ON THE MOMENTS AND THE INTERFACE OF THE SYMBIOTIC BRANCHING MODEL

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In this paper we introduce a critical curve separating the asymptotic behavior of the moments of the symbiotic branching model, introduced by Etheridge and Fleischmann [*Stochastic Process. Appl.* **114** (2004) 127–160] into two regimes. Using arguments based on two different dualities and a classical result of Spitzer [*Trans. Amer. Math. Soc.* **87** (1958) 187–197] on the exit-time of a planar Brownian motion from a wedge, we prove that the parameter governing the model provides regimes of bounded and exponentially growing moments separated by subexponential growth. The moments turn out to be closely linked to the limiting distribution as time tends to infinity. The limiting distribution can be derived by a self-duality argument extending a result of Dawson and Perkins [*Ann. Probab.* **26** (1998) 1088–1138] for the mutually catalytic branching model.

As an application, we show how a bound on the 35th moment improves the result of Etheridge and Fleischmann [*Stochastic Process. Appl.* **114** (2004) 127–160] on the speed of the propagation of the interface of the symbiotic branching model.

1. Introduction. In 2004, Etheridge and Fleischmann [8] introduced a stochastic spatial model of two interacting populations known as the symbiotic branching model, parametrized by a parameter $\varrho \in [-1, 1]$ governing the correlation between the two driving noises. The model can be considered in three different spatial setups which we now explain.

First, the continuous-space symbiotic branching model is given by the system of stochastic partial differential equations

$$(1.1) \quad \text{cSBM}(\varrho, \kappa)_{u_0, v_0} : \begin{cases} \dfrac{\partial}{\partial t} u_t(x) = \dfrac{1}{2}\Delta u_t(x) + \sqrt{\kappa u_t(x) v_t(x)}\, dW_t^1(x), \\[2mm] \dfrac{\partial}{\partial t} v_t(x) = \dfrac{1}{2}\Delta v_t(x) + \sqrt{\kappa u_t(x) v_t(x)}\, dW_t^2(x), \\[2mm] u_0(x) \geq 0, \qquad x \in \mathbb{R}, \\[1mm] v_0(x) \geq 0, \qquad x \in \mathbb{R}, \end{cases}$$

where Δ denotes the Laplace operator and $\kappa > 0$ is a fixed constant known as the branching rate. $\mathbf{W} = (W^1, W^2)$ is a pair of correlated standard Gaussian white noises on $\mathbb{R}_+ \times \mathbb{R}$ with correlation $\varrho \in [-1, 1]$, that is, the unique Gaussian process with covariance structure

$$(1.2) \qquad \mathbb{E}[W_{t_1}^1(A_1) W_{t_2}^1(A_2)] = (t_1 \wedge t_2)\ell(A_1 \cap A_2),$$

$$(1.3) \qquad \mathbb{E}[W_{t_1}^2(A_1) W_{t_2}^2(A_2)] = (t_1 \wedge t_2)\ell(A_1 \cap A_2),$$

$$(1.4) \qquad \mathbb{E}[W_{t_1}^1(A_1) W_{t_2}^2(A_2)] = \varrho(t_1 \wedge t_2)\ell(A_1 \cap A_2),$$

where ℓ denotes Lebesgue measure, $A_1, A_2 \in \mathcal{B}(\mathbb{R})$ and $t_1, t_2 \geq 0$. Note that we work with a white noise $\mathbf{W}$ in the sense of Walsh [25]. Solutions of this model have been considered rigorously in the framework of the corresponding martingale problem in Theorem 4 of [8], which states that, under suitable conditions on the initial conditions $u_0(\cdot)$, $v_0(\cdot)$, a solution exists for all $\varrho \in [-1, 1]$. The martingale problem is well posed for all $\varrho \in [-1, 1)$, which implies the strong Markov property except in the boundary case $\varrho = 1$.

For a discrete spatial version we consider the system of interacting diffusions on $\mathbb{Z}^d$, with values in $\mathbb{R}_{\geq 0}$, defined by the coupled stochastic differential equations

$$(1.5) \qquad \text{dSBM}(\varrho, \kappa)_{u_0, v_0} : \begin{cases} du_t(i) = \Delta u_t(i)\, dt + \sqrt{\kappa u_t(i) v_t(i)}\, dB_t^1(i), \\ dv_t(i) = \Delta v_t(i)\, dt + \sqrt{\kappa u_t(i) v_t(i)}\, dB_t^2(i), \\ u_0(i) \geq 0, \qquad i \in \mathbb{Z}^d, \\ v_0(i) \geq 0, \qquad i \in \mathbb{Z}^d, \end{cases}$$

where now $\{B^1(i), B^2(i)\}_{i \in \mathbb{Z}^d}$ is a family of standard Brownian motions with covariances given by

$$(1.6) \qquad [B_\cdot^n(i), B_\cdot^m(j)]_t = \begin{cases} \varrho t, & i = j \text{ and } n \neq m, \\ t, & i = j \text{ and } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

In the discrete case, Δ denotes the discrete Laplacian

$$\Delta u_t(i) = \sum_{|k-i|=1} \frac{1}{2d} \big(u_t(k) - u_t(i)\big).$$

Note that in this paper we denote by $[N_\cdot, M_\cdot]_t$ the cross-variation of two martingales N, M. This is to avoid confusion with $\langle f, g \rangle$ which will be defined to be the sum (resp., integral) of the product of f and g.

Finally, the nonspatial symbiotic branching model is defined by the stochastic differential equations

$$\text{SBM}(\varrho, \kappa)_{u_0, v_0} : \begin{cases} du_t = \sqrt{\kappa u_t v_t}\, dB_t^1, \\ dv_t = \sqrt{\kappa u_t v_t}\, dB_t^2, \\ u_0 \geq 0, \\ v_0 \geq 0. \end{cases}$$

Again, the noises are correlated with $[B_\cdot^1, B_\cdot^2]_t = \varrho t$. This simple toy-model (see also [19] and [6]) can be analyzed quite simply and will be used to prove properties of the spatial models.

CONVENTION 1.1.   From time to time we skip the dependence on ϱ, κ, u_0 and v_0 if there is no ambiguity. Solutions of cSBM, SBM and dSBM for $d \leq 2$ are called symbiotic branching processes in the recurrent case whereas solutions of dSBM for $d \geq 3$ are called symbiotic branching processes in the transient case.

Interestingly, symbiotic branching models include well-known spatial models from different branches of probability theory. In the discrete spatial case (and analogously in continuous-space) interacting diffusions of the type

$$(1.7) \qquad dw_t(i) = \Delta w_t(i)\, dt + \sqrt{\kappa f(w_t(i))}\, dB_t(i)$$

have been studied extensively in the literature. Some important examples are the following:

EXAMPLE 1.   The stepping stone model from mathematical genetics: $f(x) = x(1-x)$.

EXAMPLE 2.   The parabolic Anderson model (with Brownian potential) from mathematical physics: $f(x) = x^2$.

EXAMPLE 3.   The super random walk from probability theory: $f(x) = x$.

For the super random walk, κ is the branching rate which in this case is time–space independent. In [7], a two-type model based on two super random walks with time–space dependent branching was introduced. The branching rate for one species is proportional to the value of the other species. More precisely, the authors considered

$$du_t(i) = \Delta u_t(i)\, dt + \sqrt{\kappa u_t(i)v_t(i)}\, dB_t^1(i),$$
$$dv_t(i) = \Delta v_t(i)\, dt + \sqrt{\kappa u_t(i)v_t(i)}\, dB_t^2(i),$$

where now $\{B^1(i), B^2(i)\}_{i \in \mathbb{Z}^d}$ is a family of independent standard Brownian motions. Solutions are called mutually catalytic branching processes. In the following years, properties of this model were well studied (see, e.g., [3] and [2]). The corresponding continuous-space version was also treated in [7].

For correlation $\varrho = 0$, solutions of the symbiotic branching model are obviously solutions of the mutually catalytic branching model. The case $\varrho = -1$ with the additional assumption $u_0 + v_0 \equiv 1$ corresponds to the stepping stone model. To see this, observe that in the perfectly negatively correlated case $B^1(i) = -B^2(i)$ which implies that the sum $u + v$ solves a discrete heat equation and with the

further assumption $u_0 + v_0 \equiv 1$ stays constant for all time. Hence, for all $t \geq 0$, $u(t, \cdot) \equiv 1 - v(t, \cdot)$, which shows that u is a solution of the stepping stone model with initial condition u_0 and v is a solution with initial condition v_0. Finally, suppose w is a solution of the parabolic Anderson model, then, for $\varrho = 1$, the pair $(u, v) := (w, w)$ is a solution of the symbiotic branching model with initial conditions $u_0 = v_0 = w_0$.

The purpose of this and the accompanying paper [1] is to understand the nature of the symbiotic branching model better. How does the model depend on the correlation ϱ? Are properties of the extremal cases $\varrho \in \{-1, 0, 1\}$ inherited by some parts of the parameter space? Since the longtime behavior of the super random walk, stepping stone model, mutually catalytic branching model and parabolic Anderson model is very different, one might guess that the parameter space $[-1, 1]$ can be divided into disjoint subsets corresponding to different regimes.

The focus of [1] is second moment properties. In the discrete setting, but with a more general setup, growth of second moments is analyzed in detail. A moment duality is used to reduce the problem to moment generating functions and Laplace transforms of local times of discrete-space Markov processes. A precise analysis of those is used to derive intermittency and aging results which show that different regimes occur for $\varrho < 0$, $\varrho = 0$ and $\varrho > 0$.

In contrast to [1], the present paper is not restricted to second moment properties. The aim is to understand the pathwise behavior of symbiotic branching processes better.

REMARK 1.2. In this paper, we restrict ourselves to the simplest setups which already provide the full variety of results. For the discrete spatial model we thus restrict ourselves to the discrete Laplacian instead of allowing more general transitions. This is not necessary; see [7] or [2] for a construction of solutions and main properties for more general underlying migration mechanisms in the case $\varrho = 0$. Furthermore, we mainly restrict ourselves to homogeneous initial conditions and remark where results hold more generally. Here, for nonnegative real numbers we denote by **u** the constant functions $u(\cdot) \equiv u$.

The paper is organized as follows: our main results are presented in Section 2. Before proving the results, we collect basic properties of the symbiotic branching models and discuss the dualities that we need. This is carried out in Section 3. The final sections are devoted to the proofs. In Section 4, proofs of the longtime convergence in law are given, and in Section 5 we discuss the longtime behavior of moments. Finally, in Section 6 we show how to use the results of Section 5 to strengthen the main result of [8].

2. Results. Before stating the main results, we briefly recall from [8] that the state space of cSBM is given by pairs of tempered functions, that is, pairs of functions contained in

$$M_{\text{tem}} = \left\{ u \mid u : \mathbb{R} \to \mathbb{R}_{\geq 0}, \lim_{|x| \to \infty} u(x) \phi_\lambda(x) \text{ exists and } \|u \phi_\lambda\|_\infty < \infty \ \forall \lambda < 0 \right\},$$

where $\phi_\lambda(x) = e^{\lambda|x|}$, and we think of M_{tem} as being topologized by the metric given in [8], equation (13), yielding a Polish space.

The state space for dSBM is similar. It was not discussed in [8] and so we present details in Section 3.

2.1. *Convergence in law.* We begin with a result, generalizing Theorem 1.5 of [7], on the longtime behavior of the laws of symbiotic branching processes in the recurrent case.

PROPOSITION 2.1. *Suppose (u_t, v_t) is a spatial symbiotic branching process in the recurrent case with $\varrho \in (-1, 1)$, $\kappa > 0$ and initial conditions $u_0 = \mathbf{u}$, $v_0 = \mathbf{v}$. Let B^1 and B^2 be two Brownian motions with covariance*

$$[B^1_\cdot, B^2_\cdot]_t = \varrho t, \qquad t \geq 0,$$

and initial conditions $B^1_0 = u$, $B^2_0 = v$. Further, let

$$\tau = \inf\{t \geq 0 : B^1_t B^2_t = 0\}$$

be the first exit time of the correlated Brownian motions B^1, B^2 from the upper right quadrant. Then, weakly in M^2_{tem},

$$\mathbb{P}^{\mathbf{u}, \mathbf{v}}[(u_t, v_t) \in \cdot] \Rightarrow P^{u, v}[(\bar{B}^1_\tau, \bar{B}^2_\tau) \in \cdot]$$

as $t \to \infty$. Here, $(\bar{B}^1_\tau, \bar{B}^2_\tau)$ denotes the pair of constant functions on $\mathbb{R}$, respectively, $\mathbb{Z}^d$ ($d = 1, 2$) taking the values of the stopped Brownian motions (B^1_τ, B^2_τ).

In particular, the proposition shows ultimate extinction of one species in law.

REMARK 2.2. For simplicity, Proposition 2.1 is formulated for constant initial conditions even though the result holds more generally. Theorem 1.5 of [7] (the case $\varrho = 0$) was extended in [4] to nondeterministic initial conditions: for fixed $u, v \geq 0$ let $\mathcal{M}_{u,v}$ be the set of probability measures ν on M^2_{tem} such that

$$(2.1) \qquad \sup_{x \in \mathbb{R}} \int (a^2(x) + b^2(x)) \, d\nu(a, b) < \infty$$

and

$$(2.2) \quad \lim_{t \to \infty} \int \left[(P_t a(x) - u)^2 + (P_t b(x) - v)^2 \right] d\nu(a, b) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

Here, (P_t) denotes the transition semigroup of Brownian motion (the definition for the discrete case is similar). The proof of [4] can also be applied to $\varrho \neq 0$ and, thus, Proposition 2.1 holds in the same way for initial distributions $\nu \in \mathcal{M}_{u,v}$.

The restriction to $\varrho \in (-1, 1)$ arises from our method of proof which exploits a self-duality of the process which gives no information for $\varrho \in \{-1, 1\}$. Let us briefly discuss the behavior of the limiting distributions in the boundary cases $\varrho \in \{-1, 1\}$ which are well known in the literature and fit neatly into our result. First, suppose (w_t) is a solution of the stepping stone model (see Example 1) and $w_0 \equiv w \in [0, 1]$. It was proved in [20] that

$$(2.3) \qquad \mathcal{L}^{\mathbf{w}}(w_t) \stackrel{t \to \infty}{\Rightarrow} w\delta_{\mathbf{1}} + (1 - w)\delta_{\mathbf{0}},$$

where $\delta_{\mathbf{1}}$ (resp., $\delta_{\mathbf{0}}$) denotes the Dirac distribution concentrated on the constant function $\mathbf{1}$ (resp., $\mathbf{0}$). This can be reformulated in terms of perfectly anti-correlated Brownian motions (B^1, B^2) as before: for $\varrho = -1$, the pair (B^1, B^2) takes values only on the straight line connecting $(0, 1)$ and $(1, 0)$, and stops at the boundaries. Hence, the law of (B^1_τ, B^2_τ) is a mixture of $\delta_{(0,1)}$ and $\delta_{(1,0)}$ and the probability of hitting $(1, 0)$ is equal to the probability of a one-dimensional Brownian motion started in $w \in [0, 1]$ hitting 1 before 0, which is w, and hence matches (2.3). Second, let (w_t) be a solution of the parabolic Anderson model with Brownian potential (see Example 2) and constant initial condition $w_0 \equiv w \geq 0$. In [21] it was shown that

$$(2.4) \qquad \mathcal{L}^{\mathbf{w}}(w_t) \stackrel{t \to \infty}{\Rightarrow} \delta_{\mathbf{0}}.$$

As discussed above, when viewed as a symbiotic branching process with $\varrho = 1$, this implies

$$(2.5) \qquad \mathcal{L}^{\mathbf{w},\mathbf{w}}(u_t, v_t) \stackrel{t \to \infty}{\Rightarrow} \delta_{\mathbf{0},\mathbf{0}}.$$

From the viewpoint of two perfectly positive-correlated Brownian motions, we obtain the same result since they simple move on the diagonal dissecting the upper right quadrant until they eventually get absorbed in the origin, that is, $(B^1_\tau, B^2_\tau) = (0, 0)$ almost surely.

