limsup_{n\to\infty} \frac{X_n^* - X_n}{\log n} < \infty$, $\mathbb{P}$*-a.s.*

Proof. The event $\{X_n^* - X_n \geq M\}$ implies that for some $x = 0, 1, \ldots, n-1$ the random walk after first hitting x then backtracks to $x - M$. Thus,

$$\mathbb{P}\left(X_n^* - X_n \geq M\right) \leq \sum_{x=0}^{n-1} \mathbb{P}^x(T_{x-M} < \infty) = n\mathbb{P}(T_{-M} < \infty),$$

where the last equality follows from the translation invariance of the measure P on environments. It was shown in [9], Lemma 3.3, that Assumptions 1 and 2 imply that there exist constants $C, C' > 0$ such that $\mathbb{P}(T_{-M} < \infty) \leq C\mathrm{e}^{-C'M}$. Taking $M = K\log n$ for $K > 2/C'$ we obtain that

$$\mathbb{P}\left(X_n^* - X_n \geq \delta(\log n)^2\right) \leq Cn^{-(C'K-1)},$$

which is summable over n. The claim of the lemma now follows from the Borel–Cantelli Lemma. □

We will also need the following Corollary of Theorem 1.6.

Corollary 8.2. *Let $\kappa \in (0, 2)$, and let $\mu_{\lambda,\kappa}$ be the limiting random probability measure given by the conclusion of Theorem 1.6 (that is $\mu_{n,\omega} \Longrightarrow \mu_{\lambda,\kappa}$). Then, $\mu_{n,\omega}(x,\infty) \Longrightarrow \mu_{\lambda,\kappa}(x,\infty)$ for any $x \in \mathbb{R}$.*

Proof. First of all, note that the random probability measures $\mu_{\lambda,\kappa}$ are continuous distributions with probability 1. That is, $\mathbf{P}(\mu_{\lambda,\kappa}(\{x\}) > 0) = 0$. To see this, note that on an event of probability 1, we can write $\mu_{\lambda,\kappa} = E_1(\cdot/\xi_1) * \tilde{\mu}_{\lambda,\kappa}$, where ξ_1 is the largest point of the Poisson process, E_1 is the standard exponential distribution, and $\tilde{\mu}_{\lambda,\kappa}$ is another random probability distribution. The continuity of the exponential distribution then implies that $\mu_{\lambda,\kappa}$ is also continuous.

For any $x \in \mathbb{R}$, the mapping $\pi \mapsto \pi(x,\infty)$ from $\mathcal{M}_1$ to $\mathbb{R}$ is continuous on the set $\mathcal{C}_x = \{\pi \in \mathcal{M}_1\colon\ \pi(\{x\}) = 0\}$. Since we showed above that $P(\mu_{\lambda,\kappa} \in \mathcal{C}_x) = 1$, the continuous mapping theorem implies that $\mu_{n,\omega}(x,\infty) \Longrightarrow \mu_{\lambda,\kappa}(x,\infty)$ as $n \to \infty$. □

We are now ready to give the proof of Corollary 1.8.

Proof of Corollary 1.8. We will first prove Theorem 1.8 with X_n^* in place of X_n and then use Lemma 8.1 to transfer the results to X_n. Since the centering and scaling used depends on κ we divide the proof into three cases: $\kappa \in (0,1)$, $\kappa = 1$, and $\kappa \in (1,2)$.

Case I: $\kappa \in (0,1)$. If $\kappa \in (0,1)$, then (69) implies that for $x \in \mathbb{R}$ fixed

$$\begin{aligned} P_\omega\bigl(X_n^* < xn^\kappa\bigr) &= P_\omega(T_{\lceil xn^\kappa \rceil} > n) \\ &= P_\omega\biggl(\frac{T_{\lceil xn^\kappa \rceil}}{\lceil xn^\kappa \rceil^{1/\kappa}} > \frac{n}{\lceil xn^\kappa \rceil^{1/\kappa}}\biggr) \\ &= \mu_{\lceil xn^\kappa \rceil,\omega}\biggl(\frac{n}{\lceil xn^\kappa \rceil^{1/\kappa}}, \infty\biggr). \end{aligned}$$

Corollary 8.2 implies that the last term above converges in distribution to $\mu_{\lambda,\kappa}(x^{-1/\kappa},\infty) = H(N_{\lambda,\kappa})(x^{-1/\kappa},\infty)$ (note that here we are using the monotonicity of distribution functions, the fact that $\mu_{\lambda,\kappa}$ is a continuous distribution with probability 1, and the fact that $n/\lceil xn^\kappa \rceil^{1/\kappa} \to x^{-1/\kappa}$ as $n \to \infty$). Thus, we have shown that

$$P_\omega\bigl(X_n^* < xn^\kappa\bigr) \Longrightarrow H(N_{\lambda,\kappa})\bigl(x^{-1/\kappa},\infty\bigr). \tag{70}$$

