The Annals of Probability
2011, Vol. 39, No. 4, 1243–1285
DOI: 10.1214/10-AOP579

LOOP-ERASED RANDOM WALK AND POISSON KERNEL ON PLANAR GRAPHS

BY ARIEL YADIN AND AMIR YEHUDAYOFF

University of Cambridge and Technion—IIT

Lawler, Schramm and Werner showed that the scaling limit of the loop-erased random walk on $\mathbb{Z}^2$ is SLE_2. We consider scaling limits of the loop-erasure of random walks on other planar graphs (graphs embedded into $\mathbb{C}$ so that edges do not cross one another). We show that if the scaling limit of the random walk is planar Brownian motion, then the scaling limit of its loop-erasure is SLE_2. Our main contribution is showing that for such graphs, the discrete Poisson kernel can be approximated by the continuous one.

One example is the infinite component of super-critical percolation on $\mathbb{Z}^2$. Berger and Biskup showed that the scaling limit of the random walk on this graph is planar Brownian motion. Our results imply that the scaling limit of the loop-erased random walk on the super-critical percolation cluster is SLE_2.

1. Introduction. Let G be a graph. The *loop-erased random walk* or LERW on G is obtained by performing a random walk on G, and then erasing the loops in the random walk path in chronological order. The resulting path is a self-avoiding path in the graph G, starting and ending at the same points as the random walk. LERW was invented by Lawler in [5] as a natural measure on self-avoiding paths. It was studied extensively on the graphs $\mathbb{Z}^d$. In dimensions $d \geq 4$, the scaling limit is known to be Brownian motion (see [7]). In dimension $d = 3$, Kozma proved that the scaling limit exists and that the limit is invariant under rotations and dilations (see [4]). In order to study the case $d = 2$, in [13] Schramm introduced a one-parameter family of random continuous curves, known as Schramm–Loewner evolution or SLE_κ. In [9] Lawler, Schramm and Werner proved that the scaling limit of LERW on $\mathbb{Z}^2$ is SLE_2. Their result also holds for other two-dimensional lattices. Many other processes in statistical mechanics have been shown to converge to SLE_κ for other values of κ.

In this paper, we focus on the scaling limit of LERW on planar graphs, not necessarily lattices. A planar graph is a graph embedded into the complex plane so that edges do not intersect each other; a precise definition is provided in Section 1.1. We allow weighted and directed graphs, but require them to be irreducible; that is, any two points are connected by a path in the graph.

Received October 2008.
MSC2010 subject classifications. 60F17, 60J99, 60K35.
Key words and phrases. Loop-erased random walk, Schramm–Loewner evolution, Poisson kernel, planar graphs.

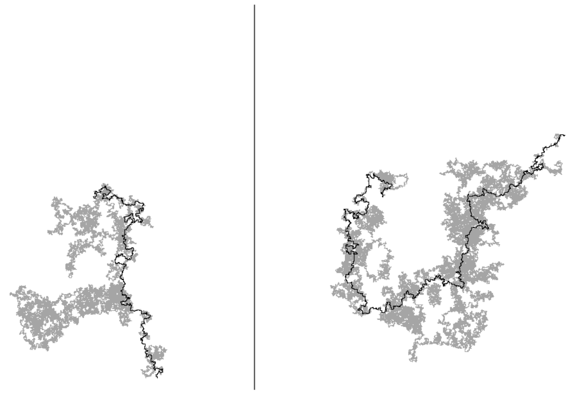

FIG. 1. *LERW* (black) *and simple random walk* (gray) *stopped on exiting the unit disc*. *The underlying graphs are* $\mathbb{Z}^2$ (left) *and the super-critical percolation cluster with parameter* 0.75 (right). *The mesh size is* $1/600$.

Our main result, Theorem 1.1, is a generalization of [9]. Let G be an irreducible graph, and let $f : G \to \mathbb{C}$ be an embedding of G into the complex plane. If $f(G)$ is planar (in the sense above), and if the scaling limit of the random walk on $f(G)$ is planar Brownian motion, then the scaling limit of LERW on $f(G)$ is SLE_2.

One interesting example is the infinite component of super-critical percolation on $\mathbb{Z}^2$. That is, consider bond percolation on $\mathbb{Z}^2$, each bond open with probability $p > 1/2$, all bonds independent. Then, a.s. there exists a unique infinite connected component. In [1] Berger and Biskup proved that a.s. the scaling limit of the random walk on this infinite component is Brownian motion. Together with our result, this implies that a.s. the scaling limit of LERW on the super-critical percolation cluster is SLE_2 (see Figure 1).

Another example of a planar graph with random walk converging to planar Brownian motion is given by Lawler in [6] (see the example following Lemma 5). For each vertex $z \in \mathbb{Z}^2$, define transition probabilities as follows: the probability to go either up or down is $p(z)/2$, and the probability to go either left or right is $(1 - p(z))/2$. Lawler proved in [6] that if $p(z)$ are all chosen i.i.d. such that $\mathbb{P}[p(z) = p] = \mathbb{P}[p(z) = 1 - p] = 1/2$, for some $0 < p < 1/2$, then a.s. the scaling limit of the random walk on this graph is planar Brownian motion. Our result implies that the LERW on this graph converges to SLE_2.

The main contribution of this work is Lemma 1.2, that states that for planar graphs, the discrete Poisson kernel can be approximated by the continuous Poisson kernel. This result holds for any bounded domain, although the boundary behavior can be arbitrary. This result also holds "pointwise," regardless of the local geometry of the graph. Perhaps it can be used to generalize other limit theorems about processes on $\mathbb{Z}^2$ (such as IDLA) to more general planar graphs (e.g., the super-critical percolation cluster).

1.1. *Definitions and notation.* For any $v, u \in \mathbb{C}$, denote $[v, u] = \{(1-t)v + tu : 0 \le t \le 1\}$.

Planar-irreducible graphs. Let $G = (V, E)$ be a directed weighted graph; that is, $E : V \times V \to [0, \infty)$. We write $(v, u) \in E$, if $E(v, u) > 0$. Let $o \in V$ be a fixed vertex. Let $f : V \to \mathbb{C}$ be an embedding of G in the complex plane such that:

(1) $f(o) = 0$.

(2) The embedding of G in $\mathbb{C}$ is a "planar" graph; that is, for every two edges $(v, u), (v', u') \in E$ such that $\{v, u\} \cap \{v', u'\} = \varnothing$, $[f(v), f(u)] \cap [f(v'), f(u')] = \varnothing$.

(3) For every compact set $K \subset \mathbb{C}$, the number of vertices $v \in V$ such that $f(v) \in K$ is finite.

We think of the graph G as its embedding in $\mathbb{C}$. For $\delta > 0$, let $G_\delta = (V_\delta, E_\delta)$ be the graph defined by

$$V_\delta = \{\delta f(v) : v \in V\} \quad \text{and} \quad E_\delta(\delta f(v), \delta f(u)) = E(v, u);$$

that is, G_δ is the embedding of G in $\mathbb{C}$ scaled by a factor of δ.

We assume that $\sum_{u \in V} E(v, u) < \infty$ for every $v \in V$. Let $P : V \times V \to [0, 1]$ be

$$P(v, u) = \frac{E(v, u)}{\sum_{w \in V} E(v, w)}.$$

We call the Markov chain induced on V_δ by P the *natural random walk on G_δ*. We assume that the natural random walk is *irreducible*; that is, for every $v, u \in V$, there exists $n \in \mathbb{N}$ such that $P^n(v, u) > 0$.

We call a graph G that satisfies all the above properties a *planar-irreducible* graph. For the remainder of this paper we consider only planar-irreducible graphs.

Loop erasure. Let $x(0), x(1), \ldots, x(n)$ be $n+1$ vertices in G_δ. Define $x[0, n]$ as the linear interpolation of $(x(0), \ldots, x(n))$; that is, for $t \in [0, n]$, set

$$x(t) = \big(1 - (t - \lfloor t \rfloor)\big)x(\lfloor t \rfloor) + (t - \lfloor t \rfloor)x(\lfloor t \rfloor + 1).$$

Define the *loop-erasure* of $x(\cdot)$ as the self-avoiding sequence induced by erasing loops in chronological order; that is, the loop-erasure of $x(\cdot)$ is the sequence $y(\cdot)$ that is defined inductively as follows: $y(0) = x(0)$, and $y(k+1)$ is defined using $y(k)$ as $y(k+1) = x(T+1)$, where $T = \max\{\ell \le n : x(\ell) = y(k)\}$ [the loop-erasure ends once $y(k) = x(n)$].

A *path* from v to u in G_δ is a sequence $v = x(0), x(1), \ldots, x(n) = u$ such that $(x(j), x(j+1)) \in E_\delta$ for all j. The *reversal* of the path $x(\cdot)$ is the sequence $x(n), x(n-1), \ldots, x(0)$. The reversal of a path is not necessarily a path.

Domains. Denote by $\mathbb{U}$ the open unit disc in $\mathbb{C}$. Let $D \subsetneq \mathbb{C}$ be a simply connected domain such that $0 \in D$. Define $V_\delta(D)$ as the set of vertices $z \in V_\delta \cap D$ such that there is a path from 0 to z in G_δ. Define

$$\partial V_\delta(D) = \{(v,u) : (v,u) \in E_\delta, v \in V_\delta(D), [v,u] \cap \partial D \neq \varnothing\},$$

the "boundary" of G_δ in D. Denote by $\varphi_D : D \to \mathbb{U}$ the unique conformal map onto the unit disc such that $\varphi_D(0) = 0$ and $\varphi'_D(0) > 0$. Define the *inner radius of* D as $\operatorname{rad}(D) = \sup\{R \geq 0 : R \cdot \mathbb{U} \subseteq D\}$.

Throughout this paper, we work with a fixed domain and its sub-domains. Fix a specific bounded domain $\mathbf{D} \subsetneq \mathbb{C}$ such that $\operatorname{rad}(\mathbf{D}) > 1/2$ (one can think of $\mathbf{D}$ as $\mathbb{U}$). Denote

$$\mathfrak{D} = \{D \subseteq \mathbf{D} : D \text{ simply connected domain}, \operatorname{rad}(D) > 1/2\}.$$

SLE. Radial SLE$_\kappa$ in $\mathbb{U}$ can be described as follows (for more details see, e.g., [8, 9, 12, 13, 15]). Let γ be a simple curve from $\partial\mathbb{U}$ to 0. Parameterize γ so that $g'_t(0) = e^t$, where g_t is the unique conformal map mapping $\mathbb{U} \setminus \gamma[0,t]$ onto $\mathbb{U}$ with $g_t(0) = 0$ and $g'_t(0) > 0$. It is known that the limit $W(t) = \lim_{z\to\gamma(t)} g_t(z)$ exists, where z tends to $\gamma(t)$ from within $\mathbb{U} \setminus \gamma[0,t]$. In addition, $W : [0,\infty) \to \partial\mathbb{U}$ is a continuous function, and the Loewner differential equation is satisfied

$$\partial_t g_t(z) = g_t(z) \frac{W(t) + g_t(z)}{W(t) - g_t(z)}$$

and $g_0(z) = z$. The function $W(\cdot)$ is called the *driving function* of γ.

Taking $W(t) = e^{iB(\kappa t)}$, where $B(\cdot)$ is a one-dimensional Brownian motion (started uniformly on $[0, 2\pi]$), one can solve the Loewner differential equation, obtaining a family of conformal maps g_t. It turns out that for $\kappa \leq 4$, the curve γ obtained from the driving function W (defined as $\gamma(0) = W(0)$ and $\gamma(0,t] = \mathbb{U} \setminus g_t^{-1}(\mathbb{U})$) is indeed a simple curve from $\partial\mathbb{U}$ to 0 (see [12]). The curve γ is called the SLE$_\kappa$ path.

Weak convergence. We define weak convergence using one of several equivalent definitions (see Chapter III in [14], e.g.). Let $\alpha, \beta : [0,1] \to \mathbb{U}$ be two continuous curves. Let Φ be the set of continuous nondecreasing maps $\phi : [0,1] \to [0,1]$. We say that α and β are equivalent if $\alpha = \beta \circ \phi$ for some $\phi \in \Phi$. Let $\mathcal{C}$ be the set of all equivalence classes under this relation. Define $\varrho(\alpha,\beta) = \inf_{\phi\in\Phi} \sup_{t\in[0,1]} |\alpha(t) - \beta(\phi(t))|$.

It is known that $\varrho(\cdot,\cdot)$ is a metric on $\mathcal{C}$. Let Σ be the Borel σ-algebra generated by the open sets of ϱ. Let μ be a probability measure on $(\mathcal{C}, \Sigma)$. We say that $A \in \Sigma$ is *μ-continuous*, if $\mu(\partial A) = 0$, where ∂A is the boundary of A.

Let $\{\mu_n\}$ be a sequence of probability measures on $(\mathcal{C}, \Sigma)$. We say that $\{\mu_n\}$ *converges weakly* to μ, if for all μ-continuous events $A \in \Sigma$, it holds that $\mu_n(A)$ converges to $\mu(A)$.

Poisson kernel. Let $D \in \mathfrak{D}$. For $a \in V_\delta(D)$ and $b \in V_\delta(D) \cup \partial V_\delta(D)$, define $H(a,b) = H^{(\delta)}(a,b;D)$ to be the probability that a natural random walk on G_δ, started at a and stopped on exiting D, visits b. That is,

$$H(a,b) = \begin{cases} \mathbb{P}[\exists 0 \le k \le \tau : S(k) = b], & b \in V_\delta(D), \\ \mathbb{P}[(S(\tau-1), S(\tau)) = b], & b \in \partial V_\delta(D), \end{cases}$$

where $S(\cdot)$ is a natural random walk on G_δ started at a, and τ is the exit time of $S(\cdot)$ from D. We sometimes denote the segment $(S(\tau-1), S(\tau))$ by $S(\tau)$; for example, instead of $(S(\tau-1), S(\tau)) = b$ we write $S(\tau) = b$, and for a set $J \subseteq \partial D$, we write $S(\tau) \in J$ instead of writing $[S(\tau-1), S(\tau)] \cap J \neq \varnothing$.

Let $e = (v,u) \in \partial V_\delta(D)$. Let $\tilde{e} \in \partial D$ be the "first" point on the $[v,u]$ that is not in D; that is, let $s = \inf\{0 \le t \le 1 : (1-t)v + tu \notin D\}$, and let $\tilde{e} = (1-s)v + su$. Define $\varphi(e) = \lim_{t \to s^-} \varphi((1-t)v + tu)$.

For $a \in V_\delta(D)$ and $b \in V_\delta(D) \cup \partial V_\delta(D)$, define the *Poisson kernel*

$$\lambda(a,b) = \lambda(a,b;D) = \frac{1 - |\varphi(a)|^2}{|\varphi(a) - \varphi(b)|^2}.$$

If $B(\cdot)$ is a planar Brownian motion started at $x \in \mathbb{U}$, τ is the exit time of $B(\cdot)$ from $\mathbb{U}$, and J is a Borel subset of $\partial\mathbb{U}$, then

$$\mathbb{P}_x[B(\tau) \in J] = \int_J \lambda(x,\zeta;\mathbb{U})\, d\zeta, \tag{1.1}$$

where $d\zeta$ is the uniform measure on $\partial\mathbb{U}$ (see Chapter 3 of [10]).

Complex analysis. Throughout the proofs we will make repeated use of three classical theorems in the theory of analytic and conformal maps: the Schwarz lemma, the Koebe distortion theorem and the Koebe 1/4 theorem. These can be found in [2] or [11].

1.2. *Main results*. Let G be a planar-irreducible graph. Let ν_δ be the law of the natural random walk on G_δ started at 0 and stopped on exiting $\mathbb{U}$. Let μ_δ be the law of the loop-erasure of the reversal of the natural random walk on G_δ started at 0 and stopped on exiting $\mathbb{U}$.

THEOREM 1.1. *Let $\{\delta_n\}$ be a sequence converging to 0. If ν_{δ_n} converges weakly to the law of planar Brownian motion started at 0 and stopped on exiting $\mathbb{U}$, then μ_{δ_n} converges weakly to the law of radial SLE_2 in $\mathbb{U}$ started uniformly on $\partial\mathbb{U}$.*

The proof of Theorem 1.1 is given in Section 6. A key ingredient in the proof is the following lemma, that shows that the discrete Poisson kernel can be approximated by the continuous one (its proof is given in Section 5).

LEMMA 1.2. *For all* $\varepsilon, \alpha > 0$, *there exists* δ_0 *such that for all* $0 < \delta < \delta_0$ *the following holds*:

Let $D \in \mathfrak{D}$, *let* $a \in V_\delta(D)$ *be such that* $|\varphi_D(a)| \leq 1 - \varepsilon$, *and let* $b \in \partial V_\delta(D)$. *Then*,

$$\left| \frac{H^{(\delta)}(a, b; D)}{H^{(\delta)}(0, b; D)} - \lambda(a, b; D) \right| \leq \alpha.$$

Lemma 1.2 holds for all graphs that are planar, irreducible and such that the scaling limit of the random walk on them is planar Brownian motion. The question arises whether a similar result holds in "higher dimensions." The answer is negative. For $d > 2$, one can construct a subgraph of $\mathbb{Z}^d$ such that Lemma 1.2 does not hold for it. The idea is to disconnect one-dimensional subsets, leaving only one edge connecting them to the rest of $\mathbb{Z}^d$. This can be done in a way so that the random walk will still converge to d-dimensional Brownian motion, but for points in these sets the discrete Poisson kernel will be far from the continuous one.

One can also ask whether Lemma 1.2 can be generalized to nonplanar graphs. The answer is again negative. Consider the underlying graph of the following Markov chain. Toss a coin; if it comes out heads, run a simple random walk on $\delta\mathbb{Z}^2$ conditioned to exit the unit disc in the upper half plane, and if the coin comes out tails, run a simple random walk on $\delta\mathbb{Z}^2$ conditioned to exit the unit disc in the lower half plane. This Markov chain converges to planar Brownian motion, but the underlying graph is not planar. In this example, for any point other than 0, the discrete Poisson kernel is supported only on one half of the unit disc (and so is far from the continuous one).

The proof of Theorem 1.1 mainly follows the proof of Lawler, Schramm and Werner in [9]. To understand the new ideas in our paper, let us first give a very brief overview of the argument in [9]. Denote by γ the loop-erasure of the reversal of the natural random walk, and let W be the driving function of γ given by Loewner's thoery.

The first step is to show that W converges to Brownian motion on $\partial\mathbb{U}$. A key ingredient in this step is showing that the discrete Poisson kernel can be approximated by the continuous Poisson kernel (see Lemma 1.2 above). The proof of the convergence of the Poisson kernel in [9] is based on lattice properties, whereas the proof here uses converges to planar Brownian motion from only one vertex, namely 0, and the planarity of the graph.

The second step of the proof is using a compactness argument to conclude a stronger type of convergence. As in [9], we show that the laws given by γ are *tight* (see definition in Section 6.3.1 below). The proof of tightness in [9] uses a "natural" family of compact sets. In our setting, it is not necessarily true that γ belongs to one of these compact sets with high probability (and so the argument of [9] fails). To overcome this difficulty, we define a "weaker" notion of tightness, which we are able to use to conclude the proof.

We now discuss the first step, the proof of Lemma 1.2, in more detail. Let a be a vertex in $\mathbb{U}$, and let b be an edge on $\partial\mathbb{U}$ (in fact, we need to consider arbitrary $D \in \mathfrak{D}$, but we ignore this here). The intuition behind Lemma 1.2 is that two independent planar Brownian motions, started at 0 and at the vertex a, conditioned on exiting $\mathbb{U}$ at a small interval around b, intersect each other with high probability. Intuitively, this should give us a way to couple a random started at 0 and a random walk started at the vertex a (conditioned on exiting $\mathbb{U}$ at a small interval around b), so that they will both exit $\mathbb{U}$ at the same point with high probability. There are several obstacles in this argument: first, we are not able to provide such a coupling, and we overcome this difficulty using harmonic functions. Second, we are not given a priori any information on the random walk starting at the vertex a. Third, we also need to consider the case where the two walks do not intersect. Finally, we are interested in what happens at a specific edge b, and not in its *neighborhood* (the local geometry around b can be almost arbitrary). The main properties of G that allow us to overcome these obstacles are its planarity and the weak convergence of the random walk started at 0 to planar Brownian motion.

