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Connected Spatial Networks over Random Points and a Route-Length Statistic

David J. Aldous and Julian Shun

Abstract. We review mathematically tractable models for connected networks on random points in the plane, emphasizing the class of *proximity graphs* which deserves to be better known to applied probabilists and statisticians. We introduce and motivate a particular statistic R measuring shortness of routes in a network. We illustrate, via Monte Carlo in part, the trade-off between normalized network length and R in a one-parameter family of proximity graphs. How close this family comes to the optimal trade-off over all possible networks remains an intriguing open question.

The paper is a write-up of a talk developed by the first author during 2007–2009.

Key words and phrases: Proximity graph, random graph, spatial network, geometric graph.

1. INTRODUCTION

The topic called *random networks* or *complex networks* has attracted huge attention over the last 20 years. Much of this work focuses on examples such as social networks or WWW links, in which edges are not closely constrained by two-dimensional geometry. In contrast, in a *spatial network* not only are vertices and edges situated in two-dimensional space, but also it is actual distances, rather than number of edges, that are of interest. To be concrete, we visualize idealized inter-city road networks, and a feature of interest is the (minimum) route length between two given cities. Because we work only in two dimensions, the word *spatial* may be misleading, but equally the word *planar* would be misleading because we do not require networks to be planar graphs (if edges cross, then a junction is created).

Our major purpose is to draw the attention of readers from the applied probability and statistics communities to a particular class of spatial network models. Recall that the most studied network model, the *random geometric graph* [40] reviewed in Section 2.1, does not permit both connectivity and bounded normalized length in the $n \to \infty$ limit. An attractive alternative is the class of *proximity graphs*, reviewed in Section 2.3, which in the deterministic case have been studied within computational geometry. These graphs are always connected. Proximity graphs on *random* points have been studied in only a few papers, but are potentially interesting for many purposes other than the specific "short route lengths" topic of this paper (see Section 6.5). One could also imagine constructions which depend on points having specifically the Poisson point process distribution, and one novel such network, which we name the *Hammersley network*, is described in Section 2.5.

Visualizing idealized road networks, it is natural to take total network length as the "cost" of a network, but what is the corresponding "benefit"? Primarily we are interested in having short route lengths. Choosing an appropriate statistic to measure the latter turns out to be rather subtle, and the (only) technical innovation of this paper is the introduction (Section 3.2) and motivation of a specific statistic R for measuring the effectiveness of a network in providing short routes.

In the theory of spatial networks over random points, it is a challenge to quantify the trade-off between network length [precisely, the normalized length L defined at (2)] and route length efficiency statistics such

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as R. Our particular statistic R is not amenable to explicit calculation even in comparatively tractable models, but in Section 4 we present the results from Monte Carlo simulations. In particular, Figure 7 shows the trade-off for the particular *β-skeleton* family of proximity graphs.

Given a normalized network length L, for any realization of cities there is some network of normalized length L which minimizes R. As indicated in Section 5, by general abstract mathematical arguments, there must exist a deterministic function $R_{\mathrm{opt}}(L)$ giving (in the "number of cities $\to \infty$" limit under the random model) the minimum value of R over all possible networks of normalized length L. An intriguing open question is as follows:

> how close are the values $R_{\beta\text{-skel}}(L)$ from the β-skeleton proximity graphs to the optimum values $R_{\mathrm{opt}}(L)$?

As discussed in Section 5.3, at first sight it looks easy to design heuristic algorithms for networks which should improve over the β-skeletons, for example, by introducing Steiner points, but in practice we have not succeeded in doing so.

This paper focuses on the random model for city positions because it seems the natural setting for theoretical study. As a complement, in [10] we give empirical data for the values of (L, R) for certain real-world networks (on the 20 largest cities, in each of 10 US States). In [8] we give analytic results and bounds on the trade-off between L and the mathematically more tractable *stretch* statistic $R_{\max}$ at (4), in both worst-case and random-case settings for city positions. Let us also point out a (perhaps) nonobvious insight discussed in Section 3.3: in designing networks to be efficient in the sense of providing short routes, the main difficulty is providing short routes between city-pairs at a specific distance (2–3 standardized units) apart, rather than between pairs at a large distance apart.

Finally, recall this is a nontechnical account. Our purpose is to elaborate verbally the ideas outlined above; some technical aspects will be pursued elsewhere.

2. MODELS FOR CONNECTED SPATIAL NETWORKS

There are several conceptually different ways of defining networks on random points in the plane. To be concrete, we call the points *cities*; to be consistent about language, we regard x_i as the *position* of city i and represent network edges as line segments (x_i, x_j).

First (Sections 2.1–2.3) are schemes which use deterministic rules to define edges for an arbitrary deterministic configuration of cities; then one just applies these rules to a random configuration. Second, one can have random rules for edges in a deterministic configuration (e.g., the probability of an edge between cities i and j is a function of Euclidean distance $d(x_i, x_j)$, as in popular *small worlds* models [39]), and again apply to a random configuration. Third, and more subtly, one can have constructions that depend on the randomness model for city positions—Section 2.5 provides a novel example.

We work throughout with reference to Euclidean distance $d(x, y)$ on the plane, even though many models could be defined with reference to other metrics (or even when the triangle inequality does not hold, for the MST).

2.1 The Geometric Graph

In Sections 2.1–2.3 we have an arbitrary configuration $\mathbf{x} = \{x_i\}$ of city positions, and a deterministic rule for defining the edge-set $\mathcal{E}$. Usually in graph theory one imagines a *finite* configuration, but note that everything makes sense for *locally finite* configurations too. Where helpful, we assume "general position," so that intercity distances $d(x_i, x_j)$ are all distinct.

For the *geometric graph* one fixes $0 < c < \infty$ and defines

$$(x_i, x_j) \in \mathcal{E} \quad \text{iff} \quad d(x_i, x_j) \le c.$$

For the *K-neighbor graph* one fixes $K \ge 1$ and defines

> $(x_i, x_j) \in \mathcal{E}$ iff x_i is one of the K closest neighbors of x_j, or x_j is one of the K closest neighbors of x_i.

A moment's thought shows these graphs are in general not connected, so we turn to models which are "by construction" connected. We remark that the connectivity threshold c_n in the *finite* n-vertex model of the random geometric graph has been studied in detail—see Chapter 13 of [40].

2.2 A Nested Sequence of Connected Graphs

The material here and in the next section was developed in graph theory with a view toward algorithmic applications in computational geometry and pattern recognition. The 1992 survey [28] gives the history of the subject and 116 citations. But everything we need is immediate from the (careful choice of) definitions. On our arbitrary configuration $\mathbf{x}$ we can define four graphs whose edge-sets are nested as follows:

(1) MST $\subseteq$ relative n'hood $\subseteq$ Gabriel $\subseteq$ Delaunay.

