T. E. HARRIS' CONTRIBUTIONS TO INTERACTING PARTICLE SYSTEMS AND PERCOLATION

BY THOMAS M. LIGGETT[1]

University of California at Los Angeles

Interacting particle systems and percolation have been among the most active areas of probability theory over the past half century. Ted Harris played an important role in the early development of both fields. This paper is a bird's eye view of his work in these fields, and of its impact on later research in probability theory and mathematical physics.

1. Introduction. Ted's passing was a great loss to me personally and professionally, as well as to probability theory in Southern California and beyond. For three decades, he and I were the primary probabilists at USC and UCLA, respectively. Our approach to mathematics and our mathematical interests were very similar. My wife, Chris, and I enjoyed many wonderful social occasions at the Harris home in Beverly Hills with Ted and Connie. They were great hosts. The conversation was always stimulating. They were interested in so many things!

Ted initiated the Southern California Probability Symposium in about 1970. It is probably the oldest meeting of its type in the US, and continues to provide an exciting place for interactions among Southern California probabilists to this day. In two of the years, 1989 and 2007, the meeting was dedicated to Ted. The first was on the occasion of his 70th birthday, and the second was in his memory. Even after his retirement, Ted attended the USC probability seminar regularly—always sitting in the front row, and asking perceptive questions.

Few mathematicians have had a greater ratio of number of ideas to number of papers. Ted wrote fewer papers (about 30) than many prominent mathematicians, but each is a jewel. In each of the several areas in which he worked, he was among the first in the field. He had an uncanny ability to sense which problems would lead to major developments. His taste was impeccable. After having a significant impact on one area, he would go on to another topic, leaving it to others to flesh out the subject. His work went a long way toward shaping the growth of probability theory in the second half of the twentieth century. I hope to do justice to his many ideas on interacting particle systems and percolation in this brief article. I will mention only a few later papers in which his influence can be seen—there are many others.

2. Percolation. In the standard percolation model, bonds in Z^d are independently labeled open with probability p and closed with probability $1 - p$. One then asks whether the subgraph of Z^d obtained by retaining only the open bonds contains an infinite connected component. The answer of course depends on the value of p—it is yes if p is sufficiently large, and no if p is small. The value at which the answer changes is known as the critical value, p_c.

The mathematical theory of percolation is generally viewed as beginning in 1957 with paper [6] by Broadbent and Hammersley. Only three years later, Ted proved in [14] that in two dimensions, $p_c \geq \frac{1}{2}$. In fact, he proved that there is no infinite cluster when $p = \frac{1}{2}$. Prior to that, the best result on the critical value was $0.35 \leq p_c \leq 0.65$. The primary tools he used were a correlation inequality (more on this in Section 5 below), the self-duality of the two-dimensional lattice, and path intersection arguments. It took another 20 years for Kesten to prove in [23] that $p_c = \frac{1}{2}$, using among other techniques, more refined path intersection arguments. It takes only a glance at Grimmett's book [13] or the more recent [4] by Bollobás and Riorden to get a sense of how big and important percolation has become since Ted's pioneering work. Recent work on SLE scaling limits for percolation and other models is one of the most exciting developments in modern probability theory—see [4, 7, 25] and [35], for example. In fact, one of the 2010 Fields Medals was awarded to Smirnov for his work in the field. Again, path intersection arguments play an important role. Ted did not return explicitly to percolation after [14], but percolation ideas were to play a major role in his later work in interacting particle systems.

3. The contact process. Contact processes constitute one of the two or three major classes of interacting particle systems. They play somewhat the same role in this area that Brownian motion plays in the theory of stochastic processes in Euclidean space: they are simple to describe, they have many of the useful properties that other systems in the field may or may not have—in this case, self-duality, attractiveness and additivity—and they lead to challenging mathematical problems. Contact processes were first introduced and studied by Ted Harris in [17]. Literally hundreds of papers have been written about them in the past 35 years. Few mathematicians have been credited with starting a field that would become as important as this one.

The basic contact process on $S = Z^d$ is a Markov process η_t on $\{0, 1\}^S$, which can be thought of as a model for the spread of infection. A configuration $\eta \in \{0, 1\}^S$ represents the state in which certain sites are infected [those for which $\eta(x) = 1$]; the others are healthy. The value at a site $x \in S$ changes from 1 to 0 at rate 1 (i.e., infected sites recover after a unit exponential time), and from 0 to 1 at a rate proportional to the number of infected neighbors. The constant of proportionality is λ. Of course, the configuration $\eta \equiv 0$ is a trap for the process—infections cannot appear spontaneously. When there are only finitely many infected sites, the state of the system is usually denoted by $A_t = \{x : \eta_t(x) = 1\}$.