2. Preliminaries. Let $D \in \mathfrak{D}$. For $z \in V_\delta(D)$, let $S_z(\cdot)$ be a natural random walk on G_δ started at z. Let $\tau_D^{(z)}$ be the exit time of $S_z(\cdot)$ from D. When D is clear, we omit the subscript from $\tau_D^{(z)}$ and use $\tau^{(z)}$. For $U \subset D$, define

$$\Theta_z(U) = \Theta_z^D(U) = \min\{0 \le t \le \tau^{(z)} : S_z(t) \in U\}.$$

For a path $\gamma[T_1, T_2]$ in D, denote by $\varphi_D \circ \gamma[T_1, T_2]$ the path in $\mathbb{U}$ that is the image of $\gamma[T_1, T_2]$ under the map φ_D.

2.1. *Encompassing a point.* For $r > 0$ and $z \in \mathbb{C}$, denote $\rho(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$, the disc of radius r centered at z.

Crossing a rectangle. Let $z_1, z_2 \in \mathbb{C}$ and $r > 0$. Define $\square(z_1, z_2, r)$ as the $4r$ by $4r + |z_2 - z_1|$ open rectangle around the interval $[z_1, z_2]$; more precisely, define $\square(z_1, z_2, r)$ as the interior of the convex hull of the four points $z_1 - 2r(u + v)$, $z_1 - 2r(u - v)$, $z_2 + 2r(u + v)$ and $z_2 + 2r(u - v)$, where $u = \frac{z_2 - z_1}{|z_2 - z_1|}$ and $v = u \cdot i$.

Let $\gamma : [T_1, T_2] \to \mathbb{C}$ be a curve. Let $t_1 = \inf\{t \ge T_1 : \gamma(t) \in \rho(z_1, r)\}$ and $t_2 = \inf\{t \ge T_1 : \gamma(t) \in \rho(z_2, r)\}$. We say that $\gamma[T_1, T_2]$ *crosses* $\square(z_1, z_2, r)$, if $t_1 < t_2 \le T_2$ and $\gamma[t_1, t_2] \subset \square(z_1, z_2, r)$.

Encompassing a point. Let $z \in \mathbb{C}$ and $r > 0$. Define $z_1, \dots, z_5 \in \mathbb{C}$ to be the following five points: let $r' = r/20$, let $z_1 = z - 8r' - 4r'i$, let $z_2 = z + 4r' - 4r'i$, let $z_3 = z + 4r' + 4r'i$, let $z_4 = z - 4r' + 4r'i$ and let $z_5 = z - 4r' - 8r'i$.

We say that $\gamma[T_1, T_2]$ *r-encompasses* z, denoted $\gamma[T_1, T_2] \circlearrowleft^{(r)} z$, if $\gamma[T_1, T_2]$ crosses all rectangles $\square(z_1, z_2, r')$, $\square(z_2, z_3, r')$, $\square(z_3, z_4, r')$, $\square(z_4, z_5, r')$.

If $\gamma[T_1, T_2] \circlearrowleft^{(r)} z$, then any path from z to infinity must intersect $\gamma[T_1, T_2]$; that is, z does not belong to the unique unbounded component of $\mathbb{C} \setminus \gamma[T_1, T_2]$. Also, if $\gamma[T_1, T_2] \circlearrowleft^{(r)} z$, there exist $\tau_1 < \tau_2 \le T_2$ such that $\gamma[\tau_1, \tau_2] \circlearrowleft^{(r)} z$ and $\gamma[\tau_1, \tau_2] \subset \rho(z, r)$.

2.2. *Compactness of* $\mathfrak{D}$. Let $D \in \mathfrak{D}$. We bound the derivative of φ_D^{-1} at 0. Using the Schwarz lemma, since $\varphi_D^{-1}(0) = 0$, we have $\mathrm{rad}(D)/|\varphi_D^{-1\prime}(0)| \le 1$. Since $\mathrm{rad}(D) > 1/2$, we have $|\varphi_D^{-1\prime}(0)| > 1/2$. Using the Schwarz lemma again, we have $|\varphi_D^{-1\prime}(0)| \le C'$, for $C' = \sup\{|x| : x \in \mathbf{D}\}$. Thus, there exists a constant $c = c(\mathbf{D}) > 0$ such that

$$c \le |\varphi_D^{-1\prime}(0)| \le c^{-1}. \tag{2.1}$$

Let $\varepsilon > 0$. Every map φ_D^{-1}, for $D \in \mathfrak{D}$, can be thought of as a continuous map on the compact domain $K = \{\xi \in \mathbb{U} : |\xi| \le 1 - \varepsilon\}$. The set of maps $\{\varphi_D^{-1}\}_{D\in\mathfrak{D}}$ is pointwise relatively compact. Let $z \in K$, then for every $z' \in K$,

$$|\varphi_D^{-1}(z) - \varphi_D^{-1}(z')| \le |\varphi_D^{-1\prime}(\zeta)| \cdot |z - z'|$$

for some $\zeta \in K$. By the Koebe distortion theorem and (2.1), there exists a constant $c_1 = c_1(\mathbf{D}) > 0$ such that $|\varphi_D^{-1\prime}(\zeta)| \le c_1 \cdot \varepsilon^{-3}$. Thus, $\{\varphi_D^{-1}\}_{D\in\mathfrak{D}}$ is equicontinuous. Hence, by the Arzelá–Ascoli theorem, $\{\varphi_D^{-1}\}_{D\in\mathfrak{D}}$ is relatively compact (as maps on K).

PROPOSITION 2.1. *For any $\varepsilon, \eta > 0$, there exist $\delta_0 > 0$ and a finite family of domains $\mathfrak{D}_{\varepsilon,\eta}$, such that for every $D \in \mathfrak{D}$ there exists $\tilde{D} \in \mathfrak{D}_{\varepsilon,\eta}$ with the following properties*:

(1) $\tilde{D} \subset D$.
(2) *For every $a \in D$ such that $|\varphi_D(a)| \le 1 - \varepsilon$, we have $|\varphi_{\tilde{D}}(a)| \le 1 - \varepsilon/2$.*
(3) *For every $\xi \in \partial\tilde{D}$, we have $|\varphi_D(\xi)| \ge 1 - \eta$.*
(4) *For every $\xi \in \mathbb{C}$ such that $|\xi| \le 1$, we have $|\varphi_D(\varphi_{\tilde{D}}^{-1}(\xi)) - \xi| \le \eta$.*
(5) *For every $\xi \in \mathbb{C}$ such that there exists z in the closure of $\tilde{D}$ with $|z - \xi| \le \delta_0$, we have $|\varphi_D(\xi) - \varphi_D(z)| \le \eta$.*

We call $\tilde{D}$ the (ε, η)-*approximation* of D.

PROOF OF PROPOSITION 2.1. Let $\varepsilon_1, \varepsilon_2 > 0$ be small enough, and let $K = \{\xi \in \mathbb{U} : |\xi| \le 1 - \varepsilon_1\}$. By the relative compactness of $\{\varphi_D^{-1}\}_{D\in\mathfrak{D}}$ (as maps on K), there exists a finite family of domains $\mathfrak{D}'$ such that for every $D \in \mathfrak{D}$ there exists $D' \in \mathfrak{D}'$ with

$$\mathrm{dist}(\varphi_D^{-1}, \varphi_{D'}^{-1}) = \max_{x\in K} |\varphi_D^{-1}(x) - \varphi_{D'}^{-1}(x)| < \varepsilon_2. \tag{2.2}$$

Set $\mathfrak{D}_{\varepsilon,\eta}$ to be the set of $\tilde{D} = \varphi_{D'}^{-1}((1-2\varepsilon_1)\mathbb{U})$ for $D' \in \mathfrak{D}'$.

Let $D \in \mathfrak{D}$, let $D' \in \mathfrak{D}'$ be the closest domain to D in $\mathfrak{D}'$ and let $\tilde{D} = \varphi_{D'}^{-1}((1-2\varepsilon_1)\mathbb{U})$. By (2.1), and by the Koebe distortion theorem, for every $z \in K$,

$$\frac{\varepsilon_1}{C} < |\varphi_D^{-1\prime}(0)| \cdot \frac{1-|z|}{8} \le |\varphi_D^{-1\prime}(z)| \le |\varphi_D^{-1\prime}(0)| \cdot \frac{2}{(1-|z|)^3} < \frac{C}{\varepsilon_1^3}, \tag{2.3}$$

where $C = C(\mathbf{D}) > 0$ is a constant.

We prove property (1). Using (2.3), for every $z_1 \in \mathbb{U}$ such that $|z_1| = 1-\varepsilon_1$ and $z_2 \in \mathbb{U}$ such that $|z_2| = 1-2\varepsilon_1$,

$$|\varphi_D^{-1}(z_1) - \varphi_D^{-1}(z_2)| = |\varphi_D^{-1\prime}(\xi)||z_1 - z_2| \ge \frac{\varepsilon_1^2}{C} \tag{2.4}$$

for some $\xi \in K$. By (2.2), for every $z \in \tilde{D}$, there exists $\zeta \in \varphi_D^{-1}((1-2\varepsilon_1)\mathbb{U})$ such that $|z-\zeta| < \varepsilon_2$. Thus, for $\varepsilon_2 < \frac{\varepsilon_1^2}{C}$, we have $\tilde{D} \subset \varphi_D^{-1}(K) \subset D$.

We prove property (2). Let $a \in D$ be such that $|\varphi_D(a)| \le 1-\varepsilon$. We first show that for $\varepsilon_1 \le \varepsilon/4$,

$$\operatorname{dist}(b, \partial\tilde{D}) \ge c \cdot \varepsilon^2$$

for a constant $c = c(\mathbf{D}) > 0$, where $b = \varphi_{D'}^{-1}(\varphi_D(a))$. Since $2\varepsilon_1 < \varepsilon$, $b \in \tilde{D}$. By the Koebe 1/4 theorem, using the Koebe distortion theorem and since $\varphi_{\tilde{D}}^{-1}(x) = \varphi_{D'}^{-1}((1-2\varepsilon_1)x)$,

$$\begin{aligned} \operatorname{dist}(b, \partial\tilde{D}) &\ge \frac{(1-|\varphi_{\tilde{D}}(b)|) \cdot |\varphi_{\tilde{D}}^{-1\prime}(\varphi_{\tilde{D}}(b))|}{4} \ge \frac{(1-|\varphi_{\tilde{D}}(b)|)^2 \cdot (1-2\varepsilon_1)}{C} \\ &= \frac{(1-|\varphi_D(a)/(1-2\varepsilon_1)|)^2 \cdot (1-2\varepsilon_1)}{C} \ge c \cdot \varepsilon^2. \end{aligned}$$

Thus, $\rho(b, \varepsilon_2) \subset \tilde{D}$, for $\varepsilon_2 < c \cdot \varepsilon^2$. Thus, by (2.2), $[a, b] \subset \tilde{D}$, which implies, using the Koebe distortion theorem,

$$\begin{aligned} |\varphi_{D'}(a) - \varphi_D(a)| &= |\varphi_{D'}(a) - \varphi_{D'}(b)| = |\varphi_{D'}'(\xi)| \cdot |b-a| \\ &\le \frac{C}{1-|\varphi_{D'}(\xi)|} \cdot \varepsilon_2 \le \frac{\varepsilon_2 \cdot C}{2\varepsilon_1} \end{aligned}$$

for some $\xi \in \tilde{D}$. Thus, for $\varepsilon_2 < \frac{\varepsilon_1 \cdot \varepsilon^2}{2C}$,

$$|\varphi_{\tilde{D}}(a)| = \frac{|\varphi_{D'}(a)|}{1-2\varepsilon_1} \le \frac{1-\varepsilon+\varepsilon_2 \cdot C/(2\varepsilon_1)}{1-2\varepsilon_1} < 1-\frac{\varepsilon}{2}. \tag{2.5}$$

We prove property (3). Let $\xi \in \partial\tilde{D}$. Let $z = \varphi_D^{-1}(\varphi_{D'}(\xi))$. By (2.2), $|z-\xi| < \varepsilon_2$. By (2.4), $\rho(z, \varepsilon_1^2/C) \subset \varphi_D^{-1}(K)$. Thus, for $\varepsilon_2 \le \varepsilon_1^2/C$, using (2.3),

$$|\varphi_D(\xi) - \varphi_D(z)| \le |\varphi_D'(\zeta)| \cdot \varepsilon_2 \le \frac{C\varepsilon_2}{\varepsilon_1} \le \varepsilon_1$$

for some $\zeta \in \varphi_D^{-1}(K)$. Since $|\varphi_D(z)| = |\varphi_{D'}(\xi)| = 1 - 2\varepsilon_1$,

$$|\varphi_D(\xi)| \geq |\varphi_D(z)| - |\varphi_D(\xi) - \varphi_D(z)| \geq 1 - 3\varepsilon_1 > 1 - \eta$$

for $\varepsilon_1 < \eta/3$.

We prove property (4). Let $\xi \in \mathbb{C}$ be such that $|\xi| \leq 1$. Using (2.2),

$$\begin{aligned}|\varphi_D(\varphi_{\tilde{D}}^{-1}(\xi)) - \xi| &\leq \left|\varphi_D\left(\varphi_{D'}^{-1}\left((1-2\varepsilon_1)\xi\right)\right) - (1-2\varepsilon_1)\xi\right| + |(1-2\varepsilon_1)\xi - \xi| \\ &= |\varphi_D'(\zeta)| \cdot \left|\varphi_{D'}^{-1}\left((1-2\varepsilon_1)\xi\right) - \varphi_D^{-1}\left((1-2\varepsilon_1)\xi\right)\right| + 2\varepsilon_1 \\ &\leq |\varphi_D'(\zeta)| \cdot \varepsilon_2 + 2\varepsilon_1\end{aligned}$$

for some $\zeta \in e = [\varphi_{D'}^{-1}((1-2\varepsilon_1)\xi), \varphi_D^{-1}((1-2\varepsilon_1)\xi)]$. Since the length of e is at most ε_2, and since $\varepsilon_2 \leq \varepsilon_1^2/C$, using (2.4), we have $e \subset \varphi_D^{-1}(K)$. Thus, $\varphi_D(\zeta) \in K$, which implies using (2.3) that $|\varphi_D'(\zeta)| \leq \frac{C}{\varepsilon_1}$. Choosing $\varepsilon_2 \leq \frac{\varepsilon_1^2}{C}$ and $3\varepsilon_1 \leq \eta$ the proof is complete.

We prove property (5). Let $\xi \in \mathbb{C}$ be such that there exists z in the closure of $\tilde{D}$ with $|z - \xi| \leq \delta_0$. As in property (4), for $\delta_0 \leq \varepsilon_1^2/C$, we have $[\xi, z] \subset \varphi_D^{-1}(K)$, which implies

$$|\varphi_D(\xi) - \varphi_D(z)| \leq |\varphi_D'(\zeta)| \cdot \delta_0 \leq \frac{C\delta_0}{\varepsilon_1} \leq \eta$$

for some $\zeta \in [\xi, z]$ and $\delta_0 \leq \eta\varepsilon_1/C$. □

3. Preliminaries for Brownian motion.

3.1. *Brownian motion measure continuity.*

PROPOSITION 3.1. *Let $D \subsetneq \mathbb{C}$ be a simply connected domain such that $0 \in D$. Let ν be the law of planar Brownian motion $B(\cdot)$ (started at some point in D and stopped on exiting D). Let τ be the exit time of $B(\cdot)$ from D. Then, the following events are ν-continuous*:

(1) *For any $r > 0$ and $z \in D$ such that $\rho(z, r) \subset D$, the event $\{B[0, \tau] \circlearrowleft^{(r)} z\}$.*
(2) *For any disc $\rho(z, r) \subset D$, the event $\{B[0, \tau] \cap \rho(z, r) \neq \varnothing\}$.*
(3) *If $D = \mathbb{U}$, for any interval $I \subset \partial\mathbb{U}$, the event $\{B(\tau) \in I\}$.*

PROOF. We use the following claim.

CLAIM 3.2. *Let $U \subset D$ be an open set, and $\tau_{\partial U} = \inf\{t \geq 0 : B(t) \in \partial U\}$. Then, if $U = \rho(z, r)$ or if $U = \square(z_1, z_2, r)$ for some $z_1, z_2 \in D$, we have $\mathbb{P}[\tau_1 > \tau_{\partial U}] = \mathbb{P}[\tau_2 > \tau_{\partial U}] = 0$, where $\tau_1 = \inf\{t \geq \tau_{\partial U} : B(t) \in U\}$ and $\tau_2 = \inf\{t \geq \tau_{\partial U} : B(t) \notin U \cup \partial U\}$.*

PROOF. We prove $\mathbb{P}[\tau_1 > \tau_{\partial U}] = 0$. The proof for τ_2 is similar. Let $\mathcal{F}(t)$ be the σ-algebra generated by $\{B(s) : 0 \le s \le t\}$, and let $\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}(s)$. Since

$$\{\tau_1 = \tau_{\partial U}\} = \bigcap_{n\in\mathbb{N}} \left\{\exists 0 < \varepsilon < \frac{1}{n} : B(\tau_{\partial U} + \varepsilon) \in U\right\} \in \mathcal{F}^+(\tau_{\partial U}),$$

by Blumenthal's 0–1 law and the strong Markov property (see, e.g., Chapter 2 in [10]), $\mathbb{P}[\tau_1 = \tau_{\partial U} \mid \mathcal{F}(\tau_{\partial U})] \in \{0, 1\}$. Since for any small enough $\varepsilon > 0$, $\mathbb{P}[\tau_1 \le \tau_{\partial U} + \varepsilon] \ge \mathbb{P}[B(\tau_{\partial U} + \varepsilon) \in U] \ge \frac{1}{10}$, we have $\mathbb{P}[\tau_1 > \tau_{\partial U}] = 0$. $\square$

The event $\{B[0, \tau] \circlearrowleft^{(r)} z\}$ is the intersection of four events of the form $\{B[0, \tau] \text{ crosses } \square(z_j, z_{j+1}, r')\}$, for appropriate $z_1, \dots, z_5$, and r'. So it suffices to prove that for any $\square(z_1, z_2, r) \subset D$, the event $\{B[0, \tau] \text{ crosses } \square(z_1, z_2, r)\}$ is ν-continuous. By definition,

$$\{B[0, \tau] \text{ crosses } \square(z_1, z_2, r)\} = \{t_1 < t_2\} \cap \{t_2 \le \tau\} \cap \{B[t_1, t_2] \subset \square(z_1, z_2, r)\},$$

where $t_1 = \inf\{t \ge 0 : B(t) \in \rho(z_1, r)\}$ and $t_2 = \inf\{t \ge 0 : B(t) \in \rho(z_2, r)\}$.

Let $\tau_1 = \inf\{t \ge 0 : B(t) \in \partial\rho(z_1, r)\}$. The boundary of the event $\{t_1 < t_2\}$ is contained in the event $\{t_1 > \tau_1\}$. Thus, by Claim 3.2, the boundary of $\{t_1 < t_2\}$ has zero ν-measure.

Let $\tau_2 = \inf\{t \ge 0 : B(t) \in \partial\rho(z_2, r)\}$. The boundary of the event $\{t_2 \le \tau\}$ is contained in the event $\{t_2 > \tau_2\}$. Thus, by Claim 3.2, the boundary of $\{t_2 \le \tau\}$ has zero ν-measure.

Let $\tau_3 = \inf\{t_1 \le t \le t_2 : B(t) \in \partial\square(z_1, z_2, r)\}$ and $\tau_4 = \inf\{t \ge \tau_3 : B(t) \notin \square(z_1, z_2, r) \cup \partial\square(z_1, z_2, r)\}$. The boundary of the event $\{B[t_1, t_2] \subset \square(z_1, z_2, r)\}$ is contained in the event $\{\tau_4 > \tau_3\}$. Thus, by Claim 3.2, the boundary of $\{B[t_1, t_2] \subset \square(z_1, z_2, r)\}$ has zero ν-measure.

This proves property (1). A similar (simpler) argument proves property (2). To prove property (2), note that the measure ν is supported on curves that intersect $\partial\mathbb{U}$ at most at one point. Hence, up to zero ν-measure, the boundary of the event $\{B(\tau) \in I\}$ is the event $\{B(\tau) \in \{w, w'\}\}$, where w and w' are the endpoints of I in $\partial\mathbb{U}$. Since $\{B(\tau) \in \{w, w'\}\}$ has zero ν-measure, we are done. $\square$

3.2. *Probability estimates.* This section contains some lemmas regarding planar Brownian motion. Some of these lemmas may be considered "folklore." For the sake of brevity, we omit the proofs.

Notation. In the following $B(\cdot)$ is a planar Brownian motion. For $x \in \mathbb{U}$, $\mathbb{P}_x$ is the measure of $B(\cdot)$ conditioned on $B(0) = x$. For $r > 0$, define $A(r)$ to be the annulus of inner radius r and outer radius $5r$ centered at 1, intersected with the unit disc; that is, $A(r) = \{1 + z : r < |z| < 5r\} \cap \mathbb{U}$. Also, define $\xi(r) = 1 - 3r \in A(r)$. Note that $\rho(\xi(r), r) \subset A(r)$ for $r < 1/25$.