Here are the definitions (for MST and Delaunay, it is easy to check these are equivalent to more familiar definitions). In each case, we write the criterion for an edge (x_i, x_j) to be present:

- *Minimum spanning tree (MST)* [24]. There does not exist a sequence $i = k_0, k_1, \dots, k_m = j$ of cities such that

$$\max(d(x_{k_0}, x_{k_1}), d(x_{k_1}, x_{k_2}), \dots, d(x_{k_{m-1}}, x_{k_m})) < d(x_i, x_j).$$

- *Relative neighborhood graph.* There does not exist a city k such that

$$\max(d(x_i, x_k), d(x_k, x_j)) < d(x_i, x_j).$$

- *Gabriel graph.* There does not exist a city inside the disc whose diameter is the line segment from x_i to x_j.
- *Delaunay triangulation* [23]. There exists some disc, with x_i and x_j on its boundary, so that no city is inside the disc.

The inclusions (1) are immediate from these definitions. Because the MST (for a finite configuration) is connected, all these graphs are connected.

Figure 1 illustrates the relative neighborhood and Gabriel graphs. Figures for the MST and the Delaunay triangulation can be found online at http://www.spss.com/research/wilkinson/Applets/edges.html.

Constructions such as the relative neighborhood and Gabriel graphs have become known loosely as *proximity graphs* in [28] and subsequent literature, and we next take the opportunity to turn an implicit definition in the literature into an explicit definition.

2.3 Proximity Graphs

Write v_- and v_+ for the points $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$. The *lune* is the intersection of the open discs of radii 1 centered at v_- and v_+. So v_- and v_+ are not in the lune but are on its boundary. Define a *template* A to be a subset of $\mathbb{R}^2$ such that:

(i) A is a subset of the lune.
(ii) A contains the open line segment (v_-, v_+).
(iii) A is invariant under the "reflection in the y-axis" map $\mathrm{Reflect}_x(x_1, x_2) = (-x_1, x_2)$ and the "reflection in the x-axis" map $\mathrm{Reflect}_y(x_1, x_2) = (x_1, -x_2)$.
(iv) A is open.

For arbitrary points x, y in $\mathbb{R}^2$, define $A(x, y)$ to be the image of A under the natural transformation (translation, rotation and scaling) that takes (v_-, v_+) to (x, y).

DEFINITION. Given a template A and a locally finite set $\mathcal{V}$ of vertices, the associated *proximity graph* G has edges defined by, for each $x, y \in \mathcal{V}$,

(x, y) is an edge of G iff $A(x, y)$ contains no vertex of $\mathcal{V}$.

From the definitions:

- if A is the lune, then G is the relative neighborhood graph;
- if A is the disc centered at the origin with radius $1/2$, then G is the Gabriel graph.

But the MST and Delaunay triangulation are not instances of proximity graphs.

Note that replacing A by a subset A' can only introduce extra edges. It follows from (1) that the proximity

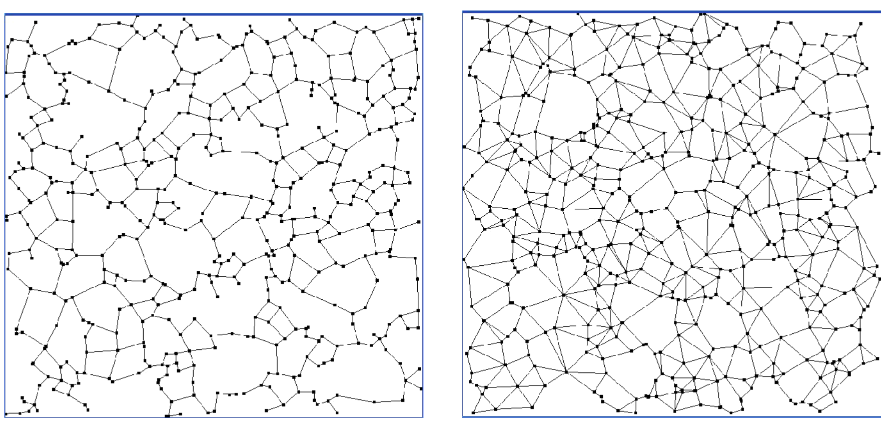

FIG. 1. *The relative neighborhood graph* (left) *and Gabriel graph* (right) *on different realizations of* 500 *random points.*

graph is always connected. The Gabriel graph is planar. But if A is not a superset of the disc centered at the origin with radius $1/2$, then G might not be a subgraph of the Delaunay triangulation, and in this case edges may cross, so G is not planar (e.g., if the vertex-set is the four corners of a square, then the diagonals would be edges).

For a given configuration $\mathbf{x}$, there is a collection of proximity graphs indexed by the template A, so by choosing a monotone one-parameter family of templates, one gets a monotone one-parameter family of graphs, analogous to the one-parameter family $\mathcal{G}_c$ of geometric graphs. Here is a popular choice [30] in which $\beta = 1$ gives the Gabriel graph and $\beta = 2$ gives the relative neighborhood graph.

DEFINITION (The *β-skeleton* family). (i) For $0 < \beta < 1$ let A_β be the intersection of the two open discs of radius $(2\beta)^{-1}$ passing through v_- and v_+.

(ii) For $1 \le \beta \le 2$ let A_β be the intersection of the two open discs of radius $\beta/2$ centered at $(\pm(\beta - 1)/2, 0)$.

2.4 Networks Based on Powers of Edge-Lengths

It is not hard to think of other ways to define one-parameter families of networks. Here is one scheme used in, for example, [38]. Fix $1 \le p < \infty$. Given a configuration $\mathbf{x}$, and a route (sequence of vertices) $x_0, x_1, \dots, x_k$, say, the cost of the route is the sum of pth powers of the step lengths. Now say that a pair (x, y) is an edge of the network $\mathcal{G}_p$ if the cheapest route from x to y is the one-step route. As p increases from 1 to ∞, these networks decrease from the complete graph to the MST. Moreover, for $p \ge 2$ the network $\mathcal{G}_p$ is a subgraph of the Gabriel graph.

