

VRRW ON COMPLETE-LIKE GRAPHS: ALMOST SURE BEHAVIOR

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By a theorem of Volkov [12] we know that on most graphs with positive probability the linearly vertex-reinforced random walk (VRRW) stays within a finite “trapping” subgraph at all large times. The question of whether this tail behavior occurs with probability one is open in general. In his thesis, Pemantle [5] proved, via a dynamical system approach, that for a VRRW on any complete graph the asymptotic frequency of visits is uniform over vertices. These techniques do not easily extend even to the setting of complete-like graphs, that is, complete graphs ornamented with finitely many leaves at each vertex. In this work we combine martingale and large deviation techniques to prove that almost surely the VRRW on any such graph spends positive (and equal) proportions of time on each of its nonleaf vertices. This behavior was previously shown to occur only up to event of positive probability (cf. Volkov [12]). We believe that our approach can be used as a building block in studying related questions on more general graphs. The same set of techniques is used to obtain explicit bounds on the speed of convergence of the empirical occupation measure.

1. Introduction. Consider a complete-like graph \mathcal{G}_d with $d \geq 2$ interior vertices (or sites) and $r_i \geq 0$ exterior vertices or *leaves* attached to the i th interior site, $i \in \{1, \dots, d\}$. More precisely, denote by $V_d = \{1, 2, \dots, d, \ell_1^1, \dots, \ell_{r_1}^1, \dots, \ell_1^d, \dots, \ell_{r_d}^d\}$ the set of sites of \mathcal{G}_d , and by E_d the set of its edges. Typically we denote the edge connecting two different sites v and w by $\{v, w\}$. Any two sites that share an edge are called *neighbors*. If v and w are neighbors we also write $v \sim w$. Then E_d consist of $d(d-1)/2$ edges connecting each pair of interior sites, as well as of the edges $\{i, \ell_r^i\}$, for each $i \in \{1, \dots, d\}$ and $r = 1, \dots, r_i$. We will refer to ℓ_r^i as the r th leaf attached to the interior vertex i . It is possible that $r_i = 0$ for some i , in which case there is no leaf attached to i . If $r_i = 0$, for all $i = 1, \dots, d$, then \mathcal{G}_d is the complete graph on d vertices. Any graph from the above class can be viewed as a “perturbation” of the complete graph.

We start by recalling the (discrete-time) linearly vertex reinforced random walk (VRRW) (see, e.g., [6]). This process can be constructed on general bounded de-

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gree graphs, but since the current work concerns VRRW on complete-like graphs given above, the definition below can be read with this special setting in mind.

The *time* t will run through positive integers. We denote by $X(t)$ the position (site) of the walk at time t . Assume that $z(0, v)$ are given positive integer quantities. For example, it could be $z(0, v) \equiv 1, v \in V_d$. Without loss of generality, we can assume that the initial time is $t_0 = \sum_{v \in V_d} z(0, v)$. Let $Z(t, v)$ equal $z(0, v)$ plus the number of visits to vertex $v \in V_d$ up to time $t, t \geq t_0$. Note that in this way we have $\sum_{v \in V_d} Z(t, v) \equiv t$ for $t \geq t_0$. Denote by $(\mathcal{F}_t, t \geq t_0)$ the filtration generated by $(X(t), t \geq t_0)$ (or equivalently by $(Z(t, v), t \geq t_0), v \in V_d)$ up to time t . Then on the event $\{X(t) = v\}$ the transitions of our process are given by

$$(1.1) \quad \mathbb{P}(X(t+1) = w | \mathcal{F}_t) = \frac{Z(t, w)}{\sum_{y \in V_d: y \sim v} Z(t, y)}$$

for all $w \in V_d, w \sim v$. In particular, when at ℓ_r^i , the walk must return to i in the next step.

Let

$$\pi(t) = \frac{1}{t} (Z(t, 1), Z(t, 2), \dots, Z(t, d), \\ Z(t, \ell_1^1), \dots, Z(t, \ell_{r_1}^1), \dots, Z(t, \ell_1^d), \dots, Z(t, \ell_{r_d}^d))$$

be the occupation measure generated by the VRRW above at time t , determined by the vector of its atoms. Let $\pi_\infty = \lim_{t \rightarrow \infty} \pi(t)$ be the asymptotic occupation measure on the event where this limit exists, and set $\pi_\infty = (0, 0, \dots, 0)$ on the complement. Note that $\pi(t) \in \mathbb{R}^{|V_d|}$, for all t , where $|V_d| := d + \sum_{i=1}^d r_i$, and we use this fact without further mention. Set

$$\pi_{\text{unif}} := \left(\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}, 0, \dots, 0 \right),$$

where the initial d coordinates are positive, and the other $\sum_{i=1}^d r_i$ are equal to 0.

The first goal of this paper is to prove

THEOREM 1. *For VRRW on $\mathcal{G}_d, d \geq 3$, we have $\mathbb{P}(\pi_\infty = \pi_{\text{unif}}) = 1$.*

The next statement is related to the slow speed of convergence noticed by Pemantle and Skyrms in [7]. Denote by $\|\cdot\| = \|\cdot\|_\infty$ the maximum norm on $\mathbb{R}^{|V_d|}$.

THEOREM 2. *Let \mathcal{G}_d be the complete-like graph on $d \geq 3$ vertices. Then for any $\delta > 0$*

$$(1.2) \quad \mathbb{P}\left(\limsup_{t \rightarrow \infty} \|\pi(t) - \pi_{\text{unif}}\| t^{1/3-\delta} < \infty\right) = 1 \quad \text{if } d = 3, 4,$$

$$(1.3) \quad \mathbb{P}\left(\limsup_{t \rightarrow \infty} \|\pi(t) - \pi_{\text{unif}}\| t^{1/(d-1)} < \infty\right) = 1 \quad \text{if } d \geq 5.$$

Moreover, for each $d \geq 3$, if $|V_d| \geq d + 1$ (there exists at least one leaf) and any $\delta > 0$

$$(1.4) \quad \mathbb{P}\left(\liminf_{t \rightarrow \infty} \|\pi(t) - \pi_{\text{unif}}\| t^{(d-2)/(d-1)+\delta} = \infty\right) = 1.$$

In particular, the empirical occupation measure converges to π_{unif} at least as fast as an inverse of a certain power function, and not faster than an inverse of another power function (provided $|V_d| > 0$). Note that (1.4) gives an upper bound on the power exponent which is strictly smaller than 1. To the best of our knowledge, this is the first rigorous result verifying “slow convergence” for this class of models. However, the problem of finding a lower bound on the speed in the case of the complete graph is still open, and we believe that the true rate of convergence is closer to the one in (1.2) and (1.3). We wish to point out that computer simulations seem to be misleading in predicting/confirming any of the above results, due to the slow speed of convergence. With this in mind, it is worth mentioning that our computer simulations seem to suggest that for $d = 3$

$$\frac{\log M(\|\pi(t) - \pi_{\text{unif}}\|)}{\log t} \rightarrow -\frac{1}{2},$$

where $M(X)$ stands for the median of a random variable X . The special case $d = 2$ will be discussed in Section 3.4.

There exist a few mathematical results on the asymptotic behavior of VRRW preceding this work. As mentioned in the abstract, Pemantle [5] proved that on any complete graph the asymptotic frequencies of visits by the VRRW are the same for all vertices. The papers [8] and [11] study the VRRW on the integers \mathbb{Z} . Pemantle and Volkov [8] prove that this VRRW cannot get trapped on a subgraph spanned by 4 sites, and moreover that it gets trapped on a random subgraph spanned by 5 subsequent sites with a positive probability. Tarrès [11] proved that this striking behavior occurs almost surely, using subtle martingale and coupling techniques.

A study by Volkov [12] exhibits a family of “trapping subgraphs” for the VRRW on a general graph, where the range of the VRRW is contained in any such subgraph. Recent results of Benaïm and Tarrès [2] show similar localization phenomenon for certain natural generalizations of VRRW. The asymptotic results in both [2] and [12] are shown to hold only on an event of *positive* probability. Volkov [13] initiated the analysis of nonlinearly reinforced VRRW. His analysis mostly concentrated on the power-law reinforcement functions and the VRRW on \mathbb{Z} . Many interesting open questions remain.

The rest of the paper is organized as follows. Sections 1.1–1.3 recall a few techniques used in related settings, and establish some preliminary results. In Section 2 we introduce a modified VRRW on a triangle with one special (more reinforced) vertex and study the asymptotics of weights on the nonspecial vertices. Section 3 contains the proof of Theorem 1 in the general (and novel) case of complete-like

graphs \mathcal{G}_d , and Section 4 discusses some generalizations for d -partite graphs with leaves. Finally, in Section 5 we show Theorem 2.

We will use the symbol \wedge (resp., \vee) to denote the operation of taking the minimum (resp., maximum) of two or more numbers. For f and g , two sequences of positive functions defined on the positive reals, we write $f(t) = O(g(t))$ if $\limsup_t f(t)/g(t)$ is finite, $g(t) \asymp f(t)$ or $f(t) = \Theta(g(t))$ if both $f(t) = O(g(t))$ and $g(t) = O(f(t))$, and $f(t) = o(g(t))$ if $\lim_t f(t)/g(t) = 0$. The above notations extend in a straightforward way to the stochastic setting.

1.1. Multi-color Pólya urns and VRRW on complete graphs. We devote this short subsection to a calculation that will hopefully both stimulate the reader's interest in the problem, and point out some of the difficulties awaiting. In addition, we will use a modification of the supermartingale below in arguments of Section 3. Fix $d \geq 2$, and let Π be the d -color Pólya urn started with one ball of each color. In particular, at each step, one ball is drawn from the urn at random, and it is placed back immediately together with another ball of the same color. As usual, let the initial time be d , and for each time $t \geq d$ denote by $\Pi_i(t)$ the number of balls of color i , $i = 1, \dots, d$ in the urn at time t . In this way $\sum_{i=1}^d \Pi_i(t) = t$ always. A slick way (see [12], Section 2.1) to prove convergence of the frequencies $\Pi_i(t)/t$, $i = 1, \dots, d$, to nontrivial (nonzero, a.s.) random variables is via the following martingale method. Using classical martingales $\Pi_i(t)/t$ for showing this convergence is not optimal for showing that the limit is nonzero, almost surely. Define

$$M_i(t) := \log(t) - \log(\Pi_i(t) - 1),$$

and then check that the drift of this process equals

$$\mathbb{E}(M_i(t+1) - M_i(t) | \mathcal{F}_t) = \log\left(1 + \frac{1}{t}\right) - \frac{\Pi_i(t)}{t} \log\left(1 + \frac{1}{\Pi_i(t) - 1}\right),$$

and is therefore almost surely negative. Thus $M_i(t)$ is a nonnegative supermartingale and it converges almost surely to a finite quantity, hence $\Pi_i(t)/t$ converges almost surely to a positive quantity.

Next consider the VRRW on complete graph with d vertices. The only difference of transitions of $(Z(t, 1), \dots, Z(t, d))$ from those of $(\Pi_1(t), \dots, \Pi_d(t))$ is that $\Pi_i(t+1)$ becomes $1 + \Pi(t)$ with probability proportional to $\Pi_i(t)$ no matter which ball was drawn at time $t-1$, while $Z(t+1, i)$ becomes $1 + Z(t, i)$ with probability proportional to $Z(t, i)$ only if the current position of the VRRW is not i ; in turn this proportion is taken with respect to the values at all but the currently visited site. If one tries simply to recycle the above supermartingale by subtracting a drift increment of order $1/t$ at each time t when $Z(t, i) = Z(t-1, i) + 1$, then on the event that $Z(t, i)$ is asymptotically of order larger than $t/\log(t)$ [this happens, since $Z(t, i) \sim t/d$, a.s.] the sum of the drift increments diverges and it not possible to conclude convergence of $M_i(t)$. One could think that there should be a simple way to overcome the above difficulty, but we are not aware of one.

1.2. *Large deviation tools.* Part of our analysis (cf. Section 3.3) will use the strategy of Volkov [12] (see also [2]).

We recall the following classical facts. Let ξ_i be i.i.d. random variables with $\mathbb{P}\{\xi_i = 1\} = 1 - \mathbb{P}\{\xi_i = 0\} = p \in (0, 1)$. Define for $a, p \in (0, 1)$,

$$(1.5) \quad H(a, p) := a \log \frac{a}{p} + (1 - a) \log \frac{1 - a}{1 - p} \geq 0.$$

Recall an elementary fact from large deviation theory (see, e.g., [9]): for any $a^+ \in [p, 1)$ and any $a^- \in (0, p]$, we have

$$(1.6) \quad \mathbb{P}\left\{\frac{1}{n} \sum_{i=1}^n \xi_i \geq a^+\right\} \leq e^{-nH(a^+, p)}, \quad \mathbb{P}\left\{\frac{1}{n} \sum_{i=1}^n \xi_i \leq a^-\right\} \leq e^{-nH(a^-, p)}.$$

It is easy to verify (see also Propositions 2.2 and 2.3 in [12]) that

$$(1.7) \quad \begin{aligned} H(a, p) &= \frac{\delta^2}{2p(1-p)} + \Theta\left(\frac{\delta^3}{p^2(1-p)^2}\right) \\ &\quad \text{if } a = p \pm \delta, \text{ where } \delta \ll 1 \quad \text{and} \\ H(a, p) &= p(r \log r - r + 1) + \Theta(p^2) \\ &\quad \text{if } a = rp, r = \Theta(1), \text{ and } a \vee p \ll 1. \end{aligned}$$

1.3. *Urn and martingale tools.* We start by recalling the results on urns from Pemantle and Volkov [8]. We will often use them directly in coupling arguments; however we will also need to generalize Theorem 3 below (see Lemma 1) during the course of our analysis.

The urn model defined below generalizes both the (original) Pólya and the Friedman urn, and it is sometimes referred to as the *generalized Pólya urn*. Consider the dynamics

$$(1.8) \quad \begin{aligned} (X_{n+1}, Y_{n+1}) &= (X_n + a, Y_n + b) && \text{with probability } \frac{X_n}{X_n + Y_n}, \\ (X_{n+1}, Y_{n+1}) &= (X_n + c, Y_n + d) && \text{with probability } \frac{Y_n}{X_n + Y_n}. \end{aligned}$$

We do not necessarily assume that the random numbers X_n, Y_n (of balls) are integer valued. When $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a multiple of the identity matrix (resp., $a = d$ and $b = c$ are all nonzero), we recover Pólya's (resp., Friedman's) urn. In all cases where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a left eigenvector (v_1, v_2) with positive components and $abcd > 0$, Friedman's analysis [3] can be carried through to show that $X_n/(X_n + Y_n)$ converges a.s. to $v_1/(v_1 + v_2)$. When $a > d, b > 0$ and $c = 0$ the urn is still Friedman like: although $(0, 1)$ is an eigenvector, it is easy to see that the principal eigenvector is $(a - d, b)$ and that $X_n/(X_n + Y_n) \rightarrow (a - d)/[(a - d) + b]$ a.s. The case $ad = bc = 0$ is trivial, so we are left with the cases $ad > 0 = b = c$ and

$ad > 0 = bc < b + c$. Multiplication of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by a constant does not affect the asymptotic behavior. Due to symmetry, the interesting behavior is captured in the following two theorems.

THEOREM 3 ([8], Theorem 2.2). *Suppose $a > d = 1$, and $b = c = 0$. Then $\log X_n / \log Y_n \rightarrow a$.*

THEOREM 4 ([8], Theorem 2.3). *Suppose $a = d = 1$, $b = 0$ and $c > 0$. Then $X_n / (cY_n) - \log Y_n$ converges to a random limit in $(-\infty, \infty)$.*

REMARK 1. (1) Theorem 3 implies that for any $\varepsilon > 0$ we have $X_n^{(1/a-\varepsilon)} \leq Y_n \leq X_n^{(1/a+\varepsilon)}$ for all large n , almost surely. Since $X_n + Y_n \asymp n$, this easily implies that X_n is equal to $a \cdot n$ plus lower order terms, while Y_n is asymptotically equal to $n^{1/a}$ multiplied by a random factor A_n , where for any $\varepsilon > 0$ $A_n \in (n^{-\varepsilon}, n^\varepsilon)$ for all large n .

(2) The result in Theorem 4 may be more surprising, in that it shows Y_n to be of the order $n / \log n$ multiplied by a specific constant, with a random lower order correction. That is, X_n is asymptotically $cY_n(A + \log Y_n)$, where A is a random constant. This class of urns was used in [8] to prove that VRRW on \mathbb{Z} cannot get trapped on a subgraph spanned by 4 subsequent points. Note that in the special case $c = 1$, the urn process corresponds to a VRRW on the graph \mathcal{G} with $V(\mathcal{G}) = \{u, v\}$, having one edge between u and v and one loop connecting u to itself, observed at the times of successive visits to vertex u . Thus VRRW on this \mathcal{G} spends roughly $n / \log n$ units of time at v up to time n .

(3) Both of the above theorems can be derived using an elegant method of Athreya and Ney [1], by embedding the urn into a continuous time multi-type branching process. However, the proof by embedding (see also [4] for recent progress) is much less robust to “variations” in dynamics than the martingale proofs of [8]. One such variation is the setting where some (or all) of the parameters a, b, c, d are perturbed about fixed values (their means), and where the distribution of these random perturbations varies over time. Section 2 is devoted to proving some extensions in this direction that turn out to be essential for our analysis.