Next, note that $X_n \le X_n^*$ implies that

$$P_\omega\bigl(X_n^* < xn^\kappa\bigr) \le P_\omega\bigl(X_n < xn^\kappa\bigr) \le P_\omega\bigl(X_n^* < xn^\kappa + (\log n)^2\bigr) + P_\omega\bigl(X_n^* - X_n > (\log n)^2\bigr). \tag{71}$$

Lemma 8.1 implies that $P_\omega(X_n^* - X_n > (\log n)^2)$ converges to 0 in L^1, and thus also in distribution. Therefore, (70) and (71) complete the proof of Theorem 1.8 when $\kappa \in (0,1)$ (here we again are using the monotonicity of distribution functions and the fact that $\mu_{\lambda,\kappa} = H(N_{\lambda,\kappa})$ is continuous with probability 1).

Case II: $\kappa = 1$. Recall from Remark 4.8 that the sequence $D(n)$ is given by

$$D(n) = \frac{\lfloor n/\bar\nu \rfloor}{n} D'\bigl(\lfloor n/\bar\nu \rfloor\bigr) = \frac{\lfloor n/\bar\nu \rfloor}{n} E_Q[\beta_1 \mathbf{1}_{\{\beta_1 \le \bar\nu \lfloor n/\bar\nu \rfloor\}}].$$

Note first of all that this implies $D(n) \sim A \log n$, where $A = C_0/\bar\nu$. Moreover, this explicit representation also gives that $D(y(n)) - D(x(n)) \to 0$ as $n \to \infty$ for any sequences $x(n), y(n) \to \infty$ with $x(n) \sim y(n)$.

We postpone for now the definition of the averaged centering term $\delta(n)$ for the random walk X_n. Whatever $\delta(n)$ is, for fixed x we let $\gamma(n) = \lceil \delta(n) + xn/(\log n)^2 \rceil$. Then, (69) implies that

$$\begin{aligned} P_\omega\biggl(\frac{X_n^* - \delta(n)}{n/(\log n)^2} < x\biggr) &= P_\omega\bigl(X_n^* < \delta(n) + xn/(\log n)^2\bigr) \\ &= P_\omega(T_{\gamma(n)} > n) \\ &= P_\omega\biggl(\frac{T_{\gamma(n)} - \gamma(n)D(\gamma(n))}{\gamma(n)} > \frac{n - \gamma(n)D(\gamma(n))}{\gamma(n)}\biggr) \\ &= \mu_{\gamma(n),\omega}\biggl(\frac{n - \gamma(n)D(\gamma(n))}{\gamma(n)}, \infty\biggr). \end{aligned} \tag{72}$$

Now, we can choose $\delta(n)$ so that

$$\delta(n)D\bigl(\delta(n)\bigr) = n + o(1), \quad \text{as } n \to \infty. \tag{73}$$

Then, recalling the definition of $\gamma(n)$ and the fact that $D(n) \sim A \log n$ as $n \to \infty$, this implies that

$$\gamma(n) \sim \delta(n) \sim \frac{n}{A \log n}, \quad \text{as } n \to \infty,$$

and

$$\lim_{n\to\infty} \frac{n - \gamma(n)D(\gamma(n))}{\gamma(n)} = -A^2 x.$$

(Note that in this last limit we used the fact that $D(\gamma(n)) - D(\delta(n)) \to 0$ since $\delta(n), \gamma(n) \to \infty$ and $\delta(n) \sim \gamma(n)$ as $n \to \infty$.)

Recalling (72) and having chosen $\delta(n)$ according to (73), Corollary 8.2 implies that

$$P_\omega\left(\frac{X_n^* - \delta(n)}{n/(\log n)^2} < x\right) \Longrightarrow \lim_{\varepsilon\to 0^+}\big(H_\varepsilon(N_{\lambda,1}) * \delta_{-c_{\lambda,1}(\varepsilon)}\big)\big(-A^2 x, \infty\big), \quad \forall x \in \mathbb{R}.$$

Replacing X_n^* with X_n in the above statement is again accomplished by using Lemma 8.1. The proof is essentially the same as in the case $\kappa \in (0,1)$ and is therefore ommitted.

