NECESSARY AND SUFFICIENT CONDITIONS FOR REALIZABILITY OF POINT PROCESSES

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We give necessary and sufficient conditions for a pair of (generalized) functions $\rho_1(\mathbf{r}_1)$ and $\rho_2(\mathbf{r}_1, \mathbf{r}_2)$, $\mathbf{r}_i \in X$, to be the density and pair correlations of some point process in a topological space X, for example, $\mathbb{R}^d$, $\mathbb{Z}^d$ or a subset of these. This is an infinite-dimensional version of the classical "truncated moment" problem. Standard techniques apply in the case in which there can be only a bounded number of points in any compact subset of X. Without this restriction we obtain, for compact X, strengthened conditions which are necessary and sufficient for the existence of a process satisfying a further requirement—the existence of a finite third order moment. We generalize the latter conditions in two distinct ways when X is not compact.

1. Introduction. A *point process* is a probability measure on the family of all locally finite configurations of points in some topological space X; for an overview see [10]. Here we will often adopt the terminology of statistical mechanics, referring to the points as *particles* and to their expected densities and correlations as *correlation functions*. In many applications, quantities of interest can be calculated from the first few correlation functions—often the first two—alone (see [14] and below). Given the process in some explicit form, for example, as a Gibbs measure, one can in principle calculate these correlation functions, although in practice this is often impossible. On the other hand, one may start with certain prescribed correlation functions; these might arise as computable approximations to those of some computationally intractable process as occurs in the study of equilibrium fluids [14] or might express some partial information about an as yet unknown process as in the study of heterogeneous materials. One would like to determine whether or not these given functions are in fact the correlation functions of *some* point process, that is, are they *realizable*?

This paper is a continuation of our previous work on the realizability problem [4, 20] to which the reader may wish to refer but is independent and may be read separately. In this Introduction we briefly summarize our approach to the problem and then discuss a few applications in the physical and biological sciences. We

summarize definitions and background in Section 2 and describe our new results formally in the remainder of the paper.

It is often convenient to view the realizability problem as a *truncated moment problem*. In that setting it is an instance of the general problem of finding a process supported on some given subset of a linear space and having specified *moments* up to some given order, that is, specified expectation values of all linear functions and their products up to that order or equivalently of all polynomial functions of that degree. (The problem is called "truncated" because not all moments are prescribed.) To identify the realizability problem as a truncated moment problem we use the interpretation of a configuration of points as a sum of Dirac point measures and thus as a Radon measure on X. In this sense the set of all point configurations becomes a subset of the linear space of all signed Radon measures and to specify the correlation functions of the process up to some order n is then just to specify moments, in the above sense, up to order n. This is an infinite-dimensional instance of a classical mathematical problem [1, 18]. For the one-dimensional moment problem there are many powerful and interesting results but for higher-dimensional truncated problems there are fewer (see [9, 11, 18] and references therein). For the infinite-dimensional truncated moment problem we are only aware of [35].

In this paper we derive several classes of conditions on correlation functions which are necessary and/or sufficient for their realizability by a point process (or, in some cases, by a point process with certain extra properties); for simplicity we suppose that we wish to realize only two moments, that is, the first and second correlation functions but the methods extend directly to the general case. The conditions we consider are obtained via a standard general technique for moment problems, that is, Riesz' method [2]: one defines a linear functional on the space of quadratic polynomial functions of configurations in such a way that its value on any polynomial coincides with the expectation of the polynomial with respect to any realizing measure (should one exist). If the polynomial in question is nonnegative on the set of point configurations then it is a necessary condition for realizability that the value of the linear functional on the polynomial be nonnegative as well. The linear functional is expressible in terms of the prescribed correlation functions alone so that this gives rise to a necessary condition for realizability. These necessary conditions are discussed in Section 3.1.

The challenge is to show that these conditions are in fact also sufficient or, if they are not, to find an appropriate strengthening. There are two general classical approaches to the construction of realizing measures, one based on dual cones (see [18], Chapters I and V) and one on an extension theorem for nonnegative functionals (see Riesz's method in [2]). We follow the latter path: we first extend the linear functional described above to an appropriate larger space of continuous functions then prove that the extended functional can be realized by a measure. The first step follows in great generality from the Riesz–Krein extension theorem, cf. Theorem 3.6.

If the set of particle configurations is compact then the second step can be established by the well-known Riesz–Markov theorem; in Section 3.2 we use these ideas to establish sufficiency of the conditions described above in this case. The set of configurations is compact if the system has a local restriction on the number of particles; such a restriction can arise naturally from an a priori restriction on the total number of particles in the system (the result in this case was already proven directly in [27] and [13]) or in a setting where the given correlation functions, by vanishing on certain sets (as would be implied by a hard-core exclusion condition), prohibit particles from being closer to each other than some given distance. Lattice systems in which there can be no more than a given number of particles per site are included in this case (see Section 3.2).

When it is not known that the support of the desired measure is compact, we use a compact function, that is, a function with compact level sets to obtain an analogue of the Riesz–Markov representation theorem from the Daniell theory of integration. In our case we may use an appropriate power of a linear function as the desired compact function. In general, however, the process obtained in this way will not automatically realize the highest prescribed moment (cf. [18], Chapter V.1); this is a feature of the truncated moment problem on noncompact spaces (in our case we must even consider nonlocally compact spaces) which does not arise if moments of all orders are prescribed because there is always a higher moment at hand to control the lower ones. This difficulty is not avoidable; in fact the conditions derived from the positive polynomials are not in general sufficient in the noncompact case (see [18] and Example 3.12 below). An alternative approach for the locally compact case is given in [18].

In this paper we propose a new and quite natural approach for infinite-dimensional moment problems. We modify the conditions in order that they become sufficient but they then cease to be necessary; rather, they are necessary and sufficient for the modified realizability problem in which one requires realizability of the first two correlation functions by a process which has a finite third (local) moment. This, to our knowledge, is the first extension of the abstract characterization of necessary and in some sense sufficient conditions for an infinite-dimensional moment problem. The technique suggested should apply also to other types of infinite-dimensional moment problems. A similar approach was exploited recently in the finite-dimensional (locally compact) case based on the the dual cone approach in [9]. We discuss this nonlocally compact case in Section 3.3, treating first the case of particle systems in finite volume (X compact) and give two alternative results in the infinite volume (X locally compact).

In Section 3.4 we derive mild conditions under which the limit of realizable correlation functions stays realizable. In Section 4 we show that correlation functions with some symmetry, for example, translation invariance can, under mild extra assumptions, be realized by a point process with the same symmetry. In Section 5 we study a particular three-parameter subfamily of the full set of necessary conditions derived earlier; we show that certain well-known realizability conditions may be

obtained from those of this subfamily and that, in fact, they subsume all conditions from the subfamily.

We now discuss briefly some applications of the realizability problem. As already mentioned, the problem has a long tradition in the theory of classical fluids [14, 27, 28]. It arises there because an important ingredient of the theory is the introduction of various approximation schemes, such as the Percus–Yevick and hyper-netted chain approximations [14], for computing the first two correlation functions of the positions of the fluid molecules. It is then of interest to determine whether or not the resulting functions in fact correspond to any point process, that is, are in some sense internally consistent. If they are, then they provide rigorous bounds for properties of the system under consideration. The realizability problem was extensively discussed in [8, 31, 32, 34] which consider the realization problem in various contexts, including a conjecture related to the problem of the maximal density of sphere packing in high dimensions [33].

The quantum mechanical variant of the realizability problem, known as representability problem for reduced density matrices, is the basis of one approach to the computation of the ground state energies of molecules [5–7, 23, 24] yielding rigorous lower bounds. Interest in this method is rising at present because improved algorithms in semi-definite programming have led to an accuracy superior to that of the traditional electronic structure method. These new methods are numerically robust and reproduce further properties of the ground state; they are, however, at present not competitive in terms of computation time [12]. In [13, 19], the authors give sufficient conditions for representability for systems with a fixed finite number of particles, based on the dual cone approach mentioned above. (Reference [27] gives corresponding classical results; see Remark 3.11(b) below.)

Applications of the problem of describing a point process from its low order correlations also occur in biological contexts; for example, in spatial ecology [25] and in the study of neural spikes [3, 16]. In this and other situations it is natural to consider a closely related problem in which the correlation functions are specified only on part of the domain X; for example, if X is a lattice then we might only specify the nearest neighbor correlations. See [17] for a similar problem in error correcting codes. This will not be considered here; see, however, [20], Section 7.

2. Definitions. We consider point processes in a locally compact space X which has a countable base of the topology. X is then a complete separable metric space under an appropriate metric d_X [10], that is, it is a Polish space. We will sometimes use the fact that such a metric exists for which closed balls of finite radius are compact [15]. Measurability in X will for us always mean measurability with respect to the Borel σ-algebra on X. We will write $\mathcal{C}_c(X)$ for the space of real-valued continuous functions with compact support on X and $\mathcal{M}_c(X)$ for the space of real-valued bounded measurable functions with compact support on X. The specific examples that we have in mind for X include the Euclidean spaces $\mathbb{R}^d$, manifolds (in particular the torus) and countable sets equipped with the discrete

topology. In the following we refer for brevity to these countable sets as *lattices*; the important special cases are $\mathbb{Z}^d$ and the discrete toruses. For the spaces $\mathbb{R}^d$, $\mathbb{Z}^d$ and the usual and the discrete toruses one has as additional structure: a natural action of the group of translations and the (uniform) measure which is invariant under this action.

Intuitively, a point process on X is a random distribution of points in X such that, with probability one, any compact set contains only finitely many of these points. To give a precise definition, recall that a *Radon measure* on X is a Borel measure which is finite on compact sets and denote by $\mathcal{N}(X)$ the space of all Radon measures η on X which take as values either a nonnegative integer (i.e., a member of $\mathbb{N}_0 = \{0, 1, \ldots\}$) or infinity. A measure $\eta \in \mathcal{N}(X)$ corresponds to a point configuration via the representation

$$(1) \qquad \eta(d\mathbf{r}) = \sum_{i \in I} \delta_{\mathbf{x}_i}(d\mathbf{r}),$$

where either I is finite or $I = \mathbb{N} := \{1, 2, \ldots\}$; $\mathbf{x}_i \in X$ for $i \in I$ and if $I = \mathbb{N}$ the sequence $(\mathbf{x}_i)_{i \in I}$ has no accumulation points in X; and $\delta_{\mathbf{x}_i}$ is the unit mass (Dirac measure) supported at $\mathbf{x}_i$. Note that in this formulation there can be several distinctly labeled points of the process at the same point of X. The correspondence between η and $(\mathbf{x}_i)_{i \in I}$ is one-to-one modulo relabeling of the points. The requirement that η be a Radon measure corresponds to the condition that any compact set contain only finitely many points of the process.