In the current work, we will repeatedly bound the \limsup (by a finite random quantity) of a process that has supermartingale increments whenever its value is sufficiently large via a separate martingale technique (see Chapter 4 of [10] for a similar idea in a somewhat simpler setting).

In our general setting, we are given $(\xi_n, n \geq 0)$, a discrete-time process (not necessarily bounded below nor above), adapted to a filtration $(\mathcal{F}_n, n \geq 0)$. In addition, suppose there exists $a, b \in \mathbb{R}$, $b > 0$ such that:

(1) ξ has supermartingale increments on $[a, \infty)$, that is,

$$(1.9) \quad \mathbb{E}((\xi_{k+1} - \xi_k)1_{\{\xi_k \geq a\}} | \mathcal{F}_k) \leq 0;$$

(2) the overshoot of ξ across a is asymptotically bounded by b , that is,

$$(1.10) \quad o^*(a) := \limsup_k 1_{\{\xi_k < a < \xi_{k+1}\}} (\xi_{k+1} - a) \leq b \quad \text{almost surely};$$

(3) the tail variance of ξ on $[a, \infty)$ is finite, that is,

$$(1.11) \quad \sum_k \mathbb{E}[(\Delta \xi_k)^2 1_{\{\xi_k \geq a\}}] < \infty \quad \text{where } \Delta \xi_k := \xi_{k+1} - \xi_k.$$

LEMMA 1. *Under the above assumptions*

$$\xi^* := \limsup_{n \rightarrow \infty} \xi_n < \infty, \quad a.s.$$

PROOF. Due to shift and scaling, without loss of generality (WLOG) we may assume that $a = -1$ and $b = 1$. Next fix a small $\delta > 0$, and define

$$B_\delta^{(n)} = \left\{ \sup_{k \geq n} 1_{\{\xi_k < -1 < \xi_{k+1}\}} (\xi_{k+1} - (-1)) \leq 1 + \delta \right\}.$$

Property (1.10) can be restated as $\lim_{n \rightarrow \infty} \mathbb{P}(B_\delta^{(n)}) = 1$. We shall now introduce an auxiliary process

$$\xi'^{(n,\delta)} \equiv xi' := (\xi'_k, k \geq n),$$

adapted to the filtration generated by $(\xi_k, k \geq n)$, and such that the three properties (1.9)–(1.11) hold for ξ' , with $a = \delta$ and $b = 0$. Moreover, the inequality in (1.9) for ξ' becomes equality

$$(1.12) \quad \mathbb{E}((\xi'_{k+1} - \xi'_k)1_{\{\xi'_k \geq \delta\}} | \mathcal{F}_k) = 0, \quad k \geq n,$$

and also

$$(1.13) \quad B_\delta^{(n)} \subset \bigcap_{k \geq n} \{\xi_k \leq \xi'_k\} \quad \text{almost surely.}$$

Define $\xi'_n \equiv \xi'^{(n,\delta)}_n := \xi_n$, and for $k \geq n$ let

$$(1.14) \quad \xi'_{k+1} := \begin{cases} \xi'_k + \Delta \xi_k - \mathbb{E}(\Delta \xi_k | \mathcal{F}_k), & \text{if } \xi_k \geq -1, \\ (\xi'_k + \Delta \xi_k) \wedge \delta, & \text{if } \xi_k < -1 \text{ and } \xi'_k < \delta, \\ \xi'_k, & \text{if } \xi_k < -1 \text{ and } \xi'_k \geq \delta. \end{cases}$$

If $\xi'_k \geq \delta$ then either $\xi_k \geq -1$ in which case the increment of ξ' is the Doob–Meyer martingale “correction” of the increment of ξ , or $\xi_k < -1$ and then ξ' does not change value. So indeed, (1.9) holds for ξ' as (1.12). The property (1.10) is immediate since a positive overshoot of ξ' across δ may occur only as a result of a jump of ξ when its current value is greater than -1 , but these jumps

are asymptotically negligible by (1.11). Similarly, (1.11) for ξ' is easy to derive from the definition (1.14), the property (1.11) for ξ , and the standard fact $\mathbb{E}((\Delta\xi_k - \mathbb{E}(\Delta\xi_k|\mathcal{F}_k))^2|\mathcal{F}_k) \leq \mathbb{E}((\Delta\xi_k)^2|\mathcal{F}_k)$, almost surely. Finally, using (1.9) and the definition of $B_\delta^{(n)}$, one can check inductively that (1.13) holds. Namely, $\xi_n \leq \xi'_n$ is the base of induction, and for $k \geq n$ either $-1 \leq \xi_k \leq \xi'_k$ (the last inequality is by induction hypothesis) in which case $\Delta\xi'_k \geq \Delta\xi_k$ due to (1.9) yielding $\xi_{k+1} \leq \xi'_{k+1}$, or $\xi_k < -1$ and $\xi'_k \geq \delta$ in which case on $B_\delta^{(n)}$ we have $\xi_{k+1} < \delta \leq \xi'_k = \xi'_{k+1}$, or finally $\xi_k < -1$ and $\xi_k \leq \xi'_k < \delta$ in which case again on $B_\delta^{(n)}$ we have $\xi_{k+1} = \xi_k + \Delta\xi_k \leq \delta \wedge (\xi'_k + \Delta\xi_k) = \xi'_{k+1}$. Therefore,

$$\mathbb{P}(\xi^* = \infty) \leq \mathbb{P}((B_\delta^{(n)})^c) + \mathbb{P}\left(\limsup_k \xi_k'^{(n,\delta)} = \infty\right).$$

We conclude that it suffices to show

$$(1.15) \quad \mathbb{P}\left(\limsup_k \xi_k'^{(n,\delta)} = \infty\right) = 0$$

for a fixed $\delta > 0$ and each $n \geq 1$.

Again by shift and scaling of space, and additional shift of time, we can henceforth assume that $a = b = 0$, and that (1.12) holds. It is clear that if the process ξ switches sign only finitely many times then it either spends all but finitely many units of time being nonnegative, in which case by the martingale convergence theorem it converges, or it spends all but finitely many units of time being nonpositive. On both events ξ^* is finite. It remains to prove the claim on the event A^\pm where ξ switches sign infinitely often. In fact we will prove here a stronger claim, namely that

$$(1.16) \quad A^\pm \cap \{\xi^* = 0\} = A^\pm \cap \{\xi^* \leq 0\} = A^\pm \quad \text{almost surely.}$$

The first identity above is clear from the definitions of A^\pm and ξ^* . Fix $\varepsilon > 0$. For $n \geq 1$, define the process

$$S_k^{(n)} := \sum_{i=n}^{k-1} (\xi_{i+1} - \xi_i) 1_{\{\xi_i \geq 0\}}, \quad k \geq n,$$

with the convention $S_n^{(n)} = 0$, and note that by assumption (1.12) on ξ , $S^{(n)}$ is a martingale started from 0 at time n .

Due to Doob's maximal inequality we have

$$\mathbb{P}\left(\sup_{k \geq n} |S_k^{(n)}| > \varepsilon\right) \leq \frac{4 \sum_{k \geq n} \mathbb{E}[(\xi_{k+1} - \xi_k)^2 1_{\{\xi_k \geq 0\}}]}{\varepsilon^2}$$

and in particular, due to (1.11), we can find $n_1 \geq 1$ such that this probability is smaller than ε , hence

$$(1.17) \quad \mathbb{P}\left(\sup_{k, j \geq n_1} |S_k^{(n_1)} - S_j^{(n_1)}| > 2\varepsilon\right) \leq 2\varepsilon.$$

Consider ξ on the event

$$A^\pm \cap \left\{ \sup_{k,j \geq n_1} |S_k^{(n_1)} - S_j^{(n_1)}| \leq 2\varepsilon \right\},$$

and note that now the maximal value of ξ on any excursion into $[0, \infty)$ that begins after time n_1 cannot exceed $\sup_{n \geq n_1} 1_{\{\xi_n < 0 < \xi_{n+1}\}} \xi_{n+1} + 2\varepsilon \leq o_{n_1}(1) + 2\varepsilon$, where $o_{n_1}(1) \rightarrow 0$, as $n_1 \rightarrow \infty$. Since ε can be taken arbitrarily small, we obtain (1.16). \square

The above result (1.16) can be improved in the following sense. Assume that ξ satisfies (1.9)–(1.11). Denote by A_a^\pm the event $\{\xi - a \text{ switches sign infinitely often}\}$.

LEMMA 2. *On A_a^\pm , we have*

$$\xi^* \leq a + b, \quad a.s.$$

PROOF. We may assume again that $a = -1$ and $b = 1$, and that $\xi_0 < -1$. Let $T_0 = 0$, and for $m \geq 1$ let T_m be the m th downward crossing time of -1 by ξ . Note that on the event A_{-1}^\pm , T_m is finite almost surely and that also $T_m \rightarrow \infty$ as $m \rightarrow \infty$. It is clear how to generalize the construction of $\xi'^{(n,\delta)}$ from the proof of Lemma 1 by replacing a fixed time n by a stopping time T_m , $m \geq 0$. Of course, the construction extends only on the event $\{T_m < \infty\}$, on the complement one can define the process as identity δ (for example). We will henceforth abbreviate $\xi''^{(m,\delta)} \equiv \xi'^{(T_m,\delta)}$.

Using (1.17) and (1.11) one can easily check, as in the proof of previous lemma, that

$$\lim_{m \rightarrow \infty} \sup_{k \geq T_m} \xi_k''^{(m,\delta)} \leq \delta.$$

Indeed, the overshoots of $\xi_k''^{(m,\delta)}$ across δ are becoming negligible as m increases, and (1.11) controls its fluctuations. In particular,

$$\xi^* 1_{A_{-1}^\pm} \leq \left(\limsup_m \sup_{k \geq T_m} \xi_k''^{(m,\delta)} \right) 1_{A_{-1}^\pm} \leq \delta.$$

Since $\delta > 0$ is arbitrary, it follows that $\mathbb{P}(A_{-1}^\pm \cap \{\xi^* > 0\}) = 0$, as claimed. \square

REMARK 2. We will sometimes consider a process ξ adapted to the filtration \mathcal{F} , where the conditions (1.9)–(1.11) apply up to additional constraint. More precisely

$$\mathbb{E}((\xi_{k+1} - \xi_k) 1_{\{\xi_k \geq a\}} | \mathcal{F}_k) 1_{E_k} \leq 0, \quad \limsup_k 1_{\{\xi_k < a < \xi_{k+1}\}} (\xi_{k+1} - a) 1_{E_k} \leq b,$$

and

$$\sum_k \mathbb{E}[(\Delta \xi_k)^2 1_{\{\xi_k \geq a\} \cap E_k}] < \infty,$$

where E_k is an \mathcal{F}_k -measurable event. In such a situation we will (nonrigorously) state that ξ satisfies (1.9)–(1.11) on $\bigcap_{k \geq n} E_k$ (for some large n) and conclude the result of Lemma 1 on the same event. The corresponding rigorous formulation of this argument is to work instead with the stopped process $\xi(T) := \{\xi_{k \wedge T}, k \geq n\}$, where a stopping time

$$T := \inf\{k \geq n : 1_{E_k} = 0\}$$

is defined precisely so that $\{T = \infty\} = \bigcap_{k \geq n} E_k$. Then $\xi(T)$ satisfies the original (1.9)–(1.11), and the asymptotics of $\xi(T)$ and ξ (as $k \rightarrow \infty$) match on the event $\{T = \infty\}$.

2. Modified VRRW on a triangle. In this section we consider a modified VRRW (MVRRW) on a triangle. Define $\tau_0^{(3)} = 0$. The transition probabilities of MVRRW are as for the VRRW on the triangle, with one difference: when the *special vertex* 3 is visited for the k th time, at the stopping time

$$(2.1) \quad \tau_k^{(3)} \equiv \tau_k := \min\{t > \tau_{k-1} : X(t) = 3\}, \quad k \geq 1,$$

its weight $Z(\tau_k, 3)$ becomes $H(k)$ rather than $Z(\tau_k - 1, 3) + 1$ [and for $t \in (\tau_k, \tau_{k+1})$ we set $Z(t, 3) = H(k)$]. Here we assume that the sequence $H(k)$ is measurable with respect to \mathcal{F}_{τ_k} , the σ -algebra generated by the process up to time τ_k , that $H(1) \geq 1$ and that for $k = 0, 1, 2, \dots$ the following property holds:

$$(2.2) \quad H(k+1) \geq H(k) + 1.$$

Thus, the special vertex 3 gets reinforced by a larger amount than nonspecial vertices 1 and 2.

We study the above MVRRW with intention of applying it several times in Section 3. A typical application is in the following context: suppose that the underlying graph is complete graph on d vertices where $d \geq 4$. If one “clumps together” all but two of the vertices (say i and j), then the VRRW generates (with the appropriate time change) a MVRRW on a triangle, where i and j correspond to 1 and 2, and the clump corresponds to the special vertex 3.

To simplify notation we will denote

$$U(t) := Z(t, 1), \quad V(t) := Z(t, 2) \quad \text{and} \quad W(t) = Z(t, 3).$$

The goal of this section is to show that $U(t) \asymp V(t)$. Before stating the main result rigorously, we do some preliminary comparisons and calculations.

First, observe that using elementary arguments (in particular, Pólya urn-like transitions of the process, when viewed from the special vertex 3) one can show

that for MVRRW both $U(t) \rightarrow \infty$ and $V(t) \rightarrow \infty$, almost surely. Similarly, it is easy to see that it is impossible that after some finite time the particle oscillates between nonspecial vertices 1 and 2. Hence $W(t) \rightarrow \infty$, and $\tau_k < \infty$, for all k , almost surely. Second, let us show that $W(t)$ cannot be too small with respect to $U(t) + V(t)$ (which seems obvious but still requires a proof). Let η_n , $n \geq 0$ be the times of the successive visits to vertices 1 or 2, that is

$$\eta_{n+1} = \inf\{t > \eta_n : X(t) \in \{1, 2\}\}.$$

Let $Y_n = W(\eta_n)$ and $X_n = U(\eta_n) + V(\eta_n)$. Then it is simple to construct a coupling of (X_n, Y_n) with the urn (X'_n, Y'_n) , featured in Theorem 4 with $a = c = d = 1$, $b = 0$, such that

$$(2.3) \quad X_n = X'_n \quad \text{and} \quad Y_n \geq Y'_n \quad \text{for all } n.$$

This yields

$$\liminf_{n \rightarrow \infty} \frac{Y_n}{X_n / \log X_n} \geq 1.$$

To simplify notation let

$$\phi(x) = x / \log x.$$

Then the above can be rewritten as

$$\liminf_{n \rightarrow \infty} \frac{W(\eta_n)}{\phi(U(\eta_n) + V(\eta_n))} \geq 1.$$

Noting that in between the consecutive times η_n the process W increases, while $U + V$ stays the same, we get

$$(2.4) \quad \liminf_{t \rightarrow \infty} \frac{W(t)}{\phi(U(t) + V(t))} \geq 1.$$

Similarly, considering the process $(U(t), V(t), W(t))$ at times when the MVRRW $X(t)$ visits vertex 1 and comparing the increments at vertices 1 and 2 [the former always increases by 1 while the latter increases by at least 1 with probability at least $V(t)/(U(t) + V(t))$] we obtain that

$$(2.5) \quad \liminf_{t \rightarrow \infty} \frac{V(t)}{\phi(U(t))} \geq 1,$$

and in a symmetric way the symmetric result

$$(2.6) \quad \liminf_{t \rightarrow \infty} \frac{U(t)}{\phi(V(t))} \geq 1.$$

To simplify notations further, recall (2.1), (2.2) and denote

$$U(\tau_k) = u, \quad V(\tau_k) = v, \quad W(\tau_k) = a = H(k), \quad n(k) = n = u + v.$$

We omit the index “ k ” from the notation in the forthcoming argument, whenever not in risk of confusion. Relations (2.4)–(2.6) imply (in a straightforward way) that for sufficiently large k we have

$$(2.7) \quad \begin{aligned} u > \phi(v)/2, \quad v > \phi(u)/2 &\implies \min\{u, v\} > \phi(n)/4 \quad \text{and} \\ &a > \phi(n)/2. \end{aligned}$$

At time $\tau_k + 1$ the walk has to visit either site 1 or 2, and moreover $\mathbb{P}(X(\tau_k + 1) = 1) = u/(u + v)$, $\mathbb{P}(X(\tau_k + 1) = 2) = v/(u + v)$.