To summarize, we have seen that the weak longtime behavior (in the recurrent case) of the classical models connected to symbiotic branching is appropriately described by correlated Brownian motions hitting the boundary of the upper right quadrant.

2.2. *Nonalmost-sure behavior.*   In contrast to extinction in law, the almost-sure behavior is very different. In the recurrent case for the mutually catalytic branching model, Cox and Klenke [3] showed that, almost surely, there is no longtime local extinction of any type, but in fact the locally predominant type changes infinitely often. It is not hard to see that the same is true for symbiotic branching with $\varrho \in (-1, 1)$. We do not give a proof since it follows from Proposition 2.1 along the same lines as in [3].

PROPOSITION 2.3.   *Let $\varrho \in (-1, 1)$, $\kappa > 0$ and suppose (u_t, v_t) is a spatial symbiotic branching process in the recurrent case with initial distribution $u_0 = \mathbf{u}$, $v_0 = \mathbf{v}$. Then, for all $(u', v') \in \{(x, 0) : x \in \mathbb{R}_{\geq 0}\} \cup \{(0, y) : y \in \mathbb{R}_{\geq 0}\}$ and $K \subset \mathbb{R}$ bounded,*

$$\mathbb{P}^{\mathbf{u}, \mathbf{v}}\Big[\liminf_{t \to \infty} \sup_{x \in K} \|(u_t(x), v_t(x)) - (u', v')\| = 0\Big] = 1,$$

respectively, for $K \subset \mathbb{Z}^d$ bounded,

$$\mathbb{P}^{\mathbf{u}, \mathbf{v}}\Big[\liminf_{t \to \infty} \sup_{k \in K} \|(u_t(k), v_t(k)) - (u', v')\| = 0\Big] = 1.$$

Again, as in Remark 2.2, the result holds for random initial conditions of the class $\mathcal{M}_{u,v}$. Note that Proposition 2.3 depends strongly on the spatial structure since in the nonspatial model almost sure convergence holds (see Proposition 4.4).

2.3. *Longtime behavior of moments.*   In [1] the second moments of symbiotic branching processes are analyzed. This particular case admits a detailed study since a moment duality (see Lemma 3.3) has a particularly simple structure which allows one to reduce the study of the moments to that of moment generating functions and Laplace transforms of local times. Here we are interested in the behavior of moments as t tends to infinity. The two available dualities (self-duality and moment duality) are combined in two steps. First, a self-duality argument combined with an equivalence between bounded moments of the exit time distribution and of the exit point distribution for correlated Brownian motions stopped on exiting the first quadrant is used to understand the effect of ϱ. It turns out that for any $p > 1$ there are critical values, independent of κ, dividing regimes in which the moments $\mathbb{E}^{1,1}[u_t^p]$, $\mathbb{E}^{1,1}[u_t(k)^p]$ and $\mathbb{E}^{1,1}[u_t(x)^p]$ are bounded in t or grow to infinity. Second, for $p \in \mathbb{N}$, a perturbation argument combined with the first step and a moment duality is used to analyze the growth to infinity in more detail.

The following critical curve captures the effect of ϱ. Note that the definition is independent of κ which will become important in the second step.

DEFINITION 2.4.   We define the critical curve of symbiotic branching models to be the real-valued function $p : (-1, 1) \to \mathbb{R}^+$, given by

$$(2.6) \qquad p(\varrho) = \frac{\pi}{\pi/2 + \arctan(\varrho/(\sqrt{1 - \varrho^2}))}.$$

Its inverse will be denoted by $\varrho(p)$ for $p > 1$.

The critical curve is plotted in Figure 1. Here, $\varrho(35)$ and $\varrho(2)$ are marked. Thirty-fifth moments are the key for the improved wavespeed result below and the special case $\varrho(2) = 0$ is discussed in [1]. We will see in Section 5 that this
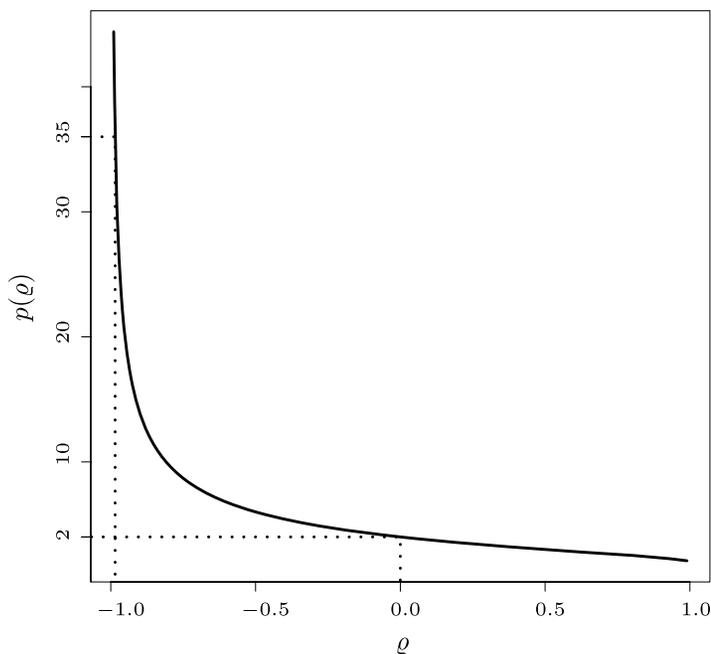
FIG. 1. *The critical curve* $p(\varrho), \varrho \in (-1, 1)$.

curve is closely connected with the exit distribution of (B_τ^1, B_τ^2) from the upper right quadrant which appeared in Proposition 2.1 above. The first main theorem states that the critical curve separates two regimes (independently of κ): that of bounded moments and that of unbounded moments.

THEOREM 2.5. *Suppose* (u_t, v_t) *is a symbiotic branching process* (*either nonspatial*, *continuous space or discrete space in arbitrary dimension*) *with initial conditions* $u_0 = v_0 = \mathbf{1}$. *If* $\varrho \in (-1, 1)$, *then, for any* $\kappa > 0$, *the following hold for* $p > 1$:

(i) *In the recurrent case,*

$$\varrho < \varrho(p) \quad \Leftrightarrow \quad \mathbb{E}^{1,1}[u_t^p], \mathbb{E}^{1,1}[u_t(k)^p] \text{ and } \mathbb{E}^{1,1}[u_t(x)^p] \text{ are bounded in } t.$$

(ii) *In the transient case,*

$$\varrho < \varrho(p) \quad \Rightarrow \quad \mathbb{E}^{1,1}[u_t(k)^p] \text{ is bounded in } t.$$

Due to symmetry the same holds for $\mathbb{E}^{1,1}[v_t^p], \mathbb{E}^{1,1}[v_t(k)^p]$ *and* $\mathbb{E}^{1,1}[v_t(x)^p]$.

Note that the theorem provides information about all positive real moments, not just integer moments. In the area below the critical curve in Figure 1, the moments remain bounded. On and above the critical curve, in the recurrent case, the moments grow to infinity.

REMARK 2.6.   For $\varrho = -1$ the curve could be extended with $p(-1) = \infty$. In terms of the previous theorem this makes sense since for $\varrho = -1$, symbiotic branching processes with initial conditions $u_0 = v_0 = \mathbf{1}$ are bounded by 2. This is justified by a simple observation: for initial conditions $u_0 = v_0 \equiv 1/2$ symbiotic branching processes with $\varrho = -1$ are solutions of the stepping stone model and, hence, bounded by 1. Uniqueness in law of solutions implies that solutions (u_t, v_t) with initial conditions (cu_0, cv_0) are equal in law to solutions c times solutions with initial conditions (u_0, v_0).

With this first understanding of the effect of ϱ on moments, we may discuss integer moments for the discrete-space model in more detail. Let us first recall some known results for solutions (w_t) of the parabolic Anderson model (see Example 2) where only the parameter κ appears. Using Itô's lemma, one sees that $m(t, k_1, \ldots, k_n) := \mathbb{E}^{\mathbf{1}}[w_t(k_1) \cdots w_t(k_n)]$ solves the (discrete-space) partial differential equation

$$\frac{\partial}{\partial t} m(t, k_1, \ldots, k_n) = \Delta m(t, k_1, \ldots, k_n) + V(k_1, \ldots, k_n) m(t, k_1, \ldots, k_n)$$

with homogeneous initial conditions. Here, the potential V is given by

$$V(k_1, \ldots, k_n) = \kappa \sum_{1 \leq i < j \leq n} \delta_0(k_i - k_j).$$

Since $H = -\Delta - V$ is an n-particle Schrödinger operator, many properties are known from the physics literature. In particular, it is well known that in the recurrent case (the potential is nonnegative) exponential growth of solutions holds for any $\kappa > 0$. By contrast, in the transient case the discrete Laplacian requires a stronger perturbation before we see exponential growth. Intuitively from the particle picture this should be true since the potential V only increases solutions if particles meet, which occurs less frequently in the transient case. For the transient case (see, e.g., [5] or [9] for more precise results), there is a decreasing sequence $\kappa(n)$ such that

$$\mathbb{E}^{\mathbf{1}}[w_t(k)^n] \text{ is bounded in } t \quad \Leftrightarrow \quad \kappa < \kappa(n)$$

and for the Lyapunov exponents

$$\gamma_n(\kappa) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}^{\mathbf{1}}[w_t(k)^n] > 0 \quad \Leftrightarrow \quad \kappa > \kappa(n).$$

These results can be proved with the n-particle path-integral representation in which solutions are expressed as

$$m(t, k_1, \ldots, k_n) = \mathbb{E}\big[e^{\kappa \int_0^t V(X_s^1, \ldots, X_s^n) \, ds}\big],$$

where $(X_t^1), \ldots, (X_t^n)$ are independent simple random walks started in $k_1, \ldots, k_n$.

Coming back to the symbiotic branching model, we ask whether or not the nth Lyapunov exponents

$$\gamma_n(\varrho, \kappa) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}^{1,1}[u_t(k)^n]$$

exist and in which cases $\gamma_n(\varrho, \kappa)$ is strictly positive. As for the parabolic Anderson model, there is a system of partial differential equations describing the moments (see Proposition 16 of [8] for the continuous-space model) and an n-particle path-integral representation of the moments. In addition to the independent motion, the particles now carry a color which randomly changes if particles of the same color stay at the same site (see Lemma 3.3). With $L_t^=$ denoting collision times of particles of same color and $L_t^{\neq}$ denoting collision times of particles of different colors, the path-integral representation of moments reads

$$\mathbb{E}^{1,1}[u_t(k)^n] = \mathbb{E}\big[e^{\kappa(L_t^= + \varrho L_t^{\neq})}\big].$$

This representation is more involved than the path-integral representation for the parabolic Anderson model since, in addition to the motion of particles, a second stochastic mechanism is included. Nonetheless, we use it to prove the following theorem which reveals that even in the recurrent case a nontrivial transition occurs.

THEOREM 2.7.   *For solutions of* dSBM$(\varrho, \kappa)_{1,1}$, *in any dimension, the following hold for* $n \in \mathbb{N}, n > 1$:

   (i) $\gamma_n(\varrho, \kappa)$ *exists for any* $\varrho \in [-1, 1]$, $\kappa > 0$,
   (ii) $\gamma_n(\varrho(n), \kappa) = 0$ *for any* $\kappa > 0$,
   (iii) *for any* $\varrho > \varrho(n)$ *there is a critical* $\kappa(n)$ *such that* $\gamma_n(\varrho, \kappa) > 0$ *if* $\kappa > \kappa(n)$.

Combined with Theorem 2.5, parts (ii) and (iii) emphasize the "criticality" of the critical curve: for $\varrho < \varrho(n)$, moments stay bounded, for $\varrho = \varrho(n)$ moments grow subexponentially fast to infinity, and for $\varrho > \varrho(n)$ moments grow exponentially fast if κ is large enough.

REMARK 2.8.   As discussed above, for the parabolic Anderson model it is natural that in the transient case perturbing the critical case does not immediately yield exponential growth, whereas perturbing the recurrent case does immediately lead to exponential growth. It is clear that in the transient case the gap in (iii) of Theorem 2.7 is really necessary: for small κ moments of the parabolic Anderson model are bounded. Since moments of symbiotic branching are dominated by moments of the parabolic Anderson model (see Lemma 3.3), for small κ moments are bounded for all ϱ.

In the case $p \notin \mathbb{N}$ there seems to be no reason why exponential growth should fail. Unfortunately, in this case there is no moment duality and hence the most useful tool to analyze exponential growth is not available.

CONJECTURE 2.9.   *In the recurrent case the moment diagram for symbiotic branching* (*Figure* 1) *describes the moments as follows*: *pairs* (ϱ, p) *below the critical curve correspond precisely to bounded moments*, *pairs at the critical curve correspond to moments which grow subexponentially fast to infinity and pairs above to the critical curve correspond to exponentially growing moments.*

A deeper understanding of the Lyapunov exponents as functions of ϱ, κ remains mainly open (for an upper bound see Proposition 5.3). For second moments $[\varrho(2) = 0]$ this is carried out in [1]. It is shown that exponential growth holds for $\varrho > 0$ and arbitrary $\kappa > 0$ in the recurrent case, whereas only for $\kappa > 2/(\varrho G_\infty(0, 0))$ in the transient case. Here G_∞ denotes the Green function of the simple random walk. The exponential (and subexponential) growth rates were analyzed in detail by Tauberian theorems.

A direct application of Theorem 2.7 is so-called intermittency of solutions. One says a spatial system with Lyapunov exponents γ_p is p-intermittent if

$$\frac{\gamma_p}{p} < \frac{\gamma_{p+1}}{p+1}.$$

Intermittent systems concentrate on few peaks with extremely high intensity (see [10]). The results above show that as ϱ tends to -1, solutions (at least for large κ) are p-intermittent for p tending to infinity. This holds since for fixed ϱ, the pth moments are bounded if (ϱ, p) lies below the critical curve. Increasing p (and κ if necessary) there is a first p such that the pth Lyapunov exponent is positive. Intermittency for higher exponents suggests that the effect gets weaker. This is to be expected since for $\varrho = -1$ solutions with homogeneous initial conditions are bounded and, hence, solutions do not produce high peaks at all. Making this effect more precise, in particular combined with the effect of Proposition 2.1, is an interesting task for the future.