We equip $\mathcal{N}(X)$ with the vague topology which is the weakest topology in which the mappings

$$(2) \qquad \eta \mapsto \langle f, \eta \rangle := \int_X f(\mathbf{r}) \eta(d\mathbf{r})$$

are continuous for all $f \in \mathcal{C}_c(X)$. $\mathcal{N}(X)$ with this topology is a Polish space [10]. Then we define a *point process* to be a Borel probability measure μ on $\mathcal{N}(X)$. If $\mathcal{N}_{\text{supp}}$ is a measurable subset of $\mathcal{N}(X)$ $\mu(\mathcal{N}_{\text{supp}}) = 1$, we will say that μ is a *point process on $\mathcal{N}_{\text{supp}}$*.

When X is a lattice, $\mathcal{N}(X)$ can be identified with $\mathbb{N}_0^X$ equipped with the product topology; $\eta \in \mathcal{N}(X)$ is then identified with the function on X for which $\eta(\mathbf{r})$ is the number of particles at the site $\mathbf{r}$. A special case is the so-called lattice gas in which there can be at most one particle per site, that is, $\eta(\mathbf{r}) \in \{0, 1\}$. On the lattice, of course, integrals in formulas like (2) become sums, the Dirac measure $\delta_{\mathbf{x}}(d\mathbf{r})$ becomes a Kronecker delta function, etc. We will not usually comment separately on the lattice case, adopting notation as in (2) without further comment.

One advantage of defining point configurations as Radon measures is the ease of then defining powers of these configurations. For $\eta \in \mathcal{N}(X)$, $\eta^{\otimes n}$ denotes the (symmetric Radon) product measure on X^n; note that from (1) we have

$$(3) \qquad \eta^{\otimes n}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) = \sum_{i_1, i_2, \ldots, i_n} \prod_{k=1}^{n} \delta_{x_{i_k}}(d\mathbf{r}_k).$$

Here we will use a notation parallel to (2): for $f_n : X^n \to \mathbb{R}$ measurable and non-negative, or for $f_n \in \mathcal{M}_c(X^n)$, we write

(4)
$$\langle f_n, \eta^{\otimes n} \rangle := \int_{X^n} f_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) \eta(d\mathbf{r}_1) \cdots \eta(d\mathbf{r}_n)$$
$$= \sum_{i_1, i_2, \ldots, i_n} f_n(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_n}).$$

By convention, $\langle f_0, \eta^{\otimes 0} \rangle = f_0$ for $f_0 \in \mathbb{R}$. We will occasionally use a similar notation for functions: if $f : X \to \mathbb{R}$, then $f^{\otimes n}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = f(\mathbf{r}_1) \cdots f(\mathbf{r}_n)$.

We will also need the *factorial nth power* $\eta^{\odot n}$ of η, the symmetric Radon measure on X^n given by

(5)
$$\eta^{\odot n}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) := \sum_{i_1, i_2, \ldots, i_n}' \prod_{k=1}^{n} \delta_{x_{i_k}}(d\mathbf{r}_k),$$

where $\sum'$ denotes a sum over distinct indices $i_1, i_2, \ldots, i_n$, so that, in parallel to (4),

(6)
$$\langle f_n, \eta^{\odot n} \rangle = \int_{X^n} f_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) \eta^{\odot n}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n)$$
$$= \sum_{i_1, i_2, \ldots, i_n}' f_n(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_n}).$$

The term "factorial power" arises because, for any measurable subset A of X,

(7) $\langle \mathbf{1}_A^{\otimes n}, \eta^{\odot n} \rangle \equiv \eta^{\odot n}(A \times \cdots \times A) = \eta(A)(\eta(A) - 1) \cdots (\eta(A) - n + 1).$

One may view $\mathcal{N}(X)$ as a subset (with the inherited topology) of the vector space of all signed Radon measures on X, equipped again with the vague topology. Motivated by this imbedding we call functions on $\mathcal{N}(X)$ of the form (2) *linear*, since they are the restrictions to $\mathcal{N}(X)$ of linear functionals. More generally, we define a *polynomial* on $\mathcal{N}(X)$ to be a function of the form

(8)
$$P(\eta) := \sum_{m=0}^{n} \langle f_m, \eta^{\odot m} \rangle,$$

where $f_0 \in \mathbb{R}$ and $f_m \in \mathcal{M}_c(X^m)$, $m = 1, \ldots, n$; without loss of generality we will assume that f_m is symmetric in its arguments when $m \geq 2$. [We would obtain the same set of polynomial functions if in (8) we replaced $\eta^{\odot m}$ by $\eta^{\otimes m}$.] We will sometimes consider *polynomials with continuous coefficients*, that is, polynomials for which $f_m \in \mathcal{C}_c(X)$, $m = 1, \ldots, n$.

2.1. *Correlation functions.* It is often convenient to study point processes through their *correlation measures*, also called *factorial moment measures* or *correlation functions*. The nth correlation measure is the expectation of the nth factorial power:

$$\rho_n(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) := \mathbb{E}_\mu[\eta^{\odot n}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n)], \tag{9}$$

that is, it is the symmetric measure ρ_n on X^n satisfying

$$\int_{X^n} f_n(\mathbf{r}_1, \ldots, \mathbf{r}_n) \rho_n(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) = \int_{\mathcal{N}(X)} \langle f_n, \eta^{\odot n} \rangle \mu(d\eta) \tag{10}$$

for all nonnegative measurable functions f_n on X^n. One may also define the nth *moment measure* of the process by replacing $\eta^{\odot n}$ by $\eta^{\otimes n}$ in (9) and (10) but these measures will not play a significant role in our discussion. The two sorts of moment measures are easily related; for example, at first order they coincide, since $\eta^{\odot 1} = \eta^{\otimes 1} = \eta$ and at second order we have

$$\int_{X \times X} f_2(\mathbf{r}_1, \mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2) = \int_{\mathcal{N}(X)} \int_X \int_X f_2(\mathbf{r}_1, \mathbf{r}_2) \eta(d\mathbf{r}_1) \eta(d\mathbf{r}_2) \mu(d\eta)$$
$$- \int_{\mathcal{N}(X)} \int_X f_2(\mathbf{r}, \mathbf{r}) \eta(d\mathbf{r}) \mu(d\eta).$$

We will usually refer to the ρ_n as *correlation functions* since this is the standard terminology in the physics literature. This usage is particularly appropriate on a lattice or when the measures are absolutely continuous with respect to Lebesgue measure, if we then gloss over the distinction between a measure and its density. From a more general viewpoint the terminology can be justified considering ρ_n as a generalized function in the sense of Schwartz. When X is a lattice, the process is a lattice gas; cf. page 1257, if and only if $\rho_2(\mathbf{r}, \mathbf{r}) = 0$ for each $\mathbf{r} \in X$.

We say that the point process μ has *finite local nth moments* if for every compact subset Λ of X,

$$\mathbb{E}_\mu[\eta(\Lambda)^n] \equiv \mathbb{E}_\mu[\langle \mathbf{1}_\Lambda, \eta \rangle^n] \equiv \mathbb{E}_\mu[\langle \mathbf{1}_\Lambda^{\otimes n}, \eta^{\otimes n} \rangle]$$
$$\equiv \int_{\mathcal{N}(X)} \eta(\Lambda)^n \mu(d\eta) < \infty. \tag{11}$$

Obviously, the point process has then also finite local mth moments for all $m \leq n$. If (11) holds for $\Lambda = X$ we say that μ has *finite nth moment*. It is easy to see that (11) is equivalent to $\rho_n(\Lambda^n) \equiv \mathbb{E}_\mu[\langle \mathbf{1}_\Lambda^{\otimes n}, \eta^{\odot n} \rangle] < \infty$ [e.g., this follows by taking $\chi = \mathbf{1}_\Lambda$ in (19) below]; in other words, the correlation measures ρ_m are σ-finite Radon measures for all $m \leq n$ if and only if (11) holds. When the process has finite local nth moment one may extend (10) to all $f_m \in \mathcal{M}_c(X^m)$ for $m \leq n$. In this paper we will assume, unless it is specifically stated otherwise, that the point processes under consideration have finite local second moments.

3. The realizability problem. In Section 2.1 we discussed how a point process μ gives rise to correlation functions ρ_n. The realizability problem is a sort of inverse problem.

DEFINITION 3.1. Given $N \in \mathbb{N}$, symmetric Radon measures ρ_n on X^n for $n = 1, \ldots, N$ and a measurable subset $\mathcal{N}_{\text{supp}}$ of $\mathcal{N}(X)$, we say that $(\rho_n)_{n=1,\ldots,N}$ is *realizable on $\mathcal{N}_{\text{supp}}$* if there exists a point process μ on $\mathcal{N}_{\text{supp}}$ which for $n = 1, \ldots, N$ has ρ_n as its nth correlation function.

Notice that, because the ρ_n in Definition 3.1 are assumed to be Radon measures, the realizing measure μ must have finite local Nth moments.

The aim of this paper is to develop necessary and sufficient conditions for realizability solely in terms of $(\rho_n)_{n=1,\ldots,N}$. We will describe these conditions in detail for the case $N = 2$; the generalization to general N is straight forward. The case $N = \infty$ was treated in [21, 22]; the problem with N finite involves certain additional difficulties, one of which is that the realizing measure is now generically nonunique (see also Example 3.12 and Remark 3.13).

3.1. *Necessary conditions.* It is rather easy to give very general necessary conditions for the realizability problem. Let $P(\eta)$ be a quadratic polynomial on $\mathcal{N}(X)$,

$$(12) \qquad P(\eta) = P_{f_0, f_1, f_2}(\eta) := f_0 + \langle f_1, \eta \rangle + \langle f_2, \eta^{\odot 2} \rangle.$$

Let μ be a point process on a given $\mathcal{N}_{\text{supp}} \subset \mathcal{N}(X)$; according to (10) the expectation $\mathbb{E}_\mu[P]$ can be computed in terms of the first two correlation functions of μ as

$$(13) \quad \mathbb{E}_\mu[P_{f_0, f_1, f_2}] = f_0 + \int_X f_1(\mathbf{r}) \rho_1(d\mathbf{r}) + \int_{X^2} f_2(\mathbf{r}_1, \mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2).$$

On the other hand, if P_{f_0, f_1, f_2} is nonnegative on $\mathcal{N}_{\text{supp}}$, that is, if for all $\eta = \sum_{i \in I} \delta_{\mathbf{x}_i} \in \mathcal{N}_{\text{supp}}$,

$$(14) \qquad f_0 + \sum_i f_1(\mathbf{x}_i) + \sum_{i \neq j} f_2(\mathbf{x}_i, \mathbf{x}_j) \geq 0,$$

then necessarily $\mathbb{E}_\mu[P_{f_0, f_1, f_2}] \geq 0$. This leads immediately to the following theorem.