For $m \geq 1$, consider the events

$$(2.8) \quad \begin{aligned} A_m(k) &= \{X(\tau_k + 1) = 1, X(\tau_k + 2) = 2, \\ X(\tau_k + 3) &= 1, X(\tau_k + 4) = 2, \dots, \\ X(\tau_k + (2m - 1)) &= 1, \text{ but } X(\tau_k + 2m) = 3\}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} B_m(k) &= \{X(\tau_k + 1) = 1, X(\tau_k + 2) = 2, \dots, X(\tau_k + 2m - 1) = 1, \\ X(\tau_k + 2m) &= 2, \text{ but } X(\tau_k + 2m + 1) = 3\}. \end{aligned}$$

Symmetrically define events $\bar{A}_m(k)$, $\bar{B}_m(k)$ where the walker starts the excursion away from vertex 3 at vertex 2, and on $\bar{A}_m(k)$ [resp., $\bar{B}_m(k)$] it visits 2 (resp., 1) immediately before returning to 3. Note that $A_m, B_m, m \geq 1$ are disjoint. On $A_m \cup B_m$, during this excursion, vertex 1 is visited exactly m times, while vertex 2 is visited $m - 1$ times on A_m and m times on B_m . Symmetric statements apply to \bar{A}_m and \bar{B}_m . It is easy to see that

$$\begin{aligned} \mathbb{P}\left(\bigcup_m (A_m \cup B_m) \mid \mathcal{F}_{\tau_k}\right) &= \mathbb{P}(X(\tau_k + 1) = 1, \tau_{k+1} < \infty \mid \mathcal{F}_{\tau_k}) \\ &= \mathbb{P}(X(\tau_k + 1) = 1 \mid \mathcal{F}_{\tau_k}) \quad \text{a.s.,} \end{aligned}$$

since $\tau_{k+1} < \infty$, almost surely. Next observe that for $m \geq 1$ (where an empty product is equal to 1)

$$\mathbb{P}(A_m \mid \mathcal{F}_{\tau_k}) = \frac{u}{u + v} \prod_{j=0}^{m-2} \left(\frac{v + j}{v + j + a} \cdot \frac{u + j + 1}{u + j + 1 + a} \right) \frac{a}{a + v + m - 1}$$

and

$$\begin{aligned} \mathbb{P}(B_m \mid \mathcal{F}_{\tau_k}) &= \frac{u}{u + v} \prod_{j=0}^{m-2} \left(\frac{v + j}{v + j + a} \cdot \frac{u + j + 1}{u + j + 1 + a} \right) \\ &\quad \times \frac{v + m - 1}{a + v + m - 1} \frac{a}{a + u + m}. \end{aligned}$$

Now define

$$C_m(k) \equiv C_m = \bigcup_{i=m}^{\infty} (A_i \cup B_i)$$

to be the event that vertex 1 is visited at least m times during the excursion (recall that there is dependence of u, v, a , and hence of A_m, B_m , and C_m on k). Then

$$\mathbb{P}(C_m | \mathcal{F}_{\tau_k}) = \frac{u}{u+v} \prod_{j=0}^{m-2} \left(\frac{v+j}{v+j+a} \cdot \frac{u+j+1}{u+j+1+a} \right).$$

If we denote

$$\lambda_u = \frac{a}{a+u}, \quad \lambda_v = \frac{a}{a+v} \quad \text{and} \quad v = (1 - \lambda_u)(1 - \lambda_v)$$

then, provided $m^2/u \ll 1$ and $m^2/v \ll 1$,

$$\begin{aligned} & \mathbb{P}(C_m(k) | \mathcal{F}_{\tau_k}) \\ &= \frac{u}{u+v} \cdot v^{m-1} \\ & \times \frac{(1+0/v)(1+1/v) \dots (1+(m-2)/v)}{(1+0/(a+v))(1+1/(a+v)) \dots (1+(m-2)/(a+v))} \\ & \times \frac{(1+1/u)(1+2/u) \dots (1+(m-1)/u)}{(1+1/(a+u))(1+2/(a+u)) \dots (1+(m-1)/(a+u))} \end{aligned} \quad (2.10)$$

$$(2.11) \quad = \frac{u}{u+v} \cdot v^{m-1} (1 + O(m^2/u) + O(m^2/v)).$$

Set $m = m(k) = \log^3 n(k) + 1$, then by (2.7) we have $m^2/u, m^2/v < 4 \log^7(n)/n = o(1)$. Similarly, by (2.7), we have

$$(2.12) \quad v = \frac{1}{(a/u+1)(a/v+1)} \leq \frac{1}{(a/n+1)^2} \leq \frac{1}{(1+1/(2 \log n))^2},$$

and so a straightforward calculus manipulation yields

$$v^{m-1} \leq n^{1-\log n}.$$

Consequently,

$$(2.13) \quad \mathbb{P}(C_{m(k)}(k) | \mathcal{F}_{\tau_k}) = \mathbb{P}(C_{(\log n)^3+1} | \mathcal{F}_{\tau_k}) < v^{m-1} (1 + o(1)) \leq \frac{1+o(1)}{n^{\log n-1}}.$$

Therefore, by the Borel–Cantelli lemma,

$$(2.14) \quad \text{only finitely many of } C_{m(k)}(k) \text{ occur, a.s.}$$

If $m \leq m(k) = \log^3 n + 1$, then we can simplify the conditional probabilities of A_m and B_m as follows:

$$(2.15) \quad \mathbb{P}(A_m | \mathcal{F}_{\tau_k}) = \frac{u}{u+v} \lambda_v v^{m-1} [1 + O(\log^7 n/n)],$$

$$(2.16) \quad \mathbb{P}(B_m | \mathcal{F}_{\tau_k}) = \frac{u}{u+v} \lambda_u (1 - \lambda_v) v^{m-1} [1 + O(\log^7 n/n)],$$

$$(2.17) \quad \mathbb{P}(\bar{A}_m | \mathcal{F}_{\tau_k}) = \frac{v}{u+v} \lambda_u v^{m-1} [1 + O(\log^7 n/n)],$$

$$(2.18) \quad \mathbb{P}(\bar{B}_m | \mathcal{F}_{\tau_k}) = \frac{v}{u+v} \lambda_v (1 - \lambda_u) v^{m-1} [1 + O(\log^7 n/n)].$$

Now let

$$\xi(t) := \frac{U(t)}{U(t) + V(t)}.$$

LEMMA 3. *We have*

$$\mathbb{P}\left(\liminf_{t \rightarrow \infty} \xi(t) > 0\right) = 1,$$

and by symmetry $\mathbb{P}(\limsup_{t \rightarrow \infty} \xi(t) < 1) = 1$.

PROOF. It suffices to restrict attention to times τ_k since by (2.14) the values of ξ during the interval (τ_k, τ_{k+1}) differ (asymptotically) from $\xi(\tau_k)$ by at most order $\log^3(U(\tau_k) + V(\tau_k))/(U(\tau_k) + V(\tau_k))$. Recall that we abbreviate $V(\tau_k) = v$, $U(\tau_k) = u$, $n = u + v$. In particular, $n \geq k + O(1)$ for each $k \geq 1$, almost surely, since between any two visits to site 3, either site 1 or 2 is visited at least once.

Define (recall the example in Section 1.1)

$$\Xi(t) = \log(U(t) + V(t)) - \log(V(t) - 1).$$

We will estimate the drift of Ξ (in the case where $v < n/3$, hence $v < u/2$) by comparing our MVRRW setting to that of the 2-color Pólya urn. In the latter case, with probability $u/(u+v)$ the new value is

$$\text{Pólya}_{\uparrow} = \log(n+1) - \log(v-1),$$

and with probability $v/(u+v)$ the new value is

$$\text{Pólya}_{\downarrow} = \log(n+1) - \log(v).$$

Thus, the drift increment of Ξ under the law of the Pólya urn is negative, since

$$(2.19) \quad \frac{u}{u+v} \log \frac{n+1}{v-1} + \frac{v}{u+v} \log \frac{n+1}{v} - \log \frac{n}{v-1} < 0,$$

see also Section 1.1.

Our goal is to bound the drift of Ξ under the modified VRRW law by its counterpart under the Pólya urn process. Intuitively, this makes sense since the shuttles pull the ratio $U/(U+V)$ closer to $1/2$, which corresponds to even more negative drift of Ξ . Note that

$$\begin{aligned}
 & \mathbb{E}(\Xi(\tau_{k+1})|\mathcal{F}_{\tau_k}) \\
 &= \sum_{m=1}^{\infty} \left(\mathbb{P}(A_m|\mathcal{F}_{\tau_k}) \log \frac{n+2m-1}{v+m-2} + \mathbb{P}(B_m|\mathcal{F}_{\tau_k}) \log \frac{n+2m}{v+m-1} \right. \\
 & \quad \left. + \mathbb{P}(\bar{A}_m|\mathcal{F}_{\tau_k}) \log \frac{n+2m-1}{v+m-1} + \mathbb{P}(\bar{B}_m|\mathcal{F}_{\tau_k}) \log \frac{n+2m}{v+m-1} \right) \\
 &= (\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})) \log \frac{n+2}{v} \\
 & \quad + \sum_{m=1}^{\infty} \left(\mathbb{P}(A_m|\mathcal{F}_{\tau_k}) \log \frac{n+2m-1}{v+m-2} + \mathbb{P}(\bar{A}_m|\mathcal{F}_{\tau_k}) \log \frac{n+2m-1}{v+m-1} \right) \\
 & \quad + \sum_{m=2}^{\infty} (\mathbb{P}(B_m|\mathcal{F}_{\tau_k}) + \mathbb{P}(\bar{B}_m|\mathcal{F}_{\tau_k})) \log \frac{n+2m}{v+m-1} \\
 &= \text{I} + \text{II} + \text{III}.
 \end{aligned}$$

Then

$$(2.20) \quad \text{II} \leq \sum_{m=1}^{\infty} \left(\log \frac{n+1}{v-1} \mathbb{P}(A_m|\mathcal{F}_{\tau_k}) + \log \frac{n+1}{v} \mathbb{P}(\bar{A}_m|\mathcal{F}_{\tau_k}) \right)$$

and

$$(2.21) \quad \text{III} \leq \sum_{m=2}^{\infty} \left(\log \frac{n+1}{v-1} \mathbb{P}(B_m|\mathcal{F}_{\tau_k}) + \log \frac{n+1}{v} \mathbb{P}(\bar{B}_m|\mathcal{F}_{\tau_k}) \right),$$

since for $m \geq 2$ and $v < n/3$

$$\frac{n+2m}{v+m-1} - \frac{n+1}{v} < 0.$$

Finally, since for $u > v$,

$$\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) = \frac{u}{n} \frac{v}{v+a} \frac{a}{a+u+1} > \frac{v}{n} \frac{u}{u+a} \frac{a}{a+v+1} = \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k}),$$

we have

$$\begin{aligned}
 (2.22) \quad \text{I} &= (\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})) \log \frac{n+2}{v} \\
 &= (\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) - \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})) \log \frac{n+2}{v} + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k}) 2 \log \frac{n+2}{v}
 \end{aligned}$$

$$\begin{aligned}
&\leq (\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) - \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})) \log \frac{n+2}{v} \\
&\quad + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k}) \left(\log \frac{n+1}{v-1} + \log \frac{n+1}{v} \right) \\
&\leq (\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) - \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})) \log \frac{n+1}{v-1} \\
&\quad + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k}) \left(\log \frac{n+1}{v-1} + \log \frac{n+1}{v} \right) \\
(2.23) \quad &= \mathbb{P}(B_1|\mathcal{F}_{\tau_k}) \log \frac{n+1}{v-1} + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k}) \log \frac{n+1}{v}.
\end{aligned}$$

For the first inequality (the third line in the display) above we use the fact that

$$\left(\frac{n+2}{v} \right)^2 \leq \frac{(n+1)^2}{v(v-1)} \quad \text{whenever } v < \frac{n}{3}.$$

Therefore,

$$\begin{aligned}
\text{I} + \text{II} + \text{III} &\leq \log \frac{n+1}{v-1} \sum_{m=1}^{\infty} (\mathbb{P}(A_m|\mathcal{F}_{\tau_k}) + \mathbb{P}(B_m|\mathcal{F}_{\tau_k})) \\
&\quad + \log \frac{n+1}{v} \sum_{m=1}^{\infty} (\mathbb{P}(\bar{A}_m|\mathcal{F}_{\tau_k}) + \mathbb{P}(\bar{B}_m|\mathcal{F}_{\tau_k})),
\end{aligned}$$

and by noting

$$\sum_{m=1}^{\infty} (\mathbb{P}(A_m|\mathcal{F}_{\tau_k}) + \mathbb{P}(B_m|\mathcal{F}_{\tau_k})) = u/(u+v)$$

and

$$\sum_{m=1}^{\infty} (\mathbb{P}(\bar{A}_m|\mathcal{F}_{\tau_k}) + \mathbb{P}(\bar{B}_m|\mathcal{F}_{\tau_k})) = v/(u+v),$$

we arrive to the following bound: provided $v < n/3$ (that is, $v < u/2$), the drift increment of the Ξ process under the modified VRRW law is smaller than the expression on the left-hand side of (2.19). In particular, Ξ has supermartingale increments whenever its value is larger than $\log 4$. It is simple to check that Ξ satisfies properties (1.9)–(1.11) with $a = \log 4$ [note that this a is different from $a \equiv a(k)$ above] and $b = 0$ (any $b \geq 0$ would suffice). Namely, we have just verified (1.9), while (1.10) is true since the steps $\Xi(\tau_{k+1}) - \Xi(\tau_k)$ are asymptotically of order at most $\log^4(n)/n$, due to the lower bound (2.7) on v and estimate (2.14). Similarly, (1.11) holds since

$$\left| \log \left(\frac{u+v+2m}{v-1+m} \right) - \log \left(\frac{u+v}{v-1} \right) \right| = O \left(\frac{m}{v} \wedge \frac{u}{v} \right) = O \left(\frac{m \log n}{n} \wedge \log n \right),$$

where the upper bound $u/v = O(\log n)$ will be useful for atypically large m . Due to (2.12), the above estimate implies the following bound:

$$\begin{aligned}
 & \mathbb{E}((\Xi(\tau_{k+1}) - \Xi(\tau_k))^2 1_{\{\Xi(\tau_k) \geq \log 4\}} | \mathcal{F}_{\tau_k}) \\
 (2.24) \quad & \leq c \left[\frac{\log^8 n}{n^2} + \log^2 n \times \mathbb{P}(C_{\log^3 n+1} | \mathcal{F}_{\tau_k}) \right] \\
 & \leq c \left(\frac{\log^8 n}{n^2} + e^{-c' \log^2 n} \right),
 \end{aligned}$$

where $c \in (0, \infty)$ and $c' \in (0, 1)$ do not depend on k . Recall that $n \geq k$, for all k , so the sequence (2.24) of upper bounds is summable in k . Now Lemma 1 yields that $\limsup_t \Xi(t)$ is finite almost surely, and this is equivalent to saying that $\liminf_t \xi(t)$ is strictly positive, almost surely. \square

3. Analysis on complete-like graphs. We will denote by $\mathcal{G} = \mathcal{G}_d$ a complete-like graph of interest. Our main goal in this section is to prove the following result leading to Theorem 1.

PROPOSITION 1. *The VRRW on \mathcal{G} satisfies: (i)*

$$\liminf_t \frac{Z(t, i)}{Z(t, j)} > 0 \quad a.s.,$$

for any two different interior sites i, j .

(ii) If ℓ_1, \dots, ℓ_r are the leaves attached to an interior site g , then

$$\begin{aligned}
 (3.1) \quad & \left\{ \liminf_t \min_{i \neq g} \frac{\sum_{j \notin \{i, g\}} Z(t, j)}{\sum_{j \neq i} Z(t, j)} > \delta \right\} \subset \left\{ \limsup_t \frac{(\sum_{j=1}^r Z(t, \ell_j))^{1+\delta}}{\sum_{i \neq g} Z(t, i)} = 0 \right\} \\
 & a.s.,
 \end{aligned}$$

where the sums above [except for $\sum_{j=1}^r Z(t, \ell_j)$] are taken over the interior sites only.

In the following subsections we prove the above proposition, treating several different cases separately. Property (ii) above will be used in the proof of Theorem 1. It gives a priori bounds on the total empirical frequency of the leaves, that simplify the large deviations estimates relative to the corresponding argument in [12] (see Section 3.3 for details).