2.4. *Speed of propagation of the interface.*   Let us conclude with a direct application of the moment bounds. Here, we will be concerned with an improved upper bound on the speed of the propagation of the interface of continuous-space symbiotic branching processes which served to some extent as the motivation for this work. To explain this, we need to introduce the notion of the interface of continuous-space symbiotic branching processes introduced in [8].

DEFINITION 2.10.   The interface at time t of a solution (u_t, v_t) of the symbiotic branching model cSBM$(\varrho, \kappa)_{u_0, v_0}$ with $\varrho \in [-1, 1]$ is defined as

$$\text{Ifc}_t = \text{cl}\{x : u_t(x)v_t(x) > 0\},$$

where cl(A) denotes the closure of the set A in $\mathbb{R}$.

In particular, we will be interested in complementary Heaviside initial conditions

$$u_0(x) = \mathbf{1}_{\mathbb{R}^-}(x) \quad \text{and} \quad v_0(x) = \mathbf{1}_{\mathbb{R}^+}(x), \qquad x \in \mathbb{R}.$$

The main question addressed in [8] is whether for the above initial conditions the so-called compact interface property holds, that is, whether the interface is compact at each time almost surely. This is answered affirmatively in Theorem 6 in [8], together with the assertion that the interface propagates with at most linear speed, that is, for each $\varrho \in [-1, 1]$ there exists a constant $c > 0$ and a finite random-time T_0 so that almost surely for all $T \geq T_0$

$$\bigcup_{t \leq T} \text{Ifc}_t \subseteq [-cT, cT].$$

Heuristically, due to the scaling property of the symbiotic branching model (Lemma 8 of [8]) one expects that the interface should move with a square-root speed. Indeed, with the help of Theorem 2.5 one can strengthen their result, at least for sufficiently small ϱ, to obtain almost square-root speed.

THEOREM 2.11. *Suppose (u_t, v_t) is a solution of* $\text{cSBM}(\varrho, \kappa)_{\mathbf{1}_{\mathbb{R}^-}, \mathbf{1}_{\mathbb{R}^+}}$ *with* $\varrho < \varrho(35)$ *and $\kappa > 0$. Then there is a constant $C > 0$ and a finite random-time T_0 such that almost surely*

$$\bigcup_{t \leq T} \text{Ifc}_t \subseteq \left[-C\sqrt{T \log(T)}, C\sqrt{T \log(T)} \right]$$

for all $T > T_0$.

The restriction to $\varrho < \varrho(35)$ is probably not necessary and only caused by the technique of the proof. Though $\varrho(35) \approx -0.9958$ is rather close to -1, the result is interesting. It shows that sub-linear speed of propagation is not restricted to situations in which solutions are uniformly bounded as they are for $\varrho = -1$. The proof is based on the proof of [8] for linear speed which carries over the proof of [24] for the stepping stone model to nonbounded processes. We are able to strengthen the result by using a better moment bound which is needed to circumvent the lack of uniform boundedness.

REMARK 2.12. We believe that, at least for $\varrho \leq 0$, the speed of propagation should be at most $C'\sqrt{t}$, for some suitable constant C', that is, for all T greater than some $T' > 0$,

$$\bigcup_{t \leq T} \text{Ifc}_t \subseteq [-C'\sqrt{T}, C'\sqrt{T}].$$

However, it seems unclear how to obtain such a refinement of Theroem 2.11 based on our moment results and the method of [24] (resp., [8]). As subexponential

bounds of higher moments cannot be avoided (see the proof of the fluctuation term estimate Lemma 6.2), our results on the behavior of higher moments show that at present, in light of Conjecture 2.9, one can only hope for stronger results for very small ϱ.

To overcome this limitation, new methods need to be employed. The authors think that a possible approach could be based on the scaling property (Lemma 8 of [8]) and recent results by Klenke and Oeler [13]. Recall that the scaling property states that if (u_t, v_t) is a solution to cSBM$(\varrho, \kappa)_{u_0, v_0}$, then

$$(u_t(x)^K, v_t(x)^K) := (u_{Kt}(\sqrt{K}x), v_{Kt}(\sqrt{K}x)), \qquad x \in \mathbb{R}, K > 0,$$

is a solution to cSBM$(\varrho, K \cdot \kappa)_{u_0^K, v_0^K}$ (with suitably transformed initial states u_0^K, v_0^K). In other words, a diffusive time–space rescaling leads to the original model with a suitably increased branching rate κ. Klenke and Oeler [13] show that, at least for the mutually catalytic model in discrete space, a nontrivial limiting process as $\kappa \to \infty$ exists. This limit is called "infinite rate mutually catalytic branching process" (see also [11, 12] for a further discusion). In particular, in Corollary 1.2 of [13] they claim that, under suitable assumptions, a nontrivial interface for the limiting process exists, which would in turn predict a square-root speed of propagation in our case. However, to make this approach rigorous is beyond the scope of the present paper.

REMARK 2.13 (Shape of the interface).   Note that our results give only limited information about the shape of the interface. For the case $\varrho = -1$, that is, with locally constant total population size, it is shown in [16] that there exists a unique stationary interface law, which may therefore be interpreted as a "stationary wave" whose position fluctuates at the boundaries, according to [24], like a Brownian motion, hence explaining the square-root speed (note that for both results, suitable bounds on fourth mixed moments are required). However, for $\varrho > -1$, the population sizes of the interface are expected to fluctuate significantly and it seems unclear how this affects the shape and speed of the interface, in particular the formation of a "stationary wave." The significance of fourth mixed moments might even lead to a phase-transition in ϱ. This gives rise to many interesting open questions.

3. Basic properties and duality.   In this section we review the setting and properties of the discrete-space model, whereas for continuous-space we refer to [8]. Note that instead of using the state space of tempered functions alternatively we may use a suitable Liggett–Spitzer space. As the results are only presented for the discrete Laplacian this does not play a crucial role. For a discussion of the mutually catalytic branching model in the Liggett–Spitzer space see [2].

3.1. *Basic properties*.   For functions $f, g : \mathbb{Z}^d \to \mathbb{R}$ we abbreviate $\langle f, g \rangle = \sum_k f(k)g(k)$. With $\phi_\lambda(k) = e^{\lambda|k|}$ the space of pairs of tempered sequences is de-

fined by

$$M_{\text{tem}}^2 = \{(u, v)|u, v : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}, \langle u, \phi_\lambda \rangle, \langle v, \phi_\lambda \rangle < \infty \ \forall \lambda < 0\}.$$

The space of continuous paths is denoted by

$$\Omega_{\text{tem}} = C(\mathbb{R}_{\geq 0}, M_{\text{tem}}^2).$$

Similarly, the space of pairs of rapidly decreasing sequences is defined by

$$M_{\text{rap}}^2 = \{(u, v)|u, v : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}, \langle u, \phi_\lambda \rangle, \langle v, \phi_\lambda \rangle < \infty \ \forall \lambda > 0\}$$

and the corresponding path space by

$$\Omega_{\text{rap}} = C(\mathbb{R}_{\geq 0}, M_{\text{rap}}^2).$$

Weak solutions are defined as in [7] for $\varrho = 0$. In much the same way as for Theorems 1.1 and 2.2 of [7], we obtain existence and the Green-function representation.

PROPOSITION 3.1. *Suppose* $(u_0, v_0) \in M_{\text{tem}}^2$ *(resp.,* M_{rap}^2*),* $\varrho \in [-1, 1]$ *and* $\kappa > 0$. *Then there is a weak solution of* $\text{dSBM}(\varrho, \kappa)_{u_0, v_0}$ *such that* $(u_t, v_t) \in \Omega_{\text{tem}}$ *(resp.,* Ω_{rap}*) and for all* $(\phi, \psi) \in M_{\text{rap}}^2$ *(resp.,* M_{tem}^2*)*

$$(3.1) \qquad \langle u_t, \phi \rangle = \langle u_0, P_t\phi \rangle + \sum_{j \in \mathbb{Z}^d} \int_0^t P_{t-s}\phi(j)\sqrt{\kappa u_s(j)v_s(j)}\, dB_s^1(j),$$

$$(3.2) \qquad \langle v_t, \psi \rangle = \langle v_0, P_t\psi \rangle + \sum_{j \in \mathbb{Z}^d} \int_0^t P_{t-s}\psi(j)\sqrt{\kappa u_s(j)v_s(j)}\, dB_s^2(j),$$

where $P_t f(k) = \sum_{j \in \mathbb{Z}^d} p_t(j, k) f(j)$ *is the semigroup associated to the simple random walk. In particular, we have*

$$(3.3) \qquad u_t(k) = P_t u_0(k) + \sum_{j \in \mathbb{Z}^d} \int_0^t p_{t-s}(j, k)\sqrt{\kappa u_s(j)v_s(j)}\, dB_s^1(j),$$

$$(3.4) \qquad v_t(k) = P_t v_0(k) + \sum_{j \in \mathbb{Z}^d} \int_0^t p_{t-s}(j, k)\sqrt{\kappa u_s(j)v_s(j)}\, dB_s^2(j).$$

The covariation structure of the Brownian motions is given by (1.6).

In fact, (3.1), (3.2) can be seen as the discrete-space versions of the martingale problem of Definition 3 in [8]. Further, (3.3), (3.4) are the discrete-space versions of the convolution form given in Corollary 20 of [8].

For the proofs of the longtime behavior of laws and moments, the key step is to transfer to the total mass processes $\langle u_t, \mathbf{1} \rangle$, $\langle v_t, \mathbf{1} \rangle$. To this end, in a similar way to Proposition 3.1, we define

$$M_F^2 = \{(u, v)|u, v : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}, \langle u, 1 \rangle, \langle v, 1 \rangle < \infty\}$$

and

$$\Omega_F = C(\mathbb{R}_{\geq 0}, M_F^2).$$

For summable initial conditions we obtain the following crucial martingale characterization.

PROPOSITION 3.2. *If* $(u_0, v_0) \in M_F^2$, *then each solution of* dSBM$(\varrho, \kappa)_{u_0, v_0}$ *has the following properties*: $(u_t, v_t) \in \Omega_F$ *and* $\langle u_t, \mathbf{1} \rangle, \langle v_t, \mathbf{1} \rangle$ *are nonnegative*, *continuous*, *square-integrable martingales with square-functions*

$$[\langle u_\cdot, \mathbf{1} \rangle]_t = [\langle v_\cdot, \mathbf{1} \rangle]_t = \kappa \int_0^t \langle u_s, v_s \rangle \, ds$$

and

$$[\langle u_\cdot, \mathbf{1} \rangle, \langle v_\cdot, \mathbf{1} \rangle]_t = \varrho \kappa \int_0^t \langle u_s, v_s \rangle \, ds.$$

We omit the proofs since they are basically standard. The only step where one needs to be careful is the existence proof. As usual for such models one first restricts the space to bounded subsets (boxes) of $\mathbb{Z}^d$, where standard Markov process theory applies. Enlarging the boxes one obtains a sequence of processes which are shown to converge to a limiting process solving dSBM$(\varrho, \kappa)_{u_0, v_0}$. To prove tightness of the approximating sequence, the moments need to be bounded uniformly in the size of the boxes. Here, more care than for $\varrho = 0$ in [7] is needed. The uniform moment bound can, for instance, be achieved using a colored particle moment duality for each box similar to the one of Lemma 3.3.

3.2. *Dualities*. The symbiotic branching model exhibits an exceptionally rich duality structure, providing powerful tools for the analysis of the longtime properties.

3.2.1. *Colored particle moment dual*. We now recall the two-colors particle moment-duality introduced in Section 3.1 of [8]. Since the dual Markov process is presented rigorously in [8] we only sketch the pathwise behavior. To find a suitable description of the mixed moment

$$\mathbb{E}^{u_0, v_0}[u_t(k_1) \cdots u_t(k_n) v_t(k_{n+1}) \cdots v_t(k_{n+m})],$$

$n + m$ particles are located in $\mathbb{Z}^d$. Each particle moves as a continuous-time simple random walk independent of all other particles. At time 0, n particles of color 1 are located at positions $k_1, \ldots, k_n$ and m particles of color 2 are located at positions $k_{n+1}, \ldots, k_{n+m}$. For each pair of particles, one of the pair changes color when the

time the two particles have spent in the same site, while both have same color, first exceeds an (independent) exponential time with parameter κ. Let

$$L_t^= = \text{total collision time of all pairs of same colors up to time } t,$$

$$L_t^{\neq} = \text{total collision time of all pairs of different colors up to time } t,$$

$$l_t^1(a) = \text{number of particles of color 1 at site } a \text{ at time } t,$$

$$l_t^2(a) = \text{number of particles of color 2 at site } a \text{ at time } t,$$

$$(u_0, v_0)^{l_t} = \prod_{a \in \mathbb{Z}^d} u_0(a)^{l_t^1(a)} v_0(a)^{l_t^2(a)}.$$

Note that since there are only $n+m$ particles, the infinite product is actually a finite product and hence well defined. The following lemma is taken from Section 3 of [8].

LEMMA 3.3. *Let (u_t, v_t) be a solution of $\mathrm{dSBM}(\varrho, \kappa)_{u_0, v_0}$, $\kappa > 0$ and $\varrho \in [-1, 1]$. Then, for any $k_i \in \mathbb{Z}^d$, $t \geq 0$,*

$$\mathbb{E}^{u_0, v_0}[u_t(k_1) \cdots u_t(k_n) v_t(k_{n+1}) \cdots v_t(k_{n+m})] = \mathbb{E}[(u_0, v_0)^{l_t} e^{\kappa(L_t^= + \varrho L_t^{\neq})}],$$

where the dual process behaves as explained above.

Note that for homogeneous initial conditions $u_0 = v_0 = \mathbf{1}$, the first factor in the expectation of the right-hand side equals 1. In the special case $\varrho = 1$, $u_0 = v_0 = \mathbf{1}$ Lemma 3.3 was already stated in [5], reproved in [9] and used to analyze the Lyapunov exponents of the parabolic Anderson model.

For $\varrho \neq 1$, the difficulty of the dual process is based on the two stochastic effects: on the one hand, one has to deal with collision times of random walks which were analyzed in [9]; additionally, particles have colors either 1 or 2 which change dynamically.

REMARK 3.4. Similar dualities hold for cSBM and SBM. For continuous-space, the random walks are replaced by Brownian motions and the collision times of the random walks by collision local times of the Brownian motions (see Section 4.1 in [8]). The simplest case is the nonspatial symbiotic branching model where the particles stay at the same site and local times are replaced by real times (see Theorem 3.2 of [19] or Proposition A5 of [6]).

