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An upper bound for front propagation velocities inside moving populations

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Abstract. We consider a two-type (red and blue or R and B) particle population that evolves on the d-dimensional lattice according to some reaction-diffusion process $R + B \to 2R$ and starts with a single red particle and a density ρ of blue particles. For two classes of models we give an upper bound on the propagation velocity of the red particles front with explicit dependence on ρ.

In the first class of models red particles evolve with a diffusion constant $D_R = 1$. Blue particles evolve with a possibly time-dependent jump rate $D_B \geq 0$, or, more generally, follow independent copies of some bistochastic process. Examples of bistochastic process also include long-range random walks with drift and various deterministic processes. For this class of models we get in all dimensions an upper bound of order $\max(\rho, \sqrt{\rho})$ that depends only on ρ and d and not on the specific process followed by blue particles, in particular that does not depend on D_B. We argue that for $d \geq 2$ or $\rho \geq 1$ this bound can be optimal (in ρ), while for the simplest case with $d = 1$ and $\rho < 1$ known as the frog model, we give a better bound of order ρ.

In the second class of models particles evolve according to Kawasaki dynamics, that is, with exclusion and possibly attraction, inside a large two-dimensional box with periodic boundary conditions (this turns into simple exclusion when the attraction is set to zero). In a low density regime we then get an upper bound of order $\sqrt{\rho}$. This proves a long-range decorrelation of dynamical events in this low density regime.

1 Models and results

1.1 A diffusion-reaction model

In [6] Kesten and Sidoravicius considered the following Markov process. A countable number of red and blue particles perform independent continuous-time simple random walks on the d-dimensional lattice $\mathbb{Z}^d$. Red particles jump at rate D_R and blue particles jump at rate D_B. When a blue particle jumps on a site occupied by a red particle, the blue particle turns red. When a red particle jumps on a site occupied by blue particles these turn red. Thinking respectively of the red and blue

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particles as individuals who have heard about a certain rumor and are ignorant of it—or as individuals who have and have not a certain contagious disease—this Markov process provides a model of rumor propagation—or epidemic diffusion—inside a moving population. This is also a reaction-diffusion dynamics of the kind $R + B \to 2R$ that can model a combustion process.

We define at each time $t \geq 0$ a *red zone* $\mathcal{R}(t)$, by the set of sites $\mathbb{Z}^d$ that have been reached by some red particle at some time $s \in [0, t]$. At any time $t \geq 0$ all the red particles stand in the red zone, but some blue particles can stand in the red zone and the red zone can contain empty sites. The red zone is the set of the sites reached by the rumor or the set of burnt sites according to one or another interpretation of the process.

Let us assume that the initial configuration was built in the following way. We put independently at each site $z \in \mathbb{Z}^d$ a random number of blue particles according to Poisson variables of mean $\rho > 0$. Then, at time $t = 0$, we choose one particle according to some probabilistic or deterministic rule, we turn it red as well as we turn red the possible other particles that stood in the same site. Then, denoting by $B(z, r)$ the Euclidean ball of center z and radius r and making a change of origin to have $\mathcal{R}(0) = \{0\}$, Kesten and Sidoravicius proved [6].

Theorem (Kesten–Sidoravicius). *If $D_B = D_R > 0$ there are two positive and finite constants $C_1 < C_2$ such that with probability* 1

$$B(0, C_1 t) \subset \mathcal{R}(t) \subset B(0, C_2 t) \tag{1.1}$$

holds for all t larger than some finite random time T_0.

If $D_R > 0$ there is a finite constant C_2 such that with probability 1

$$\mathcal{R}(t) \subset B(0, C_2 t) \tag{1.2}$$

holds for all t larger than some finite random time T_0.

Remarks. (i) Actually, no change of origin was introduced in [6]. The analogous result without change of origin is an equivalent statement, but our change of origin will be useful later.

(ii) Kesten and Sidoravicius proved the theorem in a slightly more general situation than the one described above. Instead of allowing to add red particles at a single site, they consider initial distributions obtained by adding any finite number of red particles at a finite set of sites. However, it is easy to see that the same result in this more general case is equivalent to the previous theorem. For the sake of simplicity we will restrict ourselves to discuss processes where at time $t = 0$ red particles are added at a single site. We start our discussion with the case of a single blue particle that turns red.