2.5 The Hammersley Network

There is a quite separate recent literature in theoretical probability [26, 27] defining structures such as trees and matchings directly on the infinite Poisson point process. In this spirit, we observe that the *Hammersley process* studied in [6] can be used to define a new network on the infinite Poisson point process, which we name the *Hammersley network*. This network is designed to have the feature that each vertex has exactly 4 edges, in directions NE (between North and East), NW, SE and SW. The conceptual difference from the networks in the previous section is that there is not such a simple "local" criterion for whether a potential edge (x_i, x_j) is in the network. And edges cross, creating junctions.

For a picturesque description, imagine one-eyed frogs sitting on an infinitely long, thin log, each being able to see only the part of the log to their left before the next frog. At random times and positions (precisely, as a space–time Poisson point process of rate 1) a fly lands on the log, at which instant the (unique) frog which can see it jumps left to the fly's position and eats it. This defines a continuous time Markov process (the Hammersley process) whose states are the configurations of positions of all the frogs. There is a *stationary* version of the process in which, at each time, the positions of the frogs form a Poisson (rate 1) point process on the line.

Now consider the space–time trajectories of all the frogs, drawn with time increasing upward on the page. See Figure 2. For each frog, the part of the trajectory between the completions of two successive jumps consists of an upward edge (the frog remains in place as time increases) followed by a leftward edge (the frog jumps left).

Reinterpreting the time axis as a second space axis, and introducing compass directions, that part of the trajectory becomes a North edge followed by a West edge. Now replace these two edges by a single North-West straight edge. Doing this procedure for each frog and each pair of successive jumps, we obtain a collection of NW paths, that is, a network in which each city (the reinterpreted space–time random points) has an edge to the NW and an edge to the SE. Finally, we

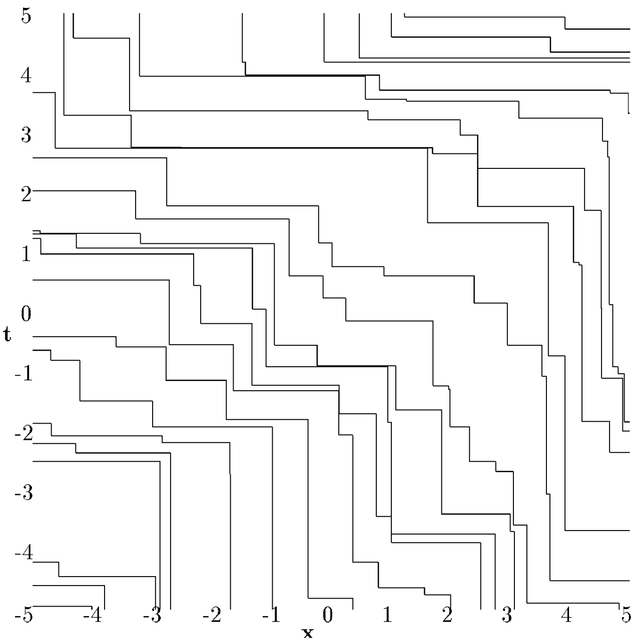

FIG. 2. *Space–time trajectories in Hammersley's process.*

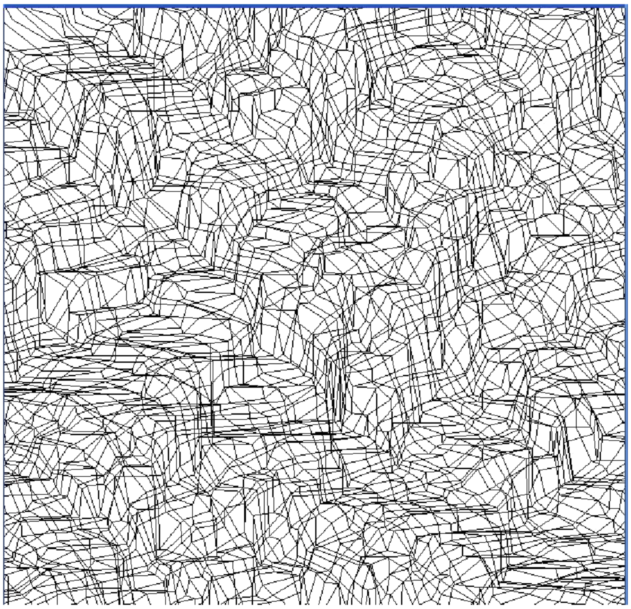

FIG. 3. *The Hammersley network on* 2500 *random points*.

repeat the construction with the same realization of the space–time Poisson point process but with frogs jumping rightward instead of leftward. This yields a network on the infinite Poisson point process, which we name the *Hammersley network*. See Figure 3.

REMARKS. (a) To draw the Hammersley network on random points in a *finite square*, one needs external randomization to give the initial (time 0) frog positions, in fact, two independent randomizations for the leftward and the rightward processes. So to be pedantic, one gets a *random* network over the given realization of cities. However, one can deduce from the theoretical results in [6] that the external randomization has effect only near the boundary of the square.

(b) The property that each vertex has exactly 4 edges, in directions NE (between North and East), NW, SE and SW, is immediate from the construction. Note, however, that while adjacent NW space–time trajectories in Figure 2 do not cross, the corresponding diagonal roads in the Hammersley network may cross, so it is not a planar graph, though this has only negligible effect on route lengths.

(c) Intuition, confirmed by Figure 7 later, says that the Hammersley network is not very efficient as a road network. It serves to demonstrate that there do exist random networks other than the familiar ones, and provides an instance where imposing deterministic constraints (the four edges, in this case) on a random network makes it much less efficient. How general a phenomenon is this?

2.6 Normalized Length

The notion of *normalized network length* L is most easily visualized in the setting of an infinite deterministic network which is "regular" in the sense of consisting of a repeated pattern. First choose the unit of length so that cities have an average density of one per unit area. Then define

(2) $L =$ average network length per unit area,

(3) $\bar{\Delta} =$ average degree (number of incident edges) of cities.

Figure 4 shows the values of L and $\bar{\Delta}$ for some simple "repeated pattern" networks. Though not directly relevant to our study of the random model, we find Figure 4 helpful for two reasons: as intuition for the interpretation of the different numerical values of L, and because we can make very loose analogies (Section 6.6) between particular networks on random points and particular deterministic networks.

3. NORMALIZED LENGTH AND ROUTE-LENGTH EFFICIENCY

3.1 The Random Model

For the remainder of the paper we work with "the random model" for city positions. The *finite model* assumes n random vertices (cities) distributed independently and uniformly in a square of area n. The *infinite model* assumes the Poisson point process of rate 1 (per unit area) in the plane. The quantities L, $\bar{\Delta}$ above and R below that we discuss may be interpreted as exact values in the infinite model or as $n \to \infty$ limits in the finite model; see Section 5. We use the word *normalized* as a reminder of the "density 1" convention—we choose the normalized unit of distance to make cities have average density 1 per unit area. After this normalization, L is the average network length per unit area.