3.2.2. *Self-duality.* Mytnik [18] introduced a self-duality for the continuous-space mutually catalytic branching model to obtain uniqueness of solutions of the corresponding martingale problem. This can be extended to symbiotic branching

models for $\varrho \in (-1, 1)$ as shown in Proposition 5 of [8]. The discrete-space self-duality for $\varrho = 0$ was proved in Theorem 2.4 of [7]. We first need more spaces of sequences:

$$E = \{(x, y) : (x, |y|) \in M_{\text{tem}}^2, |y(k)| \leq x(k) \; \forall k \in \mathbb{Z}^d\}$$

and

$$\tilde{E} = \{(x, y) \in E : x \in M_{\text{rap}}\} \supset \{(x, y) \in E : x \text{ has bounded support}\} = \tilde{E}_f.$$

In the sequel, the space E and its subspaces will be used for $(x, y) = (u_t + v_t, u_t - v_t)$. The duality function for $\varrho \in (-1, 1)$ maps $E \times \tilde{E}$ to $\mathbb{C}$ via

(3.5) $$H(u, v, \tilde{u}, \tilde{v}) = \exp\left(-\sqrt{1 - \varrho}\langle u, \tilde{u}\rangle + i\sqrt{1 + \varrho}\langle v, \tilde{v}\rangle\right).$$

With this definition the generalized Mytnik duality states:

LEMMA 3.5.   *For* $\varrho \in (-1, 1)$, $\kappa > 0$, $(u_0, v_0) \in M_{\text{tem}}^2$ *and* $(\tilde{u}_0, \tilde{v}_0) \in M_{\text{rap}}^2$ *let* (u_t, v_t) *be a solution of* $\text{dSBM}(\varrho, \kappa)_{u_0, v_0}$ *and* $(\tilde{u}_t, \tilde{v}_t)$ *be a solution of* $\text{dSBM}(\varrho, \kappa)_{\tilde{u}_0, \tilde{v}_0}$. *Then the following holds*:

$$\mathbb{E}^{u_0, v_0}[H(u_t + v_t, u_t - v_t, \tilde{u}_0 + \tilde{v}_0, \tilde{u}_0 - \tilde{v}_0)]$$
$$= \mathbb{E}^{\tilde{u}_0, \tilde{v}_0}[H(u_0 + v_0, u_0 - v_0, \tilde{u}_t + \tilde{v}_t, \tilde{u}_t - \tilde{v}_t)].$$

Analogously, the self-duality relation holds for the nonspatial model with duality function

$$H^0(u, v, \tilde{u}, \tilde{v}) = \exp\left(-\sqrt{1 - \varrho}u\tilde{u} + i\sqrt{1 + \varrho}v\tilde{v}\right),$$

mapping $(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})^2$ to $\mathbb{C}$.

4. Weak longtime convergence.   In this section we discuss weak longtime convergence of symbiotic branching models and prove Proposition 2.1. We proceed in two steps: first, we prove convergence in law to some limit law following the proof of [7] for $\varrho = 0$. Second, to characterize the limit law for the spatial models, we reduce the problem to the nonspatial model.

PROPOSITION 4.1.   *Let* $\varrho \in (-1, 1)$, $\kappa > 0$ *and* (u_t, v_t) *a solution of either* $\text{cSBM}(\varrho, \kappa)_{u_0, v_0}$ *or* $\text{dSBM}(\varrho, \kappa)_{u_0, v_0}$ *with initial conditions* $u_0 = \mathbf{u}$, $v_0 = \mathbf{v}$. *Then, as* $t \to \infty$, *the law of* (u_t, v_t) *converges weakly on* M_{tem}^2 *to some limit* (u_∞, v_∞).

PROOF.   The proof is only given for the discrete spatial case and the continuous case is completely analogous. Let us first recall the strategy of [7] for $\varrho = 0$ which can also be applied with the generalized self-duality required here. Convergence of (u_t, v_t) in M_{tem}^2 follows from convergence of $(u_t + v_t, u_t - v_t)$ in E. Using Lemma 2.3(c) of [7], it suffices to show convergence of $\mathbb{E}^{\mathbf{u}, \mathbf{v}}[H(u_t + v_t, u_t -$

$v_t, \phi, \psi)]$ for all $(\phi, \psi) \in \tilde{E}_f$. Furthermore, the limit (u_∞, v_∞) is uniquely determined by $\mathbb{E}^{\mathbf{u},\mathbf{v}}[H(u_\infty + v_\infty, u_\infty - v_\infty, \phi, \psi)]$ (see Lemma 2.3(b) of [7]). Hence, it suffices to show convergence of

$$(4.1) \quad \mathbb{E}^{\mathbf{u},\mathbf{v}}[H(u_t + v_t, u_t - v_t, \phi, \psi)] = \mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\big],$$

for all $(\phi, \psi) \in \tilde{E}_f$. Note that the technical condition of Lemma 2.3(c) of [7] is fullfilled since due to Proposition 3.1

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}[\langle u_t + v_t, \phi_{-\lambda}\rangle] = (u + v)\langle \mathbf{1}, P_t\phi_{-\lambda}\rangle < C < \infty.$$

To ensure convergence of (4.1) we employ the generalized Mytnik self-duality of Lemma 3.5 with $\tilde{u}_0 := \frac{\phi+\psi}{2}, \tilde{v}_0 := \frac{\phi-\psi}{2}$:

$$
\begin{aligned}
& \mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\big] \\
(4.2) \quad & = \mathbb{E}^{u_0,v_0}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\tilde{u}_0+\tilde{v}_0\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\tilde{u}_0-\tilde{v}_0\rangle}\big] \\
& = \mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\big[e^{-\sqrt{1-\varrho}\langle u_0+v_0,\tilde{u}_t+\tilde{v}_t\rangle+i\sqrt{1+\varrho}\langle u_0-v_0,\tilde{u}_t-\tilde{v}_t\rangle}\big] \\
& = \mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\big[e^{-\sqrt{1-\varrho}(u+v)\langle \mathbf{1},\tilde{u}_t+\tilde{v}_t\rangle+i\sqrt{1+\varrho}(u-v)\langle \mathbf{1},\tilde{u}_t-\tilde{v}_t\rangle}\big].
\end{aligned}
$$

By assumption, $\tilde{u}_0, \tilde{v}_0$ have compact support and hence by Proposition 3.2 the total-mass processes $\langle \mathbf{1}, \tilde{u}_t\rangle$ and $\langle \mathbf{1}, \tilde{v}_t\rangle$ are nonnegative martingales. By the martingale convergence theorem $\langle \mathbf{1}, \tilde{u}_t\rangle$ and $\langle \mathbf{1}, \tilde{v}_t\rangle$ converge almost surely to finite limits denoted by $\langle \mathbf{1}, \tilde{u}_\infty\rangle, \langle \mathbf{1}, \tilde{v}_\infty\rangle$. Finally, the dominated convergence theorem implies convergence of the right-hand side of (4.2) to

$$(4.3) \qquad \mathbb{E}^{\tilde{u}_0,\tilde{v}_0}\big[e^{-\sqrt{1-\varrho}(u+v)\langle \mathbf{1},\tilde{u}_\infty+\tilde{v}_\infty\rangle+i\sqrt{1+\varrho}(u-v)\langle \mathbf{1},\tilde{u}_\infty-\tilde{v}_\infty\rangle}\big].$$

Combining the above, we have proved convergence of

$$\mathbb{E}^{\mathbf{u},\mathbf{v}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t,\phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t,\psi\rangle}\big],$$

which ensures weak convergence of (u_t, v_t) in M_{tem}^2 to some limit which is uniquely determined by (4.3). □

Again, as in Remark 2.2, the previous proposition can be proved for nondeterministic initial conditions as in [4].

The rest of this section is devoted to identifying the limit (u_∞, v_∞) in the recurrent case. Before completing the proof of Theorem 2.1 we discuss a version of Knight's extension of the Dubins–Schwarz theorem (see [14], 3.4.16) for nonorthogonal continuous local martingales.

LEMMA 4.2. *Let (N_t) and (M_t) be continuous local martingales with $N_0 = M_0 = 0$ almost surely. Assume further that, for $t \geq 0$,*

$$[M., M.]_t = [N., N.]_t \quad and \quad [M., N.]_t = \varrho[M., M.]_t \qquad a.s.,$$

where $\varrho \in [-1, 1]$. If $[M., M.]_\infty = \infty$ a.s., then

$$(B_t^1, B_t^2) := (M_{T(t)}, N_{T(t)})$$

is a pair of Brownian motions with covariances $[B.^1, B.^2]_t = \varrho t$, where

$$(4.4) \qquad\qquad T(t) = \inf\{s : [M., M.]_s > t\}.$$

PROOF. It follows from the Dubins–Schwarz theorem that B^1, B^2 are each Brownian motions. Further, by the definition of $T(t)$ we obtain the claim

$$[B.^1, B.^2]_t = [M., N.]_{T(t)} = \varrho[M., M.]_{T(t)} = \varrho t. \qquad\qquad \square$$

REMARK 4.3. If $T^* := [M., M.]_\infty < \infty$ the situation becomes slightly more delicate but one can use a local version of Lemma 4.2. Indeed, define, for $t \geq 0$,

$$(4.5) \qquad\qquad B_t^1 := \begin{cases} M_{T(t)}, & \text{for } t < T^*, \\ M_{T^*}, & \text{for } t \geq T^*, \end{cases}$$

where the time-change T is given in (4.4) and define B^2 analogously for N (recall that $[M., M.]_t = [N., N.]_t$). Then the processes B^1, B^2 are Brownian motions stopped at time T^*. The covariance is again given by

$$[B.^1, B.^2]_{t \wedge T^*} = \varrho(t \wedge T^*), \qquad t \geq 0.$$

For the rest of this section let B^1, B^2 be standard Brownian motions with covariance

$$(4.6) \qquad\qquad [B.^1, B.^2]_t = \varrho t$$

started in u, v, denote their expectations by $E^{u,v}$, and let

$$\tau = \inf\{t : B_t^1 B_t^2 = 0\}.$$

The above discussion can now be used to understand the longtime behavior of symbiotic branching processes. We start by giving a proof for the nonspatial symbiotic branching model and then modify the proof to capture the corresponding result for the spatial models.

PROPOSITION 4.4. *Let (u_t, v_t) be a solution of $\mathrm{SBM}(\varrho, \kappa)_{u,v}$. Then, as $t \to \infty$, (u_t, v_t) converges almost surely to some (u_∞, v_∞). Furthermore, $\mathcal{L}^{u,v}(u_\infty, v_\infty) = \mathcal{L}^{u,v}(B_\tau^1, B_\tau^2)$ with B_τ^1, B_τ^2 from Proposition 2.1.*

PROOF. Solutions of the nonspatial symbiotic branching model are nonnegative martingales and hence converge almost surely. This implies the first part of the claim and it only remains to characterize the limit. Obviously, the L^2-martingales $(u_t), (v_t)$ satisfy the cross-variation structure assumptions of Lemma 4.2 and,

thus, $(u_t, v_t) = (B^1_{T^{-1}(t)}, B^2_{T^{-1}(t)})$. To obtain the result, we need to check that $T^{-1}(\infty) = \tau$. By definition of SBM, the time-change is given by

$$(4.7) \quad T^{-1}(t) = [u_., u_.]_t = \left[\int_0^{\cdot} \sqrt{\kappa u_s u_s} \, dB^1_s, \int_0^{\cdot} \sqrt{\kappa u_s v_s} \, dB^1_s \right]_t = \kappa \int_0^t u_s v_s \, ds.$$

To see that $T^{-1}(\infty) = \tau < \infty$, first note that $T^{-1}(t) \leq \tau$ for all $t \geq 0$. This is true since $u_t = B^1_{T^{-1}(t)}, v_t = B^2_{T^{-1}(t)}$ and solutions of SBM are nonnegative. To argue that $T^{-1}(t)$ increases to τ, more care is needed. Since the martingales converge almost surely, $T^{-1}(t)$ converges to some value $a \leq \tau$. Suppose $a < \tau$, then (u_t, v_t) converges to some (x, y) with $x, y > 0$. This yields a contradiction since $T^{-1}(t) = \kappa \int_0^t u_s v_s \, ds$ would increase to infinity. Hence, almost surely,

$$(u_t, v_t) = (B^1_{T^{-1}(t)}, B^2_{T^{-1}(t)}) \overset{t \to \infty}{\to} (B^1_{T^{-1}(\infty)}, B^2_{T^{-1}(\infty)}) = (B^1_\tau, B^2_\tau). \qquad \square$$

In particular, the proof of Proposition 4.4 provides an important relation for (B^1_τ, B^2_τ). As remarked below Lemma 3.5, the self-duality also works in the non-spatial model:

$$\mathbb{E}^{u_0, v_0}\left[e^{-\sqrt{1-\varrho}(u_t+v_t)(\tilde{u}_0+\tilde{v}_0)+i\sqrt{1+\varrho}(u_t-v_t)(\tilde{u}_0-\tilde{v}_0)}\right]$$

$$= \mathbb{E}^{\tilde{u}_0, \tilde{v}_0}\left[e^{-\sqrt{1-\varrho}(u_0+v_0)(\tilde{u}_t+\tilde{v}_t)+i\sqrt{1+\varrho}(u_0-v_0)(\tilde{u}_t-\tilde{v}_t)}\right],$$

where both (u_t, v_t) and $(\tilde{u}_t, \tilde{v}_t)$ are solutions of SBM(ϱ, κ) with different initial conditions. As shown in the proof of Proposition 4.4, (u_t, v_t) [resp., $(\tilde{u}_t, \tilde{v}_t)$] converges almost surely to (B^1_τ, B^2_τ) with initial condition (u_0, v_0) [resp., $(\tilde{u}_0, \tilde{v}_0)$]. Using dominated convergence, this shows the following duality relation for (B^1_τ, B^2_τ) when started in initial conditions (u, v), $(\tilde{u}, \tilde{v})$:

$$(4.8) \quad \begin{aligned} &E^{u,v}[H^0(B^1_\tau + B^2_\tau, B^1_\tau - B^2_\tau, \tilde{u} + \tilde{v}, \tilde{u} - \tilde{v})] \\ &= E^{\tilde{u},\tilde{v}}[H^0(B^1_\tau + B^2_\tau, B^1_\tau - B^2_\tau, u + v, u - v)]. \end{aligned}$$

PROOF OF PROPOSITION 2.1.   Again, the proof is only presented in the discrete spatial setting since the continuous case is analogous. We retain the notation of the proof of Proposition 4.1 where we showed that, as t tends to infinity,

$$\mathbb{E}^{\mathbf{u}, \mathbf{v}}\left[e^{-\sqrt{1-\varrho}\langle u_t+v_t, \phi\rangle+i\sqrt{1+\varrho}\langle u_t-v_t, \psi\rangle}\right]$$

$$\to \mathbb{E}^{(\phi+\psi)/2, (\phi-\psi)/2}\left[e^{-\sqrt{1-\varrho}(u+v)\langle\mathbf{1}, \tilde{u}_\infty+\tilde{v}_\infty\rangle+i\sqrt{1+\varrho}(u-v)\langle\mathbf{1}, \tilde{u}_\infty-\tilde{v}_\infty\rangle}\right].$$