The following proposition is a corollary of Theorem 3.15 in [10].

PROPOSITION 3.3. *Let $0 \neq x \in \mathbb{U}$ and let $0 < c < |x|$. Let τ be the exit time of $B(\cdot)$ from $\mathbb{U}$. Then,*

$$\mathbb{P}_x\bigl[\exists t \in [0,\tau] : |B(t)| \leq c\bigr] \geq \frac{1-|x|}{-\log c}.$$

PROPOSITION 3.4. *There exists $c > 0$ such that the following holds:*
Let $r > 0$ and let $z \in \mathbb{C}$. Let T be the exit time of $B(\cdot)$ from $\rho(z,r)$. Then for every $x \in \rho(z, r/2)$, $\mathbb{P}_x[B[0,T] \circlearrowleft^{(r)} z] \geq c$.

PROPOSITION 3.5. *For any $0 < \varepsilon < 1$, there exists $c > 0$ such that the following holds:*
Let $a \in \mathbb{U}$ be such that $|a| \leq 1 - \varepsilon$. Let τ be the exit time of $B(\cdot)$ from $\mathbb{U}$. Then, $\mathbb{P}_0[B[0,\tau] \circlearrowleft^{(\varepsilon)} a] \geq c$.

LEMMA 3.6. *There exists $c > 0$ such that the following holds:*
Let $0 < r < \frac{1}{25}$, let $A = A(r)$ and $\xi = \xi(r)$. Let $x \in A$ be such that $2r \leq |x - 1| \leq 4r$. Let T be the exit time of $B(\cdot)$ from A. Then,

$$\mathbb{P}_x\bigl[B[T_\xi, T_\rho] \circlearrowleft^{(r)} \xi, T_\rho < T\bigr] \geq c \cdot \frac{1-|x|}{r} \geq \frac{c}{2} \cdot \frac{1-|x|^2}{r},$$

where $T_\xi = \inf\{t > 0 : B(t) \in \rho(\xi, r/20)\}$ and $T_\rho = \inf\{t \geq T_\xi : B(t) \notin \rho(\xi, r)\}$.

LEMMA 3.7. *There exists $c > 0$ such that the following holds:*
Let $0 < \beta < \frac{1}{25\pi}$, and let $I = \{e^{it} : -\pi\beta \leq t \leq \pi\beta\}$ be the interval on the unit circle centered at 1 of measure β. Let $\pi\beta \leq r < \frac{1}{25}$, let $A = A(r)$ and $\xi = \xi(r)$. Let $x \in A$ be such that $2r \leq |x-1| \leq 4r$. Let τ be the exit time of $B(\cdot)$ from $\mathbb{U}$, and let T be the exit time of $B(\cdot)$ from A. Then,

$$\mathbb{P}_x\bigl[B[T_\xi, T_\rho] \circlearrowleft^{(r)} \xi, T_\rho < T \mid B(\tau) \in I\bigr] \geq c,$$

where $T_\xi = \inf\{t > 0 : B(t) \in \rho(\xi, r/20)\}$ and $T_\rho = \inf\{t \geq T_\xi : B(t) \notin \rho(\xi, r)\}$.

LEMMA 3.8. *For every $\eta > 0$, there exists $c > 0$ such that the following holds:*
Let $\beta, I, r, A, \xi, x, \tau$ and T be as in Lemma 3.7. Then,

$$\mathbb{P}_x[T_{\xi,\eta} < T \mid B(\tau) \in I] \geq c,$$

where $T_{\xi,\eta} = \inf\{t > 0 : B(t) \in \rho(\xi, \eta r)\}$.

LEMMA 3.9. *There exist $K, c > 0$ such that the following holds:*
Let $0 < \pi\beta < r < \frac{1}{2K}$, and let $I = \{e^{it} : -\pi\beta \leq t \leq \pi\beta\}$ be the interval on the unit circle centered at 1 of measure β. Let $\xi = \xi(r)$. Let τ be the exit time of $B(\cdot)$ from $\mathbb{U}$. Then,

$$\mathbb{P}_0\bigl[B[T_\xi, \tau] \circlearrowleft^{(r)} \xi, \tau < T_{Kr} \mid B(\tau) \in I\bigr] \geq c,$$

where $T_\xi = \inf\{t > 0 : B(t) \in \rho(\xi, r/20)\}$ and $T_{Kr} = \inf\{t > T_\xi : |B(t) - 1| \geq Kr\}$.

LEMMA 3.10. *There exist $K, c > 0$ such that the following holds*:
Let $\beta, r, I, \xi, \tau, T_\xi$ and T_{Kr} be as in Lemma 3.9. Then,

$$\mathbb{P}_0[T_\xi < \tau < T_{Kr}, B(\tau) \in I_+ \mid B(\tau) \in I] \geq c \tag{3.1}$$

and

$$\mathbb{P}_0[T_\xi < \tau < T_{Kr}, B(\tau) \in I_- \mid B(\tau) \in I] \geq c, \tag{3.2}$$

where $I_+ = \{e^{it} : \pi\beta/2 \leq t \leq \pi\beta\}$ and $I_- = \{e^{it} : -\pi\beta \leq t \leq -\pi\beta/2\}$.

4. Planarity and global behavior.

4.1. *Continuity for a fixed domain.*

PROPOSITION 4.1. *For all $\alpha > 0$, there exists $\eta > 0$ such that for all $\varepsilon > 0$, for all simply connected domains $D \subsetneqq \mathbb{C}$ such that $0 \in D$, and for all $\tilde{a} \in (1-\varepsilon)\mathbb{U}$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:
Let $y \in V_\delta(D) \cap \varphi_D^{-1}(\rho(\tilde{a}, \eta\varepsilon))$. Then, for every continuous curve g starting in $\rho(\tilde{a}, \eta\varepsilon)$ and ending outside of $\rho(\tilde{a}, \varepsilon)$, the probability that $\varphi_D \circ S_y$ does not cross g before exiting $\rho(\tilde{a}, \varepsilon)$ is at most α.

PROOF. Denote $\varphi = \varphi_D$. For $x \in D$ and $r > 0$, define

$$\tau^{(x)}(r) = \Theta_x(\varphi^{-1}(\rho(\tilde{a}, r))),$$

the time $\varphi \circ S_x$ hits $\rho(\tilde{a}, r)$, and define

$$T^{(x)}(r) = \min\{\tau^{(x)}(r/20) \leq t \leq \tau^{(x)} : \varphi(S_x(t)) \notin \rho(\tilde{a}, r)\}.$$

We use the following claim and its corollary below.

CLAIM 4.2. *There exists a universal constant $c > 0$ such that for all $0 < r < \varepsilon/40$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:
There exists $x \in V_\delta(D)$ such that $\varphi(x) \in \rho(\tilde{a}, r/20)$ and

$$\mathbb{P}\big[\varphi \circ S_x\big[0, T^{(x)}(r)\big] \circlearrowleft^{(r)} \tilde{a}, \varphi \circ S_x\big[T^{(x)}(r), T^{(x)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big] \geq c.$$

PROOF. Consider the event

$$F = \{\varphi \circ S_0\big[\tau^{(0)}(r/20), T^{(0)}(r)\big] \circlearrowleft^{(r)} \tilde{a}, \varphi \circ S_0\big[T^{(0)}(r), T^{(0)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\}.$$

Let $B(\cdot)$ be a planar Brownian motion, and let $\tau^{(B)}$ be the exit time of $B(\cdot)$ from $\mathbb{U}$. Let $\tau^{(B)}(r/20) = \inf\{0 \leq t \leq \tau^{(B)} : B(t) \in \rho(\tilde{a}, r/20)\}$, and let $T^{(B)}(r) = \inf\{\tau^{(B)}(r/20) \leq t \leq \tau^{(B)} : B(t) \notin \rho(\tilde{a}, r)\}$ [$T^{(B)}(20r)$ is defined similarly].

By weak convergence and Proposition 3.1, by the conformal invariance of Brownian motion, by the strong Markov property and by Proposition 3.4, for small enough δ_0,

$$
\begin{aligned}
\mathbb{P}[F] &\geq \frac{1}{2}\mathbb{P}_0\big[B\big[\tau^{(B)}(r/20), T^{(B)}(r)\big] \circlearrowleft^{(r)} \tilde{a},\\
&\qquad\qquad B\big[T^{(B)}(r), T^{(B)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big]\\
&\geq \frac{1}{2}\mathbb{P}_0\big[\tau^{(B)}(r/20) < \tau^{(B)}\big]\\
&\quad \times \inf_{\xi \in \rho(\tilde{a}, r/20)} \mathbb{P}_\xi\big[B\big[0, T^{(B)}(r)\big] \circlearrowleft^{(r)} \tilde{a},\\
&\qquad\qquad\qquad B\big[T^{(B)}(r), T^{(B)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big]\\
&\geq c_1 \cdot \mathbb{P}_0\big[\tau^{(B)}(r/20) < \tau^{(B)}\big],
\end{aligned}
\tag{4.1}
$$

where $c_1 > 0$ is a universal constant. In addition, by the strong Markov property,

$$
\begin{aligned}
&\mathbb{P}[F] \leq \mathbb{P}\big[\tau^{(0)}(r/20) < \tau^{(0)}\big]\\
&\qquad \times \max_x \mathbb{P}\big[\varphi \circ S_x\big[0, T^{(x)}(r)\big] \circlearrowleft^{(r)} \tilde{a}, \varphi \circ S_x\big[T^{(x)}(r), T^{(x)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big],
\end{aligned}
$$

where the supremum is over $x \in V_\delta(D) \cap \varphi^{-1}(\rho(\tilde{a}, r/20))$. Hence, since for small enough δ_0,

$$
\mathbb{P}\big[\tau^{(0)}(r/20) < \tau^{(0)}\big] \leq 2\mathbb{P}_0\big[\tau^{(B)}(r/20) < \tau^{(B)}\big],
$$

using (4.1), there exists $x \in V_\delta(D) \cap \varphi^{-1}(\rho(\tilde{a}, r/20))$ such that

$$
\mathbb{P}\big[\varphi \circ S_x\big[0, T^{(x)}(r)\big] \circlearrowleft^{(r)} \tilde{a}, \varphi \circ S_x\big[T^{(x)}(r), T^{(x)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big] \geq c. \qquad \square
$$

COROLLARY 4.3. *There exists a universal constant $c > 0$ such that for all $0 < r < \varepsilon/40$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*: *For every $w \in V_\delta(D)$ such that $\varphi(w) \in \rho(\tilde{a}, r/20)$,*

$$
\mathbb{P}\big[\varphi \circ S_w\big[0, T^{(w)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big] \geq c.
$$

PROOF. We claim that there exists a set of vertices U in $V_\delta(D)$ such that every path in G_δ that starts from w and reaches outside of $\varphi^{-1}(\rho(\tilde{a}, r))$, intersects U, and such that

$$
\mathbb{P}\big[\varphi \circ S_u\big[0, T^{(u)}(20r)\big] \circlearrowleft^{(20r)} \tilde{a}\big] \geq c \tag{4.2}
$$

for every $u \in U$, where $c > 0$ is the universal constant from Claim 4.2. This implies the corollary, since $\mathbb{P}[\varphi \circ S_w[0, T^{(w)}(20r)] \circlearrowleft^{(20r)} \tilde{a}]$ is a convex sum of $\mathbb{P}[\varphi \circ S_u[0, T^{(u)}(20r)] \circlearrowleft^{(20r)} \tilde{a}]$ for $u \in U$ (because G is irreducible).

Indeed, let U be the set of all vertices in $V_\delta(D) \cap \varphi^{-1}(\rho(\tilde{a}, r))$ such that (4.2) holds. Assume toward a contradiction that there is a path Y in G_δ starting from

w and reaching the outside of $\varphi^{-1}(\rho(\tilde{a}, r))$, such that $Y \cap U = \varnothing$. Then, every path in G_δ whose image under φ r-encompasses $\tilde{a}$, must intersect Y. Let x be the vertex guarantied by Claim 4.2. Then,

$$\begin{aligned}\mathbb{P}\bigl[\varphi \circ S_x\bigl[0, T^{(x)}(r)\bigr] \circlearrowleft^{(r)} \tilde{a}, \varphi \circ S_x\bigl[T^{(x)}(r), T^{(x)}(20r)\bigr] \circlearrowleft^{(20r)} \tilde{a}\bigr] \\ \leq \sum_{y\in Y} p(y) \cdot \mathbb{P}\bigl[\varphi \circ S_y\bigl[0, T^{(y)}(20r)\bigr] \circlearrowleft^{(20r)} \tilde{a}\bigr] < c,\end{aligned}$$

which is a contradiction to Claim 4.2 [where $\{p(y)\}_{y\in Y}$ is a distribution on the set Y]. $\square$

We continue with the proof of Proposition 4.1. Let $c > 0$ be the constant from Corollary 4.3. Let $M \in \mathbb{N}$ be large enough so that $(1 - c)^M < \alpha$. Let $\eta > 0$ be small enough so that $500^{M+1}\eta < 1/40$. For $j = 1, 2, \dots, M$, define $r_j = 500^j \eta\varepsilon$, and define $F_j = \{\varphi \circ S_y[T^{(y)}(r_j), T^{(y)}(400r_j)] \circlearrowleft^{(400r_j)} \tilde{a}\}$. By the strong Markov property and by Corollary 4.3, since $\varphi(y) \in \rho(\tilde{a}, \eta\varepsilon)$, we have $\mathbb{P}[F_j \mid \overline{F}_1, \dots, \overline{F}_{j-1}] \geq c$ for every j, which implies

$$\mathbb{P}[\overline{F}_1, \dots, \overline{F}_M] \leq (1 - c)^M < \alpha$$

(here and below $\overline{E}$ is the complement of the event E). Since G is planar-irreducible, the proposition follows. $\square$

4.2. *Starting near the boundary.* In this section we prove the version of Lemma 5.4 in [9] that is relevant to us. Part of the proof is similar to that of [9], but the setting here is more general and requires more details.

LEMMA 4.4. *For any $\varepsilon, \alpha > 0$, there exist $\eta, \delta_0 > 0$ such that for every $0 < \delta < \delta_0$ the following holds*:

Let $D \in \mathfrak{D}$, and let $x \in V_\delta(D)$ be such that $|\varphi_D(x)| \geq 1 - \eta$. Then, the probability that S_x hits the set $\{y \in D : |\varphi_D(y) - \varphi_D(x)| > \varepsilon\}$ before exiting D is at most α.

We first prove the following proposition.

PROPOSITION 4.5. *There exists $0 < \alpha < 1$ such that for any $\varepsilon > 0$, there exist $\eta, \delta_0 > 0$ such that for every $0 < \delta < \delta_0$ the following holds*:

Let $D \in \mathfrak{D}$, and let $x \in V_\delta(D)$ be such that $1 - 2\eta \leq |\varphi_D(x)| \leq 1 - \eta$. Then, the probability that S_x hits the set $\{y \in D : |\varphi_D(y) - \varphi_D(x)| > \varepsilon\}$ before exiting D is at most α.

PROOF. Let $\eta > 0$ be small enough. By (2.1), by the Koebe distortion theorem, and by the Koebe 1/4 theorem, $\mathrm{dist}(x, \partial D) \geq r_0$, where $r_0 = c \cdot \eta^2$ for some constant $c = c(\mathbf{D}) > 0$. Let $z \in \partial D$ be a point such that $r = |z - x| = \mathrm{dist}(x, \partial D)$.

Let $x' \in \mathbf{D}$ be such that $|x' - x| < r_0/C$, and let $z' \in \mathbf{D}$ be such that $|z' - z| < r_0/C$, for a large enough constant $C > 0$. We need to consider only finitely many points x' and z'.

Let $r' = |x' - z'|$, and let $R > 0$ be large enough so that $\mathbf{D} \cup \rho(x', 10r') \subset \frac{R}{2}\mathbb{U}$. Denote $A_1 = \{\xi \in \mathbb{C} : |\xi - z'| \le r'/10\} \cup [x', z'] \setminus \{x'\}$. Let Q be the connected component in $\mathbb{C}$ of $(\partial\rho(z', r')) \cap D$ that contains x'. Let A_2 and A_3 be the two connected components in $\mathbb{C}$ of $Q \setminus \{x'\}$. For large enough C, the distance from x' to ∂D is at least $3r'/4$. Thus, both A_2 and A_3 are arcs of length at least $3r'/4$. If C is large enough, $D \setminus (A_1 \cup A_2 \cup A_3)$ has three connected components in $\mathbb{C}$. For $j = 1, 2, 3$, let K_j be the connected component in $\mathbb{C}$ of $D \setminus (A_1 \cup A_2 \cup A_3)$ such that $A_j \cap \partial K_j = \varnothing$. Let $\mathcal{E}_j$ be the collection of curves $\gamma \subset R\mathbb{U}$ such that γ stays in K_j from the first time it first hits $\partial\rho(x', r'/2)$ until it exits D. By the conformal invariance of Brownian motion, there exists a universal constant $c_1 > 0$ such that for every $j = 1, 2, 3$, we have $\mathbb{P}_{x'}[B(\cdot) \in \mathcal{E}_j] > c_1$, where $B(\cdot)$ is a planar Brownian motion started at x'.

Let $A = \{y \in D : |\varphi_D(y) - \varphi_D(x)| > \varepsilon\}$. We show that there exists $j' \in \{1, 2, 3\}$ such that $A \cap K_{j'} = \varnothing$. Assume toward a contradiction that $A \cap K_j \neq \varnothing$ for all j. We prove for the case that A intersects both A_1 and A_2 (the proof for the other cases is similar). A is a connected set that intersects both A_1 and A_2, so we can choose A' to be a minimal connected subset of A that intersects both A_1 and A_2 (minimal with respect to inclusion). Thus, either A' is in the closure of K_3 or A' is in the closure of $K_1 \cup K_2$. We prove for the case that A' is in the closure of K_3 (the proof for the other case is similar).

We show that $A \cap \rho(x', r'/2) = \varnothing$. By choosing $\eta > 0$ to be small enough, and by the conformal invariance of Brownian motion, the probability that a Brownian motion started at x hits A before exiting D can be made arbitrarily small. If $A \cap \rho(x, 3r/5) \neq \varnothing$, because $\mathrm{dist}(x, \partial D) = r$ and because A is connected, the probability that a Brownian motion started at x hits A before exiting D is at least a universal constant $c_2 > 0$. This is a contradiction for a small enough η, which implies $A \cap \rho(x, 3r/5) = \varnothing$. Since $r' \le r(1 + 2/C)$ and since $|x - x'| \le r_0/C$, for large enough C we have that $\rho(x', r'/2) \subset \rho(x, 3r/5)$.

For a vertex $y \in V_\delta(\rho(x', r'/2))$, define $h(y)$ as the probability that $S_y[0, \tau_D^{(y)}]$ is in $\mathcal{E}_{j'}$. The map $h(\cdot)$ is harmonic in $V_\delta(\rho(x', r'/2))$ with respect to the law of the natural random walk on G_δ.

CLAIM 4.6. *There exist a universal constant $c_3 > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $y \in V_\delta(\rho(x', r_0/C))$ with $h(y) \ge c_3$.*

PROOF. We prove for the case $j' = 3$. The proof of the other cases is similar. The event $\mathcal{E}_3$ contains an event $\mathcal{E}$ that is independent of D; for example, there exist $x' = z_1, z_2, \ldots, z_m \in \mathbb{C}$ for $m \le 10^3$ such that $|z_{i+1} - z_i| = r'/2$, and

$$\mathcal{E} = \{\gamma \subset R\mathbb{U} : \gamma \text{ crosses } \square(z_i, z_{i+1}, r'/100) \text{ for all } i\} \subset \mathcal{E}_3.$$

Let $B(\cdot)$ be a Brownian motion, and let τ be the exit time of $B(\cdot)$ from $R\mathbb{U}$. Since $\square(z_1, z_2, r'/100), \dots, \square(z_{m-1}, z_m, r'/100)$ are $m-1$ rectangles of fixed proportions, we have $\inf_{w \in \rho(x', r'/100)} \mathbb{P}_w[B[0,\tau] \in \mathcal{E}] > c_4$ for some universal constant $c_4 > 0$. Let T be the time $B(\cdot)$ hits $\rho = \rho(x', r_0/C)$. On one hand,

$$\mathbb{P}_0\big[B[T,\tau] \in \mathcal{E}\big] \ge \mathbb{P}_0[T < \tau] \cdot c_4.$$

On the other hand, using weak convergence and Proposition 3.1, if δ_0 is small enough,

$$\begin{aligned}\mathbb{P}_0\big[B[T,\tau] \in \mathcal{E}\big] &\le 2\mathbb{P}\big[S_0\big[\Theta_0(\rho), \tau^{(0)}_{R\mathbb{U}}\big] \in \mathcal{E}\big] \\ &\le 4\mathbb{P}_0[T < \tau] \cdot \max_{y \in V_\delta(\rho)} \mathbb{P}\big[S_y\big[0, \tau^{(y)}_{R\mathbb{U}}\big] \in \mathcal{E}\big]. \qquad \square\end{aligned}$$

Let $c_3 > 0$ and let $y \in V_\delta(\rho(x', r_0/C))$ be given by Claim 4.6. Since $h(\cdot)$ is harmonic, there exists a path γ from y to $\partial\rho(x', r'/2)$ such that $h(w) \ge h(y)$ for every $w \in \gamma$. Since $h(\cdot)$ is nonnegative, harmonic and bounded,

$$h(x) \ge \mathbb{P}\big[S_x\big[0, \tau^{(x)}_{\rho(x', r'/2)}\big] \cap \gamma \ne \varnothing\big] \cdot h(y).$$

By Proposition 4.1, and by choosing large enough C, we have $\mathbb{P}[S_x[0, \tau^{(x)}_{\rho(x', r'/2)}] \cap \gamma \ne \varnothing] \ge 1/2$. Since every curve in $\mathcal{E}_{j'}$ does not intersect A, the probability that S_x hits the set A before exiting D is at most $1 - c_3/2 < 1$. $\square$

Planarity and Proposition 4.5 imply a stronger statement.