Case III: $\kappa \in (1,2)$. Let $x \in \mathbb{R}$ be fixed, and define $\psi(n) = \lceil n\mathrm{v}_P + xn^{1/\kappa}\rceil$. Then (69) implies that

$$\begin{aligned}
P_\omega\left(\frac{X_n^* - n\mathrm{v}_P}{n^{1/\kappa}} < x\right) &= P_\omega\big(X_n^* < n\mathrm{v}_P + xn^{1/\kappa}\big) \\
&= P_\omega(T_{\psi(n)} > n) \\
&= P_\omega\left(\frac{T_{\psi(n)} - \psi(n)/\mathrm{v}_P}{\psi(n)^{1/\kappa}} > \frac{n - \psi(n)/\mathrm{v}_P}{\psi(n)^{1/\kappa}}\right) \\
&= \mu_{\psi(n),\omega}\left(\frac{n - \psi(n)/\mathrm{v}_P}{\psi(n)^{1/\kappa}}, \infty\right).
\end{aligned}$$

Note that

$$\lim_{n\to\infty}\frac{n - \psi(n)/\mathrm{v}_P}{\psi(n)^{1/\kappa}} = \lim_{n\to\infty}\frac{n - \lceil n\mathrm{v}_P + xn^{1/\kappa}\rceil/\mathrm{v}_P}{\lceil n\mathrm{v}_P + xn^{1/\kappa}\rceil^{1/\kappa}} = -x\mathrm{v}_P^{-1-1/\kappa},$$

and thus Corollary 8.2 implies that

$$P_\omega\left(\frac{X_n^* - n\mathrm{v}_P}{n^{1/\kappa}} < x\right) \Longrightarrow \lim_{\varepsilon\to 0^+}\big(H_\varepsilon(N_{\lambda,\kappa}) * \delta_{-c_{\lambda,\kappa}(\varepsilon)}\big)\big(-x\mathrm{v}_P^{-1-1/\kappa}, \infty\big), \quad \forall x \in \mathbb{R}.$$

We again omit the proof that X_n^* can be replaced by X_n in the above statement. □

Appendix A: Proof of Lemma 1.2

The easiest way to see the measurability of $\bar{H}$ is to represent it as a composition of two maps, and to show that each one of these maps is measurable. We write $\bar{H} = \bar{H}_2 \circ \bar{H}_1$, where $\bar{H}_1 : \mathcal{M}_p \to l^2$ is defined by

$$\bar{H}_1(\zeta)(\cdot) = \begin{cases} (x_{(1)}, x_{(2)}, \ldots), & \text{if } \sum_{i\ge 1} x_i^2 < \infty, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $x_{(1)} \ge x_{(2)} \ge \cdots$ is the non-increasing rearrangement of the points of $\zeta = \sum_{i\ge1}\delta_{x_i}$, and $\mathbf{0}$ is the zero element in l^2, while $\bar{H}_2 : l^2 \to \mathcal{M}_1$ is defined by

$$\bar{H}_2(\mathbf{x})(\cdot) = \mathbf{P}\left(\sum_{i\ge1} x_i(\tau_i - 1) \in \cdot\right)$$

for $\mathbf{x} = (x_1, x_2, \ldots) \in l^2$, where τ_i are i.i.d. Exp(1) random variables under the measure $\mathbf{P}$. Since the Borel σ-field on l^2 coincides with its cylindrical σ-field, measurability of the map $\bar{H}_1$ will follow once we check both that for each $k = 1, 2, \ldots$ the map $\bar{H}_{1,k} : \mathcal{M}_p \to \mathbb{R}$ defined for $\zeta = \sum_{i\ge1}\delta_{x_i}$ by $\bar{H}_{1,k}(\zeta) = x_{(k)}$ is measurable, and also that the set

$$F = \left\{\zeta = \sum_{i\ge1}\delta_{x_i} : \sum_{i\ge1} x_i^2 < \infty\right\}$$

is a measurable subset of $\mathcal{M}_p$. The first statement follows since each $\bar{H}_{1,k}$ is, clearly, a continuous map. The second statement follows by writing $F = \bigcup_{m=1}^{\infty} F_m$, where for each m,

$$F_m = \left\{ \zeta = \sum_{i\ge 1} \delta_{x_i} \colon \sum_{i\ge 1} x_{(i)}^2 \le m \right\}$$

is, by the continuity of the maps $\bar{H}_{1,k}$ and Fatou's lemma, a closed set.

In order to prove measurability of the map $\bar{H}_2$, it is enough to prove its continuity. Let $\mathbf{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$, $n = 1, 2, \ldots$, be a sequence in l^2 converging to $\mathbf{y} = (y_1, y_2, \ldots) \in l^2$. Instead of proving that $\sum_{i\ge 1} x_i^{(n)}(\tau_i - 1)$ converges weakly to $\sum_{i\ge 1} y_i(\tau_i - 1)$ it is, of course, sufficient to prove convergence in probability. This latter convergence follows immediately because

$$\mathbf{E}\left(\sum_{i\ge 1} x_i^{(n)}(\tau_i - 1) - \sum_{i\ge 1} y_i(\tau_i - 1) \right)^2 = \left\| \mathbf{x}^{(n)} - \mathbf{y} \right\|_2^2.$$