(iii) The inclusion (1.2) gives a “ballistic upper bound” on $\mathcal{R}(t)$. The “ballistic lower bound” expressed in (1.1) is much harder to prove and was obtained only in the special cases $D_B = D_R > 0$ (in [6]) and $D_B = 0$ (in [1,2,8]). But it is believed that such a bound holds in the general case $D_R > 0$ (in [7]).

(iv) From a ballistic upper *and* lower bound on $\mathcal{R}(t)$ like in (1.1), Kesten and Sidoravicius deduced a “shape theorem” for the red zone: $\mathcal{R}(t)/t$ converges with probability 1 to a deterministic shape. This proves the existence of a (maybe nonisotropic) propagation velocity of the rumor or the combustion front. In this context C_1 and C_2 are respectively uniform lower and upper bounds of this possibly nonisotropic front propagation velocity.

(v) In [7] it is conjectured that in the general case $D_R > 0$ this propagation velocity does not depend on D_B (see [7], note 38).

In this paper we give an upper bound on the propagation velocity, that is, a ballistic upper bound on $\mathcal{R}(t)$ of the kind (1.2) with explicit dependence of C_2 on the density ρ and no dependence on D_B. This bound will be, in all dimensions, of order $\max(\rho, \sqrt{\rho})$. We argue that for $d \geq 2$ or $\rho \geq 1$ this bound can be optimal (in ρ), while for $d = 1$ and $\rho < 1$, we give in the simplest case $D_B = 0$ a better bound of order ρ. In addition we prove that our upper bound in $\max(\rho, \sqrt{\rho})$ holds for a larger class of models. We prove it, on the one hand, for those models in which red particles perform independent random walks while blue particles follow independent copies of any kind of bistochastic process (see below). On the other hand, we give an analogous upper bound for models in which the rumor diffuses through a “contact process” inside an interacting particle system with exclusion and possible attraction (simple exclusion, Kawasaki dynamics) when a low density limit allows for a Quasi Random Walk (QRW) approximation as introduced in [4].

1.2 One upper bound for many models

We now define the first class of models we will work with. Like previously, we start with a density $\rho > 0$ of particles, putting independently in each site $z \in \mathbb{Z}^d$ a Poissonian number of particles with mean ρ. We then put labels 1, 2, 3, … on particles. We call z_i the position of the particle i and for all $t > 0$ we will call $X_i(t) \in \mathbb{Z}^d$ and $Y_i(t) \in \{R, B\}$ its position and its color at time t. With each i we associate two continuous-time Markov processes on $\mathbb{Z}^d$, denoted Z_i^R and Z_i^B, in such a way that:

- each of these processes start at 0, and are independent;
- Z_i^R is a simple random walk process with diffusion constant or jump rate 1;
- Z_i^B is a bistochastic process, that is, satisfies

$$\forall z \in \mathbb{Z}^d, \forall t \geq 0, \qquad \sum_{z_0 \in \mathbb{Z}^d} P\big(z_0 + Z_i^B(t) = z\big) = 1. \tag{1.3}$$

  This includes simple random walks with constant or time dependent jumps rates, long-range random walks with drift, various deterministic processes; …
- The Z_i^B’s (like the Z_i^R’s) have the same law.

At time $t = 0$ we choose one particle i_0 with some probabilistic or deterministic rule, we shift the origin to i_0, we give the red color to the particles in the new origin and the blue color to the other particles so that, for all i,

$$X_i(0) = z_i - z_{i_0}, \tag{1.4}$$

$$Y_i(0) = R \qquad \text{if } X_i(0) = 0, \tag{1.5}$$

$$Y_i(0) = B \qquad \text{if } X_i(0) \neq 0. \tag{1.6}$$

Then, each particle i follows the moves of Z_i^B while $Y_i = B$, turns red when it meets a red particle and then follows the moves of Z_i^R. More formally, for all i, we define the time when blue and red particle meet by

$$\tau_i := \begin{cases} 0, & \text{if } X_i(0) = 0, \\ \inf\{t \geq 0 : Y_i(t_-) = B, \exists j \neq i, & \\ \qquad Y_j(t_-) = R, X_i(t) = X_j(t)\}, & \text{if } X_i(0) \neq 0, \end{cases} \tag{1.7}$$

with the usual convention $\inf \varnothing = +\infty$. That implies

$$X_i(t) = \begin{cases} X_i(0) + Z_i^B(t), & \text{if } t \leq \tau_i, \\ X_i(0) + Z_i^B(\tau_i) + Z_i^R(t - \tau_i), & \text{if } t > \tau_i, \end{cases} \tag{1.8}$$

$$Y_i(t) = \begin{cases} B, & \text{if } t < \tau_i, \\ R, & \text{if } t \geq \tau_i. \end{cases} \tag{1.9}$$