COROLLARY 4.7. *There exists* $0 < \alpha < 1$ *such that for any* $\varepsilon > 0$, *there exist* $\eta, \delta_0 > 0$ *such that for every* $0 < \delta < \delta_0$ *the following holds*:

Let $D \in \mathfrak{D}$, *and let* $x \in V_\delta(D)$ *be such that* $|\varphi_D(x)| \ge 1 - \eta$. *Then*, *the probability that* S_x *hits the set* $\{y \in D : |\varphi_D(y) - \varphi_D(x)| > \varepsilon\}$ *before exiting* D *is at most* α.

PROOF. Let α, η, δ_0 be given by Proposition 4.5 with $\varepsilon/10$, and let $0 < \delta < \delta_0$. For $y \in V_\delta(D)$, define $f(y)$ as the probability that S_y hits $A = \{y \in D : |\varphi_D(y) - \varphi_D(x)| > \varepsilon\}$ before exiting D. Assume toward a contradiction that $f(x) > \alpha$. The map $f(\cdot)$ is harmonic in $V_\delta(D \setminus A)$ with respect to the law of the natural random walk on G_δ. Let A' be the set of $\xi \in D$ such that $1 - 2\eta \le |\varphi_D(\xi)| \le 1 - \eta$ and $|\varphi_D(\xi) - \varphi_D(x)| \le \varepsilon/2$. By Proposition 4.5, $f(y) \le \alpha$ for all $y \in V_\delta(A')$. Thus, there exists a path γ from x to the set A in $V_\delta(D)$ that does not intersect A' such that $f(y) > \alpha$ for every $y \in \gamma$.

There exists $z' \in A'$ such that $\rho(\varphi_D(z'), \eta/10) \subset \varphi_D(A')$ and for every $\xi \in \rho(\varphi_D(z'), \eta/10)$, every path from $\varphi_D^{-1}(\xi)$ to ∂D that does not hit $\{\zeta \in D : |\varphi_D(\zeta) - \xi| > \varepsilon/10\}$ crosses γ (as a continuous curve). By the Koebe 1/4 theorem and by the Koebe distortion theorem, there exist a finite set $Z \subset \mathbb{C}$ and $\eta' > 0$, depending only on η, such that for all $\rho = \rho(\xi, \eta/10) \subset (1-\eta)\mathbb{U}$ and any $D \in \mathfrak{D}$, there exists $z \in Z$ with $\rho(z, \eta') \subset \varphi_D^{-1}(\rho)$. Thus, by weak convergence and Proposition 3.1, for small enough δ_0 (depending only on η), there exists $z \in V_\delta(D)$ such that $\varphi_D(z) \in \rho(\varphi_D(z'), \eta/10)$. The probability that S_z hits $\{\zeta \in D : |\varphi_D(\zeta) - \varphi_D(z)| > \varepsilon/10\}$ before exiting D is at least $\min_{y \in \gamma} f(y) > \alpha$. This is a contradiction to Proposition 4.5. $\square$

PROOF OF LEMMA 4.4. Let $\eta, \eta' > 0$ be small enough. We show that if δ_0 is small enough, for every $D \in \mathfrak{D}$, and for every $x \sim y \in V_\delta(D)$, we have $|\varphi_D(x) - \varphi_D(y)| < \eta'$.

By the Koebe distortion theorem, using (2.1), for every $z \in (1-\eta)\mathbb{U}$, we have $|\varphi_D^{-1\prime}(z)| \geq c\eta$ for a constant $c > 0$. By weak convergence, since G is planar-irreducible, when δ_0 tends to 0, the length of the edges of G_δ in $R\mathbb{U}$, for $R = \sup\{|z| : z \in \mathbf{D}\}$, tends to 0. This implies that if δ_0 is small enough, for every $D \in \mathfrak{D}$ and $y \sim x \in V_\delta(D)$ such that $|\varphi_D(y)|, |\varphi_D(x)| \leq 1 - \eta$, we have $|\varphi_D(y) - \varphi_D(x)| \leq \eta'$.

It remains to consider x's such that $|\varphi_D(x)| \geq 1 - \eta$. As above, for small enough δ_0, every $z \in [x, y]$ admits $|\varphi_D(z)| \geq 1 - 2\eta$. Assume toward a contradiction that $|\varphi_D(x) - \varphi_D(y)| \geq \eta'$. Thus, by Proposition 4.5 (using a similar argument to the one in Corollary 4.7), there exists $\xi \in V_\delta(D)$ such that $1 - 4\eta \leq |\varphi_D(\xi)| \leq 1 - 2\eta$ and the probability that S_ξ hits the set $\{\zeta \in D : |\varphi_D(\zeta) - \varphi_D(\xi)| > \eta'/3\}$ before exiting D is smaller than 1. However, since G is planar-irreducible, S_ξ cannot cross $[x, y]$, so the probability that S_ξ hits the set $\{\zeta \in D : |\varphi_D(\zeta) - \varphi_D(\xi)| > \eta'/3\}$ before exiting D is 1, which is a contradiction.

The proof of the lemma follows by the strong Markov property, and by applying Corollary 4.7 a finite number of times. $\square$

4.3. *Exit probabilities are correct.* Let $D \in \mathfrak{D}$. For $J \subset \partial D$, denote by $H(a, J; D)$ the probability that the natural random walk started at a exits D at J; that is, $H(a, J; D) = \sum_b H(a, b; D)$, where the sum is over all $b \in \partial V_\delta(D)$ such that $b \cap J \neq \emptyset$.

LEMMA 4.8. *For all $\varepsilon, \alpha > 0$, for all $D \in \mathfrak{D}$, and for all $J = \varphi_D^{-1}(I)$ where $I \subset \partial\mathbb{U}$ is an arc, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:

Let $a \in V_\delta(D)$ be such that $|\varphi_D(a)| \leq 1 - \varepsilon$. Then,

$$|H(a, J; D) - \mathbb{P}_a[B(\tau) \in J]| < \alpha,$$

where $B(\cdot)$ is a planar Brownian motion, and τ is the exit time of $B(\cdot)$ from D.

PROOF. Fix ε, α, D and J as above. Denote $\varphi = \varphi_D$ and denote $\tau^{(a)} = \tau_D^{(a)}$. Let $0 < \alpha_0 < 1$ be such that $\frac{(1+\alpha_0)^2}{(1-\alpha_0)^2} = 1 + \frac{\alpha}{2}$. Let $\eta > 0$ be small enough. Denote

$\mathcal{A} = \{\frac{\eta}{4}(n + m \cdot i) \in (1-\varepsilon)\mathbb{U} : n, m \in \mathbb{Z}\}$. The set $\mathcal{A}$ is finite, and there exists $\tilde{a} \in \mathcal{A}$ such that $\varphi(a) \in \rho(\tilde{a}, \eta)$. Denote $\rho = \varphi^{-1}(\rho(\tilde{a}, \eta))$.

We show that if η, δ_0 is small enough, then $\mathbb{P}[S_x(\tau^{(x)}) \in J] > (1 - \alpha_0/2) \cdot \mathbb{P}[S_y(\tau^{(y)}) \in J]$ for every $x, y \in V_\delta(\rho)$. Define $h(z)$ to be the probability that $S_z(\tau^{(z)}) \in J$. The map $h(\cdot)$ is harmonic in $V_\delta(D)$ with respect to the law of the natural random walk on G_δ. Since $h(\cdot)$ is harmonic, there exists a path γ from y to ∂D such that $h(z) \geq h(y)$ for every $z \in \gamma$. Since $h(\cdot)$ is nonnegative, harmonic and bounded,

$$h(x) \geq \mathbb{P}\big[S_x\big[0, \tau^{(x)}\big] \cap \gamma \neq \varnothing\big] \cdot h(y).$$

By Proposition 4.1, since G is planar, $\mathbb{P}[S_x[0, \tau^{(x)}] \cap \gamma \neq \varnothing] > 1 - \alpha_0/2$ for small enough η, δ_0.

Therefore, for small enough η, δ_0,

$$\left| \frac{\mathbb{P}[S_z(\tau^{(z)}) \in J]}{\mathbb{P}[S_a(\tau^{(a)}) \in J]} - 1 \right| < \alpha_0 \tag{4.3}$$

for every $z \in V_\delta(\rho)$. In addition,

$$\left| \frac{\mathbb{P}_z[B(\tau) \in J]}{\mathbb{P}_a[B(\tau) \in J]} - 1 \right| < \alpha_0 \tag{4.4}$$

for every $z \in \rho$. By weak convergence and Proposition 3.1, by the conformal invariance of Brownian motion, we can choose δ_0 so that

$$\left| \frac{\mathbb{P}[\Theta_0(\rho) < \tau^{(0)}, S_0(\tau^{(0)}) \in J]}{\mathbb{P}_0[B[0, \tau] \cap \rho \neq \varnothing, B(\tau) \in J]} - 1 \right| < \alpha_0 \tag{4.5}$$

and

$$\left| \frac{\mathbb{P}[\Theta_0(\rho) < \tau^{(0)}]}{\mathbb{P}_0[B[0, \tau] \cap \rho \neq \varnothing]} - 1 \right| < \alpha_0. \tag{4.6}$$

Combining (4.5) and (4.4),

$$\begin{aligned}
&\mathbb{P}\big[\Theta_0(\rho) < \tau^{(0)}, S_0\big(\tau^{(0)}\big) \in J\big] \\
&\quad < (1 + \alpha_0)\mathbb{P}_0\big[B[0, \tau] \cap \rho \neq \varnothing, B(\tau) \in J\big] \\
&\quad < (1 + \alpha_0)^2 \mathbb{P}_0\big[B[0, \tau] \cap \rho \neq \varnothing\big]\mathbb{P}_a[B(\tau) \in J]
\end{aligned}$$

and combining (4.3) and (4.6),

$$\begin{aligned}
&\mathbb{P}\big[\Theta_0(\rho) < \tau^{(0)}, S_0\big(\tau^{(0)}\big) \in J\big] \\
&\quad > (1 - \alpha_0)\mathbb{P}\big[\Theta_0(\rho) < \tau^{(0)}\big]\mathbb{P}\big[S_a\big(\tau^{(a)}\big) \in J\big] \\
&\quad > (1 - \alpha_0)^2 \mathbb{P}_0\big[B[0, \tau] \cap \rho \neq \varnothing\big]\mathbb{P}\big[S_a\big(\tau^{(a)}\big) \in J\big].
\end{aligned}$$

Thus, by the choice of α_0,

$$\mathbb{P}\big[S_a\big(\tau^{(a)}\big) \in J\big] < (1 + \alpha)\mathbb{P}_a[B(\tau) \in J].$$

Similarly, since $1 - \alpha < \frac{1}{1+\alpha/2}$,

$$\mathbb{P}\big[S_a\big(\tau^{(a)}\big) \in J\big] > (1-\alpha)\mathbb{P}_a[B(\tau) \in J].$$

The lemma follows, since $\mathbb{P}_a[B(\tau) \in J] \le 1$. □

Using Lemma 4.4, Lemma 4.8 yields the following.

LEMMA 4.9. *There exists a universal constant $c > 0$ such that for all $\alpha > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:

Let $D \in \mathfrak{D}$, and let $J = \varphi_D^{-1}(I)$, where $I \subset \partial\mathbb{U}$ is an arc of length at least α. Then, $H(0, J; D) \ge c \cdot \alpha$.

PROOF. Let $\eta > 0$ be small enough, and let $\tilde{D}$ be the $(1, \eta)$-approximation of D given by Proposition 2.1. Let $x \in \partial\mathbb{U}$ be the center of I, and let $A = \rho(x, \alpha/2) \cap \mathbb{U}$. Let $\mathcal{I}$ be the finite family of arcs of the form $I = \{e^{is} : \alpha j/8 \le s \le \alpha(j+1)/8\}$ for $0 \le j \le 16\pi/\alpha$.

Let $I' \in \mathcal{I}$ be so that $x \in I'$. For every $\zeta \in I'$, since $|x - \zeta| \le \alpha/8$ and since $|\varphi_D(\varphi_{\tilde{D}}^{-1}(\zeta)) - \zeta| \le \eta$, we have $|x - \varphi_D(\varphi_{\tilde{D}}^{-1}(\zeta))| \le \eta + \alpha/8 < \alpha/4$ for $\eta < \alpha/8$. Thus, $\mathrm{dist}(x, \varphi_D(\varphi_{\tilde{D}}^{-1}(I')) < \alpha/4$. As in the proof of Lemma 4.4, if δ_0 is small enough (independently of D), for every $v \sim u \in V_\delta(D)$, we have $|\varphi_D(v) - \varphi_D(u)| < \eta$. Thus, by properties (1) and (3) of Proposition 2.1,

$$H(0, J; D) \ge \mathbb{P}\big[S_0\big(\tau_{\tilde{D}}^{(0)}\big) \in \varphi_{\tilde{D}}^{-1}(I')\big] \cdot \min_y \mathbb{P}\big[S_y\big(\tau_D^{(y)}\big) \in J\big],$$

where the minimum is over $y \in V_\delta(\rho(x, \alpha/2))$ such that $|\varphi_D(y)| \ge 1 - 2\eta$. By weak convergence and Proposition 3.1, if δ_0 is small enough, we have that $\mathbb{P}[S_0(\tau_{\tilde{D}}^{(0)}) \in \varphi_{\tilde{D}}^{-1}(I')]$ is at least a universal constant times α. By Lemma 4.4, for small enough η, δ_0, we have $\min_y \mathbb{P}[S_y(\tau_D^{(y)}) \in J] \ge 1/2$. □

5. Convergence of Poisson kernel.

In this section we prove that one can approximate the discrete Poisson kernel by the continuous Poisson kernel.

5.1. *Proof of Lemma 1.2.*

We begin with a proposition that is a "special case" of Lemma 1.2 for a specific domain.

PROPOSITION 5.1. *Let $\varepsilon, \alpha > 0$ and let $D \subsetneq \mathbb{C}$ be a simply connected domain such that $0 \in D$. Then, there exists δ_0 such that for all $0 < \delta < \delta_0$ the following holds*:

Let $a \in V_\delta(D)$ be such that $|\varphi_D(a)| \le 1 - \varepsilon$, and let $b \in \partial V_\delta(D)$. Then,

$$\left| \frac{H^{(\delta)}(a, b; D)}{H^{(\delta)}(0, b; D)} - \lambda(a, b; D) \right| \le \alpha.$$

Roughly, Proposition 5.1 yields Lemma 1.2 by a compactness argument.

PROOF OF LEMMA 1.2. Let $\alpha_1 > 0$ be small enough, and let $\tilde{D}$ be the (ε, α_1)-approximation of D given by Proposition 2.1. Let $\delta_0 > 0$ be small enough, and let $0 < \delta < \delta_0$. Specifically, Proposition 5.1 holds for $\tilde{D}$ with $\varepsilon/2$ and α_1. Since $|\varphi_{\tilde{D}}(a)| \le 1 - \varepsilon/2$, for every $\tilde{b} \in \partial V_\delta(\tilde{D})$,

$$\left| \frac{H(a, \tilde{b}; \tilde{D})}{H(0, \tilde{b}; \tilde{D})} - \lambda(a, \tilde{b}; \tilde{D}) \right| \le \alpha_1.$$

Since $\tilde{D} \subset D$, for every $x \in V_\delta(\tilde{D})$,

$$H(x, b; D) = \sum_{\tilde{b}} H(x, \tilde{b}; \tilde{D}) \cdot H(\tilde{b}, b; D),$$

where the sum is over $\tilde{b} \in \partial V_\delta(\tilde{D})$, and we abuse notation and use $H(\tilde{b}, b; D)$ instead of $H(\tilde{b}_2, b; D)$, where $\tilde{b} = (\tilde{b}_1, \tilde{b}_2)$ [for every $b' \in \partial V_\delta(D)$, define $H(b', b; D) = \mathbf{1}_{\{b=b'\}}$]. Thus,

$$\begin{aligned}
&|H(a, b; D) - \lambda(a, b; D) \cdot H(0, b; D)| \\
&\quad \le \sum_{\tilde{b}} H(\tilde{b}, b; D) \cdot |H(a, \tilde{b}; \tilde{D}) - \lambda(a, b; D) \cdot H(0, \tilde{b}; \tilde{D})| \\
&\quad \le \sum_{\tilde{b}} H(\tilde{b}, b; D) \cdot H(0, \tilde{b}; \tilde{D}) \cdot |\lambda(a, \tilde{b}; \tilde{D}) - \lambda(a, b; D)| \\
&\qquad + \alpha_1 \cdot H(0, b; D).
\end{aligned} \tag{5.1}$$

Let $\alpha_2, \alpha_3 > 0$ be small enough. Let $I \subset \partial\mathbb{U}$ be an arc of length α_2 centered at $\varphi_D(b)$. Denote $\tilde{I} = \varphi_{\tilde{D}}^{-1}(I) \subset \partial\tilde{D}$. We use the following two claims.

CLAIM 5.2. *For every $\tilde{b} \in \partial V_\delta(\tilde{D})$ such that $\tilde{b} \cap \tilde{I} \neq \varnothing$, $|\lambda(a, \tilde{b}; \tilde{D}) - \lambda(a, b; D)| \le \alpha_3$.*

PROOF. By the choice of I, $|\varphi_{\tilde{D}}(\tilde{b}) - \varphi_D(b)| \le \alpha_2$. Since $a \in \tilde{D}$, by property (4) of Proposition 2.1 with $\xi = \varphi_{\tilde{D}}(a)$, we have $|\varphi_D(a) - \varphi_{\tilde{D}}(a)| = |\varphi_D(\varphi_{\tilde{D}}^{-1}(\xi)) - \xi| \le \alpha_1$. By the continuity of $\lambda(\cdot, \cdot; \mathbb{U})$, if α_1, α_2 are small enough, $|\lambda(a, \tilde{b}; \tilde{D}) - \lambda(a, b; D)| \le \alpha_3$. □

CLAIM 5.3. *For every $\tilde{b} \in \partial V_\delta(\tilde{D})$ such that $\tilde{b} \cap \tilde{I} = \varnothing$, $H(\tilde{b}, b; D) \le \alpha_3 \cdot H(0, b; D)$.*

PROOF. Assume that $\tilde{b} \notin \partial V_\delta(D)$ [otherwise, $H(\tilde{b}, b; D) = 0$, since $\tilde{b} \cap I = \varnothing$]. In this case, $H(\tilde{b}, b; D)$ is $H(\tilde{b}_2, b; D)$ where $\tilde{b}_2$ is the endpoint of $\tilde{b}$. Denote $b' = \varphi_{\tilde{D}}(\tilde{b}) \in \partial\mathbb{U}$. So $|b' - \varphi_D(b)| \ge \alpha_2/10$. By property (4) of Proposition 2.1, $|\varphi_D(\varphi_{\tilde{D}}^{-1}(b')) - b'| \le \alpha_1$. By weak convergence for small enough δ_0, the length of the edge $\tilde{b}$ is small enough. Thus, by property (5) of Proposition 2.1, for small enough δ_0, we have $|\varphi_D(\tilde{b}_2) - \varphi_D(\varphi_{\tilde{D}}^{-1}(b'))| \le \alpha_1$, which implies $|\varphi_D(\tilde{b}_2) - b'| \le 2\alpha_1$. Therefore, $|\varphi_D(\tilde{b}_2) - \varphi_D(b)| \ge \alpha_2/10 - 2\alpha_1 > \alpha_2/20$, for $\alpha_1 < \alpha_2/40$.