THEOREM 3.2 (Necessary conditions). *If the pair (ρ_1, ρ_2) is realizable by a point process on $\mathcal{N}_{\text{supp}} \subset \mathcal{N}(X)$ then for any quadratic polynomial P_{f_0, f_1, f_2} which is nonnegative on $\mathcal{N}_{\text{supp}}$,*

$$(15) \qquad f_0 + \int_X f_1(\mathbf{r}) \rho_1(d\mathbf{r}) + \int_{X^2} f_2(\mathbf{r}_1, \mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2) \geq 0.$$

Theorem 3.2 gives uncountably many necessary conditions for realizability indexed by the triples (f_0, f_1, f_2). In Section 5 we will discuss how various standard conditions for realizability are obtained from one class of such triples. Unfortunately, the practical use of the theorem is limited because it is very difficult to identify admissible triples which lead to new and useful necessary conditions.

3.2. *Sufficient conditions*: *Hard core exclusion.* The idea of a "hard core exclusion," which prevents the points of a process from being too close together, is a common one in statistical physics. To be precise:

DEFINITION 3.3. Suppose that d is a metric for the topology of X and $D > 0$. A symmetric measure ρ_2 on $X \times X$ *forces a hard core exclusion with diameter D for the metric d* if

$$(16) \qquad \rho_2\big(\{(\mathbf{r}_1, \mathbf{r}_2) \in X \times X \mid d(\mathbf{r}_1, \mathbf{r}_2) < D\}\big) = 0.$$

Condition (16) says that, with probability one, no two points of the process can lie in a distance less than D from each other. It is clear that if ρ_2 forces a hard core exclusion with diameter D then any point process with second correlation function ρ_2 must be supported on

$$\mathcal{N}_D(X) := \left\{ \eta \equiv \sum_i \delta_{\mathbf{x}_i} \mid d(\mathbf{x}_i, \mathbf{x}_j) \geq D \text{ for all } i \neq j \right\}.$$

In this subsection we show that under this hard core hypothesis the necessary condition of Section 3.1 for realizability on $\mathcal{N}_D(X)$ is also sufficient.

THEOREM 3.4. *Let (ρ_1, ρ_2) be Radon measures on X and $X \times X$, respectively, with ρ_2 symmetric and suppose that ρ_2 forces a hard core exclusion with diameter D for a metric d. Then (ρ_1, ρ_2) is realizable on $\mathcal{N}_D(X)$ if and only if for any quadratic polynomial $P_{f_0, f_1, f_2}(\eta)$ which is nonnegative on $\mathcal{N}_D(X)$, f_0, f_1 and f_2 satisfy (15).*

REMARK 3.5. (a) The hard core exclusion condition of Definition 3.3 depends on the choice of metric d. Note, however, that if ρ_2 satisfies (16) for some metric (generating the topology of X) then (ρ_1, ρ_2) will be realizable on the domain $\mathcal{N}_D(X)$ defined using that metric. In the following we will not stress the dependence of $\mathcal{N}_D(X)$ on the metric.

(b) If X is a lattice, then a point process realizing (ρ_1, ρ_2) is a lattice gas if and only if there exists a metric d and a $D > 0$ such that ρ_2 forces a hard core exclusion with diameter D for the metric d. If ρ_2 forces a hard core exclusion for some d and D, then certainly $\rho_2(\mathbf{r}, \mathbf{r}) = 0$ for all $\mathbf{r}$; on the other hand, given a lattice gas we may topologize X via the metric in which $d(\mathbf{r}_1, \mathbf{r}_2) = 1$ whenever $\mathbf{r}_1 \neq \mathbf{r}_2$ and in this metric ρ_2 forces an exclusion with diameter $D = 1/2$. Thus, for lattice

gases, Theorem 3.4 gives necessary and sufficient conditions for realizability with $\mathcal{N}_D(X)$ just the set of lattice gas configurations. Of course, other hard core restrictions are possible; for example, on $\mathbb{Z}^d$ in the standard metric we may, in this way, forbid simultaneous occupancy of two nearest neighbor sites.

(c) If X is a finite set and ρ_2 forces an exclusion via $\rho_2(\mathbf{r}, \mathbf{r}) \equiv 0$ then $\mathcal{N}(X)$ is finite and the question of realizability is one of the feasibility of a (finite) linear programming problem: to find $(p_\eta)_{\eta \in \mathcal{N}(X)}$ with $p_\eta \geq 0$ and, for $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2 \in X$ with $\mathbf{r}_1 \neq \mathbf{r}_2$,

$$\sum_\eta p_\eta = 1, \qquad \sum_{\eta(\mathbf{r})=1} p_\eta = \rho_1(\mathbf{r}) \quad \text{and} \quad \sum_{\eta(\mathbf{r}_1)=\eta(\mathbf{r}_2)=1} p_\eta = \rho_2(\mathbf{r}_1, \mathbf{r}_2).$$

By the duality theorem of linear programming the problem is feasible if and only if a certain dual minimization problem has nonnegative solution. But in fact the dual problem involves the coefficients of what we have called a quadratic polynomial in η, the constraints of the problem correspond to the positivity of this polynomial and the quantity to be minimized is just the left-hand side of (15); that is, Theorem 3.4 is equivalent in this case to the duality theorem. The realization problem on a finite set can thus be studied numerically via standard linear programming methods (see, e.g. [4]).

For convenience we collect here three standard results which will be used in proving Theorem 3.4 and in Section 3.3. In stating the first two we will let V be a vector space of real-valued functions on a set Ω. V is a *vector lattice* if for every $v \in V$ also $|v| \in V$ (equivalently $v_+ \in V$). On V we may consider the natural (pointwise) partial order; we say that a subspace V_0 of V *dominates* V if for every $v \in V$ there exist $v_1, v_2 \in V_0$ such that $v_1 \leq v \leq v_2$. Then [1, 22]:

THEOREM 3.6 (Riesz–Krein extension theorem). *Suppose that V is a vector space of functions as above and let V_0 be a subspace that dominates V. Then any nonnegative linear functional on V_0 has at least one nonnegative linear extension to all of V.*

We note that the nonuniqueness of the extension given by this theorem is the root of the nonuniqueness, mentioned above, of the realizing point process. The next result is from the Daniell theory of integration [26, 30]:

THEOREM 3.7. *Let V be a vector space of functions as above which is a vector lattice and which contains the constant functions. Let L be a nonnegative linear functional on V for which*:

(D) *If $(v_n)_{n \in \mathbb{N}}$ is a sequence of functions in V which decreases monotonically to zero then $\lim_{n \to \infty} L(v_n) = 0$.*

Then there exists one and only one measure v on (Ω, Σ_V), where Σ_V is the σ-algebra generated by V, such that for all $v \in V$,

$$L(v) = \int_\Omega v(\omega) v(d\omega).$$

Finally we give a well-known characterization of compact subsets of $\mathcal{N}(X)$ which follows from [10], Corollary A.2.6.V and the observation in Section 2 that X is metrizable with a metric for which all bounded sets have compact closure.

LEMMA 3.8. *A set $C \subset \mathcal{N}(X)$ is compact if and only if C is closed and $\sup_{\eta \in C} \eta(\Lambda) < \infty$ for every compact subset $\Lambda \subset X$.*

Our next result is the key step in the proof of Theorem 3.4.

PROPOSITION 3.9. *Let $\mathcal{N}_{\text{supp}}$ be a compact subset of $\mathcal{N}(X)$, let (ρ_1, ρ_2) be Radon measures on X and $X \times X$, respectively, with ρ_2 symmetric and suppose that any quadratic polynomial $P_{f_0, f_1, f_2}(\eta)$ which is nonnegative on $\mathcal{N}_{\text{supp}}$ satisfies (15). Then (ρ_1, ρ_2) is realizable by a point process supported on $\mathcal{N}_{\text{supp}}$.*

PROOF. Let V be the vector space of all continuous functions on $\mathcal{N}_{\text{supp}}$ and let V_0 be the vector space of all quadratic polynomials P_{f_0, f_1, f_2} with continuous coefficients; from the compactness of $\mathcal{N}_{\text{supp}}$ it is clear that V_0 dominates V. Let L be the linear form on V_0 defined by

$$L(P_{f_0, f_1, f_2}) := f_0 + \int_X f_1(\mathbf{r}) \rho_1(d\mathbf{r}) + \int_{X \times X} f_2(\mathbf{r}_1, \mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2).$$

The hypothesis of the theorem is precisely that L is nonnegative so by the Riesz–Krein extension theorem we can extend L to a nonnegative linear functional on all of V. Since $\mathcal{N}_{\text{supp}}$ is compact, the Riesz–Markov representation theorem implies that there exists a probability measure μ on $\mathcal{N}_{\text{supp}}$—that is, a point process on X—such that

$$L(F) = \int_{\mathcal{N}_{\text{supp}}} F(\eta) \mu(d\eta)$$

for all $F \in V$. In particular, taking $F_n(\eta) = \langle f_n, \eta^{\odot n} \rangle$ for $n = 1, 2$, with $f_n \in C_c(X^n)$ and f_2 symmetric, we obtain (10) for $n = 1, 2$ for continuous f_1, f_2; this suffices to imply that ρ_1 and ρ_2 are indeed the correlation functions of the process μ. □

Note that the proof shows that it suffices for realizability that (15) holds for polynomials with continuous coefficients.

PROOF OF THEOREM 3.4. If μ is a realization of ρ_1, ρ_2 then, as observed above, it must be supported on $\mathcal{N}_D(X)$, and by Theorem 3.2 must satisfy the given

condition. As the set $\mathcal{N}_D(X)$ is compact, by Lemma 3.8, the converse direction follows from Proposition 3.9. □

Hard core exclusion is not the only natural possibility for a compact $\mathcal{N}_{\text{supp}}$. If N is a natural number then the set of all configurations with exactly N particles, or at most N particles,

$$\mathcal{N}^N(X) := \{\eta \in \mathcal{N}(X) \mid \eta(X) = N\},$$
$$\mathcal{N}^{\leq N}(X) := \{\eta \in \mathcal{N}(X) \mid \eta(X) \leq N\},$$

is compact. We summarize the consequences in the following corollary.

COROLLARY 3.10.   *Let* (ρ_1, ρ_2) *be Radon measures on* X *and* $X \times X$ *with* ρ_2 *symmetric. Suppose that any quadratic polynomial* $P_{f_0, f_1, f_2}(\eta)$ *which is nonneg-ative on* $\mathcal{N}^N(X)$ [*resp.,* $\mathcal{N}^{\leq N}(X)$] *satisfies* (15). *Then* (ρ_1, ρ_2) *is realizable by a point process supported on* $\mathcal{N}^N(X)$ [*resp.,* $\mathcal{N}^{\leq N}(X)$].