We will call process of type RB any process that can be built in this way. The Kesten and Sidoravicius reaction-diffusion model is a process of type RB when $D_R = 1$. We will call it KS process. The general case $D_R > 0$ can be mapped on the KS process by a simple time rescaling.

Setting, like previously, for all $t \geq 0$,

$$\mathcal{R}(t) := \{z \in \mathbb{Z}^d : \exists i \geq 1, \exists s \in [0, t], (X_i, Y_i)(s) = (z, R)\}, \tag{1.10}$$

we will prove:

Theorem 1. *There is a positive constant δ_d that depends only on d and such that, for any RB process and for all $t \geq 0$*

$$P\left(\exists z \in \mathcal{R}(t) \setminus B\left(0, \frac{\bar{\rho} t}{\delta_d}\right)\right) \leq \frac{\bar{\rho}^2 e^{-\delta_d \rho t}}{\delta_d \rho^5} \tag{1.11}$$

with

$$\bar{\rho} := \max(\rho, \sqrt{\rho}). \tag{1.12}$$

As a consequence, using the Borel–Cantelli lemma we get:

Corollary 1.1. *There is a positive constant δ_d that depends only on d and such that for any RB process, with probability* 1

$$\mathcal{R}(t) \subset B\left(0, \frac{\max(\rho, \sqrt{\rho})t}{\delta_d}\right) \tag{1.13}$$

holds for all t larger than some finite random time T_0.

We will give an analogous result for a second class of models. In dimension $d = 2$ we consider a low-density lattice gas, with density ρ, that evolves according to the following Kawasaki dynamics at inverse temperature $\beta \geq 0$, inside a large box $\Lambda(\rho)$ (that goes to infinity when $\rho \to 0$), with periodic boundary conditions. In this dynamics the particles evolve with exclusion and attraction. More precisely, the total number of particles is,

$$N := \rho|\Lambda(\rho)|, \tag{1.14}$$

where $|\Lambda(\rho)|$ denotes the volume of $\Lambda(\rho)$. We will write $\hat{\eta}_i(t) \in \Lambda(\rho)$ for the position at time t of the particle i in $\{1, \ldots, N\}$ and $\eta_t \in \{0, 1\}^{\Lambda(\rho)}$ for the configuration of the occupied sites in $\Lambda(\rho)$, in such a way that, for all $t \geq 0$,

$$\sum_{z \in \Lambda(\rho)} \eta_t(z) = N. \tag{1.15}$$

The energy of a configuration $\eta \in \{0, 1\}^{\Lambda(\rho)}$ is

$$H(\eta) := \sum_{\substack{\{x,y\} \in \Lambda(\rho) \\ |x-y|=1}} -U\eta(x)\eta(y), \tag{1.16}$$

where $|\cdot|$ now denotes the Euclidean norm and $-U \leq 0$ is the binding energy. With each particle we associate a Poissonian clock of intensity 1. At each time t when a particle's clock rings, we choose with uniform probability a nearest neighbor site of the particle, say i. If this site is occupied by another particle then i does not move. If not, we consider the configuration η' obtained by moving i to the vacant site and then with probability

$$p = e^{-\beta[H(\eta') - H(\eta)]_+} \tag{1.17}$$

i moves to the vacant site and, with probability $1 - p$, i remains where it was at time t_-. Observe that the case $U = 0$ corresponds to the simple exclusion process.

In addition, we choose at time $t = 0$ some particle i_0 according to some probabilistic or deterministic rule and give to i_0, as well as to the particles that share with i_0 the same cluster at time $t = 0$, the color red, while all the other particles receive the blue color. A red particle will definitively remain red (like previously) and a blue particle turns red as soon as it shares some cluster with some red particle. We call RBK process this dynamics and, for all $t \geq 0$, the red zone $\mathcal{R}(t)$ is defined like above.