Denote $\xi = \varphi_D(\tilde{b}_2)$, and $A = \{x \in \mathbb{U} : |x - \xi| > \alpha_2/50\}$. Also denote $M = \max_y H(y, b; D)$, where the maximum is over $y \in V_\delta(D)$ such that $|\varphi_D(y) - \varphi_D(b)| \ge \alpha_2/50$. As in the proof of Lemma 4.4, if δ_0 is small enough, for every $v \sim u \in V_\delta(D)$, we have $|\varphi_D(v) - \varphi_D(u)| < \alpha_2/100$. Thus, $H(\tilde{b}, b; D)$ is at most M times the probability that $\varphi_D \circ S_{\tilde{b}_2}$ hits A.

Since $|\xi - \varphi_D(\varphi_{\tilde{D}}^{-1}(b'))| \le \alpha_1$, using property (3) of Proposition 2.1, $|\xi| \ge 1 - 2\alpha_1$. Let $\alpha_4 > 0$ be small enough. Using Lemma 4.4, for α_1 small enough, the probability that $\varphi_D \circ S_{\tilde{b}_2}$ hits A is at most α_4.

We show that $M \le C \cdot H(0, b; D)$, for some $C = C(\alpha_2) > 0$. Let $y \in V_\delta(D)$ be such that $|\varphi_D(y) - \varphi_D(b)| \ge \alpha_2/50$. The map $H(\cdot, b; D)$ is harmonic with respect to the law of the natural random walk on G_δ. Thus, there exists a path γ from y to b in $V_\delta(D)$ such that for every $z \in \gamma$, $H(z, b; D) \ge H(y, b; D)$. Since $H(\cdot, b; D)$ is nonnegative, harmonic and bounded,

$$H(0, b; D) \ge \mathbb{P}\big[S_0[0, \tau_D^{(0)}] \cap \gamma \ne \varnothing\big] \cdot H(y, b; D).$$

Therefore, we need to show that $p = \mathbb{P}[S_0[0, \tau_D^{(0)}] \cap \gamma \ne \varnothing]$ can be bounded from below by a function of α_2.

Think of γ as a continuous curve, and denote $\gamma' = \{\zeta \in \gamma : |\varphi_D(\zeta) - \varphi_D(b)| \le \alpha_2/50\}$. Denote $D' = D \setminus \gamma'$. By the conformal invariance of the harmonic measure, the length of the arc $\varphi_{D'}(\gamma')$ is at least a universal constant times α_2. Also, for small enough α_2, we have $\mathrm{rad}(D') \ge 1/4$. Thus, by Lemma 4.9 applied to D' (using Lemma 4.9 with $\mathfrak{D}' = \{2D : D \in \mathfrak{D}\}$), p is at least a universal constant times α_2. Set $C(\alpha_2) = \frac{1}{p}$.

Setting $\alpha_4 \cdot C(\alpha_2) \le \alpha_3$, the proof is complete. □

By Claims 5.2 and 5.3,

$$\begin{aligned}
(5.1) \le & \sum_{\tilde{b} : \tilde{b} \cap \tilde{I} \ne \varnothing} H(\tilde{b}, b; D) \cdot H(0, \tilde{b}; \tilde{D}) \cdot |\lambda(a, \tilde{b}; \tilde{D}) - \lambda(a, b; D)| \\
& + \sum_{\tilde{b} : \tilde{b} \cap \tilde{I} = \varnothing} H(\tilde{b}, b; D) \cdot H(0, \tilde{b}; \tilde{D}) \cdot |\lambda(a, \tilde{b}; \tilde{D}) - \lambda(a, b; D)| \\
& + \alpha_1 \cdot H(0, b; D) \\
\le & \alpha_3 \cdot \sum_{\tilde{b}} H(\tilde{b}, b; D) \cdot H(0, \tilde{b}; \tilde{D}) + \alpha_3 \cdot H(0, b; D) \cdot \sum_{\tilde{b}} H(0, \tilde{b}; \tilde{D}) \\
& + \alpha_1 \cdot H(0, b; D) \\
\le & (2\alpha_3 + \alpha_1) \cdot H(0, b; D).
\end{aligned}$$

Choosing $2\alpha_3 + \alpha_1 < \alpha$ completes the proof. □

5.2. *Proof of Proposition* 5.1. Fix $\varepsilon, \alpha > 0$. Let N be a large enough integer so that

$$(5.2)\qquad \big(1 - c_1(\varepsilon, \alpha)\big)^N < \frac{\alpha}{8c_2},$$

where $c_1(\varepsilon, \alpha) > 0$ is given below in Proposition 5.4, and $c_2 > 0$ is the universal constant given below in Proposition 5.5. Let $\beta > 0$ be small enough so that

$$(5.3)\qquad \beta < \frac{\varepsilon}{50\pi K 5^N} \quad \text{and} \quad \beta < \frac{\alpha\varepsilon^5}{16c_3},$$

where $K > 5$ is the universal constant from Lemmas 3.9 and 3.10, and $c_3 > 0$ is the universal constant given below in (5.5), and let $r = 2\pi\beta$. Let $\eta > 0$ be given by Proposition 4.1 with α equals $\frac{\alpha\varepsilon^2}{16}$. Let $\delta_0 > 0$ be small enough (to be determined below), and let $0 < \delta < \delta_0$.

Denote $\varphi = \varphi_D$. Denote $\mathcal{A} = \{\frac{\varepsilon}{100}(n + m \cdot i) \in (1 - \varepsilon)\mathbb{U} : n, m \in \mathbb{Z}\}$. The set $\mathcal{A}$ is finite, and there exists $\tilde{a} \in \mathcal{A}$ such that $\varphi(a) \in \rho(\tilde{a}, \varepsilon/40)$. Denote $\mathcal{B} = \{e^{\pi\beta n i/20} : 0 \le n \le 100/\beta\}$. The set $\mathcal{B}$ is finite, and there exists $\tilde{b} \in \mathcal{B}$ such that $|\varphi(b) - \tilde{b}| \le \beta/10$. Denote $I = \{\tilde{b} \cdot e^{it} : -\pi\beta \le t \le \pi\beta\}$, and denote $J = \varphi^{-1}(I)$. Roughly, b is an edge in the middle of the small interval J.

For $j = 1, 2, \ldots, N$, let $R_j = 5^j K r$, let $\xi_j = \tilde{b}(1 - 3R_j)$, and let $\rho_j = \rho(\xi_j, \eta^3 R_j)$. For $z \in V_\delta(D)$, define

$$T_j^{(z)} = \min\{t \ge 0 : |\varphi(S_z(t)) - \tilde{b}| \le R_j\}.$$

On the event $\{S_z(\tau^{(z)}) \in J\}$, we have $T_N^{(z)} \le T_{N-1}^{(z)} \le \cdots \le T_1^{(z)} \le \tau^{(z)}$. Let $E_j^{(z)}$ be the event

$$E_j^{(z)} = \{\varphi \circ S_0[T_{j+1}^{(0)}, T_j^{(0)}] \cap \rho_j \ne \varnothing\} \cap \{\varphi \circ S_z[T_{j+1}^{(z)}, T_j^{(z)}] \cap \rho_j \ne \varnothing\}.$$

Denote $E_j = E_j^{(a)}$.

We use the following three propositions.

PROPOSITION 5.4. *Let* $1 \le j \le N$. *Then,*

$$\mathbb{P}\big[E_j \mid \overline{E}_{j+1}, \ldots, \overline{E}_N, S_0\big(\tau^{(0)}\big) \in J, S_a\big(\tau^{(a)}\big) \in J\big] \ge c_1$$

for $c_1 = c_1(\varepsilon, \alpha) > 0$.

PROPOSITION 5.5. *There exists a universal constant* $c_2 > 0$ *such that for every* $z \in \{0, a\}$,

$$\begin{aligned}\mathbb{P}\big[S_z\big(\tau^{(z)}\big) = b \mid \overline{E}_1, \ldots, \overline{E}_N, S_0\big(\tau^{(0)}\big) \in J, S_a\big(\tau^{(a)}\big) \in J\big]&\\ \le c_2 \cdot \mathbb{P}\big[S_0\big(\tau^{(0)}\big) = b \mid S_0\big(\tau^{(0)}\big) \in J\big].&\end{aligned}$$

PROPOSITION 5.6. *For every* $j = 1, \dots, N$,

$$\left| \frac{\mathbb{P}[S_a(\tau^{(a)}) = b \mid \text{EXIT}, E_j, \overline{E}_{j+1}, \dots, \overline{E}_N]}{\mathbb{P}[S_0(\tau^{(0)}) = b \mid \text{EXIT}, E_j, \overline{E}_{j+1}, \dots, \overline{E}_N]} - 1 \right| \le \frac{\alpha\varepsilon^2}{4}.$$

Before proving the three propositions above, we show how they imply Proposition 5.1. Let $z \in \{0, a\}$. Write

$$\begin{aligned} H(z, b) &= H^{(\delta)}(z, b; D) \\ &= \mathbb{P}\big[S_z\big(\tau^{(z)}\big) = b \mid S_z\big(\tau^{(z)}\big) \in J\big] \cdot \mathbb{P}\big[S_z\big(\tau^{(z)}\big) \in J\big]. \end{aligned} \tag{5.4}$$

By Lemma 4.8, by (1.1), and since $|\varphi(z)| \le 1 - \varepsilon$,

$$\big|\mathbb{P}\big[S_z\big(\tau^{(z)}\big) \in J\big] - \lambda(z, b) \cdot \beta\big| \le c_3 \frac{\beta^2}{\varepsilon^3} \tag{5.5}$$

for a universal constant $c_3 > 0$, which implies

$$\left| \frac{\mathbb{P}[S_a(\tau^{(a)}) \in J]}{\mathbb{P}[S_0(\tau^{(0)}) \in J]} - \lambda(a, b) \right| < \frac{\alpha}{4}. \tag{5.6}$$

Denote $\text{EXIT} = \{S_0(\tau^{(0)}) \in J\} \cap \{S_a(\tau^{(a)}) \in J\}$, and denote $\text{INT} = E_1 \cup E_2 \cup \dots \cup E_N$. Since

$$\begin{aligned} &\mathbb{P}\big[S_z\big(\tau^{(z)}\big) = b \mid S_z\big(\tau^{(z)}\big) \in J\big] \\ &\quad = \sum_{j=1}^{N} \mathbb{P}\big[S_z\big(\tau^{(z)}\big) = b \mid \text{EXIT}, E_j, \overline{E}_{j+1}, \dots, \overline{E}_N\big] \\ &\qquad \times \mathbb{P}[E_j, \overline{E}_{j+1}, \dots, \overline{E}_N \mid \text{EXIT}] \\ &\qquad + \big[S_z\big(\tau^{(z)}\big) = b \mid \text{EXIT}, \overline{\text{INT}}\big] \cdot \mathbb{P}[\overline{\text{INT}} \mid \text{EXIT}], \end{aligned}$$

we have

$$\begin{aligned} &\big|\mathbb{P}\big[S_a\big(\tau^{(a)}\big) = b \mid S_a\big(\tau^{(a)}\big) \in J\big] - \mathbb{P}\big[S_0\big(\tau^{(0)}\big) = b \mid S_0\big(\tau^{(0)}\big) \in J\big]\big| \\ &\quad \le \sum_{j=1}^{N} \mathbb{P}[E_j, \overline{E}_{j+1}, \dots, \overline{E}_N \mid \text{EXIT}] \\ &\qquad \times \big|\mathbb{P}\big[S_a\big(\tau^{(a)}\big) = b \mid \text{EXIT}, E_j, \overline{E}_{j+1}, \dots, \overline{E}_N\big] \\ &\qquad\quad - \mathbb{P}\big[S_0\big(\tau^{(0)}\big) = b \mid \text{EXIT}, E_j, \overline{E}_{j+1}, \dots, \overline{E}_N\big]\big| \\ &\qquad + 2 \max_{z \in \{0,a\}} \big[S_z\big(\tau^{(z)}\big) = b \mid \text{EXIT}, \overline{\text{INT}}\big] \cdot \mathbb{P}[\overline{\text{INT}} \mid \text{EXIT}]. \end{aligned} \tag{5.7}$$

By Propositions 5.4 and 5.5,

$$
\begin{aligned}
&2 \max_{z \in \{0,a\}} \big[S_z\big(\tau^{(z)}\big) = b \mid \mathrm{EXIT}, \overline{\mathrm{INT}}\big] \cdot \mathbb{P}[\overline{\mathrm{INT}} \mid \mathrm{EXIT}] \\
&\quad < \mathbb{P}\big[S_0\big(\tau^{(0)}\big) = b \mid S_0\big(\tau^{(0)}\big) \in J\big] \cdot \frac{\alpha}{4}.
\end{aligned} \tag{5.8}
$$

Plugging Proposition 5.6 and (5.8) into (5.7),

$$
\begin{aligned}
&\big|\mathbb{P}\big[S_a\big(\tau^{(a)}\big) = b \mid S_a\big(\tau^{(a)}\big) \in J\big] - \mathbb{P}\big[S_0\big(\tau^{(0)}\big) = b \mid S_0\big(\tau^{(0)}\big) \in J\big]\big| \\
&\quad < \frac{\alpha\varepsilon^2}{2} \cdot \mathbb{P}\big[S_0\big(\tau^{(0)}\big) = b \mid S_0\big(\tau^{(0)}\big) \in J\big].
\end{aligned}
$$

Thus, plugging (5.6) into (5.4),

$$
\left|\frac{H(a,b)}{H(0,b)} - \lambda(a,b)\right| < \frac{\alpha}{4}\left(1 + \frac{\alpha\varepsilon^2}{2}\right) + \frac{\alpha\varepsilon^2}{2}\lambda(a,b) < \alpha.
$$

5.3. *Proof of Proposition* 5.4. For the rest of this proof denote by $E^{(z)}$ the event

$$
E^{(z)} = \overline{E}^{(z)}_{j+1} \cap \cdots \cap \overline{E}^{(z)}_{N} \cap \big\{S_0\big(\tau^{(0)}\big) \in J\big\} \cap \big\{S_z\big(\tau^{(z)}\big) \in J\big\},
$$

and denote $E = E^{(a)}$. We show that

$$
\mathbb{P}\big[\varphi \circ S_a\big[T^{(a)}_{j+1}, T^{(a)}_{j}\big] \cap \rho_j \neq \varnothing \mid E\big] \tag{5.9}
$$

is at least a constant (that may depend on ε and α). This implies the proposition, since S_0 and S_a are independent (and since the same argument holds for 0 as well).

CLAIM 5.7. *There exists a set of vertices $U \subset V_\delta(D)$ such that*:

- *Every path from $\varphi^{-1}(\rho(\tilde{a}, \varepsilon/40))$ to the boundary of $D \setminus \varphi^{-1}((1-\varepsilon/2)\mathbb{U})$ in G_δ goes through U.*
- *For every $u \in U$, we have $\mathbb{P}[\varphi \circ S_u[T^{(u)}_{j+1}, T^{(u)}_{j}] \cap \rho_j \neq \varnothing \mid E^{(u)}] \geq c_1$ with $c_1 = c_1(\varepsilon, \alpha) > 0$.*

PROOF. Assume toward a contradiction that such a set does not exist. Since G is planar-irreducible, there exists a path Y from $\varphi^{-1}(\rho(\tilde{a}, \varepsilon/40))$ to the boundary of $D \setminus \varphi^{-1}((1-\varepsilon/2)\mathbb{U})$ such that for every vertex y in Y,

$$
\mathbb{P}\big[\varphi \circ S_y\big[T^{(y)}_{j+1}, T^{(y)}_{j}\big] \cap \rho_j \neq \varnothing \mid E^{(y)}\big] < c_1(\varepsilon, \alpha). \tag{5.10}
$$

Define an auxiliary random walk L; let $L(\cdot)$ be a natural random walk started at 0 (independent of S_0), and let $\tau^{(L)}$ be the exit time of $L(\cdot)$ from D. For $j \leq k \leq N$, let

$$
T^{(L)}_k = \min\{t \geq 0 : |\varphi(L(t)) - \tilde{b}| \leq R_k\},
$$

let

$$E_k^{(L)} = \{\varphi \circ S_0[T_{k+1}^{(0)}, T_k^{(0)}] \cap \rho_k \neq \varnothing\} \cap \{\varphi \circ L[T_{k+1}^{(L)}, T_k^{(L)}] \cap \rho_k \neq \varnothing\}$$

and let

$$E^{(L)} = \overline{E}_{j+1}^{(L)} \cap \cdots \cap \overline{E}_N^{(L)} \cap \{S_0(\tau^{(0)}) \in J\} \cap \{L(\tau^{(L)}) \in J\}.$$

Consider

$$\mathbb{P}[L[0, T_N^{(L)}] \cap Y \neq \varnothing, \varphi \circ L[T_{j+1}^{(L)}, T_j^{(L)}] \cap \rho_j \neq \varnothing \mid E^{(L)}]. \tag{5.11}$$

By (5.10), and by the strong Markov property, we have (5.11) $< c_1(\varepsilon, \alpha)$. On the other hand, by weak convergence and Proposition 3.1, by Lemma 3.8, by Proposition 3.5, and by the planarity of G,

$$(5.11) \geq \mathbb{P}[\varphi \circ L[0, T'] \circlearrowleft^{(\varepsilon/2)} \tilde{a}, \varphi \circ L[T_{j+1}^{(L)}, T_j^{(L)}] \cap \rho_j \neq \varnothing \mid E^{(L)}] \geq c_2,$$

where T' is the first time $L(\cdot)$ hits the set $\{z \in D : |\varphi(z)| \geq 1 - \varepsilon/2\}$, and $c_2 = c_2(\varepsilon, \alpha) > 0$. This is a contradiction for $c_1 = c_2$. $\square$

By Claim 5.7, and by the strong Markov property, (5.9) is a convex combination of

$$\mathbb{P}[\varphi \circ S_u[T_{j+1}^{(u)}, T_j^{(u)}] \cap \rho_j \neq \varnothing \mid E^{(u)}] \qquad \text{for } u \in U,$$

which implies that (5.9) $\geq c_1(\varepsilon, \alpha)$.

5.4. *Proof of Proposition* 5.5. We use the following lemma, which is a variant of Harnack's inequality.

LEMMA 5.8. *There exists a universal constant $c > 0$ such that the following holds*:

Let $w \in V_\delta(D)$ be such that $|\varphi(w) - \tilde{b}| \geq Kr$. If $\mathbb{P}[S_w(\tau^{(w)}) \in J] > 0$, then

$$\mathbb{P}[S_w(\tau^{(w)}) = b \mid S_w(\tau^{(w)}) \in J] \leq c \cdot \mathbb{P}[S_0(\tau^{(0)}) = b \mid S_0(\tau^{(0)}) \in J].$$

Before proving the lemma, we show how the lemma implies Proposition 5.5.

PROOF OF PROPOSITION 5.5. Denote by W the set of $w \in V_\delta(D)$ such that $|\varphi(w) - \tilde{b}| \geq Kr$ and $\mathbb{P}[S_w(\tau^{(w)}) \in J] > 0$. As in the proof of Lemma 4.4, if δ_0 is small enough, for every $v \sim u \in V_\delta(D)$, we have $|\varphi(v) - \varphi(u)| < \beta$. By the strong Markov property,

$$\mathbb{P}[S_z(\tau^{(z)}) = b \mid \overline{E}_1, \ldots, \overline{E}_N, S_0(\tau^{(0)}) \in J, S_a(\tau^{(a)}) \in J]$$

is at most

$$\max_{w \in W} \mathbb{P}[S_w(\tau^{(w)}) = b \mid S_w(\tau^{(w)}) \in J]. \tag{5.12}$$

Lemma 5.8 implies the proposition. $\square$

PROOF OF LEMMA 5.8. Let

$$I_+ = \{\tilde{b} \cdot e^{it} : \pi\beta/2 \le t \le \pi\beta\} \quad \text{and} \quad I_- = \{\tilde{b} \cdot e^{it} : -\pi\beta \le t \le -\pi\beta/2\}.$$

Let $J_+ = \varphi^{-1}(I_+)$ and $J_- = \varphi^{-1}(I_-)$. Let $U = \{x \in D : |\varphi(x) - \tilde{b}| \ge Kr\}$, let $\xi = \tilde{b} \cdot (1 - 3r)$, and let $\rho = \rho(\xi, r/20)$.

We use the following claim and its corollary.