3.2 The Route-Length Efficiency Statistic *R*

In designing a network, it is natural to regard total length as a "cost". The corresponding "benefit" is having short routes between cities. Write $\ell(i, j)$ for the route length (length of shortest path) between cities i and j in a given network, and $d(i, j)$ for Euclidean distance between the cities. So $\ell(i, j) \geq d(i, j)$, and we write

$$r(i, j) = \frac{\ell(i, j)}{d(i, j)} - 1$$

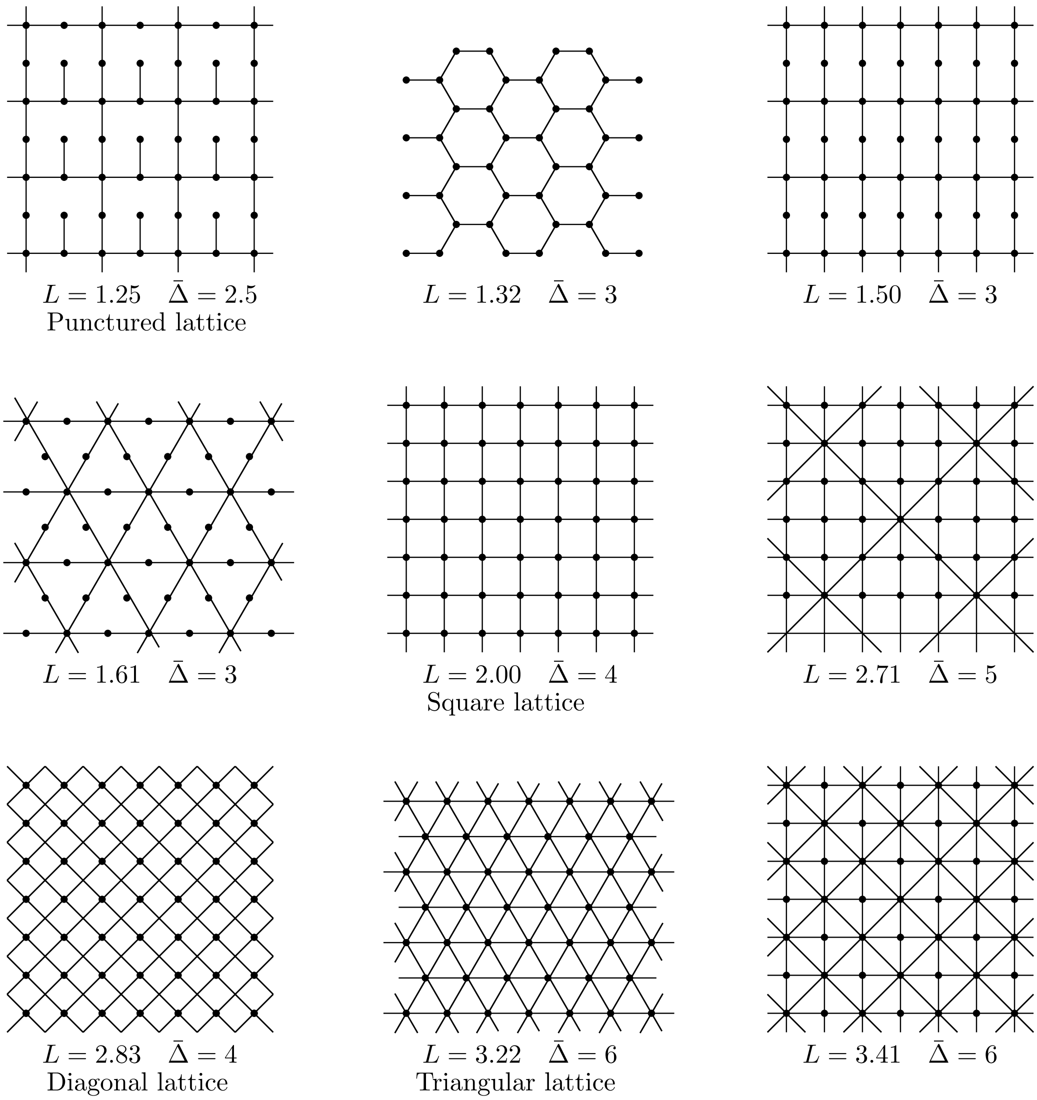

FIG. 4. *Variant square, triangular and hexagonal lattices. Drawn so that the density of cities is the same in each diagram, and ordered by value of L.*

so that "$r(i, j) = 0.2$" means that route length is 20% longer than straight line distance. With n cities we get $\binom{n}{2}$ such numbers $r(i, j)$; what is a reasonable way to combine these into a single statistic? Two natural possibilities are as follows:

$$R_{\max} := \max_{j \neq i} r(i, j), \tag{4}$$
$$R_{\text{ave}} := \text{ave}_{(i,j)}\, r(i, j),$$

where $\text{ave}_{(i,j)}$ denotes average over all distinct pairs (i, j). The statistic $R_{\max}$ has been studied in the context of the design of geometric spanner networks [37] where it is called the *stretch*. However, being an "extremal" statistic $R_{\max}$ seems unsatisfactory as a descriptor of real world networks—for instance, it seems unreasonable to characterize the UK rail network as inefficient simply because there is no very direct route between Oxford and Cambridge.

The statistic R_{ave} has a more subtle drawback. Consider a network consisting of:

- the minimum-length connected network (Steiner tree) on given cities;
- and a superimposed sparse collection of randomly oriented lines (a *Poisson line process* [45]).