A similar result would hold for X a lattice and, for some $k \geq 0$, $\mathcal{N}_{\text{supp}}$ the set of configurations with at most k particles at any site.

REMARK 3.11.   (a) The essential property for the proof of Proposition 3.9 is the compactness of $\mathcal{N}_{\text{supp}}$. Indeed, the result is false if $\mathcal{N}_{\text{supp}}$ is replaced by $\mathcal{N}(X)$; see Example 3.12 below.

(b) Corollary 3.10 was established by Percus in [28] and [27] using the tech-nique of double dual cone. This technique should give an alternative approach to prove sufficiency of the conditions but will require a careful identification of the closure of the initial cone requiring considerations similar to those above. In [13] and [19] a quantum mechanical version of Corollary 3.10 was worked out in the framework of reduced density matrices and trace class operators. A characteriza-tion of the closure of the cone was not considered.

(c) For any given (ρ_1, ρ_2) one could, of course, attempt to use Proposition 3.9 to establish realizability on some suitably chosen compact subset $\mathcal{N}_{\text{supp}} \subset \mathcal{N}(X)$. For translation invariant (ρ_1, ρ_2) in $\mathbb{R}^d$, for example, one might require that for $\Lambda \subset X$ with volume $|\Lambda|$, $\eta(\Lambda) \leq A(1 + |\Lambda|^k)$ for suitably chosen A and k. We do not, however, know of an example in which such an approach succeeds. What is significant about processes with hard cores is that the hard core constraint is of physical interest, is expressible in terms of the given datum ρ_2 and forces any realization to be on a compact set of configurations.

3.3. *Sufficient conditions without a hard core.*   We now consider the case of general (ρ_1, ρ_2) in which we have no a priori reason, such as a hard core constraint, to expect a realizing process to be supported on a compact set of configurations. In this case the necessary conditions of Theorem 3.2 are in general not sufficient, as shown by the following example.

EXAMPLE 3.12. Let $X = \mathbb{R}^d$ and consider the pair of correlation functions $\rho_1(\mathbf{r}) \equiv 0$, $\rho_2(\mathbf{r}_1, \mathbf{r}_2) \equiv 1$. This is certainly not realizable, since if it were realized by some process μ then for any measurable set Λ, $\mathbb{E}_\mu[\eta(\Lambda)] = \int_\Lambda \rho_1(d\mathbf{r}) = 0$ and hence, $\eta(\Lambda) = 0$ with probability one so that the second correlation function of μ would have to vanish. But consider the point process

$$(17) \qquad \mu^\epsilon(d\eta) := (1 - \epsilon^2)\delta_0(d\eta) + \epsilon^2 \pi_{1/\epsilon}(d\eta),$$

where $\epsilon \in (0, 1]$, π_z denotes the Poisson measure on $\mathbb{R}^d$ with density z and δ_0 is the measure concentrated on $\eta = 0$. The corresponding correlation functions $\rho_1^\varepsilon(\mathbf{r}) = \varepsilon$ and $\rho_2^\varepsilon(\mathbf{r}_1, \mathbf{r}_2) = 1$ converge as $\epsilon \to 0$ to the given (ρ_1, ρ_2), from which it follows easily that the latter fulfills the necessary condition of Theorem 3.2.

In the following subsections we give sufficient conditions for realizability in the general case. Lemma 3.8 and Proposition 3.9 indicate that difficulties in doing so will be associated with the local occurrence of an unbounded number of particles. The key idea is to control this by requiring not only realization of ρ_1 and ρ_2 but also the existence in some form of a finite third moment (a moment of order $2 + \epsilon$ would suffice). Such a requirement can be motivated by reconsidering the proof of Theorem 3.4, omitting the hard core hypothesis and trying to prove existence of a process supported on $\mathcal{N}(X)$. Defining V to include only functions of quadratic growth in η and using Theorem 3.7 rather than the Riesz–Markov theorem, one may establish the existence of a process μ realizing ρ_1 but not necessarily ρ_2. The situation in this section (see, e.g., Theorem 3.14) is similar: by controlling a third moment we can realize the first two correlation functions. The condition can also be motivated by considering Example 3.12; no similar example can be constructed in which the third moments of the processes μ^ϵ are uniformly bounded.

REMARK 3.13. (a) Even if X is a lattice one will still need to control some higher moment if there is no bound on the number of particles per site.

(b) In the case in which all correlation functions are prescribed, that is, when $N = \infty$ in the sense of Definition 3.1, the need to control an "extra" moment does not arise. See [21, 22].

Since the essential difficulties are local they will occur even for compact X; we will first discuss this case where certain technical difficulties are absent. Throughout this section we will define the function H_n^χ on $\mathcal{N}(X)$, where χ is a strictly positive bounded continuous function on X and $n \geq 0$, by

$$(18) \qquad H_n^\chi(\eta) := \langle \chi^{\otimes n}, \eta^{\odot n} \rangle = {\sum_{i_1,\ldots,i_n}}' \chi(x_{i_1}) \cdots \chi(x_{i_n})$$

with $\sum'$ as in (5). Note that since all summands in (18) are nonnegative the sum always is well defined, though it may be infinite. For $\Lambda \subset X$ we write $H_n^\Lambda := H_n^{\mathbf{1}_\Lambda}$

and we abbreviate H_n^X as H_n. In (18) we have defined H_n^χ using the factorial power $\eta^{\odot n}$ but one could equivalently work with $\eta^{\otimes n}$; this follows from the fact that for each $n \geq 0$ there exists a constant $b_n > 0$ such that for all $\eta \in \mathcal{N}(X)$,

$$(19) \qquad \frac{1}{2}\langle \chi, \eta \rangle^n - b_n \leq H_n^\chi(\eta) \leq \langle \chi, \eta \rangle^n \equiv \langle \chi^{\otimes n}, \eta^{\otimes n} \rangle.$$

To verify (19) we note that as all summands in (18) are nonnegative the upper bound is immediate. On the other hand, the difference of $\langle \chi, \eta \rangle^n - H_n^\chi(\eta)$ can be bounded by a linear combination of $\langle \chi, \eta \rangle^m$ for $m < n$ and each of these can be estimated above by $c\langle \chi, \eta \rangle^n + c'$ for $c > 0$ arbitrary small. As mentioned just below (11), the inequalities (19) implies that μ has finite local nth moments is equivalent to $\mathbb{E}_\mu[H_n^\Lambda] < \infty$ for all compact Λ, so that μ has finite nth moment if and only if $\mathbb{E}_\mu[H_n] < \infty$.

We will say that μ has *finite nth χ-moment* if $\mathbb{E}_\mu[H_n^\chi] < \infty$; in particular, μ then has support on the set of all configurations η with $\langle \chi, \eta \rangle < \infty$. By (19) and the positivity of χ, finite nth χ-moment implies finite local nth moments. Clearly the converse will not hold for general χ but we will show in Lemma 3.16 below that a measure with finite local nth moments has finite nth χ-moment for an appropriately chosen χ.

3.3.1. *Compact X.* Suppose that X is compact. In the next theorem we give a condition which is both necessary and sufficient for (ρ_1, ρ_2) to be realizable by a process with a finite third moment. As a corollary we obtain a sufficient condition for realizability of (ρ_1, ρ_2). The conditions that we will give involve cubic polynomials of a special form that we will call *restricted*. These have the form

$$(20) \qquad Q_{f_0, f_1, f_2, f_3}(\eta) = f_0 + \langle f_1, \eta \rangle + \langle f_2, \eta^{\odot 2} \rangle + f_3 H_3(\eta),$$

where $f_0, f_3 \in \mathbb{R}$, $f_1 \in \mathcal{C}_c(X)$ and $f_2 \in \mathcal{C}_c(X^2)$ with f_2 symmetric.

THEOREM 3.14. *Let X be compact. Then symmetric Radon measures ρ_1 and ρ_2 on X and $X \times X$ are realizable by a point process with a finite third moment if and only if there exists a constant $R > 0$ such that any restricted cubic polynomial Q_{f_0, f_1, f_2, f_3} which is nonnegative on $\mathcal{N}(X)$ satisfies*

$$(21) \qquad f_0 + \int_X f_1(\mathbf{x})\rho_1(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y})\rho_2(d\mathbf{x}, d\mathbf{y}) + f_3 R \geq 0.$$

We now have:

COROLLARY 3.15. *If the condition of Theorem 3.14 holds then the pair (ρ_1, ρ_2) is realizable.*

PROOF OF THEOREM 3.14. Let V be the vector space of all continuous functions F on $\mathcal{N}(X)$ such that $|F| \leq C(1 + H_3)$ for some constant $C > 0$ and let V_0

be the subspace of V consisting of all restricted cubic polynomials. For any $R \geq 0$ we may define a linear functional L_R on V_0 by

$$L_R(Q_{f_0,f_1,f_2,f_3}) := f_0 + \int_X f_1(\mathbf{r})\rho_1(d\mathbf{r})$$
$$+ \int_{X \times X} f_2(\mathbf{r}_1, \mathbf{r}_2)\rho_2(d\mathbf{r}_1, d\mathbf{r}_2) + f_3 R.$$

Then we must show that ρ_1, ρ_2 is realizable by a measure with a finite third moment if and only if L_R is nonnegative for some $R > 0$.

The condition is clearly necessary since if μ is such a realizing measure and $Q_{f_0,f_1,f_2,f_3} \geq 0$ then

(22) $$L_{\mu(H_3)}(Q_{f_0,f_1,f_2,f_3}) = \int Q_{f_0,f_1,f_2,f_3}(\eta)\mu(d\eta) \geq 0.$$

Suppose conversely then that R is such that L_R is nonnegative on V_0. It is easily seen that V_0 dominates V, so that, by Theorem 3.6, L_R has a nonnegative extension, which we will also call L_R, to all of V. It remains to show that this extended linear form is actually given by a measure.