CLAIM 5.9. *There exists a universal constant $c_1 > 0$ such that the following holds*:

(1) *There exists $x_0 \in V_\delta(D) \cap \varphi^{-1}(\rho)$ such that*

$$\mathbb{P}\big[\varphi \circ S_{x_0}\big[0, \tau^{(x_0)}\big] \circlearrowleft^{(r)} \xi, S_{x_0}\big[0, \tau^{(x_0)}\big] \cap U = \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \ge c_1.$$

(2) *There exists $x_+ \in V_\delta(D) \cap \varphi^{-1}(\rho)$ such that*

$$\mathbb{P}\big[S_{x_+}\big(\tau^{(x_+)}\big) \in J_+, S_{x_+}\big[0, \tau^{(x_+)}\big] \cap U = \varnothing \mid S_{x_+}\big(\tau^{(x_+)}\big) \in J\big] \ge c_1.$$

(3) *There exists $x_- \in V_\delta(D) \cap \varphi^{-1}(\rho)$ such that*

$$\mathbb{P}\big[S_{x_-}\big(\tau^{(x_-)}\big) \in J_-, S_{x_-}\big[0, \tau^{(x_-)}\big] \cap U = \varnothing \mid S_{x_-}\big(\tau^{(x_-)}\big) \in J\big] \ge c_1.$$

PROOF. We first prove (1). Consider

$$\begin{aligned} &\mathbb{P}\big[\varphi \circ S_0\big[\Theta_0(\varphi^{-1}(\rho)), \tau^{(0)}\big] \circlearrowleft^{(r)} \xi, \\ &\qquad S_0\big[\Theta_0(\varphi^{-1}(\rho)), \tau^{(0)}\big] \cap U = \varnothing \mid S_0\big(\tau^{(0)}\big) \in J\big]. \end{aligned} \tag{5.13}$$

First, by weak convergence and Proposition 3.1, using Lemma 3.9, we have (5.13) $\ge c_1$, for a universal constant $c_1 > 0$. Second, by the strong Markov property,

$$(5.13) \le \max_x \mathbb{P}\big[\varphi \circ S_x\big[0, \tau^{(x)}\big] \circlearrowleft^{(r)} \xi, S_x\big[0, \tau^{(x)}\big] \cap U = \varnothing \mid S_x\big(\tau^{(x)}\big) \in J\big],$$

where the maximum is over x in $V_\delta(D) \cap \varphi^{-1}(\rho)$ such that $\mathbb{P}[S_x(\tau^{(x)}) \in J] > 0$.

For the proof of property (2) we consider $\{S_x(\tau^{(x)}) \in J_+\}$ instead of $\{\varphi \circ S_x[0, \tau^{(x)}] \circlearrowleft^{(r)} \xi\}$, and use the same argument with Lemma 3.10. Similarly, for property (3) we consider $\{S_x(\tau^{(x)}) \in J_-\}$. $\square$

COROLLARY 5.10. *There exists a universal constant $c_2 > 0$ such that the following holds*:

There exists $x_0 \in V_\delta(D) \cap \varphi^{-1}(\rho)$ such that

$$\mathbb{P}\big[S_{x_0}\big(\tau^{(x_0)}\big) \in J_+, S_{x_0}\big[0, \tau^{(x_0)}\big] \cap U = \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \ge c_2$$

and

$$\mathbb{P}\big[S_{x_0}\big(\tau^{(x_0)}\big) \in J_-, S_{x_0}\big[0, \tau^{(x_0)}\big] \cap U = \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \ge c_2.$$

PROOF. Let x_0, x_+, x_- be as given in Claim 5.9. We prove the first inequality for x_0, the proof of the second one is similar. Define

$$h(z) = \mathbb{P}\big[S_z\big(\tau^{(z)}\big) \in J_+, S_z\big[0, \tau^{(z)}\big] \cap U = \varnothing \mid S_z\big(\tau^{(z)}\big) \in J\big].$$

The map $h(\cdot)$ is harmonic, and so there exists a path γ from x_+ to ∂D such that $h(z) \ge h(x_+)$ for every $z \in \gamma$. Since $h(\cdot)$ is nonnegative, harmonic and bounded, by Claim 5.9,

$$\begin{aligned} h(x_0) &\ge \mathbb{P}\big[S_{x_0}\big[0, \tau^{(x_0)}_{D \setminus U}\big] \cap \gamma \ne \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \cdot h(x_+) \\ &\ge \mathbb{P}\big[\varphi \circ S_{x_0}\big[0, \tau^{(x_0)}\big] \circlearrowleft^{(r)} \xi, \\ &\qquad S_{x_0}\big[0, \tau^{(x_0)}\big] \cap U = \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \cdot h(x_+) \\ &\ge c_2. \end{aligned}$$

□

Back to the proof of Lemma 5.8. For $y \in V_\delta(D)$, define

$$p(y) = \begin{cases} \mathbb{P}\big[S_y\big(\tau^{(y)}\big) = b \mid S_y\big(\tau^{(y)}\big) \in J\big], & \text{if } \mathbb{P}\big[S_y\big(\tau^{(y)}\big) \in J\big] > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $p(\cdot)$ is harmonic, there exists a path γ from w to b such that $p(z) \ge p(w)$ for every $z \in \gamma$. Let x_0 be the vertex given by Corollary 5.10. By the choice of $\tilde{b}$, $\varphi(b) \in I$ and $\varphi(b) \notin I_+ \cup I_-$. Thus, since $w \in U$, assume without loss of generality that every path from x_0 to J_+ that does not intersect U crosses γ (otherwise, this holds for J_-). Thus, since $p(\cdot)$ is nonnegative, harmonic and bounded, by Corollary 5.10,

$$\begin{aligned} p(x_0) &\ge \mathbb{P}\big[S_{x_0}\big[0, \tau^{(x_0)}\big] \cap \gamma \ne \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \cdot p(w) \\ &\ge \mathbb{P}\big[S_{x_0}\big(\tau^{(x_0)}\big) \in J_+, S_{x_0}\big[0, \tau^{(x_0)}\big] \cap U = \varnothing \mid S_{x_0}\big(\tau^{(x_0)}\big) \in J\big] \cdot p(w) \qquad (5.14) \\ &\ge c_2 \cdot p(w), \end{aligned}$$

where $c_2 > 0$ is a constant.

Similarly, there exists a path γ from x_0 to b (we abuse notation and use γ again) such that $p(z) \ge p(x_0)$ for every $z \in \gamma$. Since G is planar-irreducible, every path from 0 that encompasses $\varphi^{-1}(\rho)$ crosses γ. Since $p(\cdot)$ is nonnegative, harmonic and bounded,

$$\begin{aligned} p(0) &\ge \mathbb{P}\big[S_0\big[0, \tau^{(0)}\big] \cap \gamma \ne \varnothing \mid S_0\big(\tau^{(0)}\big) \in J\big] \cdot p(x_0) \\ &\ge \mathbb{P}\big[\varphi \circ S_0\big[0, \tau^{(0)}\big] \circlearrowleft^{(r)} \xi \mid S_0\big(\tau^{(0)}\big) \in J\big] \cdot p(x_0). \end{aligned}$$

By weak convergence and Proposition 3.1, and by Lemma 3.9,

$$\mathbb{P}\big[\varphi \circ S_0\big[0, \tau^{(0)}\big] \circlearrowleft^{(r)} \xi \mid S_0\big(\tau^{(0)}\big) \in J\big] \ge c_3,$$

where $c_3 > 0$ is a constant. Using (5.14),

$$p(0) \ge c_3 \cdot p(x_0) \ge c_4 \cdot p(w)$$

for a constant $c_4 > 0$. □

5.5. *Proof of Proposition* 5.6. For $y \in V_\delta(D)$, define

$$p(y) = \begin{cases} \mathbb{P}[S_y(\tau^{(y)}) = b \mid S_y(\tau^{(y)}) \in J], & \text{if } \mathbb{P}[S_y(\tau^{(y)}) \in J] > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $p(\cdot)$ is harmonic, for every $y \in V_\delta(D)$, there exists a path γ_y from y to ∂D such that $p(u) \geq p(y)$ for every $u \in \gamma_y$. Let $w, y \in V_\delta(\rho_j)$. Since $p(\cdot)$ is nonnegative, harmonic and bounded,

$$p(w) \geq \mathbb{P}[S_w[0, \tau^{(w)}] \cap \gamma_y \neq \varnothing \mid S_w(\tau^{(w)}) \in J] \cdot p(y).$$

Let $\sigma^{(w)}$ be the first time S_w exits $\varphi^{-1}(\rho(\xi_j, \eta^2 R_j))$. As in the proof of Lemma 4.4, if δ_0 is small enough, for every $v \sim u \in V_\delta(D)$, we have $|\varphi(v) - \varphi(u)| < \beta(\eta - \eta^2)$. By the strong Markov property,

$$\begin{aligned} &\mathbb{P}[S_w[0, \tau^{(w)}] \cap \gamma_y \neq \varnothing \mid S_w(\tau^{(w)}) \in J] \\ &\quad = \frac{\mathbb{P}[S_w[0, \tau^{(w)}] \cap \gamma_y \neq \varnothing, S_w(\tau^{(w)}) \in J]}{\mathbb{P}[S_w(\tau^{(w)}) \in J]} \\ &\quad \geq \frac{\mathbb{P}[S_w[0, \sigma^{(w)}] \cap \gamma_y \neq \varnothing] \cdot \min_z \mathbb{P}[S_z(\tau^{(z)}) \in J]}{\mathbb{P}[S_w(\tau^{(w)}) \in J]}, \end{aligned}$$

where the minimum is over $z \in V_\delta(D)$ such that $\varphi(z) \in \rho(\xi_j, \eta R_j)$. Define $h(z)$ to be the probability that $S_z(\tau^{(z)}) \in J$. Since $h(\cdot)$ is harmonic, there exists a path g_w from w to ∂D such that $h(u) \geq h(w)$ for every $u \in g_w$. Since $h(\cdot)$ is nonnegative, harmonic and bounded, by the choice of η,

$$h(z) \geq \mathbb{P}[S_z[0, \tau^{(z)}] \cap g_w \neq \varnothing] \cdot p(w) \geq \left(1 - \frac{\alpha\varepsilon^2}{16}\right) \cdot p(w).$$

Also by the choice of η, $\mathbb{P}[S_w[0, \sigma^{(w)}] \cap \gamma_y \neq \varnothing] \geq 1 - \frac{\alpha\varepsilon^2}{16}$. Thus,

$$p(w) \geq \left(1 - \frac{\alpha\varepsilon^2}{8}\right) \cdot p(y).$$

The strong Markov property implies the proposition.

6. Convergence of the loop-erasure. In this section we show that the scaling limit of the loop-erasure of the reversal of the natural random walk on G is SLE_2 (for a planar-irreducible graph G such that the scaling limit of the natural random walk on G is planar Brownian motion). Most of our proof follows the proof of Lawler, Schramm and Werner in [9].

6.1. *The observable.* Let $D \in \mathfrak{D}$, and let $\delta > 0$. For $v \in V_\delta(D)$, let $S_v(\cdot)$ be the natural random walk on G_δ started at v and stopped on exiting D. Denote by $\hat{S}_v(\cdot)$ the loop-erasure of the reversal of $S_v(\cdot)$.

REMARK 6.1. There is a technicality we need to address. Let $\gamma'(0), \ldots, \gamma'(T) = v$ be the loop-erasure of the reversal of $S_v(\cdot)$. The edge $e = [\gamma'(0), \gamma'(1)]$ is not contained in D. Define $\gamma(0) \in \partial D$ as the last point on e not in D (see the definition of Poisson kernel in Section 1.1), and define $\gamma(i) = \gamma'(i)$ for $i = 1, \ldots, T$.

Let $\gamma(\cdot)$ be the loop-erasure of the reversal of a natural random walk started at 0 and stopped on exiting D; that is, $\gamma(\cdot)$ has the same distribution as $\hat{S}_0(\cdot)$, but is independent of $S_0(\cdot)$ [from the time $\gamma(\cdot)$ hits 0 it stays there].

PROPOSITION 6.2. *Let $v \in V_\delta(D)$. For $n \in \mathbb{N}$, define the random variable*

$$M_n = \frac{\mathbb{P}[\hat{S}_v[0,n] = \gamma[0,n]]}{\mathbb{P}[\hat{S}_0[0,n] = \gamma[0,n]]}.$$

Then, M_n is a martingale with respect to the filtration generated by $\gamma[0,n]$.

PROOF. By the definition of $\gamma(\cdot)$, for every $w \in V_\delta(D)$,

$$\mathbb{P}\big[\gamma(n+1) = w \mid \gamma[0,n]\big] = \mathbb{P}\big[\hat{S}_0(n+1) = w \mid \hat{S}_0[0,n] = \gamma[0,n]\big].$$

Thus,

$$\begin{aligned}
&\mathbb{E}[M_{n+1} \mid \gamma[0,n]] \\
&\quad = \sum_w \mathbb{P}\big[\gamma(n+1) = w \mid \gamma[0,n]\big] \\
&\qquad\quad \times \frac{\mathbb{P}[\hat{S}_v[0,n] = \gamma[0,n], \hat{S}_v(n+1) = w]}{\mathbb{P}[\hat{S}_0[0,n] = \gamma[0,n], \hat{S}_0(n+1) = w]} \\
&\quad = \sum_w \mathbb{P}\big[\hat{S}_v(n+1) = w \mid \hat{S}_v[0,n] = \gamma[0,n]\big] \frac{\mathbb{P}[\hat{S}_v[0,n] = \gamma[0,n]]}{\mathbb{P}[\hat{S}_0[0,n] = \gamma[0,n]]} \\
&\quad = M_n.
\end{aligned}$$

□

Let $\mathcal{E}_n^{(v)}$ be the event that $S_v(\cdot)$ hits the set $\partial D \cup \gamma[0,n]$ at $\gamma(n)$, where we think of $S_v(\cdot)$ as a continuous curve (linearly interpolated on the edges of G_δ). Denote

$$H_n(v, \gamma(n)) = \mathbb{P}\big[\mathcal{E}_n^{(v)}\big].$$

PROPOSITION 6.3. *For $v \in V_\delta(D)$,*

$$\frac{H_n(v, \gamma(n))}{H_n(0, \gamma(n))}$$

is a martingale with respect to the filtration generated by $\gamma[0,n]$.

PROOF. Define

$$M_n = \frac{\mathbb{P}[\hat{S}_v[0,n] = \gamma[0,n]]}{\mathbb{P}[\hat{S}_0[0,n] = \gamma[0,n]]}$$

as in Proposition 6.2. Since M_n is a martingale, it suffices to show that

$$\frac{\mathbb{P}[\mathcal{E}_n^{(v)}]}{\mathbb{P}[\mathcal{E}_n^{(0)}]} = M_n.$$

Let $z \in \{v, 0\}$, and let $S(\cdot)$ be the path $S_z[\Theta_z(\gamma[0,n]), \tau_D^{(z)}]$. Since $\{\hat{S}_z[0,n] = \gamma[0,n]\} = \{\hat{S}[0,n] = \gamma[0,n]\}$, by the strong Markov property,

$$\begin{aligned}\mathbb{P}\bigl[\hat{S}_z[0,n] = \gamma[0,n], \mathcal{E}_n^{(z)}\bigr] &= \mathbb{P}\bigl[\hat{S}[0,n] = \gamma[0,n], \mathcal{E}_n^{(z)}\bigr] \\ &= \mathbb{P}\bigl[\hat{S}_{\gamma(n)}[0,n] = \gamma[0,n]\bigr]\mathbb{P}\bigl[\mathcal{E}_n^{(z)}\bigr],\end{aligned}$$

which implies

$$\mathbb{P}\bigl[\hat{S}_z[0,n] = \gamma[0,n] \mid \mathcal{E}_n^{(z)}\bigr] = \mathbb{P}\bigl[\hat{S}_{\gamma(n)}[0,n] = \gamma[0,n]\bigr]. \tag{6.1}$$

In addition, since $\{\hat{S}_z[0,n] = \gamma[0,n]\} \subseteq \mathcal{E}_n^{(z)}$,

$$\mathbb{P}\bigl[\hat{S}_z[0,n] = \gamma[0,n]\bigr] = \mathbb{P}\bigl[\mathcal{E}_n^{(z)}\bigr]\mathbb{P}\bigl[\hat{S}_z[0,n] = \gamma[0,n] \mid \mathcal{E}_n^{(z)}\bigr]. \tag{6.2}$$

Combining (6.1) and (6.2),

$$\begin{aligned}\frac{\mathbb{P}[\mathcal{E}_n^{(v)}]}{\mathbb{P}[\mathcal{E}_n^{(0)}]} &= \frac{\mathbb{P}[\mathcal{E}_n^{(v)}]}{\mathbb{P}[\mathcal{E}_n^{(0)}]} \cdot \frac{\mathbb{P}[\hat{S}_{\gamma(n)}[0,n] = \gamma[0,n]]}{\mathbb{P}[\hat{S}_{\gamma(n)}[0,n] = \gamma[0,n]]} \\ &= \frac{\mathbb{P}[\hat{S}_v[0,n] = \gamma[0,n]]}{\mathbb{P}[\hat{S}_0[0,n] = \gamma[0,n]]} = M_n.\end{aligned}$$

□

6.2. *The driving process.* Here are some known facts about the Schramm–Loewner evolution (for more details, see [9]). Let $D \in \mathfrak{D}$, and let $\delta > 0$. Let $\gamma(\cdot)$ be the loop-erasure of the reversal of a natural random walk started at 0 and stopped on exiting D (independent of S_0). For $s \geq 0$, define $\gamma[0,s]$ as the continuous curve that is the linear interpolation of $\gamma(\cdot)$ on the edges of G_δ. For $s \geq 0$ such that $0 \notin \gamma[0,s]$, define $\varphi_s : D \setminus \gamma[0,s] \to \mathbb{U}$ to be the unique conformal map satisfying $\varphi_s(0) = 0$ and $\varphi_s'(0) > 0$. Let $t_s = \log \varphi_s'(0) - \log \varphi_D'(0)$, the *capacity* of $\gamma[0,s]$ from 0 in D. Let

$$U_s = \lim_{z \to \gamma(s)} \varphi_s(z),$$

where z tends to $\gamma(s)$ from within $D \setminus \gamma[0,s]$. Let $W : [0,\infty) \to \partial\mathbb{U}$ be the unique continuous function such that solving the radial Loewner equation with driving function $W(\cdot)$ gives the curve $\varphi_D \circ \gamma$. Loewner's theory gives us the relation

$U_s = W(t_s)$. Let $\theta(\cdot)$ be the function such that $W(t) = W(0)e^{i\theta(t)}$. Let $\Delta_s = \theta(t_s)$, so we get that $U_s = U_0 e^{i\Delta_s}$. Since t_s is a strictly increasing function of s, we can define $\xi(r)$ to be the unique s such that $t_s = r$ [by this definition, $\xi(t_r) = r$]. By the Loewner differential equation, for every $z \in D \setminus \gamma[0, \xi(r)]$,

$$\partial_r g_r(z) = g_r(z) \frac{U_{\xi(r)} + g_r(z)}{U_{\xi(r)} - g_r(z)}, \tag{6.3}$$

where $g_r(z) = \varphi_{\xi(r)}(z)$.

PROPOSITION 6.4. *There exists $c > 0$ such that for all $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:

Let $D \in \mathfrak{D}$. Let $m = \min\{1 \le j \in \mathbb{N} : t_j \ge \varepsilon^2$ or $|\Delta_j| \ge \varepsilon\}$. Then, a.s.,

$$|\mathbb{E}[\Delta_m \mid \gamma(0)]| \le c\varepsilon^3$$

and

$$\left|\mathbb{E}[\Delta_m^2 - 2t_m \mid \gamma(0)]\right| \le c\varepsilon^3.$$

PROOF. Fix $v \in V_\delta(D)$ such that $|\varphi_D(v)| \le 1/12$. Let $Z = \varphi_0(v)$ and $U = U_0$. We follow the proof of Proposition 3.4 in [9], using our Lemma 1.2 (used with inner radius $c_1/8$) to replace Lemma 2.2 in [9]. This culminates to show that a.s.

$$\begin{aligned} &\mathrm{Re}\left(\frac{ZU(U+Z)}{(U-Z)^3}\right)\mathbb{E}[2t_m - \Delta_m^2 \mid \gamma(0)] \\ &\quad + \mathrm{Im}\left(\frac{2ZU}{(U-Z)^2}\right)\mathbb{E}[\Delta_m \mid \gamma(0)] = O(\varepsilon^3). \end{aligned} \tag{6.4}$$

Let $\eta = 1/20$. Let $f(z) = \mathrm{Re}(\frac{zU(U+z)}{(U-z)^3})$ and $g(z) = \mathrm{Im}(\frac{2zU}{(U-z)^2})$. We have $f(\eta U) > 1/100$, $g(\eta U) = 0$, and $g(i\eta U) > 1/100$. There exists $\varepsilon' > 0$ such that for every $z, w \in \frac{1}{12}\mathbb{U}$, if $|z - w| \le \varepsilon'$, then $|f(z) - f(w)| \le \varepsilon^3$ and $|g(z) - g(w)| \le \varepsilon^3$.