3.1. Graphs with leaves at a single vertex. We start by considering the simplest noncomplete graph from the class of graphs described in the Introduction. Here there are three “interior” sites 1, 2 and 3, forming a triangle, and there

is an additional leaf $\ell_1^3 = \ell \sim 3$. As in the study of MVRRW we will denote $U(t) = Z(t, 1)$, $V(t) = Z(t, 2)$, $W(t) = Z(t, 3)$ and, moreover,

$$L(t) = Z(t, \ell).$$

Clearly, the process (U, V, W) , observed only at times $(\sigma_k)_{k \geq 0}$, where $\sigma_0 = t_0$ (assume without loss of generality that $X_{t_0} \in \{1, 2, 3\}$) and

$$\sigma_k := \min\{j > \sigma_{k-1} : X_j \neq X_{\sigma_{k-1}}, X_j \in \{1, 2, 3\}\}, \quad k \geq 1,$$

has the law of $(Z(t, 1), Z(t, 2), Z(t, 3))$ generated by the motion of a particle according to a MVVRW with a special vertex 3. Therefore, Lemma 3 insures that $U(t) \asymp V(t)$, or equivalently, that both

$$(3.2) \quad \limsup_{t \rightarrow \infty} \frac{U(t)}{V(t)} \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{V(t)}{U(t)}$$

are finite random variables, almost surely. As in (2.1), denote by $\tau_k^{(g)}$ the time of the k th successive visit to site g , where $g \in \{1, 2, 3\}$. Easy comparison of $(L(\tau_k^{(3)}), U(\tau_k^{(3)}) + V(\tau_k^{(3)}))$ with the Pólya urn ensures preliminary estimate

$$(3.3) \quad \limsup_k \frac{L(\tau_k^{(3)})}{U(\tau_k^{(3)}) + V(\tau_k^{(3)})} < \infty, \quad \text{a.s.}$$

As we will soon see, $L(\tau_k^{(3)}) \ll U(\tau_k^{(3)}) + V(\tau_k^{(3)})$ as a lower (random) power. First note that for any t

$$W(t) \leq U(t+1) + V(t+1) + L(t+1) + W(t_0),$$

so that (3.2) and (3.3) imply

$$(3.4) \quad \limsup_t \frac{W(t)}{U(t)} < \infty \quad \text{almost surely,}$$

and in turn that

$$(3.5) \quad \min \left\{ \liminf_t \frac{U(t)}{t}, \liminf_t \frac{V(t)}{t} \right\} > 0 \quad \text{almost surely.}$$

Given (3.4), it is now plausible that W has the same asymptotic order as U , since its increase is “helped” by the existence of the leaf ℓ . Soft arguments based on comparison with a generalized urn yield

$$(3.6) \quad \limsup_t \frac{\phi(U(t))}{W(t)} < \infty,$$

but not more, and comparison with the VRRW on the pure triangle does not seem to be useful either in proving the complement to (3.4). However, the drift increment comparison argument of Lemma 3 is robust enough. Namely, denote by \widetilde{W} the

process that starts as $\widetilde{W}(t_0) = W(t_0)$, and that increases by amount 1 at time $t + 1$ if $X(t) \in \{1, 2\}$ and $X(t + 1) = 3$ (i.e., whenever the site 3 is visited from another interior site), and that otherwise remains unchanged. Then

$$(3.7) \quad W(t) = \widetilde{W}(t) + Z(t, \ell) - Z(t_0, \ell) = \widetilde{W}(t) + L(t) - L(t_0)$$

in particular, $\widetilde{W}(t) \leq W(t)$ for all t . Consider the process

$$(3.8) \quad \Xi(k) := \log(U(\tau_k^{(2)}) + \widetilde{W}(\tau_k^{(2)})) - \log(\widetilde{W}(\tau_k^{(2)}) - 1), \quad k \geq 1,$$

adapted to the σ -field \mathcal{F}_{τ_k} , $k \geq 1$ where $\tau_k \equiv \tau_k^{(2)}$. Let $u = U(\tau_k)$, $v = W(\tau_k)$, $\tilde{v} = \widetilde{W}(\tau_k)$, $a = V(\tau_k)$, $n = u + \tilde{v}$, and note that the drift of Ξ at time k (provided $v < u/2$) is still less or equal to expression (2.19); in particular it is negative, as we reason next. It is necessary to interchange sites 2 and 3 in the definitions (2.8) and (2.9) and the rest of this argument. While the conditional probabilities of A_m , \bar{A}_m , $m \geq 1$ and B_m , \bar{B}_m , $m \geq 2$ are different in the current setting where ℓ exists, the estimates in (2.20) and (2.21) only concern the number of shuttles m between the two sites. Therefore,

$$(3.9) \quad \begin{aligned} & \mathbb{E}(\Xi(k+1)|\mathcal{F}_{\tau_k}) \\ & \leq (\mathbb{P}(B_1|\mathcal{F}_{\tau_k}) + \mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})) \log \frac{n+2}{\tilde{v}} \\ & \quad + \sum_{m=1}^{\infty} \left(\log \frac{n+1}{\tilde{v}-1} \mathbb{P}(A_m|\mathcal{F}_{\tau_k}) + \log \frac{n+1}{\tilde{v}} \mathbb{P}(\bar{A}_m|\mathcal{F}_{\tau_k}) \right) \\ & \quad + \sum_{m=2}^{\infty} \left(\log \frac{n+1}{\tilde{v}-1} \mathbb{P}(B_m|\mathcal{F}_{\tau_k}) + \log \frac{n+1}{\tilde{v}} \mathbb{P}(\bar{B}_m|\mathcal{F}_{\tau_k}) \right). \end{aligned}$$

Next observe that $\mathbb{P}(B_1|\mathcal{F}_{\tau_k})$ does not change under the new law, since possible shuttles between site 3 and its leaf ℓ before the step from 3 to another interior site, do not influence the conditional law of this step. Finally, observe that $\mathbb{P}(\bar{B}_1|\mathcal{F}_{\tau_k})$ is smaller than $(v/n)(u/(u+a))(a/(a+v+1))$ under the new law, since possible shuttles between site 3 and its leaf ℓ that happen before the step from 3 to 1, make the probability of the move from 1 to 2 smaller than $a/(a+v+1)$. Thus the estimates (2.22) and (2.23) can be carried out verbatim. Due to (3.9), and the fact $\tilde{v} \leq v$, we obtain

$$\begin{aligned} \mathbb{E}(\Xi(k+1)|\mathcal{F}_{\tau_k}) & \leq \log \frac{n+1}{\tilde{v}-1} \cdot \frac{u}{u+v} + \log \frac{n+1}{\tilde{v}} \cdot \frac{v}{u+v} \\ & \leq \log \frac{n+1}{\tilde{v}-1} \cdot \frac{u}{u+\tilde{v}} + \log \frac{n+1}{\tilde{v}} \cdot \frac{\tilde{v}}{u+\tilde{v}}, \end{aligned}$$

as claimed. In order to apply Lemma 1, it remains to estimate the quantities in (1.10) and (1.11). Before doing so, we show that L is a smaller power of $U + V$, and therefore of W . So fix $\beta \geq 1$ and consider again the times $\tau_k^{(3)}$, $k \geq 1$ of

successive visits to site 3. Note that $\tau_k^{(3)}$ is different from σ_k above, and from $\tau_k \equiv \tau_k^{(2)}$ linked to the definition of Ξ . Abbreviate

$$L_k := L(\tau_k^{(3)}), \quad U_k := U(\tau_k^{(3)}), \quad V_k := V(\tau_k^{(3)}), \quad W_k := W(\tau_k^{(3)}) = k.$$

Then, if $\delta \in (0, 1)$, on

$$P_k^\delta := \left\{ \frac{U_k}{U_k + W_k} \wedge \frac{V_k}{V_k + W_k} > \delta \right\},$$

we have

$$\begin{aligned} \mathbb{E} \left(\frac{L_{k+1}^\beta}{U_{k+1} + V_{k+1}} \middle| \mathcal{F}_{\tau_k^{(3)}} \right) &\leq \frac{(L_k + 1)^\beta}{U_k + V_k} \cdot \frac{L_k}{U_k + V_k + L_k} \\ (3.10) \quad &+ \frac{(L_k)^\beta}{U_k + V_k + 1} \cdot \frac{(1 - \delta)(U_k + V_k)}{U_k + V_k + L_k} \\ &+ \frac{(L_k)^\beta}{U_k + V_k + 2} \cdot \frac{\delta(U_k + V_k)}{U_k + V_k + L_k}. \end{aligned}$$

Namely, either the walk visits the leaf ℓ at time $\tau_k^{(3)} + 1$ and steps back to site 3 at time $\tau_k^{(3)} + 2 = \tau_{k+1}^{(3)}$, or it visits $\{1, 2\}$ at time $\tau_k^{(3)} + 1$, and given this, it revisits the same set at time $\tau_k^{(3)} + 2$ with probability larger than δ .

Using (3.2) and (3.4) one easily sees that

$$(3.11) \quad \mathbb{P} \left(\lim_{\delta \searrow 0} \liminf_k P_k^\delta \right) = 1.$$

From now on we take δ small and think about the behavior of the process $(L_k)^\beta / (U_k + V_k)$ on $\bigcap_{k \geq n_0} P_k^\delta$, where n_0 is a large finite integer.

REMARK 3. The part (a) of the next lemma will not be used in the sequel of the current argument; however its argument will be needed in the next section.

LEMMA 4. (a) *Estimate (3.3) and $\liminf_t (U(t) \wedge V(t)) / \phi(t) > 0$ are already sufficient for*

$$(3.12) \quad \lim_t \frac{L(t)}{U(t) + V(t)} = 0 \quad a.s.$$

(b) *On $\bigcap_{k \geq n_0} P_k^\delta$, for any $\beta < 1 + \delta$ we have that*

$$(3.13) \quad \lim_t \frac{L(t)^\beta}{U(t) + V(t)} = 0 \quad a.s.$$

PROOF. (a) We need a slightly more precise estimate than (3.10). Namely, keeping track of which interior vertex (1 or 2) the walk visits first, one obtains that

$$\begin{aligned}
 \mathbb{E}\left(\frac{L_{k+1}^\beta}{U_{k+1} + V_{k+1}} \middle| \mathcal{F}_{\tau_k^{(3)}}\right) &\leq \frac{(L_k + 1)^\beta}{U_k + V_k} \cdot \frac{L_k}{U_k + V_k + L_k} \\
 &\quad + \frac{(L_k)^\beta}{U_k + V_k + 1} \cdot \frac{U_k W_k}{(U_k + V_k + L_k)(V_k + W_k)} \\
 &\quad + \frac{(L_k)^\beta}{U_k + V_k + 2} \cdot \frac{U_k V_k}{(U_k + V_k + L_k)(V_k + W_k)} \\
 &\quad + \frac{(L_k)^\beta}{U_k + V_k + 1} \cdot \frac{V_k W_k}{(U_k + V_k + L_k)(U_k + W_k)} \\
 &\quad + \frac{(L_k)^\beta}{U_k + V_k + 2} \cdot \frac{V_k U_k}{(U_k + V_k + L_k)(U_k + W_k)}.
 \end{aligned}
 \tag{3.14}$$

The right-hand side in (3.15) equals

$$\frac{L_k^\beta}{(U_k + V_k)}(1 + R_k),
 \tag{3.15}$$

with $\beta = 1$, and with

$$\begin{aligned}
 R_k &= 1/(U_k + V_k + L_k) \\
 &\quad \times \left\{ 1 - \left(\frac{U_k W_k}{(U_k + V_k + 1)(W_k + V_k)} + \frac{V_k W_k}{(U_k + V_k + 1)(W_k + U_k)} \right) \right. \\
 &\quad \left. - 2 \left(\frac{U_k V_k}{(U_k + V_k + 2)(W_k + V_k)} + \frac{U_k V_k}{(U_k + V_k + 2)(W_k + U_k)} \right) \right\}.
 \end{aligned}$$

The last expression equals to

$$- \left(\frac{U_k V_k}{(U_k + V_k + 2)(W_k + V_k)} + \frac{U_k V_k}{(U_k + V_k + 2)(W_k + U_k)} \right) + o\left(\frac{1}{U_k + V_k}\right).$$

Now due to hypotheses of part (a) we conclude that $U_k + V_k \asymp k$ and $U_k \wedge V_k \geq ck/\log k$ for some positive random c . Hence the leading term above has absolute value larger than a term of order $1/\log k$. In particular, the process $L_k/(U_k + V_k)$ is a positive super-martingale, so it converges almost surely to a finite limit. However, the limit must be 0, since on the event $\lim_k L_k/(U_k + V_k) > 0$ the drift increment above is of the order at least $1/(k \log k)$, so the drift would not be summable otherwise. In this way one can also see that the asymptotic order of L_k may not be of the form k/a_k , if a_k converge to infinity sufficiently slowly so that $\sum_k 1/(k \log k \times a_k) = \infty$. The last observation will not be used in the sequel.

(b) Note that on $\bigcap_{k \geq n_0} P_k^\delta$, for any $\beta < 1 + \delta$ we have the same expression (3.15) for the right-hand side in (3.15), except that now R_k is smaller than

$$\frac{1}{U_k + V_k + L_k} \left(\beta - (1 - \delta) - 2\delta + O\left(\frac{1}{L_k}\right) + O\left(\frac{1}{U_k + V_k}\right) \right).$$

This can be seen already from (3.10), since $(L_k + 1)^\beta / L_k^\beta = 1 + \beta/L_k + O(L_k^{-2})$. Consequently, R_k is again negative for all sufficiently large k , and therefore $L_k^\beta / (U_k + V_k)$ converges to a finite random quantity. In particular, for any $\beta' < \beta$ the limit in (3.13) is 0 on the event $\bigcap_{k \geq n_0} P_k^\delta$, and due to (3.11), after letting $\delta \rightarrow 0$, one obtains (3.13), hence part (ii) of Proposition 1 for the triangle ornamented with a single leaf. \square

In order to prove (1.10) and (1.11) for the process Ξ from (3.8), we will derive analogues to (2.13) and (2.14). The reader can check that in the special case where the leaves are attached to 3 only (that is, no leaves are attached at 1 or 2), one does not need (3.13) to obtain sufficiently good estimates. Nevertheless, we will soon consider the general case, hence doing the calculations while accounting for (3.13) will prove useful.

Due to Lemma 4(b) and (3.6) and (3.7), we have $\{\bigcap_{k \geq n_0} P_k^\delta\} \subset \{\widetilde{W}(t) \asymp W(t)\}$, and therefore

$$(3.16) \quad \left\{ \bigcap_{k \geq n_0} P_k^\delta \right\} \subset \left\{ \limsup_t \frac{\phi(U(t))}{\widetilde{W}(t)} < \infty \right\} \quad \text{almost surely.}$$

Suppose that $\beta > 1$ and that $(p_k^m)_{m \geq 1, k \geq 1}$ is a table of numbers in $(0, 1)$ such that

$$1 - p_k^m \leq \frac{c(m, k)}{k^{1-1/\beta}}, \quad m, k \geq 1,$$

where, for each finite integer s ,

$$(3.17) \quad \limsup_k \max_{m \leq s} c(m, k) < \infty.$$

Let $(G_k, k \geq 0)$ be a random process (adapted to a filtration $(\mathcal{H}_k, k \geq 0)$) taking values in the nonnegative integers, and assume that it satisfies conditional “geometric-like” relations

$$(3.18) \quad \mathbb{P}(G_k > m + 1 | G_k > m, \mathcal{H}_{k-1}) = 1 - p_k^{m+1}, \quad m \geq 0.$$

Then $\mathbb{P}(G_k > s | \mathcal{H}_{k-1}) = \prod_{m \leq s} (1 - p_k^m) \leq (\max_{m \leq s} c(m, k))^s / k^{s(1-1/\beta)}$, and therefore, under the assumption (3.17), we have

$$(3.19) \quad \lim_{j \rightarrow \infty} \mathbb{P} \left(\bigcap_{k \geq j} \{G_k \leq 2/(1 - 1/\beta)\} \right) = 1.$$

Consider the behavior of VRRW on $\bigcap_{k \geq n_0} P_k^\delta$ and fix some $\beta \in (1, 1 + \delta)$. Following each time $\tau_k^{(3)} = \sigma_{k'}$ when VRRW visits site 3 from another interior site,

the particle will make a nonnegative (possibly 0) number \tilde{N}_k of shuttles to ℓ before visiting the next (different) interior site at time $\sigma_{k'+1}$. Note that \tilde{N}_k in fact stands for $W(\sigma_{k'+1}) - W(\sigma_{k'}) = W(\sigma_{k'+1}) - k$. Let j be a large integer. Since $W(\tau_k^{(3)}) = W_k = k$ we have on $\bigcap_{k \geq n_0} P_k^\delta$ that $U_k + V_k \geq 2\delta k/(1 - \delta)$, and due to (3.13) that $L_k \leq k^{1/\beta}$, for all $k \geq j$ (with an overwhelming probability as $j \rightarrow \infty$). As a consequence, one can construct a process G satisfying (3.17) and (3.18) [where $c(m, k)$ can be taken as $2\delta/(1 - \delta)$ for all $k \geq j$ and $m \leq s$, so the lim sup in (3.17) is bounded by $2\delta/(1 - \delta)$] such that $\tilde{N}_k \leq G_k$ (note that G is defined for all k , but the coupling of \tilde{N}_k and G_k is necessary only for k such that $\tau_k^{(3)} = \sigma_{k'}$). Due to (3.19), we conclude that

$$(3.20) \quad \{\tilde{N}_k \leq 2/(1 - 1/\beta)\} \quad \text{for all sufficiently large } k,$$

with an overwhelming probability on $\bigcap_{k \geq n_0} P_k^\delta$.

Therefore, one can redo the calculation (2.10), this time writing instead of the third term an analogous

$$(3.21) \quad \frac{(1 + 0/v)(1 + s_1/v) \cdots (1 + s_{m-2}/v)}{(1 + 0/(a + v))(1 + s_1/(a + v)) \cdots (1 + s_{m-2}/(a + v))},$$

where $s_{i+1} - s_i \geq 1$ and $s_{i+1} - s_i \leq 2/(1 - 1/\beta)$ for all i , and for all large k . The estimate (2.11) holds as before, with different constants comprised in $O(m^2/u) + O(m^2/v)$. Together with (3.5), this immediately implies (2.13) and (2.14), and thus (1.10) and (1.11) for Ξ , as at the end of the proof of Lemma 3. Note that in this step we also make use of the preliminary estimate (3.16).