Let W be the subspace of V consisting of those functions $F \in V$ such that $|F| \leq C(1 + H_2)$ for some $C > 0$. W is a lattice which generates the σ-algebra corresponding to the vague topology because it contains all functions of the form $\langle f, \cdot \rangle$ with f continuous. We wish to apply Theorem 3.7 to L_R on W and so must verify that L_R satisfies (D). Let $(F_n)_{n \in \mathbb{N}}$ be a monotonically decreasing sequence in W which converges pointwise to 0 and let $\epsilon > 0$ be given. The sets $K_n := \{\eta \in \mathcal{N}(X) \mid F_n(\eta) \geq \epsilon[1 + H_3(\eta)]\}$ are closed because F_n and H_3 are continuous. Moreover, K_n is compact because, since $F_n \in W$, K_n is for some $C > 0$ a subset of $\{\eta \in \mathcal{N}(X) \mid C[1 + H_2(\eta)] \geq \epsilon[(1 + H_3(\eta))]\}$, and the latter set is compact by Lemma 3.8 since $\eta(X)$ is bounded on it. Because the F_n decrease to zero pointwise, $\bigcap_n K_n = \varnothing$, so there must exist an $N \in \mathbb{N}$ with $K_n = \varnothing$ for $n \geq N$, that is, with $F_n \leq \epsilon(1 + H_3)$ for all $n \geq N$. This, with the positivity of L_R, implies that for $n \geq N$,

$$L_R(F_n) \leq L_R(Q_{\epsilon,0,0,\epsilon}) = \epsilon(1 + R).$$

As ϵ was arbitrary, (D) holds and, therefore, Theorem 3.7 implies that there exists a probability measure μ on $\mathcal{N}(X)$ such that for $F \in W$,

$$L_R(F) = \int_{\mathcal{N}(X)} F(\eta)\mu(d\eta).$$

In particular, for all $f_0 \in \mathbb{R}$ and continuous functions f_1 and f_2 on X and $X \times X$,

$$f_0 + \int_X f_1(\mathbf{x})\rho_1(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y})\rho_2(d\mathbf{x}, d\mathbf{y})$$
$$= \int_{\mathcal{N}(X)} P_{f_0,f_1,f_2}(\eta)\mu(d\eta),$$

which implies that μ realizes (ρ_1, ρ_2).

Finally, if for $n \in \mathbb{N}$ we define $H_3^{(n)}(\eta) = \min\{H_3(\eta), n\}$ then $H_3^{(n)} \in W$ and so $\int H_3^{(n)}(\eta) \, d\mu(\eta) = L_R(H_3^{(n)})$. But by the positivity of L_R on V, $L_R(H_3^{(n)}) \leq L_R(H_3) = L_R(Q_{0,0,0,1}) = R$ and so the monotone convergence theorem implies that $\int H_3 \, d\mu \leq R$, that is, μ has finite third moment. $\square$

3.3.2. *Noncompact X.* For the case in which X is not compact we give, in Theorems 3.17 and 3.20, two distinct sufficient conditions for realizability which generalize Theorem 3.14 in two different ways. In this section we will finally assume that the metric d_X is such that bounded sets have compact closure; cf. the beginning of Section 2. With some fixed $\mathbf{x}_0 \in X$ define $\Lambda_N = \{\mathbf{x} \in X \mid d_X(\mathbf{x}, \mathbf{x}_0) \leq N\}$. Throughout this section we let χ be a strictly positive continuous function on X. One should think of χ as a function which vanishes at infinity; for example, if $X = \mathbb{R}^d$ we might take $\chi(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-k}$ for some $k > 0$.

LEMMA 3.16. *A point process μ on X has finite local nth moments if and only if there exists a positive continuous χ such that μ has finite nth χ-moment.*

PROOF. If μ has finite nth χ-moment then, using the continuity and positivity of χ, it follows immediately that μ has finite local nth moments. Suppose conversely that μ has finite local nth moments. Let χ_k be a nonnegative function on X with compact support satisfying $\mathbb{1}_{\Lambda_k} \leq \chi_k \leq \mathbb{1}_X$; then $\int_{\mathcal{N}(X)} \langle \chi_k, \eta \rangle^n \mu(d\eta) < \infty$ for all k. Define

$$\chi(x) := \sum_{k=1}^{\infty} c_k \chi_k(x) \qquad \text{with } c_k := \frac{2^{-k}}{1 + \sqrt[n]{\int \langle \chi_k, \xi \rangle^n \mu(d\xi)}}.$$

Then

$$\int_{\mathcal{N}(X)} H_n^\chi(\eta) \mu(d\eta) \leq \int_{\mathcal{N}(X)} \langle \chi, \eta \rangle^n \mu(d\eta)$$

$$= \sum_{k_1, k_2, \ldots, k_n = 1}^{\infty} \prod_{i=1}^{n} c_{k_i} \int_{\mathcal{N}(X)} \prod_{i=1}^{n} \langle \chi_{k_i}, \eta \rangle \mu(d\eta)$$

$$\leq \left(\sum_{k=1}^{\infty} c_k \sqrt[n]{\int_{\mathcal{N}(X)} \langle \chi_k, \eta \rangle^n \mu(d\eta)} \right)^n \leq 1,$$

where we have used Hölder's inequality. $\square$

The sufficiency criteria of the next theorem are stated in terms of χ-*restricted cubic polynomials*,

$$(23) \qquad Q_{f_0, f_1, f_2, f_3}^\chi(\eta) = f_0 + \langle f_1, \eta \rangle + \langle f_2, \eta^{\odot 2} \rangle + f_3 H_3^\chi(\eta),$$

where $f_0, \ldots, f_3$ are as in (20).

THEOREM 3.17. *Symmetric Radon measures ρ_1 and ρ_2 on X and $X \times X$ are realizable by a point process with finite local third moments if and only if there exists a constant $R > 0$ and a positive function χ such that any χ-restricted cubic polynomial $Q^\chi_{f_0, f_1, f_2, f_3}$ which is nonnegative on $\mathcal{N}(X)$ satisfies*

$$(24) \qquad f_0 + \int_X f_1(\mathbf{x})\rho_1(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y})\rho_2(d\mathbf{x}, d\mathbf{y}) + f_3 R \geq 0.$$

PROOF. According to Lemma 3.16 it suffices to show that ρ_1 and ρ_2 are realizable by a point process with finite third χ-moment if and only if the condition is satisfied. The proof of this is very similar to that of Theorem 3.14, with H_2 and H_3 replaced by H_2^χ and H_3^χ throughout, so we content ourselves here with commenting on the technical modifications necessitated by the noncompact character of X.

One source of difficulties is that H_n^χ is not a continuous function on $\mathcal{N}(X)$. This means that if the vector space used in the proof was to be defined in parallel to the V of the earlier proof then it would not contain all χ-restricted polynomials. The problem may be avoided by replacing V throughout by $V^\chi := V_0^\chi + V_1^\chi$, where V_0^χ is the space of all χ-restricted cubic polynomials (which plays the same role as did V_0 earlier) and V_1^χ, defined in parallel to the earlier V, is the vector space of all continuous functions F on $\mathcal{N}(X)$ such that $|F| \leq C(1 + H_3^\chi)$ for some constant $C > 0$.

The set K_n is replaced by $K_n^\chi = \{\eta \in \mathcal{N}(X) \mid F_n(\eta) \geq \epsilon[(1 + H_3^\chi(\eta))]\}$; the argument that K_n was closed used the continuity of H_3^χ but lower semi-continuity suffices and we establish this in the next lemma. K_n^χ is for some $C > 0$ a subset of $\{\eta \in \mathcal{N}(X) \mid C[1 + H_2^\chi(\eta)] \geq \epsilon[1 + H_3^\chi(\eta)]\}$ and this set is precompact by Lemma 3.8, since H_1^χ is bounded on it and for any compact $\Lambda \subset X$ there is a constant c_Λ with $\mathbb{1}_\Lambda \leq c_\Lambda \chi$. The sequence $H_3^{(n)}$ used in the last step of the proof is replaced by any sequence of bounded continuous functions increasing to H_3^χ; the existence of such a sequence follows from the lower semicontinuity of H_3^χ. □

LEMMA 3.18. *For any $n > 0$ the function H_n^χ is lower semi-continuous.*

PROOF. We must show that sets of the form $S := \{\eta \in \mathcal{N}(X) \mid H_n^\chi(\eta) \leq C\}$ are closed. Let (η_k) be a sequence in S converging vaguely to $\eta \in \mathcal{N}(X)$ and let (χ_m) be an increasing sequence of nonnegative continuous functions with compact support on X such that $\chi_m \nearrow \chi$. By the vague convergence $\langle \chi_m^{\otimes n}, \eta_k^{\odot n} \rangle \to \langle \chi_m^{\otimes n}, \eta^{\odot n} \rangle$ as $k \nearrow \infty$, for any fixed m, and by the monotone convergence of χ_m also $\langle \chi_m^{\otimes n}, \eta^{\odot n} \rangle \nearrow H_n^\chi(\eta)$ as $m \nearrow \infty$. Since $\langle \chi_m^{\otimes n}, \eta_k^{\odot n} \rangle \leq H_n^\chi(\eta) \leq C$, also $H_n^\chi(\eta) \leq C$, and so S is closed. □

REMARK 3.19. By taking $\chi(\mathbf{x}) \equiv 1$ in Theorem 3.17 we see that in fact Theorem 3.14 holds even when X is not compact. For typical problems, however, this

result is not very interesting since a realizing measure with finite third moment would be impossible if, for example, $\langle \eta(X) \rangle = \int_X \rho_1(dx)$ were infinite, as would be true for any nonzero translation invariant ρ_1 in $\mathbb{R}^d$.

We now give the second sufficient condition.

THEOREM 3.20.   *Let $X = \mathbb{R}^d$. Then symmetric Radon measures ρ_1 and ρ_2 on X and $X \times X$ are realizable by a point process with finite local third moments if and only if the condition of Theorem* 3.14 *holds in every Λ_N, $N \in \mathbb{N}$.*

PROOF.   The necessity of the condition follows as in the proof of Theorem 3.14. Suppose conversely that the condition of Theorem 3.14 holds in every Λ_N so that for each N there exists a measure μ_N on $\mathcal{N}(\Lambda_N)$ which realizes (ρ_1, ρ_2) in Λ_N. If $N \geq n$ then μ_N defines in the obvious way a marginal measure μ_N^n on $\mathcal{N}(\mathring{\Lambda}_n)$, where $\mathring{\Lambda}_n$ denotes the interior of Λ_n; all the measures μ_N^n, $N \geq n$, have the same one- and two-point correlation functions ρ_1 and ρ_2 on $\mathring{\Lambda}_n$. Since

$$(25) \qquad c_n := \langle \eta(\mathring{\Lambda}_n) \rangle_{\mu_N^n} = \int_{\mathring{\Lambda}_n} \rho_1(dx)$$

is independent of N, Markov's inequality implies that these measures satisfy $\mu_N^n[(K_n(M)) \geq 1 - c_n/M$, where $K_n(M) = \{\eta \mid \eta(\mathring{\Lambda}_n) \leq M\}$. Since $K_n(M)$ is compact by Lemma 3.8, the sequence of measures $(\mu_N^n)_{N \geq n}$ is tight and any subsequence of this sequence itself contains a convergent subsequence. We may thus obtain recursively sequences $(N_{n,k})_{k \in \mathbb{N}}$ such that $(N_{n+1,k})$ is a subsequence of $(N_{n,k})$ and such that $(\mu_{N_{n,k}}^n)_{k \in \mathbb{N}}$ converges weakly to a measure μ^n on $\mathcal{N}(\mathring{\Lambda}_n)$. The measure μ^n realizes (ρ_1, ρ_2) on $\mathring{\Lambda}_n$. The μ^n are compatible, in the sense that μ^n is the marginal of μ^{n+1} on $\mathcal{N}(\mathring{\Lambda}_n)$, because the projections from $\mathcal{N}(\mathring{\Lambda}_{n+1})$ onto $\mathcal{N}(\mathring{\Lambda}_n)$ are continuous since $\mathring{\Lambda}_n$ is open. Thus a realizing measure on $\mathcal{N}(X)$ exists by Kolmogorov's projective limit theorem. $\square$