See Figure 5. By choosing the density of lines to be sufficiently low, one can make the normalized network length be arbitrarily close to the minimum needed for connectivity. But it is easy to show (see [7] for careful analysis and a stronger result) that one can construct

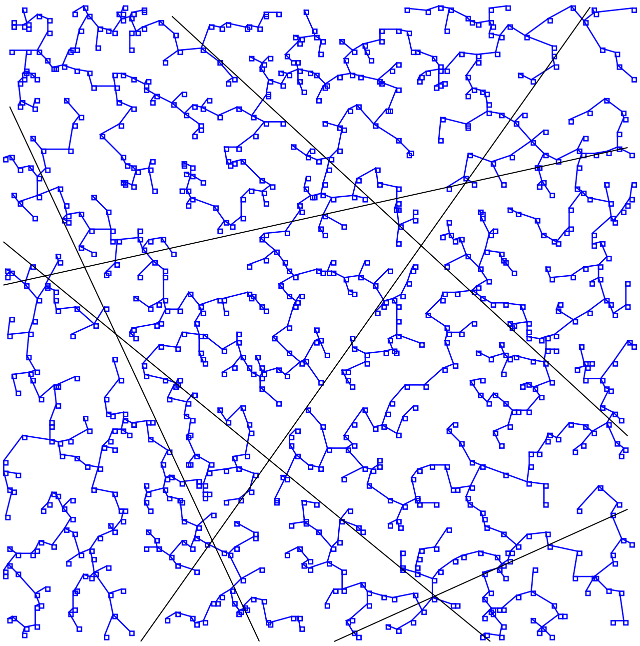

FIG. 5. *Efficient or inefficient? R_{ave} would judge this network efficient in the $n \to \infty$ limit.*

such networks so that $R_{\text{ave}} \to 0$ as $n \to \infty$. Of course no one would build a road network looking like Figure 5 to link cities, because there are many pairs of nearby cities with only very indirect routes between them. The disadvantage of R_{ave} as a descriptive statistic is that (for large n) most city-pairs are far apart, so the fact that a given network has a small value of R_{ave} says nothing about route lengths between *nearby* cities.

We propose a statistic R which is intermediate between R_{ave} and $R_{\max}$. First consider (see discussion below for details)

$$\rho(d) := \text{mean value of } r(i, j) \text{ over}$$
$$\text{city-pairs with } d(i, j) = d$$

and then define

$$R := \max_{0 \le d < \infty} \rho(d). \tag{5}$$

In words, $R = 0.2$ means that on every scale of distance, route lengths are on average at most 20% longer than straight line distance.

On an intuitive level, R provides a sensible and interpretable way to compare efficiency of different networks in providing short routes. On a technical level, we see two advantages and one disadvantage of using R instead of R_{ave}.

Advantage 1. Using R to measure efficiency, there is a meaningful $n \to \infty$ limit for the network length/efficiency trade-off [the function $R_{\text{opt}}(L)$ discussed in Section 5], and so, in particular, it makes sense to compare the values of R for networks with different n.

Advantage 2. A more realistic model for traffic would posit that volume of traffic between two cities varies as a power-law $d^{-\gamma}$ of distance d, so that in calculating R_{ave} it would be more realistic to weight by $d^{-\gamma}$. This means that the optimal network, when using R_{ave} as optimality criterion, would depend on γ. Use of R finesses this issue; the value of γ does not affect R. A related issue is that volume of traffic between two cities should depend on their populations. Intuitively, incorporating random population sizes should make the optimal R *smaller* because the network designer can create shorter routes between larger cities. We see this effect in data [10]; R calculated via population-weighting is typically slightly smaller. But we have not tried theoretical study.

Disadvantage. The statistic R is tailored to the infinite model, in which it makes sense to consider two cities at exactly distance d apart (then the other city positions form a Poisson point process). For finite n we need to discretize. For the empirical data in [10], where $n = 20$, we average over intervals of width 1 unit (recall the unit of distance is taken such that the density of cities is 1 per unit area), that is, for $d = 1, 2, \dots, 5$, we calculate

$$\tilde{\rho}(d) := \text{mean value of } r(i, j) \text{ over city-pairs}$$
$$\text{with } d - \tfrac{1}{2} < d(i, j) < d + \tfrac{1}{2}, \tag{6}$$
$$\tilde{R} := \max_{1 \le d < \infty} \tilde{\rho}(d)$$

and use $\tilde{R}$ as proxy for R. For larger n we can use shorter intervals. Thus, there is, in principle, a certain fuzziness to the notion of R for finite networks, and, in particular, it is not clear how to assign a value of R to regular networks such as those in Figure 4. But in practice, for networks we have studied on real-world data and on random points, this is not a problem, as explained next.

3.3 Characteristic Shape of the Function $\rho(d)$

For the connected networks on random points (excluding the Hammersley network) we are discussing, the function $\rho(d)$ has a characteristic shape (see Figure 6) attaining its maximum between 2 and 3 and slowly decreasing thereafter. We suspect that "this characteristic shape holds for any reasonable model," but we do not know how to turn that phrase into a precise conjecture. Note that "smoothness near the maximum" implies that any calculated value $\tilde{R}$ at (6) is quite insensitive to the choice of discretization.

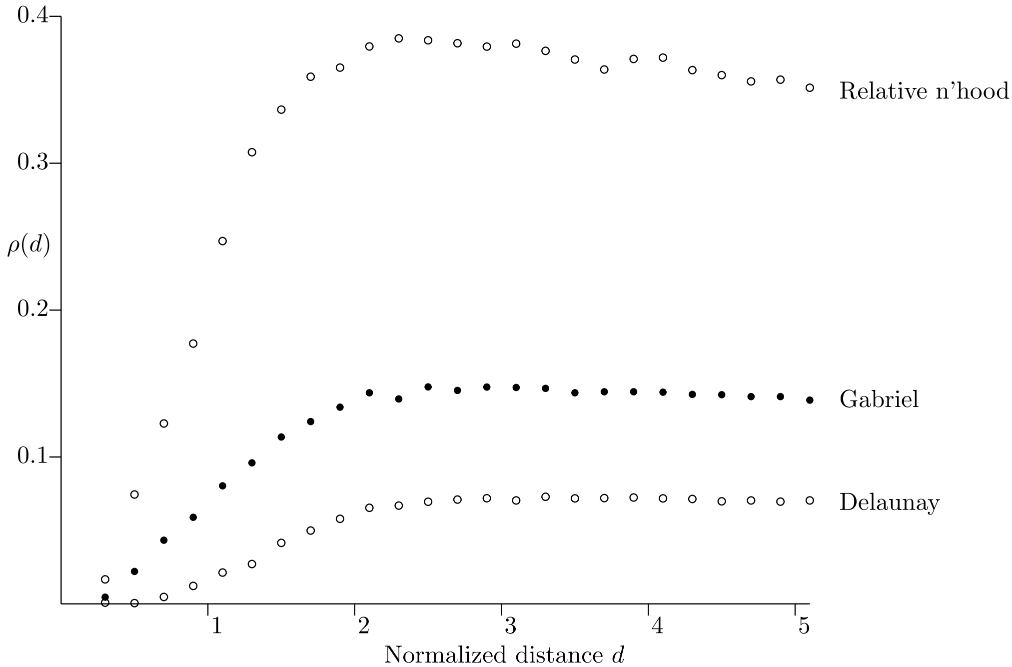

FIG. 6. *The function $\rho(d)$ for three theoretical networks on random cities. Irregularities are Monte Carlo random variation.*

This characteristic shape has a common-sense interpretation. Any efficient network will tend to place roads directly between unusually close city-pairs, implying that $\rho(d)$ should be small for $d < 1$. For large d the presence of multiple alternate routes helps prevent $\rho(d)$ from growing. At distance $2 - 3$ from a typical city i there will be about $\pi 3^2 - \pi 2^2 \approx 16$ other cities j. For some of these j there will be cities k near the straight line from i to j, so the network designer can create roads from i to k to j. The difficulty arises where there is no such intermediate city k: including a direct road (x_i, x_j) will increase L, but not including it will increase $\rho(d)$ for $2 < d < 3$.