Let $\mathfrak{D}_{1,\varepsilon'/2}$ be the finite family of domains given by Proposition 2.1. By weak convergence, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and any $\tilde{D} \in \mathfrak{D}_{1,\varepsilon'/2}$, there exist $v_1, v_2 \in V_\delta(\tilde{D})$ such that $|\varphi_{\tilde{D}}(v_1) - \eta U| < \varepsilon'/2$ and $|\varphi_{\tilde{D}}(v_2) - i\eta U| < \varepsilon'/2$.

Let $D \in \mathfrak{D}$, and let $\tilde{D} \in \mathfrak{D}_{1,\varepsilon'/2}$ be the $(1, \varepsilon'/2)$-approximation of D. Then, $\tilde{D} \subseteq D$ and $|\varphi_D(v_1) - \varphi_{\tilde{D}}(v_1)| \le \varepsilon'/2$, which implies that $f(\varphi_D(v_1)) = f(\eta U) + O(\varepsilon^3)$ and $g(\varphi_D(v_1)) = O(\varepsilon^3)$. Similarly, $g(\varphi_D(v_2)) = g(i\eta U) + O(\varepsilon^3)$. Applying (6.4) to the vertices v_1 and v_2, we have a.s.

$$\left|\mathbb{E}[2t_m - \Delta_m^2 \mid \gamma(0)]\right| = O(\varepsilon^3) \quad \text{and} \quad |\mathbb{E}[\Delta_m \mid \gamma(0)]| = O(\varepsilon^3). \qquad \square$$

The following theorem shows that $\theta(\cdot)$ converges to one-dimensional Brownian motion.

THEOREM 6.5. *For all $D \in \mathfrak{D}$, and all $\alpha, T > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:

Let $u \in [0, 2\pi]$ be a uniformly distributed point, and let $B_1(\cdot)$ be one-dimensional Brownian motion started at u. Then, there is a coupling of $\gamma(\cdot)$ and $B_1(\cdot)$ such that

$$\mathbb{P}\Big[\sup_{0 \le t \le T} |\theta(t) - B_1(2t)| > \alpha\Big] < \alpha.$$

PROOF. The proof follows the proof of Theorem 3.7 in [9], using our Proposition 6.4 to replace Proposition 3.4 in [9]. $\square$

6.3. *Weak convergence*. In this section we show that the scaling limit of the loop-erasure of the reversal of the natural random walk on G is SLE_2. It would seem natural to follow the proofs in Section 3.4 of [9]. However, as stated in the Introduction there is a difficulty with this approach. The proof of tightness in [9] uses a "natural" family of compact sets. In our setting, it is not necessarily true that γ belongs to one of these compact sets with high probability (and so the argument of [9] fails). To overcome this difficulty, we define a "weaker" notion of tightness, which we are able to use to conclude the proof.

6.3.1. *A sufficient condition for tightness*. For a metric space $\mathcal{X}$, and a set $A \subseteq \mathcal{X}$, define $A^\varepsilon = \bigcup_{a \in A} \rho(a, \varepsilon)$, where $\rho(a, \varepsilon)$ is the ball of radius ε centered at a. The following are Theorems 11.3.1, 11.3.3 and 11.5.4 in [3].

THEOREM 6.6. *Let $\mathcal{X}$ be a metric space. For any two laws μ, ν on $\mathcal{X}$, let*

$$d(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \le \nu(A^\varepsilon) + \varepsilon \text{ for all Borel sets } A \subset \mathcal{X}\}.$$

Then, $d(\cdot, \cdot)$ is a metric on the space of laws on $\mathcal{X}$ [$d(\cdot, \cdot)$ is called the Prohorov metric].

THEOREM 6.7. *Let $\mathcal{X}$ be a separable metric space. Let $\{\mu_n\}$ and μ be laws on $\mathcal{X}$. Then, $\{\mu_n\}$ converges weakly to μ if and only if $d(\mu_n, \mu) \to 0$, where $d(\cdot, \cdot)$ is the Prohorov metric.*

Let $\{\mu_\delta\}$ be a family of laws on a metric space $\mathcal{X}$. We say that $\{\mu_\delta\}$ is *tight* if for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathcal{X}$ such that for all δ, $\mu_\delta(K_\varepsilon) \ge 1 - \varepsilon$.

THEOREM 6.8. *Let $\mathcal{X}$ be a complete separable metric space. Let $\{\mu_\delta\}$ be a family of laws on $\mathcal{X}$. Then, $\{\mu_\delta\}$ is tight if and only if every sequence $\{\mu_{\delta_n}\}_{n \in \mathbb{N}}$ has a weakly-converging subsequence.*

We use these theorems to prove an equivalent condition for tightness of measures on a separable metric space.

LEMMA 6.9. *Let $\mathcal{X}$ be a complete separable metric space. Let $\{\mu_m\}_{m\in\mathbb{N}}$ be a sequence of laws on $\mathcal{X}$ with the following property: for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathcal{X}$ such that for any $\alpha > 0$, there exists $M > 0$ such that for all $m \geq M$,*

$$\mu_m(K_\varepsilon^\alpha) \geq 1 - \varepsilon.$$

Then, the sequence $\{\mu_m\}$ is tight.

PROOF. Let $\{K_n\}$ be a sequence of compact sets such that for all $\alpha > 0$, there exists $M > 0$ such that for all $m \geq M$, $\mu_m(K_n^\alpha) \geq 1 - n^{-1}$.

Define

$$M(\alpha, n) = \min\{j \in \mathbb{N} : \forall m \geq j\, \mu_m(K_n^\alpha) \geq 1 - n^{-1}\}.$$

For $k \in \mathbb{N}$, define $M_0(1/k, n) = \max\{M(1/k, n), k\}$, and for $\frac{1}{k} \leq \alpha < \frac{1}{k-1}$, define $M_0(\alpha, n) = M_0(1/k, n)$. For fixed n, the function $M_0(\cdot, n)$ has the following properties: (i) The function $M_0(\alpha, n)$ is right-continuous in α. (ii) The function $M_0(\alpha, n)$ is a monotone nonincreasing function of α. (iii) $\lim_{\alpha\to 0} M_0(\alpha, n) = \infty$. (iv) For every $0 < \alpha < 1$, $M_0(\alpha, n) \geq M(\alpha, n)$.

For every m, define $\alpha_n(m) = \inf\{0 < \beta < 1 : M_0(\beta, n) \leq m\}$. For every $\eta > 0$, $\alpha_n(M_0(\eta, n)) \leq \eta$, which implies that

$$\lim_{m\to\infty} \alpha_n(m) = 0.$$

In addition, $M_0(\alpha_n(m), n) \leq m$, which implies that for all $m > 0$,

$$\mu_m\bigl(K_n^{\alpha_n(m)}\bigr) \geq 1 - n^{-1}. \tag{6.5}$$

For m and $n \geq 2$, define

$$\mu_{m,n}(A) = \frac{\mu_m(A \cap K_n^{\alpha_n(m)})}{\mu_m(K_n^{\alpha_n(m)})}$$

for all Borel $A \subset \mathcal{X}$. We show that for any fixed $n \geq 2$, the sequence $\{\mu_{m,n}\}_{m\in\mathbb{N}}$ is tight. Let $X_{m,n}$ be a random variable with law $\mu_{m,n}$. Since $X_{m,n} \in K_n^{\alpha_n(m)}$ a.s., we can define a random variable $\hat{X}_{m,n} \in K_n$ such that a.s. the distance between $X_{m,n}$ and $\hat{X}_{m,n}$ is at most $2\alpha_n(m)$. Let $\hat{\mu}_{m,n}$ be the law of $\hat{X}_{m,n}$. The Prohorov distance between $\mu_{m,n}$ and $\hat{\mu}_{m,n}$ is at most $2\alpha_n(m)$. Thus, if a sequence $\{\hat{\mu}_{m_k,n}\}_{k\in\mathbb{N}}$ converges to some limit in the Prohorov metric, then the sequence $\{\mu_{m_k,n}\}_{k\in\mathbb{N}}$ has a converging subsequence as well. Since $\{\hat{\mu}_{m,n}\}$ is compactly supported, it is a tight family of measures. By Theorem 6.8, $\{\mu_{m,n}\}$ is also tight.

Thus, for any $n \geq 2$ and any $\varepsilon > 0$, there exists a compact set $K_{n,\varepsilon} \subset \mathcal{X}$ such that for all $m > 0$, $\mu_{m,n}(K_{n,\varepsilon}) \geq 1 - \varepsilon$. Let $\varepsilon > 0$, and let $n = \lceil 2/\varepsilon \rceil$. For all $m > 0$, by (6.5), $\mu_m(K_n^{\alpha_n(m)}) \geq 1 - \varepsilon/2$. Thus,

$$\begin{aligned}\mu_m(K_{n,\varepsilon/2}) &\geq \mu_m\bigl(K_{n,\varepsilon/2} \cap K_n^{\alpha_n(m)}\bigr) = \mu_{m,n}(K_{n,\varepsilon/2}) \cdot \mu_m\bigl(K_n^{\alpha_n(m)}\bigr) \\ &\geq (1 - \varepsilon/2)^2 > 1 - \varepsilon,\end{aligned}$$

which implies that the sequence $\{\mu_m\}$ is tight. $\square$

6.3.2. *Quasi-loops*. Here we give some probability estimates needed for proving tightness.

CLAIM 6.10. *Let $z \in \mathbb{U}$. For all $\beta > 0$, there exist $c > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and for all $x \in V_\delta(\mathbb{U})$ such that $|x - z| \geq 2\beta$,*

$$\mathbb{P}[S_x[0, \tau_{3\mathbb{U}}^{(x)}] \cap \rho(z, \beta) = \varnothing] \geq c.$$

PROOF. It suffices to prove that there exists a set of vertices $U \subseteq V_\delta(\mathbb{U})$ such that every path starting at x and reaching $\partial\rho(x, \beta)$ intersects U, and such that

$$\mathbb{P}[S_u[0, \tau_{3\mathbb{U}}^{(u)}] \cap \rho(z, \beta) = \varnothing] \geq c$$

for every $u \in U$.

Denote $\mathcal{A} = \{\frac{\beta}{100}(n + m \cdot i) \in \mathbb{U} : n, m \in \mathbb{Z}\}$. The set $\mathcal{A}$ is finite, and there exists $\tilde{x} \in \mathcal{A}$ such that $x \in \rho(\tilde{x}, \beta/40)$.

Assume toward a contradiction that such a set U does not exist. By the planarity of G, there exists a path $Y \subseteq V_\delta(\mathbb{U})$ in G starting inside $\rho(\tilde{x}, \beta/40)$ and reaching $\partial\rho(\tilde{x}, \beta/2)$ such that

$$\mathbb{P}[S_y[0, \tau_{3\mathbb{U}}^{(y)}] \cap \rho(z, \beta) = \varnothing] < c$$

for every $y \in Y$. On one hand, by weak convergence and Proposition 3.1, and by Proposition 3.5 (and the conformal invariance of Brownian motion),

$$\begin{aligned} &\mathbb{P}[S_0[0, \tau_{3\mathbb{U}}^{(0)}] \cap Y \neq \varnothing, S_0[0, \tau_{3\mathbb{U}}^{(0)}] \cap \rho(z, \beta) = \varnothing] \\ &\quad \geq \mathbb{P}[S_0[0, \tau_{3\mathbb{U}}^{(0)}] \circlearrowleft^{(\beta/2)} \tilde{x}, S_0[0, \tau_{3\mathbb{U}}^{(0)}] \cap \rho(z, \beta) = \varnothing] > c. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathbb{P}[S_0[0, \tau_{3\mathbb{U}}^{(0)}] \cap Y \neq \varnothing, S_0[0, \tau_{3\mathbb{U}}^{(0)}] \cap \rho(z, \beta) = \varnothing] \\ &\quad \leq \max_{y \in Y} \mathbb{P}[S_y[0, \tau_{3\mathbb{U}}^{(y)}] \cap \rho(z, \beta) = \varnothing] < c, \end{aligned}$$

which is a contradiction. $\square$

CLAIM 6.11. *There exist universal constants $c_1, c_2 > 0$ such that for every $\varepsilon > 0$ there exists $0 < C \leq c_1 \varepsilon^{-c_2}$ such that for every $\beta > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds*:

Let $y \in V_\delta(\mathbb{U})$ and let $g : [0, \infty] \to \mathbb{C}$ be a curve such that $g(0) \in \rho(y, \beta/C)$ and $g(\infty) \notin \rho(y, \beta)$. Let τ_β be the exit time of $S_y(\cdot)$ from $\rho(y, \beta)$. Then,

$$\mathbb{P}[S_y[0, \tau_\beta] \cap g = \varnothing] < \varepsilon.$$

PROOF. Let $c > 0$ be the universal constant from Corollary 4.3 with the domain $2\mathbb{U}$. Let $N > 1$ be large enough so that $(1-c)^N < \varepsilon$, and let $C = 8 \cdot 500^N$. Denote $\mathcal{A} = \{\frac{\beta}{100C}(n + m \cdot i) \in 2\mathbb{U} : n, m \in \mathbb{Z}\}$. There exists $\tilde{y} \in \mathcal{A}$ such that $y \in \rho(\tilde{y}, \frac{\beta}{40C})$. For $j = 0, 1, \ldots, N$, let $r_j = 2 \cdot 500^j \beta/C$, let T_j be the first time $S_y(\cdot)$ exits $\rho(\tilde{y}, 400r_j)$ and let $\mathcal{E}_j$ be the complement of the event $\{S_y[T_j, T_{j+1}] \circlearrowleft^{(400r_{j+1})} \tilde{y}\}$.

By Corollary 4.3, there exists $\delta_0 > 0$ (independent of y, since $|\mathcal{A}| < \infty$) such that for all $0 < \delta < \delta_0$, we have $\mathbb{P}[\mathcal{E}_0] \leq 1 - c$ and $\mathbb{P}[\mathcal{E}_j | \mathcal{E}_0, \ldots, \mathcal{E}_{j-1}] \leq 1 - c$ for all $j = 1, \ldots, N-1$. Since g is a continuous curve from $\rho(y, \beta/C) \subset \rho(\tilde{y}, 2\beta/C)$ to the exterior of $\rho(y, \beta) \supset \rho(\tilde{y}, \beta/2)$, and since $r_N < \beta/2$, for all $0 < \delta < \delta_0$,

$$\mathbb{P}\big[S_y[0, \tau_\beta] \cap g = \varnothing\big] \leq \mathbb{P}[\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{N-1}] \leq (1-c)^N < \varepsilon. \qquad \square$$

Let $\gamma = \gamma_\delta$ be the loop-erasure of the reversal of the natural random walk on $V_\delta(\mathbb{U})$, started at 0 and stopped on exiting $\mathbb{U}$ (γ is a simple curve from $\partial\mathbb{U}$ to 0). For $\alpha, \beta > 0$, we say that γ has a *quasi-loop*, denoted $\gamma \in \mathcal{QL}(\alpha, \beta)$, if there exist $0 \leq s < t < \infty$ such that $|\gamma(s) - \gamma(t)| \leq \alpha$ and $\operatorname{diam}(\gamma[s, t]) \geq \beta$.

PROPOSITION 6.12. *For all $\varepsilon > 0$ and all $\beta > 0$, there exists $\alpha > 0$ such that for all $\delta > 0$,*

$$\mathbb{P}[\gamma \in \mathcal{QL}(\alpha, \beta)] < \varepsilon.$$

PROOF. Fix $\varepsilon, \beta > 0$. For $z \in \mathbb{U}$ and $\alpha > 0$, let $\mathcal{QL}(z, \alpha, \beta)$ be the set of all curves g such that there exist $0 \leq s < t < \infty$ such that $g(s), g(t) \in \rho(z, \beta)$, $|g(s) - g(t)| \leq \alpha$ and $g[s, t] \not\subseteq \rho(z, 2\beta)$. Let $\mathcal{A} = \{\frac{\beta}{100}(n + m \cdot i) \in \mathbb{U} : n, m \in \mathbb{Z}\}$.

CLAIM 6.13. *For any $z \in \mathcal{A}$ and for any $\eta > 0$, there exist $\alpha_1 > 0$ and $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$ the following holds*:

Let g be the loop-erasure of $S_0[0, \tau_{\mathbb{U}}^{(0)}]$ (g is not *the loop-erasure of the reversal). Then,*

$$\mathbb{P}[g \in \mathcal{QL}(z, \alpha_1, \beta)] \leq \eta.$$

PROOF. Fix $z \in \mathcal{A}$ and $\eta > 0$. Let $s_1 \geq 0$ be the first time $S_0(\cdot)$ hits $\rho(z, \beta)$, and let $t_1 \geq s_1$ be the first time after s_1 that $S_0(\cdot)$ is not in $\rho(z, 2\beta)$. For $j \geq 2$, let $s_j \geq t_{j-1}$ be the first time after t_{j-1} that $S_0(\cdot)$ hits $\rho(z, \beta)$, and let $t_j \geq s_j$ be the first time after s_j that $S_0(\cdot)$ is not in $\rho(z, 2\beta)$. Define g_j as the loop-erasure of $S_0[0, t_j]$, and let Y_j be the event that $g_j \in \mathcal{QL}(z, \alpha_1, \beta)$. Let $\tau = \tau_{3\mathbb{U}}^{(0)}$, and let $\mathcal{T}_j$ be the event that $t_j \leq \tau$.

Let x be the first point on g_j that is in $S_0[t_j, t_{j+1}]$. Then, g_{j+1} is g_j up to the point x, and then continues as the loop-erasure of $S_0[\sigma_x, t_{j+1}]$, where σ_x is the first time $S_0[t_j, t_{j+1}]$ hits x.

Denote $\mathcal{IO}_j = \{t_j \leq \tau < t_{j+1}\}$. The event $\{s_j < \tau\}$ implies the event $\{t_j < \tau\}$. Thus, $\mathcal{IO}_j \cap \{g \in \mathcal{QL}(z, \alpha_1, \beta)\} \subseteq Y_j$, which implies that for every $m \geq 1$,

$$\begin{aligned} \{g \in \mathcal{QL}(z, \alpha_1, \beta)\} &\subseteq \mathcal{T}_{m+1} \cup \bigcup_{j=1}^{m} (\{g \in \mathcal{QL}(z, \alpha_1, \beta)\} \cap \mathcal{IO}_j) \\ &\subseteq \mathcal{T}_{m+1} \cup \bigcup_{j=1}^{m} Y_j. \end{aligned} \tag{6.6}$$

By Claim 6.10, there exist $c > 0$ and $\delta_2 > 0$ such that for all $0 < \delta < \delta_2$ and for all $x \in V_\delta(\mathbb{U})$ such that $|x - z| \geq 2\beta$, we have $\mathbb{P}[S_x[0, \tau_{3\mathbb{U}}^{(x)}] \cap \rho(z, \beta) = \varnothing] \geq c$, which implies that

$$\mathbb{P}[\mathcal{T}_{m+1}] \leq (1 - c)^m < \varepsilon/2 \tag{6.7}$$

for large enough m.

Fix $1 \leq j \leq m$. Let h_{j+1} be the loop-erasure of $S_0[0, s_{j+1}]$. Let Q_j be the set of connected components of $h_{j+1} \cap \rho(z, 2\beta)$ that intersect $\rho(z, \beta)$ and are not connected to $S_0(s_{j+1})$. By the definition of s_{j+1}, the size of Q_j is at most j.