Let us specify the limit law as for the nonspatial symbiotic branching process. As seen in Proposition 3.2 the total-mass processes $\bar{u}_t := \langle \tilde{u}_t, \mathbf{1} \rangle$ and $\bar{v}_t := \langle \tilde{v}_t, \mathbf{1} \rangle$ are nonnegative continuous L^2-martingales with cross-variations $[\bar{u}_., \bar{v}_.]_t = \varrho[\bar{u}_., \bar{u}_.]_t = \varrho[\bar{v}_., \bar{v}_.]_t, t \geq 0$. Thus, by Lemma 4.2, reasoning as in (4.7), $(\bar{u}_t, \bar{v}_t) = (B^1_{T^{-1}(t)}, B^2_{T^{-1}(t)})$, where B^1, B^2 are Brownian motions started in $\bar{u}_0 = \langle \frac{\phi+\psi}{2}, \mathbf{1} \rangle$,

$\bar{v}_0 = \langle \frac{\phi-\psi}{2}, \mathbf{1} \rangle$ with covariance $[B^1, B^2]_t = \varrho t$ and $T^{-1}(t) = \kappa \int_0^t \langle u_s, v_s \rangle \, ds$. Again, we need to show that $T^{-1}(\infty) = \tau$. This is much more subtle than in the nonspatial case since the quadratic variation might level off even if both total-mass processes $\bar{u}_t, \bar{v}_t$ are strictly positive. In [7] it was shown that for $\varrho = 0$, almost surely, this does not happen in the recurrent case [cf. the proof of their Theorem 1.2(b)]. Their proof can be used directly for $\varrho \in (-1, 1)$. Hence, almost surely,

$$(4.9) \qquad (\langle \tilde{u}_t, \mathbf{1} \rangle, \langle \tilde{v}_t, \mathbf{1} \rangle) \overset{t \to \infty}{\Rightarrow} (B_\tau^1, B_\tau^2).$$

Combining the above discussion with (4.3), we are able to determine the limit. First, we derived

$$\mathbb{E}^{\mathbf{u}, \mathbf{v}} \big[e^{-\sqrt{1-\varrho} \langle u_t + v_t, \phi \rangle + i \sqrt{1+\varrho} \langle u_t - v_t, \psi \rangle} \big]$$
$$\overset{t \to \infty}{\Rightarrow} E^{\langle (\phi+\psi)/2, \mathbf{1} \rangle, \langle (\phi-\psi)/2, \mathbf{1} \rangle} \big[e^{-\sqrt{1-\varrho}(u+v)(B_\tau^1 + B_\tau^2) + i \sqrt{1+\varrho}(u-v)(B_\tau^1 - B_\tau^2)} \big].$$

To use Lemma 2.3(c) of [7] we manipulate the right-hand side using (4.8):

$$E^{\langle (\phi+\psi)/2, \mathbf{1} \rangle, \langle (\phi-\psi)/2, \mathbf{1} \rangle} \big[e^{-\sqrt{1-\varrho}(u+v)(B_\tau^1 + B_\tau^2) + i \sqrt{1+\varrho}(u-v)(B_\tau^1 - B_\tau^2)} \big]$$
$$= E^{\langle (\phi+\psi)/2, \mathbf{1} \rangle, \langle (\phi-\psi)/2, \mathbf{1} \rangle} [H^0(B_\tau^1 + B_\tau^2, B_\tau^1 - B_\tau^2, u+v, u-v)]$$
$$= E^{u,v} [H^0(B_\tau^1 + B_\tau^2, B_\tau^1 - B_\tau^2, \langle \phi, \mathbf{1} \rangle, \langle \psi, \mathbf{1} \rangle)]$$
$$= E^{u,v} [H(\bar{B}_\tau^1 + \bar{B}_\tau^2, \bar{B}_\tau^1 - \bar{B}_\tau^2, \phi, \psi)],$$

where, as in Proposition 2.1, $\bar{B}_\tau^1$ (resp., $\bar{B}_\tau^2$) denotes the constant function taking only the (random) value B_τ^1 (resp., B_τ^2). In total we have

$$\mathbb{E}^{\mathbf{u}, \mathbf{v}} [H(u_t + v_t, u_t - v_t, \phi, \psi)] \overset{t \to \infty}{\Rightarrow} E^{u,v} [H(\bar{B}_\tau^1 + \bar{B}_\tau^2, \bar{B}_\tau^1 - \bar{B}_\tau^2, \phi, \psi)],$$

which implies weak convergence in M_{tem}^2 of (u_t, v_t) to $(\bar{B}_\tau^1, \bar{B}_\tau^2)$ by Lemma 2.3(c) of [7]. □

5. Moments. In this section we prove Theorems 2.5 and 2.7. Before giving the proofs we prove an equivalence for moments of correlated Brownian motions.

5.1. *Moments of the exit-point and exit-time distribution of correlated Brownian motions in a quadrant.* Let $\varrho \in (-1, 1)$, $u, v > 0$ and B^1, B^2 be Brownian motions started in u, v with

$$(5.1) \qquad \langle B_\cdot^1, B_\cdot^2 \rangle_t = \varrho t.$$

The starting points of Brownian motions will be indicated by superscripts in probabilities and expectations. Further, let

$$(5.2) \qquad \tau^B = \inf\{t \geq 0 : B_t^1 B_t^2 = 0\}.$$

THEOREM 5.1. *Let $p > 0$ and $u, v > 0$. Under the above assumptions, the following conditions are equivalent*:

(i)

$$p < \frac{\pi}{\pi/2 + \arctan(\varrho/(\sqrt{1 - \varrho^2}))},$$

(ii)

$$E^{u,v}[(\tau^B)^{p/2}] < \infty,$$

(iii)

$$E^{u,v}[|(B^1_{\tau^B}, B^2_{\tau^B})|^p] < \infty.$$

PROOF. We start with the proof of the equivalence of (i) and (ii). Define a cone in the plane with angle $\theta \in (0, 2\pi)$ by

$$C(\varphi) = \{re^{i\phi} : r \geq 0, 0 \leq \phi \leq \varphi\}$$

and denote its boundary by $\partial C(\varphi)$. Note that with this definition, the positive real line is always contained in $C(\varphi)$. Further, we define, for $\varrho \in (-1, 1)$, a sector in $\mathbb{R}^2$ by

$$S(\varrho) = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq -\frac{\varrho}{\sqrt{1 - \varrho^2}}x \right\}$$

and denote by $\partial S(\varrho)$ its boundary. Note that this time, the positive imaginary axis is always in $S(\varrho)$ and that the angle of the sector at the origin is given by

$$\theta := \frac{\pi}{2} + \arctan\left(\frac{\varrho}{\sqrt{1 - \varrho^2}} \right).$$

To transform the correlated Brownian motions B^1, B^2 to planar Brownian motion we use the simple fact that $W^1 := B^1, W^2 := (\frac{B^2 - \varrho B^1}{\sqrt{1 - \varrho^2}})$ defines a pair of independent Brownian motions started in $u, (\frac{v - \varrho u}{\sqrt{1 - \varrho^2}})$ satisfying $(B^1, B^2) = (W^1, \varrho W^1 + \sqrt{1 - \varrho^2}W^2)$. By the definition of $S(\varrho)$, the planar Brownian motion (W^1, W^2) started in $(u, (\frac{v - \varrho u}{\sqrt{1 - \varrho^2}}))$ hits $\partial S(\varrho)$ if and only if the correlated Brownian motions B^1, B^2 started in u, v hit $\partial C(\frac{\pi}{2})$. Hence, for τ^B as in (5.2), we have

(5.3) $$\tau^B = \tau^W := \inf\{t \geq 0 : (W^1_t, W^2_t) \in \partial S(\varrho)\}.$$

Since planar Brownian motion is rotation invariant, $S(\varrho)$ may be rotated to agree with the cone $C(\theta)$, without changing the exit time. Obviously, with the corresponding rotated initial conditions, the law of the first exit time $\tau_{C(\theta)}$ from the

cone $C(\theta)$ agrees with the law of τ^W. For planar Brownian motion in a cone $C(\theta)$ it is well known (see [23], Theorem 2) that

$$
(5.4) \qquad E^{x,y}\big[(\tau_{C(\theta)})^{p/2}\big] < \infty \quad \Leftrightarrow \quad p < \frac{\pi}{\theta},
$$

independently of x, y. (5.3) and (5.4) now imply the equivalence of (i) and (ii) and independence of u, v.

The proof of the equivalence of (i) and (iii) is via conformal transformation of the cone $C(\theta)$ to the upper half-plane. Indeed, we are going to calculate the densities of the exit-point distributions

$$
(5.5) \qquad P^{u,v}(B^1_{\tau_B} = 0, B^2_{\tau_B} \geq y), \qquad P^{u,v}(B^1_{\tau_B} \leq x, B^2_{\tau_B} = 0).
$$

We proceed in three steps: after reducing to independent Brownian motions in $S(\varrho)$ as for the exit time, we rotate $S(\varrho)$ to $C(\theta)$ and, finally, stretch the cone to end up with the upper half-plane.

Recall that the first exit of (B^1, B^2) happens at position $(0, y) \in \partial C(\frac{\pi}{2})$ if and only if the first exit of (W^1, W^2) takes place at $(0, \frac{y}{\sqrt{1-\varrho^2}}) \in \partial S(\varrho)$. Hence, (5.5) transforms to

$$
(5.6) \qquad
\begin{aligned}
&P^{u,v}(B^1_{\tau_B} = 0, B^2_{\tau_B} \geq y) \\
&\qquad = P^{u,(v-\varrho u)/\sqrt{1-\varrho^2}}\Big(W^1_{\tau_W} = 0, W^2_{\tau_W} \geq \frac{y}{\sqrt{1-\varrho^2}} \Big).
\end{aligned}
$$

In a similar fashion one obtains

$$
(5.7) \qquad
\begin{aligned}
&P^{u,v}(B^1_{\tau_B} \leq x, B^2_{\tau_B} = 0) \\
&\qquad = P^{u,(v-\varrho u)/\sqrt{1-\varrho^2}}\Big(W^1_{\tau_W} \leq x, W^2_{\tau_W} = -\frac{\varrho}{\sqrt{1-\varrho^2}} W^1_{\tau_W} \Big).
\end{aligned}
$$

We represent the transformed initial conditions $(z_1, z_2) = (u, \frac{v-\varrho u}{\sqrt{1-\varrho^2}}) \in S(\varrho)$ in polar coordinates, that is,

$$
z_1 = \sqrt{u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}} \cos\Big(\arctan\Big(\frac{v-\varrho u}{u\sqrt{1-\varrho^2}} \Big) \Big),
$$

$$
z_2 = \sqrt{u^2 + \frac{(v-\varrho u)^2}{1-\varrho^2}} \sin\Big(\arctan\Big(\frac{v-\varrho u}{u\sqrt{1-\varrho^2}} \Big) \Big).
$$

For the rotation we add the angle $\arctan(\frac{\varrho}{\sqrt{1-\varrho^2}})$ to get the new initial condition. Finally, to map the cone $C(\theta)$ conformally to the upper half-plane $\mathbb{H}$, we apply the map $z \mapsto z^{\pi/\theta}$ which maps $C(\theta)$ onto $\mathbb{H}$. Using conformal invariance of Brownian motion (e.g., Lemma 7.19 of [15]), the problem is reduced to the computation

of the exit distribution of planar (time-changed) Brownian motion from the upper half-plane. Indeed, due to the random time change the (almost surely finite) exit time changes but not the distribution of the exit points, which is Cauchy (see Theorem 2.37 of [15]). Thus, to obtain the distribution of the exit points explicitly it, only remains to specify the transformed initial condition $\tilde{z}_1, \tilde{z}_2$, which is given by

$$\tilde{z}_1 = \left(u^2 + \frac{(v - \varrho u)^2}{1 - \varrho^2}\right)^{\pi/(2\theta)}$$
$$\times \cos\left(\frac{\pi}{\theta}\left(\arctan\left(\frac{v - \varrho u}{\sqrt{1 - \varrho^2}u}\right) + \arctan\left(\frac{\varrho}{\sqrt{1 - \varrho^2}}\right)\right)\right),$$

$$\tilde{z}_2 = \left(u^2 + \frac{(v - \varrho u)^2}{1 - \varrho^2}\right)^{\pi/(2\theta)}$$
$$\times \sin\left(\frac{\pi}{\theta}\left(\arctan\left(\frac{v - \varrho u}{\sqrt{1 - \varrho^2}u}\right) + \arctan\left(\frac{\varrho}{\sqrt{1 - \varrho^2}}\right)\right)\right).$$

Now, let $\tilde{W}^1, \tilde{W}^2$ be two independent Brownian motions with $\tilde{W}_0^1 = \tilde{z}_1, \tilde{W}_0^2 = \tilde{z}_2$ and

$$\tau^{\tilde{W}} := \inf\{t > 0 : \tilde{W}_t^2 = 0\}.$$

Then, by (5.6), (5.7),

$$P^{u,v}(B_{\tau_B}^1 = 0, B_{\tau_B}^2 \geq y) = P^{u,(v-\varrho u)/\sqrt{1-\varrho^2}}\left(W_{\tau_W}^1 = 0, W_{\tau_W}^2 \geq \frac{y}{\sqrt{1 - \varrho^2}}\right)$$

$$= P^{\tilde{z}_1, \tilde{z}_2}\left(\tilde{W}_{\tau_{\tilde{W}}}^1 \leq -\left(\frac{y}{\sqrt{1 - \varrho^2}}\right)^{\pi/\theta}\right),$$

$$P^{u,v}(B_{\tau_B}^1 \leq x, B_{\tau_B}^2 = 0) = P^{u,(v-\varrho u)/\sqrt{1-\varrho^2}}\left(W_{\tau_W}^1 \leq x, W_{\tau_W}^2 = -\frac{\varrho}{\sqrt{1 - \varrho^2}}W_{\tau_W}^1\right)$$

$$= P^{\tilde{z}_1, \tilde{z}_2}\left(0 \leq \tilde{W}_{\tau_{\tilde{W}}}^1 \leq \left(x\left(1 + \frac{\varrho^2}{1 - \varrho^2}\right)^{1/2}\right)^{\pi/\theta}\right)$$

$$= P^{\tilde{z}_1, \tilde{z}_2}\left(0 \leq \tilde{W}_{\tau_{\tilde{W}}}^1 \leq \left(\frac{x}{\sqrt{1 - \varrho^2}}\right)^{\pi/\theta}\right).$$

Explicit manipulations of the Cauchy distribution yield

(5.8)
$$P^{u,v}(B_{\tau_B}^1 = 0, B_{\tau_B}^2 \geq y)$$
$$= \int_y^\infty \frac{1}{\pi \tilde{z}_2 \sqrt{1 - \varrho^2}^{\pi/\theta}} \frac{\pi/\theta r^{\pi/\theta - 1}}{1 + (((r/\sqrt{1 - \varrho^2})^{\pi/\theta} + \tilde{z}_1)/\tilde{z}_2)^2} dr,$$

$$(5.9) \quad P^{u,v}(B^1_{\tau_B} \leq x, B^2_{\tau_B} = 0)$$

$$= \int_0^x \frac{1}{\pi \tilde{z}_2 \sqrt{1 - \varrho^2}^{\pi/\theta}} \frac{\pi/\theta r^{\pi/\theta - 1}}{1 + (((r/\sqrt{1 - \varrho^2})^{\pi/\theta} - \tilde{z}_1)/\tilde{z}_2)^2} \, dr.$$

Finally, noting that $\int_0^\infty \frac{x^{p+\alpha-1}}{1+x^{2\alpha}} \, dx < \infty$ if and only if $p < \alpha$, we deduce from (5.8) and (5.9) that

$$E^{u,v}[|(B^1_{\tau_B}, B^2_{\tau_B})|^p] < \infty \quad \text{if and only if} \quad p < \frac{\pi}{\theta}. \qquad \square$$

5.2. *Proof of Theorem* 2.5. The proof relies on a combination of the self-duality based technique of the proof of Proposition 2.3 and the close relation between the moments of the exit-time and exit-point distribution of correlated Brownian motions obtained in Theorem 5.1.