The above reasoning applied on the event $\bigcap_{k \geq n_0} P_k^\delta$ only (see also Remark 2), but due to (3.11) we conclude the following lemma.

LEMMA 5.

$$\limsup_t \Xi(t) < \infty \quad a.s.$$

As a consequence, $\liminf \tilde{W}(t)/(U(t) + \tilde{W}(t)) > 0$, almost surely, and since $W(t) \geq \tilde{W}(t)$,

$$\liminf W(t)/(U(t) + W(t)) > 0 \quad a.s.,$$

completing the proof of Proposition 1(i) in the special case of the graph with three interior vertices and one leaf.

As the reader will quickly check, the proof above carries over to any \mathcal{G} with the same interior sites $\{1, 2, 3\}$ and finitely many leaves $\{\ell_1, \dots, \ell_r\}$, all attached to the interior site 3. Namely, for the purposes of the calculation in Lemmas 4 and 5 all the leaves can be combined into one “super-leaf” so that, in particular, Proposition 1 holds via the same argument.

Moreover, suppose that \mathcal{G} has interior sites $\{1, 2, \dots, d\}$, $d \geq 4$, and finitely many leaves $\{\ell_1, \dots, \ell_r\}$, all attached to the interior site d . Let the initial position $X(t_0)$ take value in $\{1, \dots, d\}$, almost surely. Fix two different sites $i, j \in \{1, \dots, d-1\}$, and define three classes

$$(3.22) \quad C_1 := \{i\}, \quad C_2 := \{j\} \quad \text{and} \quad C_3 := \{1, \dots, d\} \setminus \{i, j\}$$

of interior vertices. Consider $S(t) = \sum_{h=1}^3 h 1_{\{X(t) \in C_h\}}$, and a sequence of stopping times $\sigma_0 := t_0$,

$$\sigma_k := \min\{s > \sigma_{k-1} : S(s) \neq S(\sigma_{k-1}), S(s) \neq 0\}, \quad k \geq 1.$$

Note that the process

$$(3.23) \quad X' \equiv (X'(k), k \geq 0) = (S(\sigma_k), k \geq 0)$$

is identical in law to the position process X of a MVRRW, with a special vertex 3. Indeed, $\{S(t) = h\} = \{X(t) \in C_h\}$, for $h = 1, 2, 3$, and $(\sigma_k)_{k \geq 0}$ are the successive times when X jumps from one class of interior vertices to another. Therefore, setting

$$Z'(k, h) := \sum_{v \in C_h} Z(\sigma_k, v), \quad h = 1, 2, 3,$$

it is simple to check that the transitions of X' are driven by (1.1), with X' (resp., Z') replacing X (resp., Z). Moreover, $Z'(k+1, 1) - Z'(k, 1)$ [resp., $Z'(k+1, 2) - Z'(k, 2)$] equals 1 if $X'(k) = 1$ (resp., $=2$), while $Z'(k+1, 3) - Z'(k, 3) = H(k) \geq 1$ if $X'(k) = 3$. A careful reader will note that the measurability requirement on H (see the beginning of Section 2) necessitates considering X' with respect to stopped filtration $(\mathcal{F}_{\sigma_k})_{k \geq 0}$ generated by X . As before, these observations ensure that $Z'(k, 1) \asymp Z'(k, 2)$ as $k \rightarrow \infty$. Since $Z(t, i) = Z'(k, 1)$ and $Z(t, j) = Z'(k, 2)$, where $t \in [\sigma_k, \sigma_{k+1})$, we conclude that $Z(t, i)$ and $Z(t, j)$ are asymptotically comparable, for all $i, j \in \{1, \dots, d-1\}$, almost surely. It is again easy to verify that

$$\limsup_t \frac{Z(t, d)}{\sum_{i=1}^{d-1} Z(t, i)} < \infty \quad \text{and} \quad \limsup_t \frac{\phi(\sum_{i=1}^{d-1} Z(t, i))}{Z(t, d)} < \infty,$$

almost surely. Since the walk necessarily returns to d after each visit to a leaf, we have $L(t) \leq Z(t_d) + L(t_0)$, and therefore by the first estimate above we conclude

$$t = Z(t, d) + \sum_{i=1}^{d-1} Z(t, i) + L(t) = O\left(\sum_{i=1}^{d-1} Z(t, i)\right) \quad \text{almost surely.}$$

This implies readily that $\sum_{i=1}^{d-1} Z(t, i) \asymp t$, and therefore that $Z(t, 1) \asymp t$ (or equivalently, $Z(t, i) \asymp t$, $\forall i = 1, \dots, d-1$), almost surely. Again combine all the leaves into a single super-leaf $\ell \sim d$. The calculation of Lemma 4(b), for the process observed at successive times $\tau_k^{(d)}$ of visit to site d , yields as before Proposition 1(ii).

Finally, let $U(t) = Z(t, 1)$, $V(t) = \sum_{g=2}^{d-1} Z(t, g)$ and $W(t) = Z(t, d)$, and consider the process at the successive times

$$(3.24) \quad \sigma'_k := \min\{j > \sigma'_{k-1} : X_j \neq X_{\sigma'_{k-1}}, X_j \in \{2, \dots, d-1\}\}, \quad k \geq 1,$$

of visit to the subset $\{2, \dots, d-1\}$. Set $\widetilde{W}(t_0) = \widetilde{W}(t_0)$ and let

$$\widetilde{W}(t) := W(t) - (Z(t, \ell) - Z(t_0, \ell)), \quad t \geq t_0.$$

Then the process Ξ defined as in (3.8) (with σ'_k in place of $\tau_k^{(2)}$) again satisfies (1.9)–(1.11) with $a = \log 4$ and $b = 0$, so Lemma 5 follows, implying Proposition 1(i) as before.

3.2. General complete-like graphs with $d \geq 3$. Assume that we are given a general complete-like graph $\mathcal{G} = \mathcal{G}_d$ from Introduction. Here the argument is somewhat more delicate, due to the fact that we cannot anymore use the MVRRW to easily obtain $Z(t, i) \asymp t$ for most (all but one) sites, which was essential in applying Lemma 4.

We start again by making some soft observations. If $\ell \sim g$, then $Z(t, \ell) \leq Z(t+1, g) + Z(t_0, \ell)$ implies that $t = \sum_{v \in V(\mathcal{G})} Z(t, v) \leq \sum_{i=1}^d (r_i + 1)Z(t+1, i) + O(1)$, and in particular that

$$(3.25) \quad \liminf_t \sum_{i=1}^d Z(t, i)/t > 0,$$

almost surely. Moreover, Pólya's urn comparisons, as in Section 2, imply that

$$\sup_t Z(t, v) = \infty, \quad v \in V(\mathcal{G}),$$

and, for each i ,

$$(3.26) \quad \limsup_t \frac{\sum_{j=1}^{r_i} Z(t, \ell_j^i)}{\sum_{g=1, g \neq i}^d Z(t, g)} < \infty \quad \text{almost surely.}$$

Here we recall that ℓ_j^i , $j = 1, \dots, r_i$, are the leaves attached at the interior site i . Soon we will see that the limit in (3.26) is 0. Since

$$(3.27) \quad Z(t, i) \leq \sum_{j=1}^{r_i} Z(t+1, \ell_j^i) + \sum_{g=1, g \neq i}^d Z(t+1, g) + Z(t_0, i),$$

after adding $\sum_{g=1, g \neq i}^d Z(t, g)$ to both sides, (3.25) and (3.26) yield

$$(3.28) \quad \liminf_t \sum_{g=1, g \neq i}^d Z(t, g)/t > 0 \quad \text{for each interior site } i, \text{ almost surely.}$$

Without loss of generality assume that $X(t_0) \in \{1, \dots, d\}$. Moreover, as already noted, each visit to a leaf of i is immediately followed by a visit to i . Therefore, if $Z(0, i) > \sum_{j=1}^{r_i} Z(0, \ell_j^i)$, then

$$(3.29) \quad Z(t, i) > \sum_{j=1}^{r_i} Z(t, \ell_j^i), \quad t \geq t_0,$$

and provided (3.29) holds at some time t , it will continue to hold at all later times. We claim that, for each $i = 1, \dots, d$, (3.29) holds starting from some finite time. Indeed, due to (3.28) the walk will almost surely (eventually) make at least $(\sum_{j=1}^{r_i} Z(0, \ell_j^i) - Z(0, i))^+ + 1$ steps from i to another interior vertex, and this ensures (3.29) upon the next return to i . Starting from the finite (stopping) time at which (3.29) holds for all $i \in \{1, \dots, d\}$, one can compare (as in Section 2) the process $(\sum_{g=1, g \neq i}^d Z(\sigma_k, g), Z(\sigma_k, i))$, where σ_k is the time of k th return to the subset of sites $\{1, \dots, d\} \setminus \{i\}$, with the generalized urn (X'_k, Y'_k) of Theorem 4 (again here $a = c = d = 1$, $b = 0$), so that $Z(\sigma_k, i) \geq Y'_k$ and $\sum_{g=1, g \neq i}^d Z(\sigma_k, g) \leq X'_k$. In particular, for each $i = 1, \dots, d$,

$$(3.30) \quad \liminf_t \frac{Z(t, i)}{\phi(\sum_{g=1, g \neq i}^d Z(t, g))} > 0 \quad \text{hence} \quad \liminf_t \frac{Z(t, i)}{\phi(t)} > 0$$

almost surely.

Due to (the argument of) Lemma 4(a), estimates (3.26) [namely, its consequence (3.28)] and (3.30) are sufficient to conclude that almost surely, for each $i = 1, \dots, d$,

$$(3.31) \quad \lim_t \frac{\sum_{j=1}^{r_i} Z(t, \ell_j^i)}{\sum_{g=1, g \neq i}^d Z(t, g)} = \lim_t \frac{\sum_{j=1}^{r_i} Z(t, \ell_j^i)}{t} = 0.$$

Indeed, the reader can quickly check that $\sum_{j=1}^{r_i} Z(t, \ell_j^i)$ [resp., $\sum_{g=1, g \neq i}^d Z(t, g)$], observed at the times of return to i , corresponds to $L(t)$ [resp., $U(t) + V(t)$], observed at the times of return to 3. The possible presence of leaves at sites $g \neq i$, corroborates inequality (3.10).

However, we wish to strengthen (3.31) to an analogue of Lemma 4(b). In order to be able to recycle its argument, it suffices to show that for any $i \neq g$, $i, g \in \{1, \dots, d\}$ we have

$$\liminf_t \frac{\sum_{l=1, l \notin \{i, g\}}^d Z(t, l)}{t} > 0,$$

or equivalently, that the third most frequently visited interior site has positive asymptotic frequency. Let $(Z_{(1)}(t), \dots, Z_{(d)}(t))$ be the vector of order statistics for $Z(t, g)$, $g = 1, \dots, d$, and set

$$S(t) = Z_{(d)}(t), \quad P(t) = Z_{(d-1)}(t) \quad \text{and} \quad R(t) = \sum_{j=1}^{d-2} Z_{(j)}(t).$$

Clearly $S(t) \asymp t$, and due to (3.28) also $P(t) \asymp t$. Moreover, due to (3.31) it must be

$$(3.32) \quad \liminf_t \frac{P(t)}{t} \geq \frac{1}{2(d-1)}.$$

Indeed, (3.31) implies that $\limsup_t S(t)/t \leq 1/2$, and hence, the identity $S(t) + P(t) + R(t) + \sum_{i=1}^d \sum_{j=1}^{r_i} Z(t, \ell_j^i) \equiv t$ and (3.31) together imply $\liminf_t (P(t) + R(t))/t \geq 1/2$, and (3.32) in turn.

It suffices to show that R is asymptotically comparable to $S + P$. Let $a(t) = \min\{i : Z_{(d)}(t) = Z(t, i)\}$ and $b(t) = \min\{i \neq a(t) : Z_{(d-1)}(t) = Z(t, i)\}$. Consider the process $\tilde{\eta}(t) := (S(t) + P(t))/R(t)$ at successive times of visit to the set $\{a(t), b(t)\}$. Without risk of confusion, let us denote by $(\tilde{\eta}_k, k \geq 0)$ the process $\tilde{\eta}$ viewed only on this restricted collection of times.

LEMMA 6. $\limsup_k \tilde{\eta}_k < \infty$, *almost surely*.

PROOF. Let τ be the time of the k th visit to the set of vertices $\{a(\cdot), b(\cdot)\}$. For concreteness suppose that the current position $X(\tau) = b(\tau)$, the calculation below is similar if $X(\tau) = a(\tau)$. Let s, p, r denote the values of $S(\tau), P(\tau), R(\tau)$, respectively, and let l denote the corresponding “total leaf weight” at $b(\tau)$. Without loss of generality we may assume that $r \geq 4(d-1) \geq 4$. Assume in addition that $s + p \geq 2r$, or equivalently, that $\tilde{\eta}_k = (s + p)/r \geq 2$. Then, on $\{Z_{(d-1)}(\tau) > Z_{(d-2)}(\tau)\}$, $\tilde{\eta}_{k+1}$ will either take value $(s + p + 1)/r$ with probability $(s + l)/(s + l + r)$, or a value smaller than $(s + p + 1)/(r + 1)$ (here we use the fact that $s + p \geq 2r$ and $r \geq 4$) with probability $r/(s + l + r)$. A careful reader will note that this includes transitions that change values of a or b . On the opposite event $\{Z_{(d-1)}(\tau) = Z_{(d-2)}(\tau)\}$ it could be that the particle jumps from $b(\tau)$ to another site with the same frequency thus increasing $s + p$ by 1 without changing r . However, if

$$(3.33) \quad r \leq \frac{1/(3(d-1))}{1 - 1/(3(d-1))} (p + s) \implies r \leq \frac{1}{3(d-1)} \tau,$$

then due to (3.32) we have $Z_{(d-2)}(\tau) < r \ll p$, whenever τ is sufficiently large. In particular, $\{Z_{(d-1)}(\tau) = Z_{(d-2)}(\tau)\}$ happens at most finitely often, almost surely. Hence, provided $\tilde{\eta}_k \geq 3(d-1) \geq 2$, the drift increment of $\tilde{\eta}$ is bounded by

$$\frac{1}{r} \cdot \frac{s + l}{s + r + l} - \frac{1}{r + 1} \frac{s + p - r}{s + r + l},$$

and since $r \geq 4(d-1)$, it will be negative for all sufficiently large τ due to (3.31)–(3.33). It is particularly easy to check the other two hypotheses of Lemma 1. Namely, the absolute value of the increment $\tilde{\eta}_{k+1} - \tilde{\eta}_k$ is of the order $1/r = 1/\sum_{g, g \neq a(\tau), b(\tau)} Z(\tau, g)$, so clearly diminishing at the time instances when $\tilde{\eta}_k$

traverses the threshold $3(d-1)$. Furthermore, due to (3.30), the sum of square increments is finite, a.s. The conclusion is now due to Lemma 1. \square

It is easy to see that Lemma 6 implies $\liminf_t R(t)/t > 0$, and that this is equivalent to having

$$(3.34) \quad \liminf_t \min_{i,j=1}^d \frac{\sum_{g=1, g \notin \{i,j\}}^d Z(t, g)}{t} > 0 \quad \text{almost surely.}$$

In analogy to the setting of the previous subsection, for each $g = 1, \dots, d$, define

$$P_k^{\delta, g} := \left\{ \min_{i=1}^d \frac{\sum_{j=1, j \notin \{i, g\}}^d Z(\tau_k^{(i)}, j)}{\sum_{j=1, j \neq i}^d Z(\tau_k^{(i)}, j)} \geq \delta \right\},$$

where, as usual, $\tau_k^{(i)}$ is the k th return time to i . The argument of Lemma 4(b) gives

$$(3.35) \quad \bigcap_{k \geq n_0} P_k^{\delta, g} \subset \left\{ \limsup_t \frac{(\sum_{j=1}^r Z(t, \ell_j^{(g)}))^{\beta}}{\sum_{i \neq g} Z(t, i)} = 0 \right\}$$

for any $\beta < 1 + \delta$, and this in turn yields Proposition 1(ii). Due to (3.34), we have, moreover,

$$(3.36) \quad \mathbb{P} \left(\lim_{\delta \rightarrow 0} \liminf_k \bigcap_{i=1}^d P_k^{\delta, i} \right) = 1.$$

Finally, consider two different interior sites i and j , the classes (3.22) and the process X' from (3.23). In analogy to (3.8) and (3.24), for $g \in \{i, j\}$, define

$$\tilde{Z}(t, g) := Z(t, g) - \sum_{j=1}^{r_g} (Z(t, \ell_j^g) - Z(t_0, \ell_j^g)), \quad t \geq t_0.$$

Then $\tilde{Z}(t, g) \leq Z(t, g)$, $t \geq t_0$, $g \in \{i, j\}$, and, moreover,

$$(3.37) \quad \left\{ \liminf_k \bigcap_{i=1}^d P_k^{\delta, i} \right\} \subset \{ \tilde{Z}(t, j) \asymp Z(t, j), \tilde{Z}(t, i) \asymp Z(t, i) \} \quad \text{almost surely.}$$

Let σ_k be the time of k th visit to class C_3 from i or from j (in particular, not accounting for the steps from C_3 to itself, and the steps from the leaves into C_3). Now consider

$$(3.38) \quad \tilde{\Xi}(k) := \log(\tilde{Z}(\sigma_k, i) + \tilde{Z}(\sigma_k, j)) - \log(\tilde{Z}(\sigma_k, j) - 1), \quad k \geq 1.$$

Fix $\delta \in (0, 1)$ and $\beta < 1 + \delta$. The asymptotics (3.35) ensures [see the discussion comprising (3.17)–(3.19)] the existence of a finite n_1 such that with an overwhelming probability there are at most $2/(1 - 1/\beta)$ repeated shuttles from i (resp., j) to

its leaves following any step into i (resp., j) from another interior site that occurs during the time interval (σ_k, σ_{k+1}) , for all $k \geq n_1$.