REMARK 3.21.   In checking the sufficient conditions for realizability given in Theorems 3.4, 3.14, 3.17 and 3.20 it may be advantageous to choose the coefficients of the quadratic polynomials (12) from a class of functions other than $\mathcal{C}_c(X^m)$. Suppose then that we can verify the conditions (13) when the coefficients f_m are chosen from $\mathcal{F}_m$, a subspace of $\mathcal{C}_c(X^m)$ [with $\mathcal{F}_0 \equiv \mathcal{C}_c(X^0) \equiv \mathbb{R}$]. By straightforward modifications of the proofs of Theorems 3.4, 3.14 and 3.17 one sees that this will suffice for realizability if $\mathcal{F}_m$ identifies measures on X^m, that is, if whenever Radon measures ν and ν' satisfy $\int_{X^m} f_m(\mathbf{r})\nu(d\mathbf{r}) = \int_{X^m} f_m(\mathbf{r})\nu'(d\mathbf{r})$ for all $f_m \in \mathcal{F}_m$, necessarily $\nu = \nu'$. For an analogously modified version of Theorem 3.20 slightly more is needed: for each N the functions from $\mathcal{F}_m$ with $\operatorname{supp} \mathcal{F}_m \subset \Lambda_N$ must identify measures on $\Lambda_N{}^m$. These conditions are fulfilled

if $\mathcal{F}_m$ forms an algebra which separates points. For example, if X is a manifold without boundary then one may take $\mathcal{F}_m$ to be $\mathcal{C}_c^\infty(X^m)$, the space of infinitely differentiable functions with compact support.

3.4. *Stability of realizability under limits.* The sufficient conditions obtained above can be used to derive general results about realizing measures. In this subsection we discuss sufficient conditions for the limit of a sequence of realizable correlation functions to be itself realizable. Each of the Theorems 3.4, 3.14, 3.17 and 3.20 will give rise to a different variant. Recall that $(\rho_1^{(n)}, \rho_2^{(n)})$ converges in the vague topology to (ρ_1, ρ_2) if for any $f_1 \in \mathcal{C}_c(X)$ and $f_2 \in \mathcal{C}_c(X \times X)$,

$$(26) \qquad \int_X f_1(\mathbf{r})\rho_1^{(n)}(d\mathbf{r}) \to \int_X f_1(\mathbf{r})\rho_1(d\mathbf{r})$$

and

$$(27) \qquad \int_{X2} f_2(\mathbf{r}_1, \mathbf{r}_2)\rho_2^{(n)}(d\mathbf{r}_1, \mathbf{r}_2) \to \int_X f_2(\mathbf{r}_1, \mathbf{r}_2)\rho_2(d\mathbf{r}_1, d\mathbf{r}_2).$$

For the hard core case we have to require a uniform exclusion diameter.

PROPOSITION 3.22. *Let* $(\rho_1^{(n)}, \rho_2^{(n)})$ *be a sequence of realizable pairs of symmetric Radon measures which converges in the vague topology to* (ρ_1, ρ_2) *and for which there exists a* $D > 0$ *such that* $\rho_2^{(n)}(\{(\mathbf{r}_1, \mathbf{r}_2) \mid d(\mathbf{r}_1, \mathbf{r}_2) < D\}) = 0$ *for all* n. *Then* (ρ_1, ρ_2) *is also realizable.*

PROOF. If P_{f_0, f_1, f_2} is a nonnegative quadratic polynomial on $\mathcal{N}_D(X)$ then the hypotheses imply that

$$(28) \qquad f_0 + \int_X f_1(\mathbf{x})\rho_1^{(n)}(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y})\rho_2^{(n)}(d\mathbf{x}, d\mathbf{y}) \geq 0$$

for all n. Taking the $n \to \infty$ limit then gives (15). By the portmanteau theorem, the limiting correlation functions force also a hard core exclusion. $\square$

For lattice gases this implies a very natural result:

COROLLARY 3.23. *Let* X *be a lattice and let* $(\rho_1^{(n)}, \rho_2^{(n)})$ *be a sequence of realizable pairs with* $\rho_2^{(n)}(\mathbf{r}, \mathbf{r}) = 0$ *for all* n *and* $\mathbf{r}$. *If* $(\rho_1^{(n)}, \rho_2^{(n)})$ *converges pointwise to* (ρ_1, ρ_2), *then* (ρ_1, ρ_2) *is realizable.*

From Theorem 3.14 we have the following.

PROPOSITION 3.24. *Let* X *be compact and let* $(\rho_1^{(n)}, \rho_2^{(n)})$ *be a sequence of realizable pairs of symmetric Radon measures which converges in the vague topology to* (ρ_1, ρ_2) *and is such that the condition of Theorem 3.14 holds for* $(\rho_1^{(n)}, \rho_2^{(n)})$ *for some* $R_n \geq 0$ *with* $\liminf_{n \to \infty} R_n < \infty$. *Then* (ρ_1, ρ_2) *is also realizable.*

The proof is similar to the proof of the next theorem, which arises from Theorem 3.17.

PROPOSITION 3.25. *Let $(\rho_1^{(n)}, \rho_2^{(n)})$ be a sequence of realizable pairs of symmetric Radon measures which converges in the vague topology to (ρ_1, ρ_2) and is such that the condition of Theorem 3.17 holds for $(\rho_1^{(n)}, \rho_2^{(n)})$ for some fixed χ and $R_n \geq 0$ with $\liminf_{n\to\infty} R_n < \infty$. Then (ρ_1, ρ_2) is also realizable.*

PROOF. We will show that (ρ_1, ρ_2) fulfills the sufficiency condition of Theorem 3.17. Without loss of generality we may replace $(\rho_1^{(n)}, \rho_2^{(n)})$ by a subsequence such that R_n converges to a finite limit R. If $Q^\chi_{f_0, f_1, f_2, f_3}$ is a nonnegative χ-restricted polynomial then the hypotheses imply that

$$(29) \qquad f_0 + \int_X f_1(\mathbf{x})\rho_1^{(n)}(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y})\rho_2^{(n)}(d\mathbf{x}, d\mathbf{y}) + f_3 R_n \geq 0$$

for all n. Taking the $n \to \infty$ limit then gives (24). □

It is easy to see that the conditions of Proposition 3.25 may be replaced by the requirement that the pairs $(\rho_1^{(n)}, \rho_2^{(n)})$ can be realized by processes μ_n in such a way that $\liminf_{n\to\infty} \int_{\mathcal{N}(X)} H_3^\chi(\eta)\mu_n(d\eta) < \infty$.

There is an analogous consequence of Theorem 3.20 whose statement we omit.

4. Realizability for stationary processes. In this section we use a variant of the previous results to consider the question of whether correlation functions having some symmetry can be realized by a point process having the same symmetry. Throughout this section we take G to be a topological group acting transitively on X in such a way that the action, considered as a map $G \times X \to X$, is continuous. The group action can then be extended to an action on the Radon measures on X and hence, on $\mathcal{N}(X)$ and thus finally to an action on point processes; the latter is continuous and linear. We call a point process *stationary* if it is invariant under this action. A stationary point process has stationary correlation functions, that is, these functions are also invariant under the action of the group. Here we address the converse question of whether or not stationary correlation functions can be realized by stationary point processes.

For simplicity we will consider only the possibilities that G be Abelian or compact, or a semi-direct product of an Abelian and a compact group.

Typical cases are $X = \mathbb{R}^d$, $\mathbb{Z}^d$, etc. As described earlier, there is then a natural action of the translation group on X. In this context for a stationary point process there necessarily exists a real number ρ such that $\rho_1(d\mathbf{r}) = \rho\, d_I\mathbf{r}$, where $d_I\mathbf{r}$ denotes the invariant measure on X: Lebesgue measure on $\mathbb{R}^d$ and the torus and counting measure on $\mathbb{Z}^d$ and the discrete torus. In general, however, it may not be true that ρ_2 has a density with respect to the Lebesgue measure on X^2; for example, consider on $\mathbb{R}$ the point process defined by $\mu(d\eta) := \int_0^1 \delta_{\bar{\eta}_y}(d\eta)\, dy$, where

$\bar{\eta}_y(dr) := \sum_{x \in \mathbb{Z}} \delta_{y+x}(dr)$. However, one can show that there must exist a Radon measure g_2 on $X = \mathbb{R}^d$ such that for any $f_2 \in \mathcal{C}_c(X^2)$,

$$(30) \qquad \int_X \int_X f_2(\mathbf{r}_1, \mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2) = \int_X \int_X f_2(\mathbf{r}, \mathbf{r} + \bar{\mathbf{r}}) \rho^2 g_2(d\bar{\mathbf{r}}) \, d_I \mathbf{r}.$$

The form in which we have written the right-hand side of (30), isolating a factor of ρ^2 in the two-point function, is natural in certain applications (see, e.g., [4, 20]).

We first consider the case in which G is Abelian. To be concrete we will fix a strictly positive bounded continuous function χ on X and work in the spirit of Theorem 3.17, considering processes with finite third χ-moments but similar results could be given in the spirit of Theorem 3.20. The key idea is to work with processes satisfying a bound on the third χ-moment which is uniform under the group action. More precisely, denoting by $g\chi$ the transformed function $\chi(g\cdot)$, we require a bound for $H_3^{g\chi}$ uniform in g. Proposition 4.1 establishes the existence of a stationary process given the existence of one process satisfying such a uniform bound and Theorem 4.3 gives sufficient conditions, solely in terms of the given moments, for the realizability by a process satisfying such a bound.