PROOF OF THEOREM 2.5. We proceed in several steps. First, the result for the nonspatial model is proved and thereafter the results for the discrete-space and the continuous-space models. Finally, we present the argument in the transient case. In the following we use the definition of B^1, B^2 and τ from Proposition 2.1.

Step 1. Suppose (u_t, v_t) is a solution of SBM$(\varrho, \kappa)_{1,1}$.

"⇒": We first assume $\varrho < \varrho(p)$, in which case Theorem 5.1 implies $E^{1,1}[\tau^{p/2}] < \infty$. As argued in the proof of Proposition 4.4, u_t is a nonnegative martingale and due to the same arguments satisfies $\mathbb{E}^{1,1}[[u.]_t^{p/2}] \leq E^{1,1}[\tau^{p/2}] < \infty$ for all $t \geq 0$ and $\kappa > 0$. Considering $\bar{u}_t = u_t - u_0 = u_t - 1$, we apply the Burkholder–Davis–Gundy inequality to get

$$\begin{aligned}
\mathbb{E}^{1,1}[u_t^p] &= \mathbb{E}^{1,1}[(\bar{u}_t + 1)^p] \\
&= \mathbb{E}^{1,1}[\mathbf{1}_{\{\bar{u}_t \leq 1\}}(\bar{u}_t + 1)^p] + \mathbb{E}^{1,1}[\mathbf{1}_{\{\bar{u}_t > 1\}}(\bar{u}_t + 1)^p] \\
&\leq C_p + C_p \mathbb{E}^{1,1}[\bar{u}_t^p] \\
&\leq C_p + C_p \mathbb{E}^{1,1}\Big[\sup_{0 \leq s \leq t} \bar{u}_s^p\Big] \\
&\leq C_p + C_p' \mathbb{E}^{1,1}[[\bar{u}.]_t^{p/2}] < \infty
\end{aligned}$$

independently of t and κ.

"⇐." Conversely, for $\varrho \geq \varrho(p)$, Theorem 5.1 implies that $E^{1,1}[(B^1_\tau)^p] = \infty$. Using Fatou's lemma and almost sure convergence of u_t to B^1_τ, the proof for the nonspatial case is finished with

$$\liminf_{t \to \infty} \mathbb{E}^{1,1}[u_t^p] \geq \mathbb{E}^{1,1}[u_\infty^p] = E^{1,1}[(B_\tau)^p] = \infty.$$

Again, this lower bound is independent of κ.

Step 2. The proof for dSBM$(\varrho, \kappa)_{\mathbf{1},\mathbf{1}}$ is started by reducing the moments for homogeneous initial conditions to finite initial conditions. Indeed, employing Lemma 3.5 with $\phi = \psi = \frac{\theta}{2}\mathbf{1}_k$, where $\mathbf{1}_k$ denotes the indicator function of site $k \in \mathbb{Z}^d$, gives

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[e^{-\sqrt{1-\varrho}\theta(u_t(k)+v_t(k))}\big] = \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[e^{-\sqrt{1-\varrho}\langle u_t+v_t, \phi+\psi\rangle}\big]$$
$$= \mathbb{E}^{\phi,\psi}\big[e^{-\sqrt{1-\varrho}\langle\mathbf{1}+\mathbf{1},\tilde{u}_t+\tilde{v}_t\rangle}\big]$$
$$= \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\big[e^{-\sqrt{1-\varrho}\theta\langle\mathbf{1},\tilde{u}_t+\tilde{v}_t\rangle}\big],$$

where we used the argument of Remark 2.6. Note that, due to our choice of initial conditions, the complex part of the self-duality vanishes. Since the above is a Laplace transform identity, we have

$$\mathcal{L}^{\mathbf{1},\mathbf{1}}\big(u_t(k) + v_t(k)\big) = \mathcal{L}^{\mathbf{1}_k,\mathbf{1}_k}(\langle\mathbf{1}, \tilde{u}_t\rangle + \langle\mathbf{1}, \tilde{v}_t\rangle)$$

and hence

$$(5.10) \qquad \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(k) + v_t(k))^p\big] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[(\langle\mathbf{1}, \tilde{u}_t\rangle + \langle\mathbf{1}, \tilde{v}_t\rangle)^p].$$

We are now prepared to finish the proof of the theorem for the discrete case.

"$\Rightarrow$." Suppose $\varrho < \varrho(p)$. Let $M_t = \langle\mathbf{1}, \tilde{u}_t\rangle + \langle\mathbf{1}, \tilde{v}_t\rangle$, which due to Lemma 3.2 is a square-integrable martingale with quadratic variation

$$[M.]_t = [\langle\mathbf{1}, \tilde{u}.\rangle]_t + [\langle\mathbf{1}, \tilde{v}.\rangle]_t + 2[\langle\mathbf{1}, \tilde{u}.\rangle, \langle\mathbf{1}, \tilde{v}.\rangle]_t = (2 + 2\varrho)[\langle\mathbf{1}, \tilde{u}.\rangle]_t.$$

To apply the Burkholder–Davis–Gundy inequality, we switch again from M to $\bar{M}_t = M_t - M_0$, which is a martingale null at zero. Hence,

$$\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[M_t^p] = \mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[(\bar{M}_t + M_0)^p] \le C_p + C_p\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[\bar{M}_t^p].$$

Then we get from (5.10) and the Burkholder–Davis–Gundy inequality

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(k) + v_t(k))^p\big] \le C_p + C_p\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[\bar{M}_t^p]$$
$$\le C_p + C_p\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}\Big[\sup_{0\le s\le t}\bar{M}_s^p\Big]$$
$$\le C_p + C_p'\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[[\bar{M}.]_t^{p/2}]$$
$$= C_p + C_p'(2 + 2\varrho)^{p/2}\mathbb{E}^{\mathbf{1}_k,\mathbf{1}_k}[[\langle\mathbf{1}, \tilde{u}.\rangle]_t^{p/2}]$$

for some constants C_p, C_p' independent of t and κ. As in the proof of Theorem 2.1, the random time change which makes the pair of total masses a pair of correlated Brownian motions is bounded by τ, that is, $[\langle\mathbf{1}, \tilde{u}.\rangle]_t \le \tau$ for all $t \ge 0$. This yields by Theorem 5.1

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}[u_t(k)^p] \le \mathbb{E}^{\mathbf{1},\mathbf{1}}\big[(u_t(k) + v_t(k))^p\big] \le C_p + C_p'(2 + 2\varrho)^{p/2}E^{\mathbf{1},\mathbf{1}}[\tau^{p/2}] < \infty.$$

"⇐." Suppose $\varrho \geq \varrho(p)$. As in the proof of Theorem 2.1 we use the almost sure convergence of $(\langle \mathbf{1}, \tilde{u}_t \rangle, \langle \mathbf{1}, \tilde{v}_t \rangle)$ to (B_τ^1, B_τ^2). Combining this with Fatou's lemma gives

$$\liminf_{t \to \infty} \mathbb{E}^{\mathbf{1}_k, \mathbf{1}_k}[(\langle \mathbf{1}, \tilde{u}_t \rangle + \langle \mathbf{1}, \tilde{v}_t \rangle)^p] \geq \liminf_{t \to \infty} \mathbb{E}^{\mathbf{1}_k, \mathbf{1}_k}[\langle \mathbf{1}, \tilde{u}_t \rangle^p]$$

$$\geq \mathbb{E}^{\mathbf{1}_k, \mathbf{1}_k}\left[\liminf_{t \to \infty} \langle \mathbf{1}, \tilde{u}_t \rangle^p\right]$$

$$= E^{1,1}[(B_\tau^1)^p].$$

The right-hand side is infinite due to Theorem 5.1 and hence $\mathbb{E}^{\mathbf{1}_k, \mathbf{1}_k}[(\langle \mathbf{1}, \tilde{u}_t \rangle + \langle \mathbf{1}, \tilde{v}_t \rangle)^p]$ diverges. Equation (5.10) now shows that $\mathbb{E}^{1,1}[(u_t(k) + v_t(k))^p]$ also grows without bound. Since symbiotic branching processes are nonnegative, this is also true for $\mathbb{E}^{1,1}[u_t(k)^p]$ as can be seen as follows:

$$\mathbb{E}^{1,1}\big[(u_t(k) + v_t(k))^p\big] \leq \mathbb{E}^{1,1}\big[(2u_t(k))^p \mathbf{1}_{\{u_t(k) \geq v_t(k)\}}\big]$$

$$+ \mathbb{E}^{1,1}\big[(2v_t(k))^p \mathbf{1}_{\{u_t(k) < v_t(k)\}}\big]$$

$$\leq 2^p \mathbb{E}^{1,1}[u_t(k)^p] + 2^p \mathbb{E}^{1,1}[v_t(k)^p]$$

$$= 2^{p+1} \mathbb{E}^{1,1}[u_t(k)^p],$$

where we used Lemma 3.3 to see that $\mathbb{E}^{1,1}[u_t(k)^p] = \mathbb{E}^{1,1}[v_t(k)^p]$.

Step 3. The proof for cSBM$(\varrho, \kappa)_{\mathbf{1},\mathbf{1}}$ is slightly more involved since we cannot use the indicator $\mathbf{1}_x$ to get $u_t(x) = \langle u_t, \mathbf{1}_x \rangle$, where now $\langle f, g \rangle = \int_\mathbb{R} f(x)g(x)\,dx$. Instead we use a standard smoothing procedure. For fixed $x \in \mathbb{R}$ let

$$p_\varepsilon(y) = \frac{1}{\sqrt{2\pi \varepsilon}} e^{-(x-y)^2/(2\varepsilon)},$$

where we skip the dependence on x. The main part is to show that

(5.11)
$$\big\|(u_t(x) + v_t(x)) - (\langle u_t, p_\varepsilon \rangle + \langle v_t, p_\varepsilon \rangle)\big\|_{L^p}$$

$$\leq \|u_t(x) - \langle u_t, p_\varepsilon \rangle\|_{L^p} + \|v_t(x) - \langle v_t, p_\varepsilon \rangle\|_{L^p} \overset{\varepsilon \to 0}{\to} 0,$$

which implies

(5.12)
$$\lim_{\varepsilon \to 0} \|\langle u_t, p_\varepsilon \rangle + \langle v_t, p_\varepsilon \rangle\|_{L^p} = \|u_t(x) + v_t(x)\|_{L^p}.$$

Due to symmetry we only consider $\|u_t(x) - \langle u_t, p_\varepsilon \rangle\|_{L^p}$. To prove (5.11) we first observe that, due to the Green function representation provided in Corollary 19 of [8],

$$\|u_t(x) - \langle u_t, p_\varepsilon \rangle\|_{L^p}$$

$$= \bigg\| P_t u_0(x) - \langle P_{t+\varepsilon}, u_0 \rangle + \int_0^t \int_\mathbb{R} p_{t-s}(x-b) M(ds, db)$$

$$- \int_0^t \int_\mathbb{R} P_{t-s} p_\varepsilon(x-b) M(ds, db) \bigg\|_{L^p},$$

where $M(ds, db)$ is a zero-mean martingale measure with quadratic variation

$$\left[\int_0^\cdot \int_{\mathbb{R}} f(s, b) M(ds, db)\right]_t = \kappa \int_0^t \int_{\mathbb{R}} f^2(s, b) u_s(b) v_s(b)\, ds\, db$$

for test functions f such that the integral is well defined (see Lemma 18 of [8] for details).

For homogeneous initial conditions, the first difference vanishes and it suffices to concentrate on the difference of the stochastic integrals. By the Burkholder–Davis–Gundy inequality the difference of the integrals can be estimated as

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} p_{t-s}(x - b) M(ds, db) - \int_0^t \int_{\mathbb{R}} P_{t-s} p_\varepsilon(x - b) M(ds, db)\right)^p\right]$$

$$\leq C\kappa^{p/2} \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} (p_{t-s}(x - b) - p_{\varepsilon+t-s}(x - b))^2 u_s(b) v_s(b)\, ds\, db\right)^{p/2}\right].$$

Now expanding $(p_{t-s}(x - b) - p_{\varepsilon+t-s}(x - b))^2 u_s(b) v_s(b)$ as

$$(p_{t-s}(x - b) - p_{\varepsilon+t-s}(x - b))^{2(p-1)/p}$$

$$\times (p_{t-s}(x - b) - p_{\varepsilon+t-s}(x - b))^{2/p} u_s(b) v_s(b),$$

we get the upper bound (taking the expectation under the integral is valid since the integrands are nonnegative)

$$C\kappa^{p/2}\left[\left(\int_0^t \int_{\mathbb{R}} (p_{t-s}(x - b) - p_{\varepsilon+t-s}(x - b))^2\, ds\, db\right)^{p-1}\right.$$

$$\left.\times \int_0^t \int_{\mathbb{R}} (p_{t-s}(x - b) - p_{\varepsilon+t-s}(x - b))^2 \mathbb{E}[(u_s(b) v_s(b))^p]\, ds\, db\right],$$

where we have used that, for $f, g \in L^p$,

$$\left(\int (f^{2(p-1)/p})(f^{2/p} g)\, dx\right)^p \leq \left(\int f^2\, dx\right)^{p-1} \int f^2 g^p\, dx$$

by Hölder's inequality. As in [8], page 153, the second term can now be bounded from above by a constant depending only on p and t. The first factor can be estimated by $\varepsilon^{(p-1)/2}$ due to [22], Lemma 6.2. Hence, for fixed $p > 1$, $x \in \mathbb{R}$ and $t \geq 0$, (5.11) holds and thus we obtain (5.12). The rest of the proof is similar to the discrete case but slightly more technical. Since $p_\varepsilon(x - \cdot)$ is rapidly decreasing, we have

$$\mathbb{E}^{\mathbf{1},\mathbf{1}}\big[e^{-2\theta\sqrt{1-\varrho}\langle u_t + v_t, p_\varepsilon\rangle}\big] = \mathbb{E}^{\theta p_\varepsilon, \theta p_\varepsilon}\big[e^{-2\sqrt{1-\varrho}\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t\rangle}\big]$$

$$= \mathbb{E}^{p_\varepsilon, p_\varepsilon}\big[e^{-\sqrt{1-\varrho}2\theta\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t\rangle}\big].$$

Thus, we get

$$\mathcal{L}^{\mathbf{1},\mathbf{1}}(\langle u_t + v_t, p_\varepsilon\rangle) = \mathcal{L}^{p_\varepsilon, p_\varepsilon}(\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t\rangle)$$

and in particular

$$\mathbb{E}^{1,1}[(\langle u_t + v_t, p_\varepsilon \rangle)^p] = \mathbb{E}^{p_\varepsilon, p_\varepsilon}[(\langle \mathbf{1}, \tilde{u}_t \rangle + \langle \mathbf{1}, \tilde{v}_t \rangle)^p].$$

We may now finish the proof in a similar way to the discrete case.