Thus, Figure 6 offers a minor insight into spatial network design: that it is city pairs at normalized distance $2 - 3$ specifically that enforce the constraints on efficient network design.

The characteristic shape—at least, the flatness over $2 \le d \le 5$—is also visible in the real-world data [10].

For the Hammersley network, the graph of $\rho(d)$ is quite different; $\rho(d)$ increases to a maximum of 0.35 around $d = 0.8$ and then decreases more steeply to a value of 0.21 at $d = 5$. This arises from the particular structure (from each city there is one road in each quadrant) resembling the deterministic "diagonal lattice" of Figure 4, in which the route between some nearby pairs will be via two diagonal roads and a junction.

4. LENGTH-EFFICIENCY TRADE-OFF FOR TRACTABLE NETWORKS

Recall that our overall theme is the trade-off between network length and route-length efficiency, and that in this paper we focus on $n \to \infty$ limits in the random model and the particular statistics L and R.

The models described in Section 2 are "tractable" in the specific sense that one can find exact analytic formulas for normalized length L. Unfortunately R is not amenable to analytic calculation, and we resort to Monte Carlo simulation to obtain values for R. Table 1 and Figure 7 show the values of (L, R) in the models. We explain below how the values of L are calculated.

Notes on Table 1. (a) Values of R from our simulations with $n = 2500$.

TABLE 1
Statistics of tractable networks on random points

Network	L	$\bar{\Delta}$	R
Minimum spanning tree	0.633	2	∞
Relative n'hood	1.02	2.56	0.38
Gabriel	2	4	0.15
Hammersley	3.25	4	0.35
Delaunay	3.40	6	0.07

Notes: Integer values are exact. Recall L is normalized length (2), $\bar{\Delta}$ is average degree (3) and R is our route-length statistic (5).

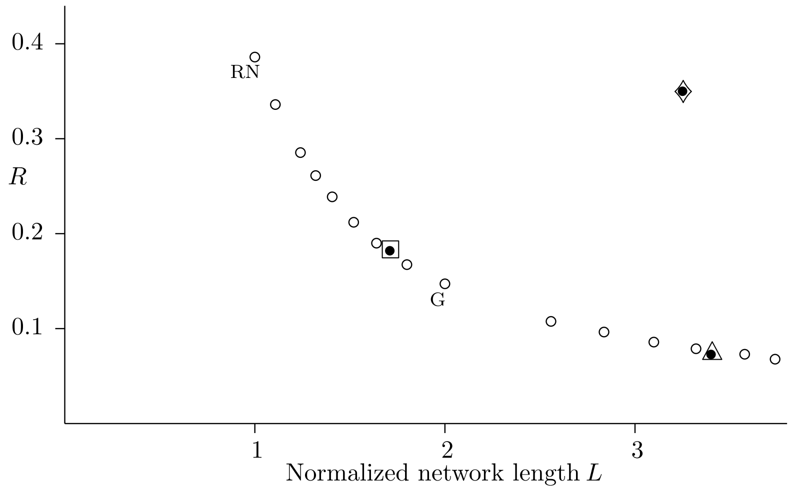

FIG. 7. *The normalized network length L and the route-length efficiency statistic R for certain networks on random points. The ∘ show the beta-skeleton family, with RN the relative neighborhood graph and G the Gabriel graph. The • are special models: △ shows the Delaunay triangulation, □ shows the network $\mathcal{G}_2$ from Section 2.4 and ◇ shows the Hammersley network.*

(b) Value of L for MST from Monte Carlo [19]. In principle, one can calculate arbitrarily close bounds [11], but apparently this has never been carried out. Of course, $\bar{\Delta} = 2$ for any tree.

(c) The Gabriel graph and the relative neighborhood graph fit the assumptions of Lemma 1 with $c = \pi/4$ and $c = \frac{2\pi}{3} - \frac{\sqrt{3}}{4}$, respectively, and their table entries for L and $\bar{\Delta}$ are obtained from Lemma 1, as are the values for β-skeletons in Figure 7.

(d) For the Hammersley network, every degree equals 4, so $L = 2 \times$ (mean edge-length). It follows from theory [6] that a typical edge, say, NE from (x, y), goes to a city at position $(x + \xi_x, y + \xi_y)$, where ξ_x and ξ_y are independent with Exponential(1) distribution. So mean edge-length equals

$$\int_0^\infty \int_0^\infty \sqrt{x^2 + y^2} e^{-x-y}\, dx\, dy \approx 1.62. \tag{7}$$

(e) For any triangulation, $\bar{\Delta} = 6$ in the infinite model. For the Delaunay triangulation, $L = ES$ where S is the perimeter length of a typical cell, and it is known ([35], page 113) that $ES = \frac{32}{3\pi}$. Note [33] that the Delaunay triangulation is in general *not* the minimum-length triangulation. Our simulation results in Figure 6 for $\rho(d)$ for the Delaunay triangulation are roughly consistent with a simulation result in [13] saying that $\rho(65) \approx 0.05$.

4.1 A Simple Calculation for Proximity Graphs

Let us give an example of an elementary calculation for proximity graphs over random points.

LEMMA 1. *For a proximity graph with template A on the Poisson point process,*

$$L = \frac{\pi^{3/2}}{4c^{3/2}}, \tag{8}$$

$$\bar{\Delta} = \frac{\pi}{c}, \tag{9}$$

where $c = \text{area}(A)$.

PROOF. Take a typical city at position x_0. For a city x at distance s the chance that (x_0, x) is an edge equals $\exp(-cs^2)$ and so

$$\text{mean-degree} = \int_0^\infty \exp(-cs^2) 2\pi s\, ds,$$

$$L = \frac{1}{2}\int_0^\infty s \exp(-cs^2) 2\pi s\, ds.$$

Evaluating the integrals gives (8) and (9). □

One can derive similar integral formulas for other "local" characteristics, for example, mean density of triangles and moments of vertex degree. See [18, 20, 21, 34] for a variety of such generalizations and specializations.