We will show that a Doob–Meyer modification of the process $\tilde{\Xi}$ still satisfies the properties (1.9)–(1.11) so that again

$$(3.39) \quad \limsup_k \tilde{\Xi}(k) < \infty \quad \text{a.s. on } \liminf_k \bigcap_{i=1}^d P_k^{\delta,i}.$$

This is equivalent to

$$\liminf_t \frac{\tilde{Z}(t, j)}{\tilde{Z}(t, i)} > 0 \quad \text{a.s. on } \liminf_k \bigcap_{i=1}^d P_k^{\delta,i}.$$

Due to (3.36) and (3.37) we can conclude Proposition 1(i).

Denote $u(k) \equiv u = Z(\sigma_k, i)$, $\tilde{u}(k) \equiv \tilde{u} = \tilde{Z}(\sigma_k, i)$, $v(k) \equiv v = Z(\sigma_k, j)$, $\tilde{v}(k) \equiv \tilde{v} = \tilde{Z}(\sigma_k, j)$, $n(k) \equiv n = \tilde{u} + \tilde{v}$ and $a(k) \equiv a = \sum_{g \in C_3} Z(\sigma_k, g)$. In fact, (1.10) and (1.11) hold for $\tilde{\Xi}$ as in the case of the graph with leaves at a single vertex only, using (3.35) instead of Proposition 1(ii). For (1.9), note first that (cf. also the next lemma)

$$\mathbb{P}(\bar{B}_1 | \mathcal{F}_{\sigma_k}) \leq \frac{v}{u+v} \cdot \frac{u}{a+u} \cdot \frac{a}{a+v+1} \quad \text{almost surely,}$$

since possible shuttles to leaves $\ell_1^j, \dots, \ell_{r_j}^j$ can only decrease the probability of return to class C_3 when stepping out of i into an interior site.

LEMMA 7. *We have*

$$(3.40) \quad \mathbb{P}(B_1 | \mathcal{F}_{\sigma_k}) \in \left[\frac{u}{u+v} \cdot \frac{v}{a+v} \cdot \frac{a(1-\varepsilon(k))}{a+u+1}, \frac{u}{u+v} \cdot \frac{v}{a+v} \cdot \frac{a}{a+u+1} \right] \\ \text{almost surely,}$$

where $\varepsilon(k)$ is \mathcal{F}_{σ_k} -measurable nonnegative random variable, such that on $\bigcap_{k \geq n_0} P_k^{\delta,i}$,

$$\varepsilon(k) = O\left(\frac{(a+v)^{1/\beta}}{a+u}\right) \quad \text{almost surely.}$$

PROOF. Recall that on B_1 the particle steps from a site in the class C_3 to i , next does a certain number $N(k; u)$ (possibly 0) of shuttles to the leaves $\ell_1^i, \dots, \ell_{r_i}^i$ before a step to j , and finally, does a number (possibly 0) of shuttles to the leaves $\ell_1^j, \dots, \ell_{r_j}^j$ before stepping back to C_3 . It is now simple to check that

$$\varepsilon(k) = \frac{u+v}{u} \mathbb{E} \left[1_{\{X(\sigma_k+1)=i\}} \mathbb{E} \left(\frac{N(k; u)}{a+u+N(k; u)+1} \middle| \mathcal{F}_{\sigma_k}, X(\sigma_k+1)=i \right) \middle| \mathcal{F}_{\sigma_k} \right],$$

so it suffices to show (recall that $v < u/2$)

$$\mathbb{E}\left(\frac{N(k; u)}{a + u + N(k; u)} \middle| \mathcal{F}_{\sigma_k}, X(\sigma_k + 1) = i\right) \leq C \frac{(a + v)^{1/\beta}}{a + u} \quad \text{almost surely,}$$

for some finite constant C . Let $q \equiv q(k) := \sum_{j=1}^{r_i} Z(\sigma_k, \ell_j^i) \equiv \sum_{j=1}^{r_i} Z(\sigma_k + 1, \ell_j^i)$ be the total weight of the leaves attached to i at time σ_k (that is, $\sigma_k + 1$). Our calculation is based on the same reasoning as the discussion comprising (3.17)–(3.19); however, the expectation bound is simpler, since the random variable $N(k; u)/(a + u + N(k; u))$ of interest is bounded by 1. Namely, $\mathbb{P}(N(k; u) \geq 2q | \mathcal{F}_{\sigma_k}, 1_{\{X(\sigma_k+1)=i\}}) \leq \mathbb{P}(N(k; u) \geq q + 1 | \mathcal{F}_{\sigma_k}, 1_{\{X(\sigma_k+1)=i\}}) = \frac{q}{a+u+q}$, and therefore

$$\mathbb{E}\left(\frac{N(k; u)}{a + u + N(k; u)} \middle| \mathcal{F}_{\sigma_k}, X(\sigma_k + 1) = i\right) \leq \frac{2q}{a + u + 2q} + \frac{q}{a + u + q} \leq \frac{3q}{a + u}.$$

The very last term is bounded by $C(v + a)^{1/\beta}/(a + u)$, provided $q \leq C(v + a)^{1/\beta}$, which happens eventually on $\bigcap_{k \geq n_0} P_k^{\delta, i}$, almost surely. \square

Note that almost surely on $\{v < u/2\}$

$$(3.41) \quad \frac{(a + v)^{1/\beta}}{a + u} = O\left(\frac{1}{(a + u)^{1-1/\beta}}\right) = O\left(\frac{1}{(\sigma_k)^{1-1/\beta}}\right),$$

where we used (3.28) for the last estimate. Due to the fact $\mathbb{P}(B_1 | \mathcal{F}_{\sigma_k}) + \varepsilon(k) \geq \mathbb{P}(\bar{B}_1 | \mathcal{F}_{\sigma_k})$ the calculations (2.22) and (2.23) can be modified to yield

$$\begin{aligned} & (\mathbb{P}(B_1 | \mathcal{F}_{\sigma_k}) + \mathbb{P}(\bar{B}_1 | \mathcal{F}_{\sigma_k})) \log \frac{n + 2}{\tilde{v}} \\ & \leq \mathbb{P}(B_1 | \mathcal{F}_{\sigma_k}) \log \frac{n + 1}{\tilde{v} - 1} + \mathbb{P}(\bar{B}_1 | \mathcal{F}_{\sigma_k}) \log \frac{n + 1}{\tilde{v}} \\ & \quad + \varepsilon(k) \left(\log \frac{n + 1}{n + 2} + \log \frac{\tilde{v}}{\tilde{v} - 1} \right). \end{aligned}$$

Denote

$$r(k) := \varepsilon(k) \left(\log \frac{n + 1}{n + 2} + \log \frac{\tilde{v}}{\tilde{v} - 1} \right) 1_{\{v < u/2\}}.$$

We therefore obtain

$$\begin{aligned} & \mathbb{E}(\tilde{\Xi}(k + 1) - \tilde{\Xi}(k) | \mathcal{F}_{\tau_k}) \\ & \leq \log \frac{n + 1}{\tilde{v} - 1} \cdot \frac{u}{u + v} + \log \frac{n + 1}{\tilde{v}} \cdot \frac{v}{u + v} - \log \frac{n}{\tilde{v} - 1} + r(k) \\ (3.42) \quad & \leq \frac{1}{u + v} \left[\frac{u + v}{\tilde{u} + \tilde{v}} - \frac{v}{\tilde{v}} \right] + O\left(\frac{1}{\tilde{v} \cdot n}\right) + r(k) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{u+v} \cdot \frac{\tilde{v}u - \tilde{u}v}{(\tilde{u} + \tilde{v})\tilde{v}} + O\left(\frac{1}{\tilde{v} \cdot n}\right) + r(k) \\
 (3.43) \quad &\leq \frac{1}{u+v} \cdot \frac{u(\tilde{v} - v) + v(u - \tilde{u})}{(\tilde{u} + \tilde{v})\tilde{v}} + O\left(\frac{1}{\tilde{v} \cdot n}\right) + r(k) \\
 &=: \tilde{r}(k),
 \end{aligned}$$

where for the second inequality we develop (recall $n = \tilde{u} + \tilde{v}$)

$$\log\left(\frac{\tilde{u} + \tilde{v} + 1}{u + v + 1}\right) - \log\left(\frac{\tilde{u} + \tilde{v}}{u + v}\right) \quad \text{and} \quad \log\left(\frac{v}{\tilde{v}}\right) - \log\left(\frac{v-1}{\tilde{v}-1}\right)$$

via Taylor's expansion up to quadratic order terms. Lemma 7, jointly with (3.30), (3.35) and (3.41), implies that, on $\bigcap_{k \geq n_0} \bigcap_{i=1}^d P_k^{\delta, i}$, $D_\infty := \sum_{l=1}^\infty \tilde{r}(l)$ is a finite random variable, almost surely. Now observe that on $\{D_\infty \leq K\} = \bigcap_{k \geq 1} \{\sum_{l=1}^k \tilde{r}(l) \leq K\}$, the process

$$\tilde{\Xi}' := \left(\tilde{\Xi}(k) - \sum_{l \leq k-1} \tilde{r}(l), \, k \geq 0 \right)$$

satisfies (1.9)–(1.11) with $a = \log 4 + K$ and $b = 0$. Indeed, as in the previous section, one can argue that (3.20) holds for both shuttles to the leaves attached at i and at j on $\bigcap_{k \geq n_0} \bigcap_{i=1}^d P_k^{\delta, i}$. Hence one can redo the calculation (2.10), where this time the third term is replaced by (3.21), and the second one by an analogous expression. Due to Lemma 1, $\limsup_t \tilde{\Xi}'(t) < \infty$, thus $\limsup_t \tilde{\Xi}(t) \leq \limsup_t \tilde{\Xi}'(t) + K < \infty$ on $\{D_\infty \leq K\}$, almost surely. By taking K arbitrarily large we obtain (3.39).

3.3. Proof of Theorem 1. For a fixed $\varepsilon > 0$ define events

$$A(t) = A_\varepsilon(t) = \left\{ \min_{i=1, \dots, d} \frac{Z(t, i)}{t} \geq \varepsilon \text{ and } \max_{i=1, \dots, d} \frac{\sum_{j=1}^{r_i} Z(t, \ell_j^i)}{t} \leq t^{-\varepsilon} \right\}.$$

Let

$$C_\varepsilon = \left\{ \exists T : \bigcap_{t=T}^\infty A_\varepsilon(t) \text{ occurs} \right\}.$$

PROPOSITION 2. *We have $C_\varepsilon \subseteq \{\pi_\infty = \pi_{\text{unit}}\}$, almost surely.*

PROOF. The argument is effectively a copy of that for Theorem 1 in [12]. The only difference is that now the event C_ε guarantees that the events $E(k)$ defined on page 73 of [12] occur for all large enough $k \geq K$ (see [12], formula (3.1)). Observe that ε_* in the definition of $E'_2(k)$ might need to be chosen quite large, yet this does not cause difficulties in applying the argument. Indeed, ε_* does not need to satisfy [12], formulas (3.23) and (3.24), since we can skip step 5 of [12]—in the current

setting it is already covered by our estimates in previous sections, hence included in the event C_ε . Consequently (see [12], pages 73–74, for the definition of $\gamma(k)$ and k_0), we have that, whenever $k_0 \geq K$,

$$\begin{aligned}\mathbb{P}(\pi_\infty = \pi_{\text{unif}} | C_\varepsilon) &\geq \mathbb{P}(\pi_\infty = \pi_{\text{unif}} | C_\varepsilon, E(k_0)) \mathbb{P}(E(k_0) | C_\varepsilon) \\ &= \mathbb{P}(\pi_\infty = \pi_{\text{unif}} | C_\varepsilon, E(k_0)) \geq \prod_{k=k_0+1}^{\infty} (1 - \gamma(k)),\end{aligned}$$

which, since $\sum_k \gamma(k) < \infty$, can be made arbitrarily close to 1 by choosing sufficiently large k_0 . \square

PROOF OF THEOREM 1. Let

$$\xi_{ij} := \liminf_{t \rightarrow \infty} \frac{Z(t, i)}{Z(t, j)}$$

and $\tilde{C}_n = \{\min_{i,j:i \neq j} \xi_{ij} > \frac{1}{n}\}$. Proposition 1(i) implies that $\mathbb{P}(\bigcup_{n=1}^{\infty} \tilde{C}_n) = 1$, or equivalently,

$$(3.44) \quad \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{C}_n) = 1.$$

On the other hand, by part (ii) of Proposition 1 and some easy algebra, we have $\tilde{C}_n \subset C_{1/(nd)}$. The claim now follows from Proposition 2 and (3.44). \square

3.4. *Case $d = 2$.* In this section, we briefly discuss a somewhat singular case, where the number of leaves attached to the two “interior” vertices 1 and 2 influences the qualitative asymptotic behavior of the corresponding VRRW.

Namely, if $r_1 = r_2 = 0$, we have trivially (deterministically) $\pi_\infty \rightarrow \pi_{\text{unif}}$, in accordance with Theorem 1. However, if $r_1 > 0$ and $r_2 = 0$ then site 2 becomes qualitatively equal to any leaf of 1, and easy (multi-color Pólya urn) arguments show that $Z(t, 1)/t \rightarrow 1/2$, while $Z(t, 2)/t \rightarrow \alpha/2$, where α is a continuous random variable taking values in $[0, 1]$. In particular, here $\pi_\infty \not\rightarrow \pi_{\text{unif}}$. Finally, the most interesting case is when $r_1 \cdot r_2 > 0$. By combining as usual all the leaves attached to the same interior vertex into a single super-vertex, we can assume $r_1 = r_2 = 1$. Then abbreviating

$$U(t) = Z(t, 1), \quad V(t) = Z(t, 2), \quad L(t) = Z(t, \ell_1^1), \quad R(t) = Z(t, \ell_1^2),$$

one can easily check that $U(t) \asymp V(t) \asymp t$ as $t \rightarrow \infty$. Moreover, the process $L/(L + V)$ is a supermartingale when observed at times of successive visits to vertex 1. The symmetric statement holds for the process $R/(R + U)$. Due to the nonnegative supermartingale convergence, the limits

$$\xi_L := \lim_{t \rightarrow \infty} \frac{L(t)}{L(t) + V(t)}, \quad \xi_R := \lim_{t \rightarrow \infty} \frac{R(t)}{R(t) + U(t)},$$

both exists, almost surely. Comparison with the Pólya urn implies $\mathbb{P}(\xi_L = 1) = \mathbb{P}(\xi_R = 1) = 0$. Using comparison with urns featured in Theorem 3, one realizes that $\{\xi_L > 0\} \subset \{\xi_R = 0\}$, almost surely, and moreover that $R(t) = o(t^{1/a})$ for any $a \in (1, 1/\xi_L)$. The same statement holds with L and R interchanged. Clearly, $\pi_\infty \not\rightarrow \pi_{\text{unif}}$ on $\{\xi_L > 0\} \cup \{\xi_R > 0\}$.

The results of [12], Theorem 1.1, indicate that each $\{\xi_L > 0\}$ and $\{\xi_R > 0\}$ happen with positive probability; however, we do not have an argument for $\mathbb{P}(\{\xi_L > 0\} \cup \{\xi_R > 0\}) = 1$.

Using the process $\tilde{\Xi}$ from (3.38), and the reasoning analogous (but simpler to that) of Section 3.2 we obtain for $\beta > 1$

$$(3.45) \quad \{L(t) = O(t^{1/\beta})\} \subset \{\xi_R > 0\}.$$

4. Consequences for d -partite graphs with leaves. Assume $d \geq 3$. The following graph $\tilde{\mathcal{G}} \equiv \tilde{\mathcal{G}}_d = (\tilde{V}_d, \tilde{E}_d)$, featured in [12] as an example of a trapping subgraph for VRRW. It is a generalization of \mathcal{G}_d from the Introduction, where \tilde{V} is partitioned into $d + 1$ equivalence classes V_1, V_2, \dots, V_d, B . The classes V_i , $i = 1, \dots, d$ are called the *generalized vertices*, and satisfy the following two (d -partite structure) properties:

- (i) if $x, y \in V_i$, for some $i \in \{1, \dots, d\}$, then $x \not\sim y$;
- (ii) if $x \in V_i$ and $y \in V_j$ for two different $i, j \in \{1, \dots, d\}$, then $x \sim y$.