"$\Rightarrow$." Due to (5.12) we are done if we can bound $\mathbb{E}^{1,1}[\langle u_t + v_t, p_\varepsilon \rangle^p]$ independently of $\varepsilon > 0$ and $t \geq 0$. This can be done as before: $\langle \mathbf{1}, \tilde{u}_t \rangle$ and $\langle \mathbf{1}, \tilde{v}_t \rangle$ are random time-changed correlated Brownian motions with initial conditions $\langle \mathbf{1}, p_\varepsilon \rangle = 1$ for all $\varepsilon > 0$. Using, as before, the auxiliary martingale

$$\bar{M}_t = \langle \mathbf{1}, \tilde{u}_t \rangle + \langle \mathbf{1}, \tilde{v}_t \rangle - \langle \mathbf{1}, \tilde{u}_0 \rangle - \langle \mathbf{1}, \tilde{v}_0 \rangle,$$

we obtain (as in the discrete case) with the help of the Burkholder–Davis–Gundy inequality

$$\begin{aligned}
\mathbb{E}^{1,1}[(u_t(x) + v_t(x))^p] &= \lim_{\varepsilon \to 0} \mathbb{E}^{1,1}[\langle u_t + v_t, p_\varepsilon \rangle^p] \\
&= \lim_{\varepsilon \to 0} \mathbb{E}^{p_\varepsilon, p_\varepsilon}[\langle \mathbf{1}, \tilde{u}_t + \tilde{v}_t \rangle^p] \\
&\leq C_p + C_p \lim_{\varepsilon \to 0} \mathbb{E}^{p_\varepsilon, p_\varepsilon}[\bar{M}_t^p] \\
&\leq C_p + C_p' \lim_{\varepsilon \to 0} \mathbb{E}^{p_\varepsilon, p_\varepsilon}[[\bar{M}.]_t^{p/2}] \\
&\leq C_p + C_p'(2 + 2\varrho)^{p/2} E^{1,1}[\tau^{p/2}].
\end{aligned}$$

The positive constants C_p, C_p' are independent of ε and t, whereas $\bar{M}$ and the random time change $[\bar{M}.]_t$ do depend on ε. However, the bound $[\bar{M}.]_t \leq \tau$ holds for all $\varepsilon > 0$ and $t \geq 0$ since $B_0^1 = B_0^2 = \langle \mathbf{1}, p_\varepsilon \rangle = 1$. For $\varrho < \varrho(p)$ the right-hand side is finite by Theorem 5.1 and independent of $t \geq 0$. Since $\mathbb{E}^{1,1}[u_t(x)^p] \leq \mathbb{E}^{1,1}[(u_t(x) + v_t(x))^p]$, the first direction is shown.

"$\Leftarrow$." First note that by translation invariance of initial condition, spatial motion and white noise

$$\mathbb{E}^{1,1}[(u_t(x) + v_t(x))^p] = \mathbb{E}^{1,1}[(u_t(y) + v_t(y))^p]$$

for fixed time $t \geq 0$ and arbitrary spatial positions $x, y \in \mathbb{R}$ implying that

$$\mathbb{E}^{1,1}[(u_t(x) + v_t(x))^p] = \int_{x-1/2}^{x+1/2} \mathbb{E}^{1,1}[(u_t(y) + v_t(y))^p] \, dy.$$

Using Fubini's theorem and Jensen's inequality we obtain for $p > 1$ the lower bound

$$\begin{aligned}
\mathbb{E}^{1,1}[(u_t(x) + v_t(x))^p] &\geq \mathbb{E}^{1,1}\left[\left(\int_{x-1/2}^{x+1/2} (u_t(y) + v_t(y)) \, dy\right)^p\right] \\
&= \mathbb{E}^{1,1}[\langle u_t + v_t, 1_{(x-1/2, x+1/2)} \rangle^p].
\end{aligned}$$

We now choose an arbitrary nonnegative (nontrivial) smooth function f with support contained in $(x - 1/2, x + 1/2)$ that is bounded by 1 and integrates to some $c \in (0, 1)$, say. A lower bound is now given by

$$\mathbb{E}^{1,1}\big[(u_t(x) + v_t(x))^p\big] \geq \mathbb{E}^{1,1}[\langle u_t + v_t, f\rangle^p]$$
$$= \mathbb{E}^{f,f}[\langle \tilde{u}_t + \tilde{v}_t, \mathbf{1}\rangle^p]$$
$$\geq \mathbb{E}^{f,f}[\langle \tilde{u}_t, \mathbf{1}\rangle^p],$$

where we utilized for the equality the self-duality of Proposition 5 of [8].

Finally, as in the discrete case, Fatou's lemma and the martingale convergence theorem imply

$$\liminf_{t\to\infty} \mathbb{E}^{1,1}\big[(u_t(x) + v_t(x))^p\big] \geq E^{c,c}[(B_\tau^1)^p] = \infty$$

by Theorem 5.1 and due to nonnegativity of solutions as well

$$\liminf_{t\to\infty} \mathbb{E}^{1,1}[u_t(x)^p] = \infty$$

proving the claim.

Step 4. The first direction of the above proof for dSBM$(\varrho, \kappa)_{1,1}$ also works for the transient case since $\mathbb{E}^{1_k, 1_k}[[\bar{M}.]_\infty^{p/2}] \leq E^{1,1}[\tau^{p/2}]$ is independent of recurrence/transience. $\square$

5.3. *Proof of Theorem* 2.7. We now study the "criticality" of the critical curve in more detail. As a preliminary result (mixed) moments of the nonspatial model are analyzed. The idea is to combine three different techniques: the martingale argument which led to Theorem 2.5 for $\mathbb{E}^{1,1}[u_t^n]$, a perturbation argument based on the moment duality which allows us to deduce exponential increase/decrease of $\mathbb{E}^{1,1}[u_t^{n-1} v_t]$, and finally moment equations which yield exponential increase/decrease for all mixed moments $\mathbb{E}^{1,1}[u_t^{n-m} v_t^m]$.

PROPOSITION 5.2. *The following hold for nonspatial symbiotic branching processes*:

(1) *For all $\kappa > 0$ and $n \in \mathbb{N}$*:
   - $\mathbb{E}^{1,1}[u_t^n]$ *grows to a finite constant if $\varrho < \varrho(n)$*,
   - $\mathbb{E}^{1,1}[u_t^n]$ *grows subexponentially fast to infinity if $\varrho = \varrho(n)$*,
   - $\mathbb{E}^{1,1}[u_t^n]$ *grows exponentially fast if $\varrho > \varrho(n)$*.
(2) *For all $\kappa > 0$, $n \in \mathbb{N}$ and $m = 1, \ldots, n-1$*:
   - $\mathbb{E}^{1,1}[u_t^{n-m} v_t^m]$ *decreases exponentially fast if $\varrho < \varrho(n)$*,
   - $\mathbb{E}^{1,1}[u_t^{n-m} v_t^m]$ *neither grows exponentially fast nor decreases exponentially fast if $\varrho = \varrho(n)$*,
   - $\mathbb{E}^{1,1}[u_t^{n-m} v_t^m]$ *grows exponentially fast if $\varrho > \varrho(n)$*.

PROOF.   *Step* 1. Martingale arguments based on the connection of moments of exit times and exit points of correlated Brownian motions were carried out in the proof of Theorem 2.5. This led to the first part of (1). Applying Hölder's inequality with $p = \frac{n}{n-m}$, $q = \frac{n}{m}$, we get the bound

$$(5.13) \qquad \mathbb{E}^{1,1}[u_t^{n-m}v_t^m] \leq \mathbb{E}^{1,1}[u_t^n]^{(n-m)/n}\mathbb{E}^{1,1}[v_t^n]^{m/n} = \mathbb{E}^{1,1}[u_t^n]$$

by symmetry. This implies that for $\varrho < \varrho(n)$ all mixed moments stay bounded as well.

Step 2. We apply the moment duality for the nonspatial model as explained in Remark 3.4. Combining the duality with the martingale argument of the first step we can understand the case $\varrho < \varrho(n)$ for mixed moments in a simple way. Note that for mixed moments the dual process starts with $n - m$ particles of one color and m particles of the other color at time 0. Note that for mixed moments $L_t^{\neq} \geq t$, since there is always at least one pair of different color. Now suppose $\varrho < \varrho(n)$, then for $0 < \varepsilon < \varrho(n) - \varrho$ we get

$$\mathbb{E}^{1,1}[u_t^{n-m}v_t^m] = \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big] = \mathbb{E}\big[e^{\kappa(L_t^{=}+(\varrho+\varepsilon)L_t^{\neq})}e^{-\kappa\varepsilon L_t^{\neq}}\big]$$

$$\leq \mathbb{E}\big[e^{\kappa(L_t^{=}+(\varrho+\varepsilon)L_t^{\neq})}\big]e^{-\kappa\varepsilon t}.$$

Since the first factor of the right-hand side is just the moment $\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]$ for $\varrho + \varepsilon$ strictly smaller than $\varrho(n)$, this is bounded for all t and κ. Hence, for $\varrho < \varrho(n)$ all mixed moments decrease exponentially fast proving the first part of (2). Note that since u_t^n is a submartingale, the moment $\mathbb{E}^{1,1}[u_t^n]$ is nondecreasing.

For $\varrho = \varrho(n)$ we first consider the pure moments. Again, for the critical case, Theorem 2.5 implies

$$\mathbb{E}\big[e^{\kappa(L_t^{=}+(\varrho(n)-\varepsilon)L_t^{\neq})}\big] < C(\varepsilon) < \infty$$

for all $\varepsilon > 0$ and $t \geq 0$. With the crude estimate $L_t^{\neq} \leq \binom{n}{2}t$ we get

$$C(\varepsilon) > \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}e^{-\kappa\varepsilon L_t^{\neq}}\big] \geq \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}\big]e^{-\kappa\varepsilon\binom{n}{2}t}.$$

Since ε is arbitrary this implies subexponential growth to infinity of $\mathbb{E}^{1,1}[u_t(k)^n]$ at the critical point. Hence, the second part of (1) is proven and combined with (5.13) so is the upper bound of the second part of (2).

Step 3. A direct application of Itô's lemma and Fubini's theorem yields

$$\mathbb{E}^{1,1}[u_t^n] = 1 + \kappa\binom{n}{2}\int_0^t \mathbb{E}^{1,1}[u_s^{n-1}v_s]\,ds.$$

Since we already know from the martingale arguments that $\mathbb{E}^{1,1}[u_t^n]$ increases to infinity in the critical case, the mixed moment $\mathbb{E}[u_t^{n-1}v_t]$ cannot decrease exponentially fast proving the lower bound of part two of (2). Furthermore, with the same arguments as above, for $\varrho > \varrho(n)$, this leads to

$$\mathbb{E}^{1,1}[u_t^{n-1}v_t] = \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}e^{\kappa(\varrho-\varrho(n))L_t^{\neq}}\big] \geq \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho(n)L_t^{\neq})}\big]e^{\kappa(\varrho-\varrho(n))t}.$$

Since the first factor of the right-hand side equals $\mathbb{E}[u_t^{n-1}v_t]$ at the critical point, it does not decrease exponentially fast. Hence, the product increases exponentially fast. In particular, due to (5.13), this also implies the third part of (1). Now it only remains to prove exponential increase for the other mixed moments. Again, using Itô's lemma and Fubini's theorem yields the following moment equations for the mixed moments:

$$\mathbb{E}^{1,1}[u_t^{n-2}v_t^2] = 1 + \kappa \int_0^t \mathbb{E}^{1,1}[u_s^{n-1}v_s]\,ds + \varrho(n-2)\kappa \int_0^t \mathbb{E}^{1,1}[u_s^{n-2}v_s^2]\,ds$$
$$+ \binom{n-2}{2}\kappa \int_0^t \mathbb{E}^{1,1}[u_s^{n-3}v_s^3]\,ds$$

and similarly for all other mixed moments. Since we already know that $\mathbb{E}^{1,1}[u_t^{n-1}v_t]$ grows exponentially fast in t, this implies exponential growth of $\mathbb{E}^{1,1}[u_t^{n-2}v_t^2]$. Iterating this argument gives exponential growth of all mixed moments for $\varrho > \varrho(n)$. This shows the third part of (2) and the proof is finished. $\square$

Now it only remains to prove Theorem 2.7, where some ideas for the nonspatial case are recycled.

PROOF OF THEOREM 2.7. First, due to Lemma 3.3, for homogeneous initial conditions, the moments of $u_t(k)$ and $v_t(k)$ are equal for all $t \geq 0$. For the existence of the Lyapunov exponents we use a standard subadditivity argument. Hence, it suffices to show

$$\mathbb{E}^{1,1}[u_{t+s}(k)^n] \leq \mathbb{E}^{1,1}[u_t(k)^n]\mathbb{E}^{1,1}[u_s(k)^n].$$

Using Lemma 3.3, we reduce the problem to $\mathbb{E}[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}]$, where the dual process (n_t) starts with n particles of the same color all placed at site k. By the tower property and the strong Markov property, we obtain

$$\mathbb{E}^{n_0}\big[e^{\kappa(L_{t+s}^{=}+\varrho L_{t+s}^{\neq})}\big] = \mathbb{E}^{n_0}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\mathbb{E}^{n_t}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big]\big].$$

We are done if we can show that

(5.14) $$\mathbb{E}^{n'}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big] \leq \mathbb{E}^{n_0}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big]$$

for any given initial configuration n' of the dual process consisting of n particles. The general initial conditions of the dual process consist of n^1 particles of one color and n^2 particles of the other color $(n^1 + n^2 = n)$ distributed arbitrarily in space at positions $k_1, \ldots, k_n$. Using the duality relation of Lemma 3.3, we obtain

$$\mathbb{E}^{n'}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big] = \mathbb{E}^{1,1}[u_s(k_1)\cdots u_s(k_{n^1})v_s(k_{n^1+1})\cdots v_s(k_{n^1+n^2})]$$
$$\leq \mathbb{E}^{1,1}[u_s(k)^n] = \mathbb{E}^{n_0}\big[e^{\kappa(L_s^{=}+\varrho L_s^{\neq})}\big],$$

where, in the penultimate step, we have used the generalized Hölder inequality.