PROPOSITION 4.1. *Let G be Abelian and let (ρ_1, ρ_2) be stationary correlation functions realizable by a process μ satisfying $\sup_{g \in G} \mathbb{E}_\mu H_3^{g\chi} \leq R$. Then (ρ_1, ρ_2) can be realized by a stationary point process.*

PROOF. Let K_R denote the set of all point processes μ which realize (ρ_1, ρ_2) and satisfy $\sup_{g \in G} \mathbb{E}_\mu H_3^{g\chi} \leq R$; K_R is nonempty by hypothesis. The action of G on point processes leaves K_R invariant. In Lemma 4.2 we prove that K_R is convex and compact. Then by the Markov–Kakutani fixed point theorem (see, e.g., [29], Theorem V.20) there exists a $\mu \in K_R$ which is invariant with respect to the action of G. $\quad\square$

LEMMA 4.2. *The set K_R introduced in the proof of Proposition 4.1 is convex and compact in the weak topology.*

PROOF. The convexity of K_R is obvious. To show that K_R is compact in the weak topology, we first show that it is tight and hence precompact. From Lemma 3.8 it follows easily that $S_N := \{\eta \in \mathcal{N}(X) \mid \langle \eta, \chi \rangle \leq N\}$ is compact and if $\mu \in K_R$ then from $\mathbb{E}_\mu H_3^\chi \leq R$ and (19) it follows via Markov's inequality that for $\epsilon > 0$ there is a choice of N, depending only on ϵ and R, such that $\mu(S_N) > 1 - \epsilon$, verifying tightness.

It remains to prove that K_R is closed. Let μ_n be a sequence in K_R which converges weakly to a point process μ. Approximating $H_3^{g\chi}$ by an increasing sequence of bounded continuous functions and using the convergence of the sequence μ_n on such functions and the monotone convergence theorem for μ, we find that $\int_{\mathcal{N}(X)} H_3^{g\chi}(\eta) \mu(d\eta) \leq R$. It remains to show that μ_n converges also on

every quadratic polynomial $P = P_{f_0, f_1, f_2}$ with $f_1 \in \mathcal{C}_c(X)$ and $f_2 \in \mathcal{C}_c(X^2)$, which guarantees that μ has the correct first and second correlation functions. But by (19), $|P(\eta)| \leq A + B\langle \chi, \eta \rangle^2$ for some A, $B \geq 0$ and so for $M \geq 2A$, $|P(\eta)| \geq M$ implies $|P(\eta)| \leq 2B\langle \chi, \eta \rangle^2$ and so for any $\nu \in K_R$,

$$\int_{P \geq M} |P(\eta)| \nu(d\eta) \leq 2B \int_{\langle \chi, \eta \rangle^2 \geq M/2B} \langle \chi, \eta \rangle^2 \nu(d\eta)$$

$$(31) \qquad\qquad\qquad \leq \frac{(2B)^{3/2}}{M^{1/2}} \int_X \langle \chi, \eta \rangle^3 \nu(d\eta)$$

$$\leq \frac{(2B)^{3/2}}{M^{1/2}} 2(b_3 + R),$$

where we have used (19) again. But if $P^{(M)}(\eta) := \text{sign}[P(\eta)] \min\{|P(\eta)|, M\}$ then for any fixed M,

$$\int_X P^{(M)}(\eta) \mu_n(d\eta) \longrightarrow \int_X P^{(M)}(\eta) \mu(d\eta) \qquad \text{as } n \to \infty$$

and with (31) the proof is complete. $\quad\square$

Our sufficient condition for the existence of a process, analogous to Theorem 3.17, involves polynomials of the form

$$(32) \qquad Q^{\chi}_{f_0, f_1, f_2, (f_{3,1}, g_1), \ldots, (f_{3,n}, g_n)}(\eta)$$

$$= f_0 + \langle f_1, \eta \rangle + \langle f_2, \eta^{\odot 2} \rangle + \sum_{i=1}^{n} f_{3,i} H_3^{g_i \chi}(\eta),$$

where χ is as above, f_0 and $f_{3,1}, \ldots, f_{3,n}$ are real numbers, f_1 and f_2 are continuous symmetric functions with compact support on X and $X \times X$, respectively, and $g_1, \ldots, g_n \in G$. The term $\sum_{i=1}^{n} f_{3,i} H_3^{g_i \chi}$ in (32) controls moments involving $H_3^{g \chi}$ and also makes the set of all the Q^{χ} into a vector space.

THEOREM 4.3. *Let G be Abelian and let ρ_1 and ρ_2 be symmetric G-stationary Radon measures on X and $X \times X$, respectively. Then ρ_1 and ρ_2 are realizable by a stationary point process μ with $\sup_{g \in G} \int H_3^{g \chi}(\eta) \mu(d\eta) < \infty$ if and only if there is a constant $R > 0$ such that if $Q^{\chi}_{f_0, f_1, f_2, (f_{3,1}, g_1), \ldots, (f_{3,n}, g_n)}$ is nonnegative on $\mathcal{N}(X)$ then*

$$(33) \qquad f_0 + \int_X f_1(\mathbf{x}) \rho_1(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y}) \rho_2(d\mathbf{x}, d\mathbf{y}) + \sum_{i=1}^{n} f_{3,i} R \geq 0.$$

PROOF. The proof is completely parallel to the proofs of Theorems 3.14 and 3.17 and we mention only a few details. Let V be the vector space of all functions which have the form $F + \sum_{i=1}^{n} \alpha_i H_3^{g_i \chi}$, where F is a continuous function

on $\mathcal{N}(X)$ satisfying $|F| \leq C(1 + H_3^\chi)$ for some constant $C > 0$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $g_1, \ldots, g_n \in G$. Let V_0 be the subspace of V consisting of all polynomials $Q_{f_0, f_1, f_2, (f_{3,1}, g_1), \ldots, (f_{3,n}, g_n)}^\chi$. For any $R \geq 0$ we define a linear functional L_R on V_0 by

$$L_R\big(Q_{f_0, f_1, f_2, (f_{3,1}, g_1), \ldots, (f_{3,n}, g_n)}\big)$$

$$:= f_0 + \int_X f_1(\mathbf{r}) \rho_1(d\mathbf{r}) + \int_{X \times X} f_2(\mathbf{r}_1, \mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2) + \sum_{i=1}^n f_{3,i} R$$

and show that ρ_1, ρ_2 is realizable by a process μ with $\sup_{g \in G} \int H_3^{g\chi}(\eta) \mu(d\eta) < \infty$ if and only if L_R is nonnegative for some $R > 0$. The condition is clearly necessary. Conversely, if R is such that L_R is nonnegative on V_0, we may extend L_R to V using Theorem 3.6. To show that this extended linear form is given by a measure we let W be the subspace of V consisting of all continuous functions $F \in V$ such that $|F| \leq C(1 + H_2^\chi)$ for some $C > 0$ and apply Theorem 3.7 to L_R on W. The verification that L_R satisfies (D) on W is the same as the corresponding verification for Theorem 3.17 and we conclude that there exists a probability measure μ on $\mathcal{N}(X)$ such that for $F \in W$,

$$L_R(F) = \int_{\mathcal{N}(X)} F(\eta) \mu(d\eta).$$

As W includes all Q_{f_0, f_1, f_2} the measure μ realizes (ρ_1, ρ_2). Finally, for $n \in \mathbb{N}$ and $g \in G$ the lower semi-continuous function $H_3^{g\chi}$ can be approximated from below by an increasing sequence of continuous bounded functions $H_{3,k}^{g\chi}$. By the positivity of L_R on V, $\int H_{3,k}^{g\chi} d\mu = L_R(H_{3,k}^{g\chi}) \leq L_R(H_3^{g\chi}) = L_R(Q_{0,0,0,(1,g)}) = R$ and so the monotone convergence theorem implies that $\int H_3^{g\chi} d\mu \leq R$. The result follows from Proposition 4.1. $\square$

Next we consider the case of compact groups.

PROPOSITION 4.4. *Let G be a compact and let ρ_1 and ρ_2 be symmetric G-stationary Radon measures on X and $X \times X$. Then ρ_1 and ρ_2 are realizable by a stationary point process μ if and only if they are realizable.*

PROOF. Let μ be a realizing point process for ρ_1 and ρ_2. Denote by ν the Haar measure on G and by $g\mu$ the point process transformed via the action of g. Then define $\tilde{\mu} := \int_G (g\mu) \nu(dg)$ in the sense that

$$\int F(g\eta) \tilde{\mu}(d\eta) := \int_G F(g\eta) \mu(d\eta) \nu(dg) \qquad \text{for all } F \in L^1(\mathcal{N}(X), \mu),$$

$\tilde{\mu}$ is a stationary realizing point process. $\square$

Finally, we may easily combine the previous two cases and, in particular, cover the important special case of the Euclidean group acting on $\mathbb{R}^n$.

PROPOSITION 4.5.   *Let G be the semi-direct product $N \rtimes H$ of an Abelian group N and a compact topological group H and let ρ_1 and ρ_2 be symmetric G-stationary Radon measures on X and $X \times X$. Then ρ_1 and ρ_2 are realizable by a stationary point process μ with $\sup_{g \in G} \int H_3^{g\chi}(\eta)\mu(d\eta) < \infty$ if and only if there exists a constant $R > 0$ such that if $Q^{\chi}_{f_0, f_1, f_2, (f_{3,1}, g_1), \ldots, (f_{3,n}, g_n)}$, $g_i \in N$, is nonnegative on $\mathcal{N}(X)$ then*

$$(34) \qquad f_0 + \int_X f_1(\mathbf{x})\rho_1(d\mathbf{x}) + \int_{X \times X} f_2(\mathbf{x}, \mathbf{y})\rho_2(d\mathbf{x}, d\mathbf{y}) + \sum_{i=1}^{n} f_{3,i} R \geq 0.$$

PROOF.   Applying Theorem 4.3 to the action of N we obtain an N-stationary point process. Using the construction in Proposition 4.4 we arrive at point process also stationary under the action of H and hence, stationary for the action of G. The particular structure of the multiplication in the semi-direct product does not play any role.   □

As a closing remark, we note that in this section we have concentrated on extensions of the results of Section 3.3.2 to stationary processes but that corresponding extensions for the results in Sections 3.2 and 3.3.1 can be obtained similarly and in fact more easily. The next proposition gives extensions of Theorems 3.4 and 3.14.

PROPOSITION 4.6.   *Let G be as in Proposition 4.5 and let ρ_1 and ρ_2 be symmetric G-stationary Radon measures on X and $X \times X$. Then:*

(a) If ρ_2 forces a hard core exclusion for a metric d and the action of G leaves d invariant, then ρ_1 and ρ_2 are realizable by a stationary point process μ if and only if they are realizable by a point process.

(b) If X is compact, then ρ_1 and ρ_2 are realizable by a stationary point process μ with finite third moment if and only if they are realizable by a point process with finite third moment.