4.2 Other Tractable Networks

We do not know any other ways of defining networks on random points which are both "natural" and are tractable in the sense that one can find exact analytic formulas for L. In particular, we know no tractable way of defining networks with deliberate junctions as in Figure 8. Note also that, while it is easy to make ad hoc

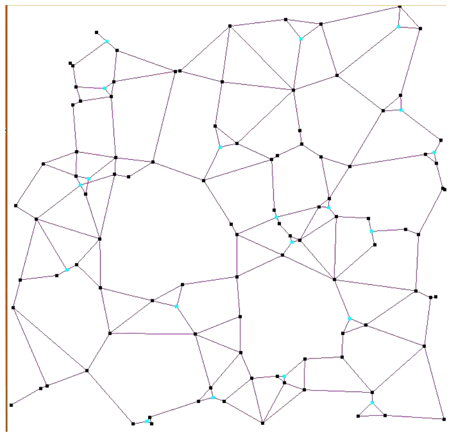

FIG. 8. *An ad hoc modification of the relative neighborhood graph, introducing junctions.*

modifications to the geometric graph to ensure connectivity, these destroy tractability. On the other hand, one can construct "unnatural" networks (see, e.g., [8]) designed to permit calculation of L.

5. OPTIMAL NETWORKS AND $N \to \infty$ LIMITS

5.1 Tractable Models

As mentioned earlier, the quantities $L, \bar{\Delta}, R$ we discuss may be interpreted as exact values in the infinite model or as $n \to \infty$ limits in the finite model. To elaborate briefly, in a realization of the finite model (n cities distributed independently and uniformly in a square of area n), a network in Table 1 has a normalized length $L_n = n^{-1} \times$ (network length) and an average degree $\bar{\Delta}_n$ which are random variables, but there is convergence (in probability and in expectation)

$$L_n \to L, \quad \bar{\Delta}_n \to \bar{\Delta} \quad \text{as } n \to \infty \tag{10}$$

to limit constants definable in terms of the analogous network on the infinite model (rate 1 Poisson point process on the infinite plane). For the proximity graphs or Delaunay triangulation, the network definition applies directly to the infinite model and proof of (10) is straightforward. For the Hammersley network, (10) is implicit in [6], and for the MST detailed arguments can be found in [9, 43].

5.2 Optimal Networks

We now turn to consideration of *optimal* networks. Given a configuration $\mathbf{x}$ of n cities in the area-n square, and a value of L which is greater than $n^{-1} \times$ (length of Steiner tree), one can define a number

$$\begin{aligned} R_n(\mathbf{x}, L) = {} & \text{min of } \tilde{R} \text{ over all networks} \\ & \text{on } \mathbf{x} \text{ with normalized length} \le L, \end{aligned} \tag{11}$$

where $\tilde{R}$ is the discretized version (6) calculated using intervals of some suitable length δ_n. Applying this to a random configuration $\mathbf{X}$ in the finite model gives, for each L, a random variable

$$\Xi_n(L) := R_n(\mathbf{X}, L).$$

One intuitively expects convergence to some deterministic limit

$$\Xi_n(L) \to R_{\text{opt}}(L) \quad \text{say, as } n \to \infty. \tag{12}$$

The analogous result for $R_{\max}$ will be proved carefully in [8], and the same "superadditivity" argument could be used to prove (12). See [43, 44, 47] for general background to such results. The point is that we do not have any explicit description of the optimal [i.e., attaining the minimum in (11)] networks in the finite or infinite models, so it seems very challenging to prove the natural stronger supposition that the finite optimal networks themselves converge (in some appropriate sense) to a *unique* infinite optimal network for which the value $R = R_{\text{opt}}(L)$ is attained.

5.3 The Curve $R_{\text{opt}}(L)$

Every possible network on the infinite Poisson point process defines a pair (L, R), and the curve $R = R_{\text{opt}}(L)$ can be defied equivalently as the lower boundary of the set of possible values of (L, R). There is no reason to believe that proximity graphs are exactly optimal, and, indeed, Figure 7 shows that the Delaunay triangulation is slightly more efficient than the corresponding β-skeleton. But our attempts to do better by ad hoc constructions (e.g., by introducing degree-3 junctions—see Figure 8 for an example) have been unsuccessful. And, indeed, the fact that the two special models in Figure 7 lie close to the β-skeleton curve lends credence to the idea that this curve is almost optimal. We therefore speculate that the function R_{opt} looks something like the curve in Figure 9, which we now discuss.

What can we say about $R_{\text{opt}}(L)$? It is *a priori* nonincreasing. It is known [47] that there exists a *Euclidean Steiner tree constant* L_{ST} representing the limit normalized Steiner tree length in the random model, and clearly $R_{\text{opt}}(L) = \infty$ for $L < L_{\text{ST}}$. The facts

$$\begin{aligned} & R_{\text{opt}}(L) < \infty \quad \text{for all } L > L_{\text{ST}}; \\ & R_{\text{opt}}(L) \to 0 \quad \text{as } L \to \infty \end{aligned} \tag{13}$$

are not trivial to prove rigorously, but follow from the corresponding facts for $R_{\max}$ proved in [8]. But we are unable to prove rigorously that $R_{\text{opt}}(L)$ is *strictly* decreasing or that it is continuous.

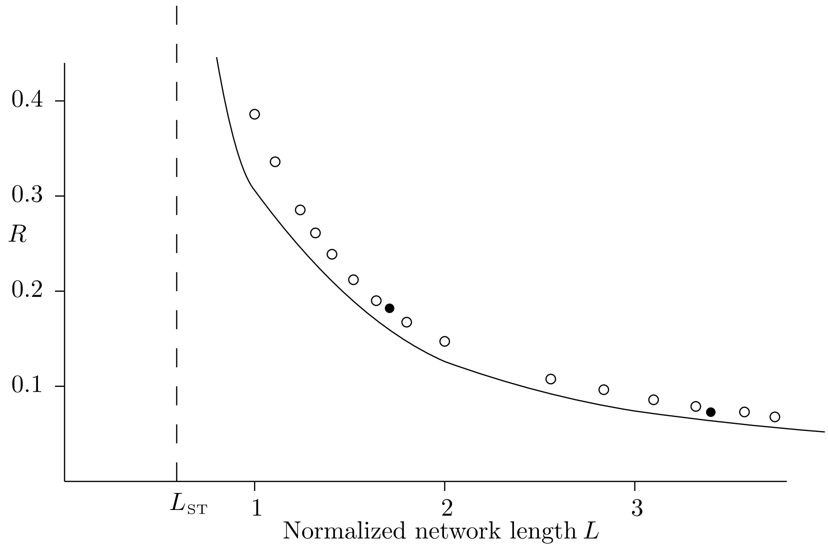

FIG. 9. *Speculative shape for the curve $R_{\mathrm{opt}}(L)$, with ∘ and • values from tractable networks in Figure 7.*

6. FINAL REMARKS

6.1 Toy Models for Road Networks

The idea of using proximity graphs as toy models for road networks has previously been noted [30] but not investigated very thoroughly. It is an intuitively natural idea to a network designer: whether or not to place a direct road from city i to a nearby city j depends (partly) on whether some other city k is close to the line between them.