Moreover, $B = \bigcup_{i=1}^d B_i$, where B_i contains the “leaves” of V_i , $i \in \{1, \dots, d\}$,

(iii) if $x \in B$ then there exists a unique $i \in \{1, \dots, d\}$ such that $x \sim y$ for at least one $y \in V_i$.

Let X be a VRRW on $\tilde{\mathcal{G}}_d$. Then X' defined by

$$X'(t) = \begin{cases} i, & X(t) \in V_i, i = 1, \dots, d, \\ \ell_i, & X(t) \in B_i, i = 1, \dots, d, \end{cases}$$

$$Z'(t, i) := \sum_{x \in V_i} Z(t, x), \quad Z'(t, \ell_i) := \sum_{y \in B_i} Z(t, y), \quad t \geq t_0,$$

is very closely related to VRRW on graph \mathcal{G}_d with $r_1 = \dots = r_d = 1$. Namely, the only difference is that on $\{X'(t) = i\}$ (that is, on $\{X(t) \in V_i\}$) some of the weight $Z'(t, \ell_i)$ may not be accounted for when computing the probability of the step to $X'(t+1)$, since $X(t)$ may equal $x \in V_i$ that is not connected to all the leaves in B_i .

Our methodology of Sections 2 and 3 carries over to the current setting and we obtain the almost sure convergence of local time frequencies for X' to π_{unif} defined for \mathcal{G}_d . Moreover, as in Proposition 1, the leaves $\ell_1^1, \dots, \ell_d^1$ are asymptotically visited a lower power order of times compared to the interior vertices.

This translates to the following almost sure behavior of the VRRW on $\tilde{\mathcal{G}}_d$: the asymptotic proportion of time spent in V_i is $1/d$ for each $i \in \{1, \dots, d\}$, while the

number of visits to B up to time t is of the order t^α , for some random α such that $\mathbb{P}(\alpha \in (0, 1)) = 1$.

We end this discussion with the following observation. If $x, y \in V_i$, for some $i \in \{1, \dots, d\}$, then

$$(4.1) \quad \lim_{t \rightarrow \infty} \frac{Z(t, x)}{Z(t, y)} \in (0, 1) \quad \text{almost surely.}$$

Note that if $B_i = \emptyset$, (4.1) is a trivial consequence of the Pólya urn convergence (see Section 1.1). Namely, in this case the returns to class V_i can happen only from $\bigcup_{j \neq i} V_j$ and they clearly have the (multi-color) Pólya urn distribution. To see (4.1) if $B_i \neq \emptyset$, first note that as before one can use simple coupling with the urn of Theorem 4 to obtain preliminary estimates

$$(4.2) \quad \liminf_{t \rightarrow \infty} \frac{Z(t, x)}{\phi(Z(t, y))} \geq 1 \quad \forall x, y \in V_i.$$

Let $L(t) = \sum_{i=1}^d Z'(t, \ell_i)$ count the visits to all the leaves combined. Due to the observations made two paragraphs above, we have that $\mathbb{P}(\bigcup_{\beta > 1} G_\beta) = 1$, where $G_\beta := \{Z'(t, i) \rightarrow 1/d, L(t) = O(t^{1/\beta})\}$. The asymptotics of $Z'(\cdot, i)$, combined with (4.2), now imply that

$$(4.3) \quad \bigcap_{x \in V_i} \{Z(t, x) \geq \phi(t)/(2|V_i|)\} \quad \text{for all sufficiently large } t, \text{ almost surely.}$$

Assume WLOG that $X(t_0) \in \bigcup_{j \neq i} V_j$, let $\tau_0 = t_0$ and for $k \geq 1$ let $\sigma_k := \inf\{t > \sigma_{k-1} : X(t-1) \in V_i, X(t) \in \bigcup_{j \neq i} V_j\}$ be the k th time of return to $\bigcup_{j \neq i} V_j$ from the class V_i . Let

$$\begin{aligned} \tilde{Z}(t, x) &:= \tilde{Z}(t-1, x) + 1_{\{X(t-1) \in \bigcup_{j \neq i} V_j, X(t)=x\}}, \\ \tilde{Z}(t, y) &:= \tilde{Z}(t-1, y) + 1_{\{X(t-1) \in \bigcup_{j \neq i} V_j, X(t)=y\}}, \quad t \geq t_0, \end{aligned}$$

counts the visits to x and y , respectively, made from interior points *exclusively* (due to definition of \tilde{G} , these points are necessarily contained in generalized vertices different from V_i). Note that $0 \leq Z(t, x) - \tilde{Z}(t, x) \leq L(t)$, so that

$$(4.4) \quad \bigcap_{t \geq t_0} \bigcap_{x \in V_i} \left\{ \left| \frac{\tilde{Z}(t, x)}{Z(t, x)} - 1 \right| \leq \frac{L(t)}{Z(t, x)} \right\} \quad \text{almost surely.}$$

Due to (4.3), we conclude that $Z(t, x)/\tilde{Z}(t, x) \rightarrow 1$ on G_β , and by letting $\beta \searrow 1$ that $Z(t, x)/\tilde{Z}(t, x) \rightarrow 1$, almost surely. Therefore, in order to show (4.1) it suffices to prove

$$(4.5) \quad \liminf_{t \rightarrow \infty} \frac{\tilde{Z}(t, x)}{\sum_{y \in V_i} \tilde{Z}(t, y)} = \limsup_{t \rightarrow \infty} \frac{\tilde{Z}(t, x)}{\sum_{y \in V_i} \tilde{Z}(t, y)} > 0 \quad \forall x \in V_i.$$

Define an “analogue” of (3.38)

$$\tilde{\Xi}(k) := \log \left(\tilde{Z}(\sigma_k, x) + \sum_{y \in V_i \setminus \{x\}} \tilde{Z}(\sigma_k, y) \right) - \log(\tilde{Z}(\sigma_k, x) - 1), \quad k \geq 1,$$

and note that estimates (4.2)–(4.4) ensure that (on each G_β) $\tilde{\Xi}$ is a supermartingale up to a summable drift. In particular, it is converging to a finite (random) limit. This setting is quite similar to that mentioned at the very end of Section 3.4, as the estimates are simpler than those of (3.42) and (3.43) due to the following fact: there is no extra term $r(k)$ in (3.42) in the current setting, since there are no direct “shuttles” from x to y on the interval $(\sigma_k, \sigma_{k+1}]$, indirect “communication” of x and y via a common leaf is atypical—its occurrence is accounted for by the differences $Z(t, x) - \tilde{Z}(t, x)$, $Z(t, y) - \tilde{Z}(t, y)$, that are both bounded by $L(t)$. Letting $\beta \searrow 1$ establishes (4.5). Let $Z_m(t)$ count the number of visits to site m up to time t for VRRW on five (or fewer, at least three) points $\{-2, -1, 0, 1, 2\}$. Then the process $(Z(t, x), Z(t, y))$ can be closely matched (coupled) to the process $(Z_{-1}(t), Z_1(t))$ on the event $\{Z_{-2}(t) = O(t^{1/\beta_1}), Z_2(t) = O(t^{1/\beta_2})\}$, where β_1, β_2 are two random quantities strictly greater than 1. The “middle point” 0 corresponds to $\bigcup_{j \neq i} (V_j \cup B_j)$, while the “boundary” -2 (resp., 2) corresponds to the set of leaves in B_i connected to x (resp., y). Recall once again the process $\tilde{\Xi}$ from (3.38) and note that we are in the situation of type (3.45) where $\tilde{\Xi}$ will be a supermartingale up to a summable drift, and, moreover, where $\tilde{Z}_{-1}(t)/Z_{-1}(t) \rightarrow 1$ and $\tilde{Z}_1(t)/Z_1(t) \rightarrow 1$. This implies that $\lim_{t \rightarrow \infty} \frac{Z_{-1}(t)}{Z_1(t)} \in (0, 1)$, almost surely, hence (4.1).

5. Speed of convergence. We first show a preliminary statement, which can be viewed as a refinement of Proposition 3.2, page 80 in [12].

LEMMA 8. *Suppose that we are given a sequence $(\eta_k)_{k \geq 1}$ such that for some $\varepsilon > 0$ we have*

$$(5.1) \quad 0 \leq \eta_k \leq 1 - \varepsilon \quad \text{and} \quad \eta_{k+1} \leq \eta_k \left[1 - \frac{C(1 - \eta_k)}{k} \right] + \frac{D}{k^{1+\tilde{\beta}}} \quad \forall k \geq k_0,$$

where $C > 0$, $D > 0$, and $\tilde{\beta} \in [0, 1]$. Then $\limsup_{k \rightarrow \infty} \eta_k h(k) < \infty$, where

$$h(k) = \begin{cases} k^{\tilde{\beta}}, & \text{if } \tilde{\beta} < C, \\ k^{\tilde{\beta}} / \log k, & \text{if } \tilde{\beta} = C, \\ k^C, & \text{if } \tilde{\beta} > C. \end{cases}$$

PROOF. First of all, let us show that $\eta_k \rightarrow 0$. Indeed, fix a positive $\tilde{\varepsilon} < \min\{C\varepsilon, \tilde{\beta}\}$, and suppose that

$$(5.2) \quad \eta_k \leq \frac{A}{k^{\tilde{\varepsilon}}}$$

for some $A > 0$. Then

$$\begin{aligned}\eta_{k+1} &\leq \frac{A}{k^{\tilde{\varepsilon}}} \left(1 - \frac{C\varepsilon}{k}\right) + \frac{D}{k^{1+\tilde{\beta}}} \\ &= \frac{A}{(k+1)^{\tilde{\varepsilon}}} - \frac{A(C\varepsilon - \tilde{\varepsilon}) - Dk^{\tilde{\varepsilon}-\tilde{\beta}} - \Theta(k^{-1})}{k^{1+\tilde{\varepsilon}}} \leq \frac{A}{(k+1)^{\tilde{\varepsilon}}},\end{aligned}$$

provided A and k are sufficiently large. We obtain by induction that (5.2) holds for all large k . Therefore, one can, in fact, assume that ε in (5.1) is arbitrarily close to 1. Hence, if $\tilde{\beta} < C$, we can set $\tilde{\varepsilon} = \tilde{\beta}$ and, assuming that $\varepsilon \in (0, 1)$ is sufficiently large so that $C\varepsilon > \tilde{\varepsilon}$, we obtain (5.2) for any A larger than $D/(C\varepsilon - \tilde{\varepsilon}) = D/(C\varepsilon - \tilde{\beta})$. This implies the claim of the lemma in the case $\tilde{\beta} < C$.

From now on assume $\tilde{\beta} \geq C$. The above arguments imply that for $\tilde{\varepsilon} = 2C/3$, we have $\eta_k \leq Ak^{-\tilde{\varepsilon}}$, for all large k and some $A < \infty$, hence

$$\eta_{k+1} \leq \eta_k \left[1 - \frac{C}{k}\right] + \frac{C\eta_k^2}{k} + \frac{D}{k^{1+\tilde{\beta}}} \leq \eta_k \left[1 - \frac{C}{k}\right] + \frac{\bar{D}}{k^{1+\tilde{\beta}}},$$

where $\bar{\beta} = \min\{\tilde{\beta}, 4C/3\}$ and $\bar{D} = D + A^2C$. If

$$\mu_k = \eta_k k^C,$$

then the last estimate together with Taylor's expansion of $(k+1)^C$ about k yields

$$\begin{aligned}\mu_{k+1} &\leq \frac{\mu_k(k+1)^C}{k^C} \left[1 - \frac{C}{k}\right] + \frac{\bar{D}(1 + \Theta(1/k))}{k^{1+\tilde{\beta}-C}} \\ &\leq \mu_k \left[1 - \frac{C(1+C)}{2k^2} + \Theta(k^{-3})\right] + \frac{2\bar{D}}{k^{1+\tilde{\beta}-C}}.\end{aligned}$$

By summing over k , this immediately implies $\limsup_k \mu_k < \infty$ if $\tilde{\beta} > C$ (that is, $1 + \tilde{\beta} - C > 1$) and $\limsup_k \mu_k / \log k < \infty$ if $\tilde{\beta} = C$, finishing the proof of the lemma. \square

PROOF OF THEOREM 2. Denote by

$$\eta(t) := 1 - d \min_{j=1,\dots,d} \frac{Z(t, j)}{t} \in [0, 1]$$

another measure of distance between the empirical occupation measure $\pi(t) = (Z(t, 1)/t, \dots, Z(t, d)/t)$ and $\pi_{\text{unif}} = (1/d, \dots, 1/d)$. Due to Theorem 1, we have $\sum_j \pi_j(t) = 1 - o(1)$, so $\eta(t)/d \leq \|\pi(t) - \pi_{\text{unif}}\| (1 + o(1)) \leq \eta(t)$. Thus it suffices to study the asymptotic behavior of $\eta(t)$.

Fix some constants $m > 1$ and $\beta \in (0, (m-1)/2)$, and let $\nu = \frac{m-1}{2} - \beta > 0$. Now consider VRRW at times $t_k = k^m$, set $N_k = t_{k+1} - t_k$ and $\alpha_j^{(k)} = Z(t_k, j)/t_k$,

$j \in \{1, \dots, d\}$, $k \in \mathbb{N}$ (here we use notations similar to those in the proof of Theorem 1 in [12]; also in order to simplify expressions we will often omit the superscript $^{(k)}$ on α 's). Define events

$$D_t(\varepsilon) := \bigcap_{i=1}^d \left\{ \frac{Z(t, i)}{t} \in \left(\frac{1}{d} - \varepsilon, \frac{1}{d} + \varepsilon \right) \right\}, \quad t \geq t_0,$$

and note that Theorem 1 can be rephrased as

$$(5.3) \quad P\left(\forall \varepsilon \in (0, 1/d) \text{ there is } K = K(\varepsilon) < \infty \text{ s.t. } \bigcap_{k \geq K} D_k(\varepsilon) \text{ occurs}\right) = 1.$$

Fix some small positive $\varepsilon < 1/d$. Due to (5.3) we can assume from now on that $\min_j \alpha_j^{(k)} \geq \varepsilon$.

It is simple to check that if we were to “freeze” the configuration at time t_k , ignore the visits to the leaves and let the VRRW evolve as a Markov chain on state space $\{1, \dots, d\}$ with transition probabilities specified by the weights $(\alpha_j^{(k)})_{j=1}^d$ [or equivalently, by $(Z(t_k, j))_{j=1}^d$], then this Markov chain would have its reversible measure proportional to $(\alpha_1^{(k)}(1 - \alpha_1^{(k)}), \dots, \alpha_d^{(k)}(1 - \alpha_d^{(k)}))$. As in the proof of [12], Theorem 1, one uses the large deviation estimates (1.6) and (1.7) to see that the number $N_{k,i}$ of visits to vertex i during $[t_k, t_{k+1})$ concentrates about its “almost” expected value (i.e., the expectation according to the above frozen measure)

$$(5.4) \quad \frac{\alpha_i(1 - \alpha_i)}{\sum_{j=1}^d \alpha_j(1 - \alpha_j)} \times N_k = \frac{\alpha_i(1 - \alpha_i)}{1 - \sum_{j=1}^d \alpha_j^2} \times N_k.$$

More precisely, let

$$(5.5) \quad E_k = \{\text{simultaneously for all } i \in \{1, \dots, d\}, \text{ the quantity } N_{k,i} \text{ does not differ from (5.4) by more than } k^{(m-1)/2+\nu} \asymp k^\nu \sqrt{N_k}\}.$$

Then (see [12], display (3.16), page 76),

$$P(E_k^c) \leq \gamma'_k := \text{Const}_1(d) \exp(-\text{Const}_2(\varepsilon, d)k^{2\nu}),$$

so we have $\sum_k \gamma'_k < \infty$. Therefore only finitely many E_k^c occur. Consequently, a.s. there is a $k_0 = k_0(\omega)$ such that $\bigcap_{k \geq k_0} E_k$ occurs. From now on, we will implicitly assume that $k \geq k_0$.

We next recall that VRRW may also visit the leaves between times t_k and t_{k+1} . We already know from Proposition 1 that $\max_i \sum_{j=1}^{r_i} Z(t, \ell_j^i) \leq t^{1-\varepsilon'}$ for some $\varepsilon' > 0$. Let us now strengthen this statement.

LEMMA 9. *Let $L(t, i) := \sum_{j=1}^{r_i} Z(t, \ell_j^i)$ be the total cumulative weight of all the leaves attached to i at time t , where $i \in \{1, \dots, d\}$. Then, if $r_i > 0$, for any $\delta > 0$ we have*

$$(5.6) \quad \mathbb{P}\left(\liminf_{t \rightarrow \infty} \frac{L(t, i)}{t^{1/(d-1)-\delta}} = \infty\right) = 1$$

and (trivially if $r_i = 0$)

$$(5.7) \quad \mathbb{P}\left(\limsup_{t \rightarrow \infty} \frac{L(t, i)}{t^{1/(d-1)+\delta}} = 0\right) = 1.$$

PROOF. We will prove only the first part of the statement, since the second one follows by an analogous argument.