PROOF.   In each case one first verifies the result for G Abelian and then extends to the semi-direct product case as in the proof of Proposition 4.5. When G is Abelian the proof of (a) follows the proof of Proposition 4.1 but now instead of K_R we consider the set K of all measures realizing (ρ_1, ρ_2). K is obviously convex; to show that K is compact we note that since $\mathcal{N}_D(X)$ is compact so is the set of all probability measures on $\mathcal{N}_D(X)$ [26], of which K is a subset. Moreover, K is closed since, because quadratic polynomials on $\mathcal{N}_D(X)$ are bounded and continuous, weak limit points of K give the same expectation values of quadratic polynomials as do points in K and thus also realize (ρ_1, ρ_2). Part (b) follows from Proposition 4.1 itself by taking $\chi = 1$ there and using the fact that then $g\chi = \chi$ for all $g \in G$.   □

5. Classes of necessary conditions. If X is finite then, as indicated in Remark 3.5(c), the necessary and sufficient conditions of Theorem 3.4 give rise to a finite linear programming problem. In this section we allow X to be infinite and consider the problem of isolating useful necessary conditions from among the uncountably infinite class of Theorem 3.2. In the latter case, as in the former, the conditions arising from distinct functions may be related; in particular, some of them may imply others. For practical purposes it would be desirable to identify a class of functions, as small as possible, such that the conditions arising from this class imply all the conditions but for this presumably very hard problem we have no solution at the moment. In this section we will, however, for a certain uncountable subclass of the full class of conditions of Theorem 3.2, identify a handful of conditions which imply those of the whole subclass so that one may check all conditions arising from the subclass by checking the few selected conditions.

Suppose that we are given a pair (ρ_1, ρ_2) of correlation functions and wish to use Theorem 3.2 to show that this pair is not realizable on some $\mathcal{N}_{\text{supp}}$. If ρ_2 forces a hard core exclusion with diameter D or if we impose a bound on the number of particles as in Corollary 3.10, then we would take $\mathcal{N}_{\text{supp}}$ to be $\mathcal{N}_D(X)$, $\mathcal{N}^{\leq N}(X)$ or $\mathcal{N}^N(X)$, but otherwise we have a priori no better choice than to take $\mathcal{N}_{\text{supp}} = \mathcal{N}(X)$. The general strategy that we suggest, and will illustrate by an example, is to introduce a family of polynomials on $\mathcal{N}(X)$ depending on some finite set of parameters and then to determine a finite subset of this family such that satisfying the necessary conditions for polynomials in the subset guarantees satisfaction for all polynomials in the original family.

We work out this strategy in a particular case obtaining in the process several standard necessary conditions which have appeared in the literature (see [20] for a detailed exposition and references). We choose a fixed nonzero $f \in \mathcal{M}_c(X)$ and consider the family of all polynomials of the form

$$(35) \qquad P^{(a,b,c)}(\eta) := a \langle f, \eta \rangle^2 + b \langle f, \eta \rangle + c.$$

Note that in the notation of (12), $P^{(a,b,c)} = P_{f_0, f_1, f_2}$, with

$$f_2(\mathbf{r}_1, \mathbf{r}_2) = a f(\mathbf{r}_1) f(\mathbf{r}_2), \qquad f_1(\mathbf{r}) = b f(\mathbf{r}) + a f^2(\mathbf{r}), \qquad f_0 = c.$$

Let $F := \{\langle f, \eta \rangle \mid \eta \in \mathcal{N}_{\text{supp}}\} \subset \mathbb{R}$ be the range of $\langle f, \cdot \rangle$ and let Γ be the convex cone of all $(a, b, c) \in \mathbb{R}^3$ such that $p(x) := ax^2 + bx + c \geq 0$ for all $x \in F$. The necessary condition then is that the linear function $L = L_{\rho_1, \rho_2}$ defined by

$$(36) \qquad \begin{aligned} L(a, b, c) = {} & a \int_{X^2} f(\mathbf{r}_1) f(\mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2) \\ & + \int_X \big(b f(\mathbf{r}) + a f^2(\mathbf{r})\big) \rho_1(d\mathbf{r}) + c, \end{aligned}$$

should be nonnegative on Γ.

Before continuing we give a (nonexhaustive) discussion of possible structure of F, excluding the uninteresting case $f = 0$, in order to give some feeling for

how this structure can affect the necessary conditions. If $\mathcal{N}_{\text{supp}} = \mathcal{N}_D(X)$ then F is bounded above and below, for example, by $\pm M_D \sup |f|$, where M_D is the maximum number of disjoint balls of diameter D which can be placed so that their centers lie in the support of f. Similar bounds hold if $\mathcal{N}_{\text{supp}}$ is $\mathcal{N}^{\leq N}(X)$ or $\mathcal{N}^N(X)$. Otherwise F is unbounded and is bounded below (by 0) if and only if $f \geq 0$ and above (again by 0) if and only if $f \leq 0$. If f takes only a finite number of values then F will consist of certain linear combinations, with integer coefficients, of these values and F may then be discrete or may be dense in $\mathbb{R}$; in the simplest case, when $f = \mathbb{1}_\Lambda$ for some $\Lambda \subset X$, F is just a set of nonnegative integers. If $\mathcal{N}_{\text{supp}} = \mathcal{N}(X)$, f is nonnegative and the range of f contains some interval $(0, \delta)$, then $F = \mathbb{R}_+$, or if the range of f contains some interval $(-\delta, \delta)$, then $F = \mathbb{R}$.

We make two more preliminary remarks. First, if a realizing measure μ exists then $E(f) := \mathbb{E}_\mu \langle f, \cdot \rangle$ and $V(f) := \text{Var}_\mu (\langle f, \cdot \rangle)$ may be calculated from ρ_1 and ρ_2 as

$$E(f) = \int_X f(\mathbf{r}) \rho_1(d\mathbf{r}),$$

$$V(f) = \int_{X^2} f(\mathbf{r}_1) f(\mathbf{r}_2) \rho_2(d\mathbf{r}_1, d\mathbf{r}_2)$$

$$+ \int_X f(\mathbf{r})^2 \rho_1(d\mathbf{r}) - \left(\int_X f(\mathbf{r}) \rho_1(d\mathbf{r}) \right)^2,$$

so that

$$(37) \qquad L(a, b, c) = a V(f) + p[E(f)].$$

Second, due to the homogeneity in (a, b, c) of the problem it suffices to consider conditions arising from polynomials with either $a = 0$ or $a = \pm 1$.

Case 1. $a = 1$. In this case, (37) implies that the constraint on ρ_1, ρ_2 will be of the form $V(f) \geq -p[E(f)]$; by taking $p(x) = [x - E(f)]^2$ we recover the obvious requirement that $V(f) \geq 0$. The condition that $V(f) \geq 0$ for all $f \in \mathcal{C}_c(X)$ is equivalent to the so-called variance condition; cf., for example, [20]. If $E(f) \in F$ then $p[E(f)] \geq 0$ whenever $p \in \Gamma$ so that (37) implies that for no choice of b and c can $L(1, b, c) \geq 0$ impose further restrictions on ρ_1, ρ_2. Otherwise, $E(f) \in (x_-, x_+)$ for some maximal open interval (x_-, x_+) disjoint from F; then the choice $p_0(x) = (x - x_-)(x - x_+)$ implies the constraint

$$(38) \qquad V(f) \geq (x_+ - E(f))(E(f) - x_-)$$

$$\text{for } x_- := \sup\{x \in F \mid x \leq E(f)\}, \qquad x_+ := \inf\{x \in F \mid x \geq E(f)\}.$$

An easy computation shows that any monic quadratic polynomial p with $p(x_-)$, $p(x_+) \geq 0$ satisfies $p[E(f)] \geq p_0[E(F)]$, so that (38) includes all restrictions arising in Case 1 [note that as written the constraint (38) includes the case $E(f) \in F$]. If $f = \mathbb{1}_\Lambda$ for $\Lambda \subset X$ then $F = \mathbb{N}_0$ and (38) was found by Yamada [36]. Whether for other choices of f one obtains additional restrictions is unknown.

In the case $x_-, x_+ \in F$, $x_- < x_+$ the choice of x_-, x_+ in (38) corresponds to an extremal ray in the cone Γ. The cone can be defined as intersection $\bigcap_{y \in F} H_y$ with $H_y := \{(a, b, c) \in \mathbb{R}^3 \mid ay^2 + by + c \geq 0\}$. Hence, to each pair $x_1 \leq x_2 \in F$ there corresponds a ray $\{(a, b, c) \in \mathbb{R}^3 \mid ax_1^2 + bx_1 + c = 0 \text{ and } ax_2^2 + bx_2 + c = 0\}$. This ray will be in the cone and hence, an extremal ray only if $(x_1, x_2) \cap F = \varnothing$. Hence, the choice of x_-, x_+ in (38) corresponds to a particular extremal ray of Γ.

Case 2. $a = -1$. In this case, $p(x)$ can be nonnegative on F only if F is bounded and reasoning as in the previous case shows that the constraint obtained from $p(x) = (\sup F - x)(x - \inf F)$,

$$(39) \qquad V(f) \leq [\sup F - E(f)][E(f) - \inf F]$$

implies all others.

Case 3. $a = 0$. We assume $b \neq 0$ since a constant polynomial conveys no restriction; then we may take $b = \pm 1$ and thus consider $p(x) = \pm(x - x_0)$. Such a linear function can be nonnegative on F only if either (i) F is bounded below, in which case the constraint from $p(x) = x - \inf F$ implies all others, or (ii) F is bounded above, in which case a similar conclusion holds for $p(x) = \sup F - x$. If f is nonnegative then $\inf F = 0$ and the condition in (i), $E(f) \geq 0$, just asserts the positivity of the measure ρ_1. Case (ii), namely, $E(f) \leq \sup F$, can occur if ρ_2 enforces a hard core exclusion or if we impose an a priori bound on the number of particles as in Corollary 3.10. A simple interpretation can be given when X is compact and $f = \mathbb{1}_X$; then $\sup F$ is the maximum number M of points which can be contained in X under the hard core or a priori condition and the condition imposed on ρ_1 by the constraint $E(\mathbb{1}_X) = \int_X \rho_1(dx) \leq M$ is that the expected number of points be less than this maximum. If we further assume that X is a torus with Lebesgue measure ν and that $\rho_1(dx) \equiv \rho\nu(dx)$ is invariant under translations then this condition is $\rho \leq M/\nu(X)$. We can then see that no constraint arising from another choice of f in Case 3 gives further restrictions on ρ_1, ρ_2; indeed, since picking an $\eta \in \mathcal{N}_{\text{supp}}(X)$ with $\eta(X) = M$ (which one does not matter) we find from $\rho \leq M/\nu(X)$ that for any f

$$(40) \qquad E(f) = \rho \int_X f(x)\nu(dx) = \rho \int_X \frac{1}{M} \sum_{y \in \eta} f(y + x)\, d\nu(x) \leq \sup F,$$

because $\sum_{y \in \eta} f(y + x) \leq \sup F$.

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