Having established existence of the Lyapunov exponents, we now turn to the more interesting question of positivity. The boundedness for $\varrho < \varrho(n)$ in Theorem 2.5 immediately implies that in this case $\gamma(\varrho, \kappa) = 0$. Now suppose $\varrho = \varrho(n)$, that is, (ϱ, n) lies on critical curve. We use the perturbation argument which we already used for the nonspatial case combined with Lemma 3.3 and Theorem 2.5 to prove that in this case moments only grow subexponentially fast. This implies that the Lyapunov exponents are zero. Again we switch from $\mathbb{E}^{1,1}[u_t(k)^n]$ to $\mathbb{E}[e^{\kappa(L_t^= + \varrho L_t^{\neq})}]$, where the dual process is started with all particles at the same site and the same color. Since moments below the critical curve are bounded, we can proceed as for the nonspatial model. For any $\varepsilon > 0$, we get

$$\infty > C(\varepsilon) > \mathbb{E}\big[e^{\kappa(L_t^= + \varrho L_t^{\neq})} e^{-\kappa \varepsilon L_t^{\neq}}\big] \geq \mathbb{E}\big[e^{\kappa(L_t^= + \varrho L_t^{\neq})}\big] e^{-\kappa \varepsilon \binom{n}{2} t}$$
$$\geq \mathbb{E}^{1,1}[u_t(k)^n] e^{-\kappa \varepsilon \binom{n}{2} t},$$

where we estimated the collision time of particles of different colors by the collision time of all particles which is bounded from above by $\binom{n}{2} t$. Since ε on the right-hand side is arbitrary, $\gamma(\varrho, \kappa)$ cannot be positive.

Finally, we assume $\varrho > \varrho(n)$. The idea is to reduce the problem to the nonspatial case which we already discussed in Proposition 5.2. Actually, we prove more than stated in the theorem since we also show that mixed moments $\mathbb{E}^{1,1}[u_t(k)^{n-m} v_t(k)^m]$ grow exponentially fast. For $m = 1, \ldots, n-1$ the perturbation argument leads to

$$\mathbb{E}^{1,1}[u_t(k)^{n-m} v_t(k)^m] = \mathbb{E}\big[e^{\kappa(L_t^= + \varrho L_t^{\neq})}\big] = \mathbb{E}\big[e^{\kappa(L_t^= + \varrho(n) L_t^{\neq})} e^{\kappa(\varrho - \varrho(n)) L_t^{\neq}}\big].$$

The idea is to obtain a lower bound by conditioning on the event that all particles have not changed their spatial positions before time t (but, of course, have changed their colors). Under this condition the particle dual is precisely the particle dual of the nonspatial model. More precisely, we get the lower bound

$$\mathbb{E}\big[e^{\kappa(L_t^= + \varrho(n) L_t^{\neq})} e^{\kappa(\varrho - \varrho(n)) L_t^{\neq}} ; \text{no spatial change of particles before time } t\big]$$

$$= \mathbb{E}\big[e^{\kappa(L_t^= + \varrho(n) L_t^{\neq})} e^{\kappa(\varrho - \varrho(n)) L_t^{\neq}} | \text{no spatial change of particles before time } t\big]$$

$$\times \mathbb{P}[\text{no spatial change of particles before time } t]$$

$$= \mathbb{E}\big[e^{\kappa(L_t^= + \varrho(n) L_t^{\neq})} e^{\kappa(\varrho - \varrho(n)) L_t^{\neq}} | \text{no spatial change of particles}\big] e^{-nt},$$

where the final equality is valid since the event {no spatial change of particles before time t} has probability e^{-nt}. This is true since the event is precisely the event that n independent exponential clocks with parameter 1 did not ring before time t. For $1 \leq m \leq n-1$ there is always at least one pair of particles of different colors and, hence, we get the lower bound

$$\mathbb{E}\big[e^{\kappa(L_t^= + \varrho(n) L_t^{\neq})} | \text{no spatial change of particles until time } t\big] e^{\kappa(\varrho - \varrho(n)) t} e^{-nt},$$

which equals

$$\mathbb{E}^{1,1}[u_t^{n-m}v_t^m]e^{\kappa(\varrho-\varrho(n))t}e^{-nt}$$

for a nonspatial symbiotic branching process with critical correlation $\varrho = \varrho(n)$. Choosing κ such that $\kappa(\varrho - \varrho(n)) > n$ the result now follows from Proposition 5.2. $\square$

As mentioned in the course of the proof, we actually proved that for $\varrho > \varrho(n)$ and $m = 0, \dots, n$

$$\mathbb{E}^{\mathbf{1,1}}[u_t(k)^{n-m}(k)v_t^m(k)]$$

grows exponentially in t. As for the nonspatial model one could ask whether, and if so how fast, mixed moments decrease for $\varrho < \varrho(n)$. For the second moments it was shown in [1] that for $\varrho < \varrho(2) = 0$

$$\mathbb{E}^{\mathbf{1,1}}[u_t(k)v_t(k)] \approx \begin{cases} \dfrac{1}{\sqrt{t}}, & d = 1, \\[2mm] \dfrac{1}{\log(t)}, & d = 2, \\[2mm] 1, & d \geq 3, \end{cases}$$

where $\approx$ denotes weak asymptotic equivalence as $t \to \infty$. It would be interesting to see whether or not different rates of decrease appear for moments.

A detailed quantitative study of the Lyapunov exponents as functions of ϱ and κ has so far only been carried out for second moments (see [1]). In contrast to the parabolic Anderson model, where higher Lyapunov exponents are well studied (see [9]), we do not have much insight. Only a first upper bound for the Lyapunov exponents in κ and the distance to the critical curve can be obtained from the perturbation argument of the previous proof.

PROPOSITION 5.3. *If $\varrho > \varrho(n)$, then $\gamma_n(\varrho, \kappa) \leq \kappa\binom{n}{2}(\varrho - \varrho(n))$.*

PROOF.   By Lemma 3.3 and Theorem 2.5 for $\varrho > \varrho(n)$, there are constants $C(\varepsilon)$ such that

$$C(\varepsilon) > \mathbb{E}\big[e^{\kappa(L_t^{=}+(\varrho-(\varrho-\varrho(n))-\varepsilon)L_t^{\neq})}\big]$$

$$= \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}e^{-\kappa(\varrho-\varrho(n)+\varepsilon)L_t^{\neq}}\big]$$

$$\geq \mathbb{E}\big[e^{\kappa(L_t^{=}+\varrho L_t^{\neq})}\big]e^{-\kappa(\varrho-\varrho(n)+\varepsilon)\binom{n}{2}t}.$$

Hence, for all $\varepsilon > 0$

$$\mathbb{E}^{\mathbf{1,1}}[u_t(k)^n] \leq C(\varepsilon)e^{\kappa(\varrho-\varrho(n)+\varepsilon)\binom{n}{2}t},$$

yielding the result. $\square$

6. Speed of propagation of the interface. In this section we show how to use the moment bounds of Theorem 2.5 to obtain an improved upper bound on the speed of propagation of the interface as defined in Definition 2.10. We will only sketch the crucial parts in the proof of Theorem 6 of [8] that need modification. Note that the method used here is based on Mueller's "dyadic grid technique" introduced in [17].

PROOF OF THEOREM 2.11.   To prove that the interface will eventually be contained in

$$\left[-C\sqrt{T\log(T)}, C\sqrt{T\log(T)}\right]$$

(for suitable $C > 0$), by symmetry, it suffices to show that the right endpoint of the interface

$$R(u_t) := \sup\{x \in \mathbb{R} | u_t(x) > 0\}$$

up to time T can eventually be bounded by $C\sqrt{T\log(T)}$. To this end we define

$$A_n := \left\{\sup_{t \leq n} R(u_t) > C\sqrt{n\log(n)}\right\}$$

and show that, for suitably chosen C, $\mathbb{P}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}(\limsup_{n\in\mathbb{N}} A_n) = 0$. By the Borel–Cantelli lemma, this follows from

$$\text{(6.1)} \qquad \sum_{n=0}^{\infty} \mathbb{P}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}(A_n) < \infty.$$

In the following we modify the arguments of [8] to obtain an upper bound for $\mathbb{P}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}(A_n)$ which is sumamble over n.

LEMMA 6.1.   *For any integer n there is a finite constant c_n such that for $\varrho < \varrho(4n-1)$*

$$\mathbb{E}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}[(u_t(x)v_t(x))^n] \leq c_n\sqrt{P_t 1_{\mathbb{R}^-}(x)}, \qquad x \in \mathbb{R}, t \geq 0.$$

PROOF.   First recall from (87) of [8] that $\mathbb{E}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}[u_t(x)] = P_t 1_{\mathbb{R}^-}(x)$. We now use Hölder's inequality and Theorem 2.5 to reduce the mixed moment to the first moment:

$$\mathbb{E}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}[(u_t(x)v_t(x))^n]$$
$$= \mathbb{E}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}[u_t(x)^{1/2}u_t(x)^{n-1/2}v_t(x)^n]$$
$$\leq (\mathbb{E}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}[u_t(x)])^{1/2}(\mathbb{E}^{1,1}[u_t(x)^{4n-1}])^{(2n-1)/(8n-2)}$$
$$\times (\mathbb{E}^{1,1}[v_t(x)^{4n-1}])^{n/(4n-1)}.$$

This follows from the generalized Hölder inequality with exponents $2, (8n - 2)/(2n - 1)$ and $(4n - 1)/n$. The first factor yields the heat flow and Theorem 2.5 shows that the latter two factors are bounded by constants for $\varrho < \varrho(4n - 1)$. $\quad \square$

We now strengthen the estimate of Lemma 23 of [8] of the stochastic part

$$N_t(b) = \int_0^t \int_{\mathbb{R}} p_{t-s}(b - a) M(ds, da)$$

of the convolution representation of solutions of Corollary 20 of [8].

LEMMA 6.2. *For $\varrho < \varrho(35)$ there is a constant C_3 such that for $\varepsilon \in (0, 1)$, $A, T \geq 1$, the following estimate holds*:

$$\mathbb{P}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}} \big(|N_t(b)| \geq \varepsilon \text{ for some } t \leq T \text{ and } b \geq A \big) \leq C_3 \varepsilon^{-18} \frac{T^{22}}{\sqrt{A}} p_{2T}(A).$$

PROOF.   The proof is along the same lines of [8] replacing only in (116) the weaker (exponentially growing) moment bound of [8] by our stronger (bounded) moment bound. In the following we sketch the arguments to show where the moments appear. Before performing the "dyadic grid technique," increments of N_t need to be estimated. First, by definition

$$\mathbb{E}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}} [|N_t(a) - N_{t'}(a')|^{2q}]$$

$$= \mathbb{E}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}} \left[\left| \int_0^t \int_{\mathbb{R}} \big(p_{t-s}(b - a) - p_{t'-s}(b - a') \big) M(ds, db) \right|^{2q} \right],$$

which by Burkholder–Davis–Gundy and Hölder's inequality gives the upper bound

$$C_1 \left| \int_0^t \int_{\mathbb{R}} [p_{t-s}(b - a) - p_{t'-s}(b - a')]^2 \, db \, ds \right|^{q-1}$$

$$\times \int_0^t \int_{\mathbb{R}} [p_{t-s}(b - a) - p_{t'-s}(b - a')]^2 \mathbb{E}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}} [(u_s(b) v_s(b))^q] \, db \, ds.$$

Using Lemma 6.1 and classical heat kernel estimates we can derive (see the calculation on pages 153, 154 of [8]) the upper bound

$$\mathbb{E}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}} [|N_t(a) - N_{t'}(a')|^{2q}]$$

$$\leq C_2 \big((|t' - t|^{1/2} + |a' - a|) \wedge t^{1/2} \big)^{q-1} \big(\sqrt{t P_t 1_{\mathbb{R}^-}(a)} + \sqrt{t' P_{t'} 1_{\mathbb{R}^-}(a')} \big).$$

This upper bound corresponds to (119) of [8] where they have an additional exponentially growing factor coming from their moment bound. The dyadic grid technique is now carried out as in [8], choosing $q = 9$, without carrying along their exponential factor. Hence, we may delete the exponential term from their final estimate (110). Note that the necessity of $\varrho < \varrho(35)$ comes from our choice $q = 9$ and Lemma 6.1. $\quad \square$

The following lemma corresponds to Proposition 24 of [8].

LEMMA 6.3. *If $\varrho < \varrho(35)$ then, for some constants C_4, C_5, the following estimate holds for $T \geq 1$ and $r \geq C_4\sqrt{T}$:*

$$\mathbb{P}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}}\left(\sup_{t \leq T} R(u_t) > r\right) \leq C_5 T^{22} p_{16T}(r).$$

PROOF. All we need to do is to argue that Proposition 24 of [8] is valid for $r \geq C_4\sqrt{T}$ instead of $r \geq 9^4(1 \vee \kappa)T$. We perform the same decomposition and note that the estimates of Step 2 of [8] are already given for $r \geq C_4\sqrt{T}$ if C_4 is large enough. The only trouble occurs in their Step 3. Up to the estimate (154), this step works for $r \geq C_4\sqrt{T}$ but here their (weaker) Lemma 23 produces an exponential in T. More precisely, they need to justify

$$e^{9^5 \kappa^2 T/c} \frac{T^{22}}{\sqrt{r}} p_{8T}(r) \leq T^{22} p_{16T}(r),$$

which is only valid for $r \geq 9^4(1 \vee \kappa)T$. As our Lemma 6.2 avoids the exponential on the left-hand side the estimate holds for $r \geq C_4\sqrt{T}$ with suitably chosen C_4 and C_5. □

The significant distinction of the previous lemma to the result of [8] is that the inequality is not only valid for $r \geq 9^4(1 \vee \kappa)T$ but for $r \geq C_4\sqrt{T}$. At this point one might hope to obtain a square-root upper bound for the growth of the interface but this fails in the final step in which we validate (6.1):

$$\sum_{n=0}^{\infty} \mathbb{P}^{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}}(A_n) \leq \sum_{n=0}^{\infty} C_5 n^{22} p_{16n}\left(C\sqrt{n\log(n)}\right)$$

$$= \sum_{n=0}^{\infty} C_5 n^{22} \frac{1}{\sqrt{\pi 32n}} e^{-C^2 n\log(n)/(32n)}$$

$$= \frac{C_5}{\sqrt{32\pi}} \sum_{n=0}^{\infty} n^{22 - C^2/32 - 1/2},$$

which is finite for C large enough. □

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