A principal reason for interest in the contact process is that it, like the percolation model, can have two different types of behavior, depending on the value of a parameter—λ in this case. It can survive, in the sense that the survival probability is positive,

$$(1) \qquad \pi(A) \equiv P^A(A_t \neq \varnothing \text{ for all } t) > 0, \qquad A \neq \varnothing,$$

even for initial configurations with finitely many infected sites, or it can die out. In [17], Ted did not talk in terms of the critical value λ_c that separates these two regimes, or even note that this value was well defined. However, he did prove that the process survives if λ is large enough, so that $\lambda_c < \infty$, and that $\lambda_c \geq \frac{1}{2d-1}$, with an improvement to $\lambda_c \geq 1.18$ in one dimension. By now, much more is known: $1.53 \leq \lambda_c < 2$ in one dimension, and $\lambda_c \sim 1/2d$ in high dimensions. The actual value of λ_c is thought to be about 1.65 in one dimension, but nothing close to this is rigorously known. (Most later results that are mentioned in this article can be found in [28] or [29].)

Ted was a fan of inequalities, as we will see in Section 5. In [17], he proved some inequalities that are not of the correlation type discussed there. It is fairly clear that the survival probability $\pi(A)$ is increasing in A. It is less obvious that it is submodular in the sense that

$$(2) \qquad \pi(A \cup B) + \pi(A \cap B) \leq \pi(A) + \pi(B).$$

This inequality played a role in his derivation of lower bounds for the critical value. Only much later was this result used in an essential way in [31] to compare a contact process with mutations to the basic contact process, showing that the former process dies out whenever the latter does. This comparison apparently cannot be carried out via more common and intuitive coupling arguments. As far as I know, (2) was not used in the intervening 34 years, even though it was generalized to some extent, and has been used in some related contexts—see [33]. Ted was again ahead of his time.

One of the most useful techniques in interacting particle systems is duality, which expresses probabilities related to one process in terms of probabilities related to another (dual) process. Forms of duality had been used quite early in the study of Brownian motion and birth and death chains. In the context of symmetric exclusion processes, duality was discovered by Spitzer in [36], and has played an essential role in that theory. In particular, it made possible a complete description of the stationary distributions of the system. Such a classification in the asymmetric case remains elusive.

In [18], Ted looked at duality more generally, and discovered in particular that the contact process is self-dual. (He used the word "associate" rather than "dual.") Self-duality for the contact process is the identity

$$P^\eta(\eta_t \equiv 0 \text{ on } A) = P^A(\eta \equiv 0 \text{ on } A_t), \qquad |A| < \infty,$$

which relates the contact process with infinitely many infections to the process with finitely many infections. Letting 1 denote the configuration with all sites infected, this says in particular that

$$P^1\big(\eta_t(x) = 1\big) = P^{\{x\}}(A_t \neq \varnothing),$$

so that survival in the sense of (1) is equivalent to survival in the sense that

$$\lim_{t\to\infty} P^1\big(\eta_t(x) = 1\big) > 0.$$

When the process survives, the "upper invariant measure" ν is defined as the limiting distribution as $t \to \infty$ of the distribution at time t of the system starting in configuration 1. Thus, duality gives

$$\nu\{\eta : \eta \not\equiv 0 \text{ on } A\} = \pi(A).$$

Duality can be used to give a simple proof of the submodularity property (2) that Ted had discovered earlier and proved by coupling. It is obtained by integrating the elementary inequality

$$1_{\{\eta \not\equiv 0 \text{ on } A \cup B\}} + 1_{\{\eta \not\equiv 0 \text{ on } A \cap B\}} \leq 1_{\{\eta \not\equiv 0 \text{ on } A\}} + 1_{\{\eta \not\equiv 0 \text{ on } B\}}$$

with respect to ν. Of course, Ted's duality was not available to him when he proved (2).

There are a number of applications of duality in [18], including a proof of the fact that every translation invariant stationary distribution for the contact process is a mixture of ν and the point mass on $\eta \equiv 0$. Now, we know that the translation invariance assumption is not needed in this statement.

Another important technique in the field is known as the graphical representation, or percolation substructure. The basic idea developed in [20] is that it is very natural and useful to construct processes like the contact process explicitly in terms of collections of independent Poisson processes. There are many advantages to this approach, including the possibility of constructing the process starting from all potential initial configurations on the same probability space. It also gives duality in an explicit way. In the space–time graphical picture, the evolution of the dual process is seen by reversing the time direction.

The graphical representation has played a crucial role in many proofs, including the 1990 proof by Bezuidenhout and Grimmett [3] that the critical contact process dies out. It is the underlying theme of Griffeath's monograph [11], and is the basis of a lot of work of Durrett and his collaborators on systems related to the contact process—see his paper [9] in the volume dedicated to Ted's 70th birthday, for example.