As observed by a referee, for the kind of models studied in this paper we expect route length $\ell(i, j)$ between distant cities to be roughly proportional to graph distance (number of edges), which is a more relevant quantity in some contexts. However, when one considers design of optimal networks, replacing or partially replacing route length by graph distance leads to quite different optimal networks [1, 22]. For some other cost/benefit functionals leading to yet different optimal networks see [2, 14].

6.2 Rigorous Proof of Finite R in Random Proximity Graphs

Table 1 presented the Monte Carlo numerical value ≈ 0.38 of R for the relative neighborhood graph on random points. From a rigorous viewpoint, the assertion that a random network has $R < \infty$ is essentially the assertion that $\rho(d) = O(d)$ as $d \to \infty$. This is often nontrivial to prove. A general sufficient condition for this property, which applies to the relative neighborhood graph (and hence all proximity graphs), is proved in [3]. The related fact that the limit $\lim_{d\to\infty} \rho(d)/d$ exists is proved in [4].

6.3 Real-World Trade-Off Between Network Length and Route-Length Efficiency

Recall that our central theme is seeking to quantify the trade-off between normalized network length l and route-length efficiency R. Figure 9 suggests that for optimal networks the "law of diminishing returns" sets in around $L = 2$ (for comparison, this is the value of L corresponding to the square grid network), in that $R_{\mathrm{opt}}(L)$ decreases rapidly to around 0.13 as L increases to 2 but decreases only slowly as L increases further. This suggests a kind of "economic prediction" for the lengths of real-world networks which are perceived by users to be efficient in providing short routes:

> the length of an efficient network linking n cities in a region of area A will be roughly $2\sqrt{An}$.

Here the $\sqrt{An}$ arises from undoing the normalization and the "2" is the value of L. Of course, this is *rough*: we mean "closer to 2 than to 1 or 3."

6.4 Other Results for the Random Network Models

There is substantial literature on the networks (MST, proximity graphs, Delaunay triangulation) in the deterministic setting. In the random case, central limit theorems for total network length have been studied in many models: for the MST in [29, 31, 32], and for the Delaunay triangulation, Voronoi tessellation, relative neighborhood and Gabriel graphs in [12, 25, 42]. Large deviation estimates for total network length are given for the Gabriel graph in [46], Section 11.4, and presumably could be extended to other models. Otherwise the literature for the random case is rather diffuse, with different focuses for different networks. For instance, work on MSTs has focused on connections with critical continuum percolation [17]. For the relative neighborhood graph and the Gabriel graph, [20] calculates $\bar{\Delta}$ and [18] shows that, in the finite model, in a certain range the β-skeletons have

$$(14) \qquad R_{\max} \text{ grows as order } \sqrt{\log n / \log\log n}$$

and [21] shows the same order for maximum vertex degree in the Gabriel graph. As for the Delaunay triangulation, there has been surprisingly little follow-up to the seminal analysis by Miles [35] (various maximal statistics are studied in [16]), though the closely related Voronoi tessellation has been studied in more detail [36].

6.5 Speculative Applications of Random Proximity Graphs

Random proximity graphs seem an interesting object of study from many viewpoints, in particular, as an attractive alternative to random geometric graphs for modeling spatial networks that are connected by design. It is remarkable that results such as (14) are the only nonelementary results about them that we can find in the literature. As well as being natural models for road networks, proximity graphs might be useful in modeling communication networks suffering line of sight interference.

At a more mathematical level, for questions such as spread-out percolation [41] or critical value of contact processes [15], random proximity graphs with small A are an interesting alternative to the usual lattice- or random graph-based models. For instance, it is natural to conjecture that the critical value p_A^* for edge percolation on a random proximity graph with template A satisfies

$$(15) \qquad p_A^* \sim \pi^{-1} \operatorname{area}(A) \quad \text{as } \operatorname{area}(A) \to 0$$

[the right side $= 1/\bar{\Delta}$ from (9)] and that the critical value λ_A^* for the contact process has the same asymptotics.

6.6 Analogies Between Deterministic and Random Networks

As mentioned earlier, we may make very loose analogies between particular networks on random points and particular deterministic networks in Figure 4, based in part on exact equality of $\bar{\Delta}$ in the latter three cases:

$$\begin{aligned} \text{Relative n'hood graph} &\leftrightarrow \text{punctured lattice},\\ \text{Gabriel graph} &\leftrightarrow \text{square lattice},\\ \text{Hammersley network} &\leftrightarrow \text{diagonal lattice},\\ \text{Delaunay triangulation} &\leftrightarrow \text{triangular lattice}.\end{aligned}$$

6.7 Scale Invariant Continuum Networks

Introducing the statistic R can be viewed as one approach to resolving the "paradox" from [7], discussed in Section 3.2, that the more natural statistic R_{ave} does not lead to realistic optimal networks in the $n \to \infty$ limit. This particular approach was prompted by visualizing real-world road networks—cf. discussion in Section 3.3. Let us mention a mathematically more sophisticated alternative, under study as a work in progress [5]. Instead of a discrete Poisson process of cities, we imagine a continuum limit. That is, for each finite set $(z_1, \dots, z_k)$ of points in the plane, there is a random network $\mathcal{S}(z_1, \dots, z_k)$ linking the points, consistent as more points are added. Mathematically natural structural properties for the distribution of such a process are as follows:

(i) translation and rotation invariance,
(ii) scale invariance,

where the latter means that routes, *as point-sets in* $\mathbb{R}^2$, are invariant in distribution under Euclidean scaling. This implies that the quantity $\rho(d)$ analogous to (5), assumed finite, is a constant, which we can call R'. The analog L' of L is defined by

> the expected length of the network on n uniform random points in the area-n square grows $\sim L'n$ as $n \to \infty$.

In this setting we can study the optimal trade-off between L' and R', and the kind of "paradoxical" Figure 5 network cannot arise because it violates scale-invariance.

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