Appendix B: Proof of Lemma 4.6

The tail decay of $E_\omega S$ was analyzed in [13], but for completeness we will briefly outline the argument here. By using h-transforms one can compute a formula for the transition probabilities of the random walk conditioned on exiting the interval $(0, \nu)$ to the right. Given these conditional transition probabilities one can apply the formula (13) for the quenched expectation of the amount of time to move one step to the right. Before giving this formula we need to introduce some notation. Recall that $M_1 = \max\{\Pi_{0,j} \colon 0 \le j < \nu\}$. Let $i_0 = \max\{i \in [1, \nu] \colon \Pi_{0,i-1} = M_1\}$, and denote

$$M^- = \min\{\Pi_{i,j} \colon 0 < i \le j < i_0\} \wedge 1 \quad \text{and} \quad M^+ = \max\{\Pi_{i,j} \colon i_0 < i \le j < \nu\} \vee 1.$$

Then, following the proof of Corollary 4.2 in [13], one can show that for any $0 < i < \nu$,

$$E_\omega^i[T_{i+1} | T_\nu < T_0] \le 1 + \frac{2\nu^3 M^+}{(M^-)^3} \le \frac{3\nu^3 M^+}{(M^-)^3}. \tag{74}$$

This immediately implies that $E_\omega S \le \frac{3\nu^4 M^+}{(M^-)^3}$. The proof of the tail decay (36) of $E_\omega S$ is then accomplished by recalling (16) and the following Lemma from [13].

Lemma B.1 (Lemma 4.1 in [13]). *For any* $0 < \varepsilon < 1$ *and* $\varepsilon', \delta > 0$,

$$Q\left(M^+ > n^\delta, M_1 > n^{(1-\varepsilon)/\kappa}\right) = \mathrm{o}\left(n^{-1+\varepsilon-\delta\kappa+\varepsilon'}\right)$$

and

$$Q\left(M^- < n^{-\delta}, M_1 > n^{(1-\varepsilon)/\kappa}\right) = \mathrm{o}\left(n^{-1+\varepsilon-\delta\kappa+\varepsilon'}\right).$$

Applying this lemma and recalling from (16) that ν has exponential tails, we obtain that for any $0 < \varepsilon < 1$ and $\varepsilon', \delta > 0$,

$$\begin{aligned} Q\left(E_\omega S > n^{5\delta}, M_1 > n^{(1-\varepsilon)/\kappa}\right) &\le Q\left(\nu^4 > n^\delta\right) + Q\left(M^+ > n^\delta, M_1 > n^{(1-\varepsilon)/\kappa}\right) \\ &\quad + Q\left(M^- < n^{-\delta}, M_1 > n^{(1-\varepsilon)/\kappa}\right) \\ &= \mathrm{o}\left(n^{-1+\varepsilon-\delta\kappa+\varepsilon'}\right). \end{aligned}$$

Choosing $5\delta = 6\varepsilon/\kappa$ completes the proof of (36).

The proof of (37) is similar. We note first of all that

$$\begin{aligned} E_\omega F^{(1)} &= 1 + E_\omega^{-1}[T_0] P_\omega\big(X_1 = -1 | T_0^+ < T_\nu\big) + E_\omega^1[T_0 | T_0 < T_\nu] P_\omega\big(X_1 = 1 | T_0^+ < T_\nu\big) \\ &\le 1 + E_\omega^{-1}[T_0] + E_\omega^1[T_0 | T_0 < T_\nu] \\ &= 2 + 2W_{-1} + E_\omega^1[T_0 | T_0 < T_\nu]. \end{aligned}$$

It was shown in [14], Lemma 2.2, that W_{-1} has exponential tails under the measure Q, so we only need to anlayze the tails of the $E_\omega^1[T_0 | T_0 < T_\nu]$. To this end, the proof of (74) can be modified by instead conditioning on exiting the interval $(0, \nu)$ to the left in order to obtain that

$$E_\omega^i[T_{i-1} | T_0 < T_\nu] \le \frac{3\nu^3 (M^+)^3}{M^-} \quad \text{for any } 0 < i < \nu.$$

Then, as was done above for $E_\omega S$, we can use (16) and Lemma B.1 to obtain that for any $0 < \varepsilon < 1$ and $\varepsilon', \delta > 0$,

$$Q\big(E_\omega^1[T_0 | T_0 < T_\nu] > n^{5\delta}, M_1 > n^{(1-\varepsilon)/\kappa}\big) = o\big(n^{-1+\varepsilon-\delta\kappa+\varepsilon'}\big).$$

Choosing again $5\delta = 6\varepsilon/\kappa$ proves (37).

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