In his paper, Ted proved a number of results using the graphical representation. Here are two:

(a) Linear growth: for sufficiently large λ,

$$P^A\left(\inf_{t>0} \frac{|A_t|}{t} > 0 \,\Big|\, A_t \neq \varnothing \text{ for all } t\right) = 1.$$

He points out that the t in the denominator can probably be replaced by the more plausible t^d.

(b) The individual ergodic theorem: for a large class of initial η and all continuous functions f,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\eta_t) \, dt = \int f \, d\nu \qquad \text{a.s.}$$

Now we know that (a) holds (with t^d in the denominator) for any $\lambda > \lambda_c$. Statement (b) has also been improved.

4. Exclusion processes.

This represents another large part of the field of interacting particle systems. An exclusion process is described via the transition probabilities $p(x, y)$ for a discrete time Markov chain on a countable set S. The process is again a continuous time Markov process on $\{0, 1\}^S$. This time, however, $\eta(x) = 1$ means that site x is occupied; $\eta(x) = 0$ means that it is vacant. A particle at x waits a unit exponential time, and then chooses a $y \in S$ with probabilities $p(x, y)$. If y is vacant, the particle moves there, while if y is occupied, it stays at x. The process was introduced by Spitzer in [36], and has been the subject of a very large number of papers in both mathematics and physics—both rigorous and nonrigorous—since then.

Again, Ted was in the game from the beginning. Prior to my general existence theorem in [27], three mathematicians constructed particle systems under various assumptions—Dobrushin [8], Holley [22] and Harris [16]. In his paper, Ted was concerned with nearest neighbor exclusion processes on Z^d. He used percolation ideas in his construction. Noting that it suffices to construct a Markov process for an arbitrarily short interval of time—the Markov property allows for an extension to all time—he showed that for such short time periods, Z^d breaks up into random finite subsets that do not interact with one another during that time period. On each of these finite subsets, the process is of course well defined.

A useful point of view in the study of symmetric [i.e., those satisfying $p(x, y) = p(y, x)$] exclusion processes was pioneered by Ted, and is known as "stirring." This is closely related to the graphical representation he introduced for contact-like processes in [20]. The idea is that a Poisson process of rate $p(x, y)$ is associated with each pair of sites x, y. At the event times of the Poisson process, the "contents" of the two sites are exchanged. If both were empty or both were occupied, nothing happens, since the particles are indistinguishable. If exactly one site is occupied, the result is that the particle at the occupied site moves to the other site. This again constructs all exclusion process with arbitrary initial configurations on the probability space of the Poisson processes. With this construction, the system is realized as a collection of interacting copies of the original Markov chain. Ted wrote about stirring in [21], and one of his students used it in [26].

An important application of stirring occurred in a paper with another connection to Ted's work. In [15], he considered a system of reflecting Brownian motions,

one starting at each point of a unit Poisson process on the line, and an extra one starting at the origin. He defined reflection by saying that when two Brownian paths meet, they interchange their paths, so that the particles maintain their original ordering. He then proved that the position of the particle originally at the origin satisfies a central limit theorem, but with scaling $t^{1/4}$, rather than $t^{1/2}$. This led Spitzer in [36] to make a conjecture on the behavior of a "tagged" particle in a symmetric exclusion process. Ted's USC colleague R. Arratia then proved the conjecture in [1], using stirring in an essential way. Here is the result. Consider the exclusion process with $S = Z^1$ and

$$p(x, x + 1) = p(x, x - 1) = \tfrac{1}{2}.$$

Initially there is a particle at the origin, and particles are placed at other sites independently with probability $\rho \in (0, 1)$. The particle that started at the origin is the tagged particle. Its position satisfies a central limit theorem, but again with the nonstandard normalization. It turns out that this is (apparently) the only case in which an unusual scaling occurs. Central limit theorems for tagged particles in exclusion systems have been proved by Varadhan and others with normalization $t^{1/2}$ in many cases in which $p(x, y)$ is translation invariant on Z^d, including systems with mean zero in [24] and [37] in any dimension (excluding Arratia's case), and systems with nonzero mean in dimensions $d \geq 3$ in [34].

5. Correlation inequalities.
Perhaps Ted's best known and most influential result is the correlation inequality in [14]. Amazingly, it is not even mentioned in the MathSciNet review of that paper. Perhaps that is not so amazing after all. Who would have known 50 years ago what an effect it would have?