Assume that the event Y_j does not occur. If for every $K \in Q_j$, the distance between $S_0[s_{j+1}, t_{j+1}]$ and $K \cap \rho(z, \beta)$ is more than α_1, then the event Y_{j+1} does not occur. Otherwise, let K be the first component in Q_j (according to the order defined by time) such that the distance between $S_0[s_{j+1}, t_{j+1}]$ and $K \cap \rho(z, \beta)$ is at most α_1. If $S_0[s_{j+1}, t_{j+1}]$ intersects K, then the event Y_{j+1} does not occur. Thus, the event $Y_{j+1} \setminus Y_j$ implies that there exists $K \in Q_j$ such that the distance between $S_0[s_{j+1}, t_{j+1}]$ and $K \cap \rho(z, \beta)$ is at most α_1, and $S_0[s_{j+1}, t_{j+1}]$ does not intersect K. By Claim 6.11, if α_1 is small enough, there exists $\delta_3 > 0$ such that for all $0 < \delta < \delta_3$, since a.s. $|Q_j| \leq m$,

$$\mathbb{P}[Y_{j+1} \setminus Y_j] < \frac{\varepsilon}{2m}.$$

Using (6.6) and (6.7), there exist $\alpha_1 > 0$ and $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$,

$$\mathbb{P}[g \in \mathcal{QL}(z, \alpha_1, \beta)] < \varepsilon. \qquad \square$$

For every $z \in \mathbb{U}$, there exists $\tilde{z} \in \mathcal{A}$ such that $z \in \rho(\tilde{z}, \beta/40)$. Thus, for $\alpha < \beta/100$,

$$\mathcal{QL}(\alpha, 8\beta) \subset \bigcup_{z \in \mathcal{A}} \mathcal{QL}(z, \alpha, \beta). \tag{6.8}$$

Since the size of $\mathcal{A}$ does not depend on α, by Claim 6.13, there exist $\alpha_1 > 0$ and $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, and for every $z \in \mathcal{A}$,

$$\mathbb{P}[g \in \mathcal{QL}(z, \alpha_1, \beta)] < \frac{\varepsilon}{|\mathcal{A}|},$$

which implies

$$\mathbb{P}[g \in \mathcal{QL}(\alpha_1, 8\beta)] < \varepsilon,$$

where g is the loop-erasure of $S_0[0, \tau_{\mathbb{U}}^{(0)}]$.

Let $\alpha_2 > 0$ be small enough so that for all $z \in \mathcal{A}$ and all $\delta \geq \delta_1$, we have that $\rho(z, \alpha_2)$ contains at most one vertex from G_δ. Set $\alpha = \min\{\alpha_1, \alpha_2\}$. This implies that for any $\delta \geq \delta_1$, $\mathbb{P}[g \in \mathcal{QL}(\alpha, 8\beta)] = 0$. Therefore, for any $\delta > 0$,

$$\mathbb{P}[g \in \mathcal{QL}(\alpha, 8\beta)] < \varepsilon.$$

By Lemma 1.1 in [15], g and γ have the same law, which completes the proof. □

PROPOSITION 6.14. *For every $\varepsilon > 0$, there exists a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$ such that for all $\delta > 0$,*

$$\mathbb{P}\bigl[\exists 0 \leq s < t < \infty : \mathrm{dist}(\gamma[0, s], \gamma[t, \infty]) < f(\mathrm{diam}(\gamma[s, t]))\bigr] < \varepsilon.$$

PROOF. By Proposition 6.12, for all $n \geq 1$, there exists $\alpha_n > 0$ such that for all $\delta > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}[\gamma \in \mathcal{QL}(\alpha_n, 2^{1-n})] < \varepsilon. \tag{6.9}$$

Let $f : (0, \infty) \to (0, 1]$ be a monotone nondecreasing function such that

$$f(2^{2-n}) < \alpha_n \qquad \text{for all } n \geq 1. \tag{6.10}$$

Let $\delta > 0$. Assume that there exist $0 \leq s < t < \infty$ such that

$$\mathrm{dist}(\gamma[0, s], \gamma[t, \infty]) < f(\mathrm{diam}(\gamma[s, t])).$$

Then, there exist $0 \leq s' < t' < \infty$ such that $|\gamma(s') - \gamma(t')| < f(\mathrm{diam}(\gamma[s', t']))$. Since $\gamma \subset \mathbb{U}$, there exists $n \geq 1$ such that $2^{1-n} < \mathrm{diam}(\gamma[s', t']) \leq 2^{2-n}$. By (6.10), there exists $n \geq 1$ such that $|\gamma(s') - \gamma(t')| < f(2^{2-n}) < \alpha_n$ and $\mathrm{diam}(\gamma[s', t']) > 2^{1-n}$, which implies that $\gamma \in \mathcal{QL}(\alpha_n, 2^{1-n})$. The proposition follows by (6.9). □

PROPOSITION 6.15. *For every $\varepsilon > 0$, there exists a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$ such that for every $\eta > 0$, there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$,*

$$\mathbb{P}\bigl[\exists t \geq 0 : \eta < 1 - |\gamma(t)| < f(\mathrm{diam}(\gamma[0, t]))\bigr] < \varepsilon.$$

PROOF. By Claim 6.11 and the strong Markov property, there exist universal constants $c_1, c_2 > 0$ such that for every $m \geq 1$, there exists $0 < C_m \leq c_1 \varepsilon^{-c_2} 2^{c_2 m}$ and $\delta_m > 0$ such that for every $0 < \delta < \delta_m$,

$$\mathbb{P}\bigl[\mathrm{diam}\bigl(S_0\bigl[T(2^{1-m^2}), \tau_{\mathbb{U}}^{(0)}\bigr]\bigr) > C_m 2^{1-m^2}\bigr] < \varepsilon 2^{-m}, \tag{6.11}$$

where

$$T(\xi) = \inf\{t \geq 0 : 1 - |S_0(t)| \leq \xi\}.$$

Since $C_m \cdot 2^{-m^2}$ tends to 0 as m tends to infinity, we can define a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$ such that $f(C_m 2^{1-m^2}) < 2^{1-(m+1)^2}$ for all $m \geq 1$.

Denote by $\mathcal{Y}$ the event that there exists $t \geq 0$ such that $\eta < 1 - |\gamma(t)| < f(\operatorname{diam}(\gamma[0, t]))$. Let M be large enough so that $2^{1-M^2} < \eta$. The event $\mathcal{Y}$ implies that there exists $1 \leq m < M$ such that

$$2^{1-(m+1)^2} < 1 - |\gamma(t)| \leq 2^{1-m^2},$$

which implies

$$2^{1-(m+1)^2} < 1 - |\gamma(t)| < f(\operatorname{diam}(\gamma[0, t])) \leq f\big(\operatorname{diam}\big(S_0\big[T(2^{1-m^2}), \tau_{\mathbb{U}}^{(0)}\big]\big)\big).$$

By the definition of f, this implies that $\operatorname{diam}(S_0[T(2^{1-m^2}), \tau_{\mathbb{U}}^{(0)}]) > C_m 2^{1-m^2}$. Using (6.11), for all $0 < \delta < \delta_0 = \min\{\delta_1, \ldots, \delta_M\}$,

$$\mathbb{P}[\mathcal{Y}] \leq \sum_{m=1}^{M} \mathbb{P}\big[\operatorname{diam}\big(S_0\big[T(2^{1-m^2}), \tau_{\mathbb{U}}^{(0)}\big]\big) > C_m 2^{1-m^2}\big] < \varepsilon. \qquad \square$$

6.3.3. *Tightness.* In this section we show that the laws of $\{\gamma_\delta\}$ are tight. Recall $\mathcal{C}$, the space of all continuous curves with the metric ϱ. Let

$$\mathcal{X}_0 = \{g \in \mathcal{C} : g(0) \in \partial\mathbb{U}, g(\infty) = 0, g(0, \infty] \subset \mathbb{U}, g \text{ is a simple curve}\}.$$

For a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$, define $\mathcal{X}_f$ to be the set of $g \in \mathcal{X}_0$ such that for all $0 \leq s < t < \infty$,

$$\operatorname{dist}(g[0, s] \cup \partial\mathbb{U}, g[t, \infty]) \geq f(\operatorname{diam}(g[s, t])).$$

The following is Lemma 3.10 from [9].

LEMMA 6.16. *Let $f : (0, \infty) \to (0, 1]$ be a monotone nondecreasing function. Then, $\mathcal{X}_f$ is compact in the topology of convergence with respect to the metric ϱ.*

For $\alpha > 0$, define

$$\mathcal{X}_f^\alpha = \{g \in \mathcal{X}_0 : \exists g' \in \mathcal{X}_f \text{ such that } \varrho(g, g') < \alpha\}.$$

LEMMA 6.17. *For every $\varepsilon > 0$, there exists a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$ such that for any $\alpha > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$,*

$$\mathbb{P}[\gamma \notin \mathcal{X}_f^\alpha] < \varepsilon.$$

PROOF. Let $g \in \mathcal{X}_0$ and let $\eta, \beta > 0$. Choose a parameterization for g and let $t_\eta(g) = \sup\{t \geq 0 : 1 - |g(t)| \leq \eta\}$. Define g^η to be the curve $g[t_\eta, \infty]$ (the curve g^η does not depend on the choice of parameterization). We say that a curve $h \in \mathcal{X}_0$ is (η, β)-*adapted* to g, if $h^\eta = g^\eta$, and $\mathrm{diam}(h[0, t_\eta(h)]) < \beta$. Let $\mathcal{A}(g, \eta, \beta)$ be the set of all curves that are (η, β)-adapted to g. Note that g is not necessarily in $\mathcal{A}(g, \eta, \beta)$, and that for any two curves $h, \tilde{h} \in \mathcal{A}(g, \eta, \beta)$,

$$\varrho(h, \tilde{h}) \leq 2\beta. \tag{6.12}$$

Define the curve $\tilde{\gamma}$ as follows. Let $x \in \partial(1 - \eta)\mathbb{U}$ be the starting point of γ^η, and let $y = \frac{x}{1-\eta} \in \partial\mathbb{U}$. Let $\tilde{\gamma}$ be the curve $[y, x] \cup \gamma^\eta$.

By Proposition 6.15, there exists a monotone nondecreasing function $f_1 : (0, \infty) \to (0, 1]$ such that for every $\eta > 0$, there exists $\delta_1 > 0$ such that for every $0 < \delta < \delta_1$,

$$\mathbb{P}\bigl[\exists t \geq 0 : \eta < 1 - |\gamma(t)| < f_1(\mathrm{diam}(\gamma[0, t]))\bigr] < \varepsilon/4. \tag{6.13}$$

By Proposition 6.14, there exists a monotone nondecreasing function $f_2 : (0, \infty) \to (0, 1]$ such that for all $\delta > 0$,

$$\mathbb{P}\bigl[\exists 0 \leq s < t < \infty : \mathrm{dist}(\gamma[0, s], \gamma[t, \infty]) < f_2(\mathrm{diam}(\gamma[s, t]))\bigr] < \varepsilon/4. \tag{6.14}$$

Define a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$ by

$$f(\xi) = \min\{\xi/2, f_1(\xi/2), f_2(\xi/2)\}.$$

Assume that there exists $t \geq 0$ such that $1 - |\tilde{\gamma}(t)| < f(\mathrm{diam}(\tilde{\gamma}[0, t]))$. Since $f(\xi) \leq \xi$, there exists $t \geq 0$ such that $\eta < 1 - |\gamma(t)| < f(\mathrm{diam}(\tilde{\gamma}[0, t]))$, and also $\mathrm{diam}(\tilde{\gamma}[0, t]) \leq \mathrm{diam}(\gamma[0, t]) + \eta$, which implies

$$\begin{aligned}\eta < f(\mathrm{diam}(\tilde{\gamma}[0, t])) &\leq \max\{f(2\,\mathrm{diam}(\gamma[0, t])), f(2\eta)\}\\ &\leq \max\{f_1(\mathrm{diam}(\gamma[0, t])), \eta\}.\end{aligned}$$

Thus, there exists $t \geq 0$ such that $\eta < 1 - |\gamma(t)| < f_1(\mathrm{diam}(\gamma[0, t]))$.

Assume that there exist $0 \leq s < t < \infty$ such that $|\tilde{\gamma}(t) - \tilde{\gamma}(s)| < f(\mathrm{diam}(\tilde{\gamma}[s, t]))$. Let $t_\eta = t_\eta(\gamma)$. Parameterize γ and $\tilde{\gamma}$ so that $\gamma(t) = \tilde{\gamma}(t)$ for every $t \geq t_\eta$. Since $f(\xi) \leq \xi$, we have that $t > t_\eta$. Assume that $s < t_\eta$. Since $\mathrm{diam}(\tilde{\gamma}[s, t]) \leq \mathrm{diam}(\gamma[t_\eta, t]) + |\tilde{\gamma}(t_\eta) - \tilde{\gamma}(s)|$,

$$\begin{aligned}|\tilde{\gamma}(t_\eta) - \tilde{\gamma}(s)| \leq |\tilde{\gamma}(t) - \tilde{\gamma}(s)| &< f(\mathrm{diam}(\tilde{\gamma}[s, t]))\\ &\leq \max\{f_2(\mathrm{diam}(\gamma[t_\eta, t])), |\tilde{\gamma}(t_\eta) - \tilde{\gamma}(s)|\},\end{aligned}$$

which implies

$$|\gamma(t) - \gamma(t_\eta)| \leq |\tilde{\gamma}(t) - \tilde{\gamma}(s)| < f_2(\mathrm{diam}(\gamma[t_\eta, t])).$$

If $s \geq t_\eta$, then $|\gamma(t) - \gamma(s)| < f_2(\mathrm{diam}(\gamma[s, t]))$.

Therefore, if $\tilde{\gamma} \notin \mathcal{X}_f$, then either there exists $t \geq 0$ such that $\eta < 1 - |\gamma(t)| < f_1(\mathrm{diam}(\gamma[0, t]))$, or there exist $0 \leq s < t < \infty$ such that $|\gamma(t) - \gamma(s)| <$

$f_2(\operatorname{diam}(\gamma[s,t]))$. By (6.13) and (6.14), for every $\eta > 0$, there exists $\delta_1 > 0$ such that for every $0 < \delta < \delta_1$,

$$\mathbb{P}[\tilde{\gamma} \notin \mathcal{X}_f] < \varepsilon/2. \tag{6.15}$$

By Claim 6.11 and the strong Markov property, for every $\alpha > 0$, there exist $\eta > 0$ and $\delta_2 > 0$ such that for all $0 < \delta < \delta_2$,

$$\mathbb{P}\big[\operatorname{diam}\big(S_0\big[T(\eta), \tau_{\mathbb{U}}^{(0)}\big]\big) \geq \alpha/2\big] < \frac{\varepsilon}{4},$$

where $T(\eta) = \inf\{t \geq 0 : 1 - |S_0(t)| \leq \eta\}$. If $1 - |\gamma(t)| \leq \eta$, then $\gamma[0,t] \subset S_0[T(\eta), \tau_{\mathbb{U}}^{(0)}]$. Thus, for every $\alpha > 0$, there exist $0 < \eta < \alpha/2$ and $\delta_2 > 0$ such that for every $0 < \delta < \delta_2$,

$$\mathbb{P}[\varrho(\gamma, \tilde{\gamma}) \geq \alpha] \leq \mathbb{P}[\gamma \notin \mathcal{A}(\gamma, \eta, \alpha/2)] < \frac{\varepsilon}{4}. \tag{6.16}$$

Using (6.15), for any $\alpha > 0$, there exist $\eta > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$,

$$\mathbb{P}[\gamma \notin \mathcal{X}_f^\alpha] \leq \mathbb{P}[\varrho(\gamma, \tilde{\gamma}) \geq \alpha] + \mathbb{P}[\tilde{\gamma} \notin \mathcal{X}_f] < \varepsilon. \qquad \square$$

Using Lemmas 6.16, 6.17 and 6.9, we have the following corollary.

COROLLARY 6.18. *Let $\{\delta_n\}$ be a sequence converging to zero, and let μ_n be the law of the curve γ_{δ_n}. Then, the sequence $\{\mu_n\}$ is tight.*

6.3.4. *Convergence.* Here we finally show that the scaling limit of the loop-erasure of the reversal of the natural random walk on G is SLE_2. We first show that any subsequential limit of $\{\gamma_\delta\}$ is a.s. a simple curve.

LEMMA 6.19. *Let $\{\delta_n\}$ be a sequence converging to zero, and let μ_n be the law of the curve γ_{δ_n}. If μ_n converges weakly to μ, then μ is supported on $\mathcal{X}_0$.*

PROOF. Let $d(\cdot,\cdot)$ be the Prohorov metric. By Theorem 6.7, $d(\mu_n, \mu) \to 0$.

As in the proof of Lemma 6.17, by (6.15) and (6.16), for every $\varepsilon > 0$, there exists a monotone nondecreasing function $f : (0, \infty) \to (0, 1]$ such that for every $\alpha > 0$, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, we can define a curve γ_δ^α such that

$$\mathbb{P}[\gamma_\delta^\alpha \notin \mathcal{X}_f] < \varepsilon \quad \text{and} \quad \mathbb{P}[\varrho(\gamma_\delta, \gamma_\delta^\alpha) \geq \alpha] < \alpha. \tag{6.17}$$

Let μ_n^α be the law of $\gamma_{\delta_n}^\alpha$. By (6.17), for all $k \in \mathbb{N}$, there exists f_k such that for every $m \in \mathbb{N}$, there exists $N_{m,k} > m + k$ such that for all $n \geq N_{m,k}$, we have $d(\mu_n, \mu_n^{1/m}) < 1/m$ and $\mu_n^{1/m}(\mathcal{X}_{f_k}) > 1 - 1/k$.

Since $d(\mu_{N_{m,k}}^{1/m}, \mu) \le d(\mu_{N_{m,k}}^{1/m}, \mu_{N_{m,k}}) + d(\mu_{N_{m,k}}, \mu)$, by Theorem 6.7, for every fixed $k \in \mathbb{N}$, the sequence $\{\mu_{N_{m,k}}^{1/m}\}_{m\in\mathbb{N}}$ converges weakly to μ. Using Lemma 6.16, the Portmanteau theorem (see Chapter III in [14]) tells us that for every $k \in \mathbb{N}$,

$$\mu(\mathcal{X}_{f_k}) \ge \limsup_{m\to\infty} \mu_{N_{m,k}}^{1/m}(\mathcal{X}_{f_k}) > 1 - 1/k.$$

Thus, since $\mathcal{X}_{f_k} \subseteq \mathcal{X}_0$ for all $k \in \mathbb{N}$,

$$\mu(\mathcal{X}_0) \ge \mu\left(\bigcup_k \mathcal{X}_{f_k}\right) = 1.$$

□

PROOF OF THEOREM 1.1. The proof follows by plugging Theorem 6.5, Corollary 6.18, and Lemma 6.19 into the proof of Theorem 3.9 in [9]. □

Acknowledgments. We thank Itai Benjamini for suggesting the problem, and for very helpful discussions. We also thank Gady Kozma for helpful discussions, and Ofer Zeitouni and Nathanael Berestycki for their help with Lemma 6.9.

REFERENCES

[1] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453

[2] CONWAY, J. B. (1995). *Functions of One Complex Variable. II. Graduate Texts in Mathematics* **159**. Springer, New York. MR1344449

[3] DUDLEY, R. M. (1989). *Real Analysis and Probability*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA. MR0982264

[4] KOZMA, G. (2007). The scaling limit of loop-erased random walk in three dimensions. *Acta Math.* **199** 29–152. MR2350070

[5] LAWLER, G. F. (1980). A self-avoiding random walk. *Duke Math. J.* **47** 655–693. MR0587173

[6] LAWLER, G. F. (1982). Weak convergence of a random walk in a random environment. *Comm. Math. Phys.* **87** 81–87. MR0680649

[7] LAWLER, G. F. (1999). Loop-erased random walk. In *Perplexing Problems in Probability. Progress in Probability* **44** 197–217. Birkhäuser, Boston, MA. MR1703133

[8] LAWLER, G. F. (2005). *Conformally Invariant Processes in the Plane. Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588

[9] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2004). Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32** 939–995. MR2044671

[10] MÖRTERS, P. and PERES, Y. (2010). *Brownian Motion*. Cambridge Univ. Press, Cambridge.

[11] POMMERENKE, C. (1992). *Boundary Behaviour of Conformal Maps. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **299**. Springer, Berlin. MR1217706

[12] ROHDE, S. and SCHRAMM, O. (2005). Basic properties of SLE. *Ann. of Math.* (2) **161** 883–924. MR2153402

[13] SCHRAMM, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084

[14] SHIRYAEV, A. N. (1996). *Probability*, 2nd ed. *Graduate Texts in Mathematics* **95**. Springer, New York. MR1368405

[15] WERNER, W. (2004). Random planar curves and Schramm–Loewner evolutions. In *Lectures on Probability Theory and Statistics*. *Lecture Notes in Math*. **1840** 107–195. Springer, Berlin. MR2079672

STATISTICAL LABORATORY
UNIVERSITY OF CAMBRIDGE
DPMMS, 3 WILBERFORCE ROAD
CAMBRIDGE, CB3 0WB
UNITED KINGDOM
E-MAIL: a.yadin@statslab.cam.ac.uk

DEPARTMENT OF MATHEMATICS
TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA 32000
ISRAEL
E-MAIL: amir.yehudayoff@gmail.com