As usual, let $\tau_k^{(i)}$ be the k th return time to the interior vertex i . Define $X'_k := \sum_{g \neq i} Z(\tau_k^{(i)}, g)$ and $Y'_k := L(\tau_k^{(i)}, i)$. Due to Theorem 1 and some simple algebra, the statement of the lemma is equivalent to the following claim: for any $\delta > 0$ we have

$$\limsup_{k \rightarrow \infty} \frac{X'_k}{(Y'_k)^{d-1+\delta}} = 0 \quad \text{almost surely.}$$

Recall (5.3). Without loss of generality we observe the process $(X', Y') := ((X'_k, Y'_k), k \geq k_1)$, where $\tau_{k_1}^{(i)} \geq K$ for some large finite K . In the spirit of Remark 2, we will modify the VRRW and in this way the process (X', Y') (note, however, that here the construction is slightly more complicated since we cannot simply “truncate” the process upon exiting the event of “good behavior”). Fix some small $\varepsilon > 0$, and define

$$D'_t(\varepsilon) := \bigcap_{i=1}^d \left\{ \frac{Z(t, i)}{\sum_{j=1}^d Z(t, j)} \in \left(\frac{1}{d} - \varepsilon, \frac{1}{d} + \varepsilon \right) \right\}, \quad t \geq t_0.$$

Due to (5.3) and Proposition 1(ii) we have that

$$(5.8) \quad P\left(\bigcap_{k \geq K} D'_k(\varepsilon)\right) \rightarrow 1 \quad \text{as } K \rightarrow \infty.$$

Define

$$T_\varepsilon(K) \equiv T_\varepsilon := \inf\{l > K : D'_l(\varepsilon) \text{ does not occur}\}.$$

If $K > 2/\varepsilon$, it is easy to see that $D'_{l-1}(\varepsilon) \subset D'_l(3\varepsilon/2)$ for $l \geq K$, so

$$(5.9) \quad \{T_\varepsilon < \infty\} \subset \bigcap_{K \leq l \leq T_\varepsilon} D'_l(3\varepsilon/2) \quad \text{almost surely.}$$

Change the dynamics of the VRRW in the following way [recall (1.1)]:

$$(5.10) \quad \begin{aligned} & \mathbb{P}(X(t+1) = w | \mathcal{F}_t) \\ &= \frac{Z(T_\varepsilon \wedge t, w)}{\sum_{y \in \{1, \dots, d, \ell_1^i, \dots, \ell_{r_i}^i\}: y \sim v} Z(T_\varepsilon \wedge t, y)} 1_{\{w \in \{1, \dots, d, \ell_1^i, \dots, \ell_{r_i}^i\}\}}. \end{aligned}$$

In words, after time T_ε the step distribution does not anymore change dynamically with the evolution of the walk; instead it is “frozen” to the configuration

$$(Z(T_\varepsilon, 1), \dots, Z(T_\varepsilon, d), Z(T_\varepsilon, \ell_1^1), \dots, Z(T_\varepsilon, \ell_{r_d}^d)),$$

and additional visits to the leaves attached at g where $g \neq i$ become impossible. Let

$$\sigma_\varepsilon := \inf\{k \geq k_1 : T_\varepsilon \leq \tau_k^{(i)}\},$$

and assume that we are given a family $\{U_k, k \geq k_1\}$ of independent uniform $[0, 1]$ random variables, and independent of the evolution of the VRRW above. Then define a modification $(\tilde{X}'_k, \tilde{Y}'_k)$ of (X', Y') by $(\tilde{X}'_{k_1}, \tilde{Y}'_{k_1}) = (X'_{k_1}, Y'_{k_1})$ and

$$(5.11) \quad (\Delta \tilde{X}'_k, \Delta \tilde{Y}'_k) := \begin{cases} (\Delta X'_k, \Delta Y'_k), & k < \sigma_\varepsilon, \\ (d-1, 0), & U_k \leq \tilde{X}'_k / (\tilde{X}'_k + \tilde{Y}'_k), \quad k \geq \sigma_\varepsilon, \\ (0, 1), & U_k > \tilde{X}'_k / (\tilde{X}'_k + \tilde{Y}'_k), \quad k \geq \sigma_\varepsilon. \end{cases}$$

In words, the evolution of (\tilde{X}', \tilde{Y}') is identical to that of (X', Y') up to time σ_ε , while (\tilde{X}', \tilde{Y}') evolves as the urn from Theorem 3 from time σ_ε onwards. In particular, the asymptotic behavior of (X', Y') and (\tilde{X}', \tilde{Y}') is the same on $\{T_\varepsilon = \infty\} = \bigcap_{l \geq K} D'_l(\varepsilon) \subset \{\sigma_\varepsilon = \infty\}$.

The point of the above construction is that (\tilde{X}', \tilde{Y}') satisfies the hypotheses of [8], Lemma 3.5, with

$$a = 1, \quad b = b(\varepsilon) = \frac{d-1+3\varepsilon d(d-3)/2}{1-3\varepsilon d/2} \quad \text{and} \\ K = K(\varepsilon) = 2 \left(\frac{d-1+3\varepsilon d(d-3)/2}{1-3\varepsilon d/2} \right)^2.$$

Indeed, suppose $k < \sigma_\varepsilon$ (otherwise the argument is trivial) and note that then with probability $Y'_k / (X'_k + Y'_k) = \tilde{Y}'_k / (\tilde{X}'_k + \tilde{Y}'_k)$ we have $X(\tau_k^{(i)} + 1) \in \{\ell_1^i, \dots, \ell_{r_i}^i\}$, so that $(\Delta \tilde{X}'_k, \Delta \tilde{Y}'_k) = (0, 1)$, while with the remaining probability $(\Delta \tilde{X}'_k, \Delta \tilde{Y}'_k) = (W_k, 0)$ where $P(W_k \geq 1) = 1$ and conditionally on $\mathcal{F}_{\tau_k^{(i)}}$, W_k is stochastically bounded from above by a Geometric random variable with success probability $(1-3\varepsilon d/2)/(d-1+3\varepsilon d(d-3)/2)$. Here we use the definition of the modified dynamics (5.10) and (5.11) together with the fact (5.9).

Due to [8], Lemma 3.5, $(\tilde{X}'_k / (\tilde{Y}'_k)^{b'}, k \geq k_1)$ is a positive supermartingale for any $b' > b(\varepsilon)$, hence converging, and its limit must be 0, almost surely (strictly speaking, the supermartingale property holds once \tilde{Y}'_{k_1} is larger than some fixed constant, but this we can assume WLOG). Note that for any δ one can choose $\varepsilon > 0$ sufficiently small so that $d-1+\delta > b(\varepsilon)$. Since $X'/(Y')^{b'}$ and $\tilde{X}'/(\tilde{Y}')^{b'}$ behave identically on $\{T_\varepsilon = \infty\} = \bigcap_{l \geq K} D'_l(\varepsilon)$, the statement of the lemma follows immediately from (5.8). \square

Now suppose that $\sum_{i=1}^d r_i > 0$, and denote by $\theta_k := \sum_{i=1}^d L(t_k, i)/t_k > 0$ the total (rescaled) weight of the leaves. Due to Lemma 9, we have $\sum_{i=1}^d L(t_k, i) =$

$o(t_k^{1/(d-1)+\delta})$, hence

$$(5.12) \quad \sum_{j=1}^d \alpha_j^{(k)} = 1 - \theta_k \quad \text{where } \theta_k = o(k^{-m[(d-2)/(d-1)-\delta]}).$$

Moreover, due to Lemma 9, we have $t_k^{1/(d-1)-\delta} = o(\sum_{i=1}^d L(t_k, i))$, therefore $t_k^{-(d-2)/(d-1)-\delta} = o(\theta_k)$

$$\begin{aligned} \|\pi(t) - \pi_{\text{unif}}\| &\geq \sum_{i=1}^d \left| \frac{Z(t_k, i)}{t_k} - \frac{1}{d} \right| \geq \left| \sum_{i=1}^d \frac{Z(t_k, i)}{t_k} - 1 \right| \\ &= \sum_{i=1}^d \frac{L(t_k, i)}{t_k} \gg t_k^{-(d-2)/(d-1)-\delta}, \quad \text{as } k \rightarrow \infty, \end{aligned}$$

yielding the lower bound claim (1.4) in Theorem 2.

We continue toward the proof of (1.2) and (1.3). Set

$$\eta_k := \eta(t_k) = 1 - d \min_{j=1, \dots, d} \alpha_j^{(k)} \geq 0,$$

and let

$$(5.13) \quad \tilde{\beta} = \min \left\{ \beta, 1, m \left(\frac{d-2}{d-1} - \delta \right) \right\},$$

where $\delta > 0$ is very small.

The following statement is a refinement of (3.28) in [12].

LEMMA 10. *On the event E_k defined by (5.5) we have*

$$(5.14) \quad \eta_{k+1} = \eta_k \left(1 - \frac{mr(1-\eta_k)}{k} \right) + \Theta \left(\frac{1}{k^{1+\tilde{\beta}}} \right),$$

where $r = r(k, \alpha^{(k)}) \in [1/(d-1), 1/(1-\eta_k)]$.

PROOF. Due to (5.12) we have

$$1 - \sum_{j=1}^d \alpha_j^2 \leq 1 - \frac{(\sum_{j=1}^d \alpha_j)^2}{d} \leq \left(1 - \frac{1}{d} \right) + \frac{2\theta_k}{d}.$$

Moreover, Theorem 1 implies in particular that $\mathbb{P}(\cap_{k \geq k_0} \{\max_{i=1}^d \alpha_i^{(k)} < 1/2\}) \rightarrow 1$ as $k_0 \rightarrow \infty$ (recall that $d \geq 3$). Since $x \mapsto x(1-x)$ is an increasing function on $[0, 1/2]$, we conclude that asymptotically

$$\begin{aligned} 1 - \sum_{j=1}^d \alpha_j^2 &= \sum_{j=1}^d \alpha_j(1-\alpha_j) + \theta_k \geq d \times \frac{1-\eta_k}{d} \left(1 - \frac{1-\eta_k}{d} \right) + \theta_k \\ &= \left(1 - \frac{1}{d} \right) - \left(1 - \frac{2-\eta_k}{d} \right) \eta_k + \theta_k. \end{aligned}$$

Thus we have shown

$$(5.15) \quad 1 - \sum_{j=1}^d \alpha_j^2 = \left(1 - \frac{1}{d}\right) - \frac{d-2}{d} \gamma \eta_k + o\left(\frac{1}{k^{m((d-2)/(d-1)-\delta)}}\right)$$

where $\gamma \in [0, 1 + \eta_k/(d-2)]$.

Note that

$$\alpha_i^{(k+1)} = \frac{\alpha_i k^m + N_k \alpha_i (1 - \alpha_i) / (1 - \sum_j \alpha_j^2) + O(k^{(m-1)/2+\nu})}{(k+1)^m},$$

where the $O(\cdot)$ term comes from the estimation of the event (5.5). Thus

$$\begin{aligned} \alpha_i^{(k+1)} &= \alpha_i \left[1 - \frac{m}{k} + \frac{m}{k} \frac{1 - \alpha_i}{1 - \sum \alpha_j^2} \right] + O\left(\frac{1}{k^{1+\tilde{\beta}}}\right) + \Theta\left(\frac{1}{k^2}\right) \\ &= \alpha_i \left[1 + \frac{m}{k} \left(\frac{1 - \alpha_i}{1 - \sum \alpha_j^2} - 1 \right) \right] + \Theta\left(\frac{1}{k^{1+\tilde{\beta}}}\right). \end{aligned}$$

Since the last expression (without the Θ part) is increasing in α_i for all sufficiently large k , it implies that if $\alpha_i^{(k)} = \min_{j=1}^d \alpha_j^{(k)}$, then $\alpha_i^{(k+1)}$ will again equal $\min_{j=1}^d \alpha_j^{(k+1)}$, unless it is “overtaken” by $\alpha_j^{(k+1)}$ for some other index j . The latter case can happen only if the difference $|\alpha_j^{(k)} - \alpha_i^{(k)}|$ is itself $O(\frac{1}{k^{1+\tilde{\beta}}})$. Hence it is always true that

$$\min_{j=1}^d \alpha_j^{(k+1)} = \min_{i=1}^d \alpha_i^{(k)} \left[1 + \frac{m}{k} \left(\frac{1 - (\min_{i=1}^d \alpha_i^{(k)})}{1 - \sum \alpha_j^2} - 1 \right) \right] + O\left(\frac{1}{k^{1+\tilde{\beta}}}\right).$$

This yields in turn

$$\begin{aligned} \eta_{k+1} &= 1 - d \left(\frac{1 - \eta_k}{d} \left[1 + \frac{m}{k} \left(\frac{1 - (1 - \eta_k)/d}{1 - \sum \alpha_j^2} - 1 \right) \right] + O\left(\frac{1}{k^{1+\tilde{\beta}}}\right) \right) \\ &= 1 - (1 - \eta_k) \left[1 + \frac{m}{k} \left(\frac{d - 1 + \eta_k}{d - 1 - (d-2)\gamma\eta_k} - 1 \right) \right] + O\left(\frac{1}{k^{1+\tilde{\beta}}}\right) \\ &= \eta_k \left(1 - \frac{m(1 - \eta_k)}{k} \times \frac{1 + \gamma(d-2)}{d - 1 - (d-2)\gamma\eta_k} \right) + O\left(\frac{1}{k^{1+\tilde{\beta}}}\right), \end{aligned}$$

where for the second equality we used (5.15). Since

$$\frac{d - 1 + \eta_k}{d - 1 - (d-2)\eta_k - \eta_k^2} < \frac{1}{1 - \eta_k},$$

we get

$$\eta_{k+1} = \eta_k \left(1 - \frac{m(1 - \eta_k)r}{k} \right) + O\left(\frac{1}{k^{1+\tilde{\beta}}}\right),$$

where $1/(d-1) \leq r \leq (1-\eta_k)^{-1}$. \square

Recalling once again fact (5.3) we can assume that for $\varepsilon = 1 - 2/d > 0$ we have $\eta_k \leq 1 - \varepsilon$, for all large k . This enables us applying Lemma 8 with $C = m/(d-1)$. Note that to get the best estimate of the speed of convergence we need to make $p(d, m) := \min\{C, \tilde{\beta}\}/m$ as large as possible, since

$$\limsup_{k \rightarrow \infty} \eta_k h(k) = \limsup_{k \rightarrow \infty} \eta(k^m) h(k) < \infty$$

for an increasing function $h(\cdot)$ a.s. implies

$$\limsup_{t \rightarrow \infty} \eta(t) h(t^{1/m}) < \infty.$$

On the other hand, recalling the definition of $\tilde{\beta}$ from (5.13), we have

$$\begin{aligned} p(d, m) &= \min\left\{\frac{1}{d-1}, \frac{1}{m}, \frac{\beta}{m}, \frac{d-2}{d-1} - \delta\right\} \\ &= \min\left\{\frac{1}{d-1}, \frac{1}{m}, \frac{1}{2} - \frac{\delta_1 + 1/2}{m}, \frac{d-2}{d-1} - \delta\right\}. \end{aligned}$$

We can make β as close as possible to $(m-1)/2$ by recalling $\beta = (m-1)/2 - \delta_1$, and taking $\delta_1 > 0$ arbitrarily small. Similarly, $\delta > 0$ can be made very small. Given a particular choice of δ, δ_1 , observe that $\max_{m>1} p(d, m)$ is achieved at $3 + 2\delta_1$, so by setting $m = 3 + 2\delta_1$ we obtain

$$\begin{aligned} p(d) &:= p(d, 3 + 2\delta_1) = \min\left\{\frac{1}{d-1}, \frac{1}{3 + 2\delta_1}, \frac{d-2}{d-1} - \delta\right\} \\ &= \min\left\{\frac{1}{d-1}, \frac{1}{3 + 2\delta_1}, \frac{1}{d-1} + \left[\frac{d-3}{d-1} - \delta\right]\right\} \\ &= \min\left\{\frac{1}{d-1}, \frac{1}{3 + 2\delta_1}\right\}. \end{aligned}$$

Consequently, $p(d)$ can be taken arbitrarily close to $1/3$ if $d \in \{3, 4\}$, while $p(d) = 1/(d-1)$ for $d \geq 5$. Setting $C = 3/(d-1)$ yields $\tilde{\beta} = \min\{1 - \delta_1, 1\} < C$ if $d \in \{3, 4\}$ and $\tilde{\beta} > C$ if $d \geq 5$. As already argued, this implies $\limsup \eta(t)t^{p(d)} < \infty$ due to Lemma 8, and completes the proof of Theorem 2. \square

REMARK 4. There is a gap in the power between the upper and lower bounds on speed of convergence in Theorem 2. One might wish to obtain further information on the lower bound using (5.14). In fact, we would be able to conclude something provided

$$\eta_{k+1} \geq \eta_k \left(1 - \frac{C(1-\eta_k)}{k}\right) + \frac{D}{k^{1+\tilde{\beta}}},$$

where both C and D are positive (or for D negative, under more complicated constraints on $C > 0$ and β that seem difficult to verify). Therefore, it is the lack of knowledge of the sign (and magnitude) of the error term in (5.14) that obstructs generalizing the above argument to obtaining lower bound estimate.

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