To state it, let S be a finite set, and consider the Bernoulli measure ν_ρ on $\{0, 1\}^S$ defined by

$$\nu_\rho\{\eta : \eta(x) = 1 \text{ for all } x \in T\} = \rho^{|T|}, \qquad T \subset S.$$

A set $A \subset \{0, 1\}^S$ is said to be increasing if $\eta \in A$ and $\eta \leq \zeta$ imply that $\zeta \in A$. [$\eta \leq \zeta$ means that $\eta(x) \leq \zeta(x)$ for all $x \in S$.] Ted's result is

(3) A, B increasing implies $\nu_\rho(A \cap B) \geq \nu_\rho(A)\nu_\rho(B)$.

Actually, he only proved this for the increasing sets that arose in his percolation problem, but that is a minor point.

Property (3) is now usually stated in terms of increasing functions rather than sets, and when applied to a general probability measure, is called "association." Thus, a probability measure μ on $\{0, 1\}^S$ is said to be associated if

(4) f, g increasing implies $\displaystyle\int fg \, d\mu \geq \int f \, d\mu \int g \, d\mu.$

Ted's theorem then states that homogeneous product measures are associated. Motivated by this result, as well as by a 1967 result of Griffiths [12], Fortuin, Kasteleyn and Ginibre [10] proved a far reaching generalization that is known as the FKG theorem: if the probability measure μ is strictly positive and satisfies

$$(5) \qquad \mu(\eta \wedge \zeta)\mu(\eta \vee \zeta) \geq \mu(\eta)\mu(\zeta), \qquad \eta, \zeta \in \{0, 1\}^S,$$

where $\eta \wedge \zeta$ and $\eta \vee \zeta$ denote the coordinate-wise minimum and maximum of η and ζ, respectively, then μ is associated. In their paper, they mentioned the understated nature of Ted's result, while recognizing its importance: "While Harris' inequality seems to have drawn less attention than it deserves, ..." Ted himself, with his usual modesty, said "perhaps the methods are also of some interest." Note that while (3) is far from obvious, the lattice condition (5) is easy to check for ν_ρ. The FKG theorem has played a pivotal role in the study of phase transitions in statistical physics over the past four decades.

Ted's other paper on correlation inequalities [19] is short, elegant, and has many consequences. It deals with implications among the following three properties for a continuous time Markov process η_t on $\{0, 1\}^S$:

(a) Preservation of association: if μ is associated, then so is μ_t, the distribution of η_t with initial distribution μ.

(b) All transitions are between comparable configurations.

(c) Attractiveness: if f is increasing on $\{0, 1\}^S$, then so is $E^\eta f(\eta_t)$.

His theorem is that in the presence of (c), (a) and (b) are equivalent. Much later, I proved in [30] that if all transitions are between configurations that differ at only one site, then (a) implies (c).

An easy consequence of Ted's result is that the upper invariant measure for the contact process ν is associated. To see this, note that the point mass at $\eta \equiv 1$ is associated, and the contact process satisfies (b) and (c). Therefore, the distribution at time t is associated. Now, let $t \to \infty$. The fact that ν is associated is not a consequence of the FKG theorem, since it is known that ν does not satisfy (5).

Ted's theorem has been extended in the case of the contact process to show that certain properties that lie between (5) and association are also preserved by the evolution—see Theorem 3.5 of [2] and Theorems 1.5 and 1.7 of [30]. One consequence of this is that the upper invariant measure ν percolates if $d \geq 2$ and λ is sufficiently large—see [32].

It is interesting to note that (3) follows from Ted's later theorem: consider the spin system in which the coordinates $\eta_t(x)$ flip independently from 1 to 0 at rate $1 - \rho$ and from 0 to 1 at rate ρ. This process satisfies (b) and (c), and has limiting distribution ν_ρ for any initial state. Therefore, ν_ρ is associated.

Paper [19] has stimulated recent work on negative correlations as well. The theory of negative correlations is more subtle than that of positive correlations. One way to see this is that in the definition of negative association, one cannot simply reverse the inequality in (4), as can be seen by taking $f = g$ there. One must

add the constraint that f and g depend on disjoint sets of coordinates. With this definition, the negative version of the FKG theorem is false: (5) with the opposite inequality does not imply negative association.

A possible version of Ted's 1977 theorem for negative association might be that systems like the symmetric exclusion process preserve the property of negative association. After all, if many particles are known to be in one part of S, then fewer can be in other parts of S. In hindsight, it is not too surprising that this is false. The restriction to functions that depend on disjoint sets of coordinates in the definition causes problems, since even if f and g satisfy this constraint, $E^\eta f(\eta_t)$ and $E^\eta g(\eta_t)$ generally will not. All is not lost, however. In [5], a property that is stronger than negative association, but still satisfied by product measures, is shown to be preserved by symmetric exclusion processes.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
520 PORTOLA PLAZA
LOS ANGELES, CALIFORNIA 90095
USA
E-MAIL: tml@math.ucla.edu