To control the propagation of the red particles in the regime $\rho \to 0$, we will use the low density to reduce the problem to simple random walks estimates. This is more challenging when ρ and β go jointly to 0 and $+\infty$: in this case we have not only a low density regime, but also a strong interaction regime. We will then deal with this more challenging regime only, setting $\rho = e^{-\Delta\beta}$ for Δ a positive parameter and sending β to infinity. We will write Λ_β for $\Lambda(\rho)$ and we will choose $|\Lambda_\beta| = e^{\Theta\beta}$ for some real parameter $\Theta > \Delta$. This regime was studied in [4] where a "Quasi Random Walk (QRW) property" was proved "up to the first time of anomalous concentration $\mathcal{T}_{\alpha,\lambda}$." For α a positive parameter that can be chosen as close as 0 as we want, and λ a slowly increasing and unbounded function such that

$$\lambda(\beta)\ln\lambda(\beta) = o(\ln\beta) \tag{1.18}$$

[e.g., $\lambda(\beta) = \sqrt{\ln\beta}$], $\mathcal{T}_{\alpha,\lambda}$ is defined as the first time there appears a square box $\Lambda \subset \Lambda_\beta$ with volume less than $e^{\beta(\Delta-\alpha/4)}$ that contains more than $\lambda/4$ particles. We will recall and use this QRW property to prove:

Theorem 2. *For the RBK process, for all $\delta > 0$ and all $C > 0$, uniformly in the starting configuration, and uniformly in $T = T(\beta) \le e^{C\beta}$,*

$$\begin{aligned} &P\big(\mathcal{T}_{\alpha,\lambda} > T \textit{ and } \exists z \in \mathcal{R}(T) \setminus B\big(0, e^{\delta\beta}\sqrt{\rho}T\big)\big) \\ &\quad \le \rho^{-3} e^{\delta\beta} \exp\{-e^{-\delta\beta}\rho T\} + SES \end{aligned} \tag{1.19}$$

where SES stands for "super exponentially small," that is, for a positive function f that does not depend neither on T and nor on the starting configuration and such that

$$\lim_{\beta\to+\infty} \frac{1}{\beta}\ln f(\beta) = -\infty. \tag{1.20}$$

We will then prove:

Corollary 1.2. *For the RBK process, for all $\delta > 0$ and all $C > 0$, uniformly in the starting configuration, and uniformly in $T = e^{K\beta}$ with K any positive parameter such that $K < C$,*

$$P\big(\mathcal{T}_{\alpha,\lambda} > T \textit{ and } \exists z \in \mathcal{R}(T) \setminus B\big(0, e^{\delta\beta}\max\big(\sqrt{T}, \sqrt{\rho}T\big)\big)\big) \le SES. \tag{1.21}$$

Of course, these results would be of no use if we were not able to have some control on $\mathcal{T}_{\alpha,\lambda}$. But in [4] we discussed the fact that, starting from a "good configuration," $\mathcal{T}_{\alpha,\lambda}$ is "very long." For example, we proved that in the case $\Delta > 2U$, starting from the canonical Gibbs measure associated with H, for all $C > 0$,

$$P(\mathcal{T}_{\alpha,\lambda} < e^{C\beta}) = SES. \tag{1.22}$$

As a consequence of these results, we will prove a long range decorrelation of dynamical events in this low density regime. Given $\Lambda^{(1)}$ and $\Lambda^{(2)}$ two square boxes contained in Λ_β, we will denote by $d(\Lambda^{(1)}, \Lambda^{(2)})$ their Euclidean distance and by $(\mathcal{F}_t^{(1)})_{t\geq 0}$ and $(\mathcal{F}_t^{(2)})_{t\geq 0}$ the filtrations generated by the restrictions $(\eta_{t\wedge\mathcal{T}_{\alpha,\lambda}}|_{\Lambda^{(1)}})_{t\geq 0}$ and $(\eta_{t\wedge\mathcal{T}_{\alpha,\lambda}}|_{\Lambda^{(2)}})_{t\geq 0}$. With these notations we will prove:

Theorem 3. *For the Kawasaki dynamics, for all $\delta > 0$ and all $C > 0$, uniformly in the starting configuration, uniformly in $T = e^{K\beta}$ with K any positive parameter such that $K < C$, uniformly in $\Lambda^{(1)}$ and $\Lambda^{(2)}$ such that*

$$d\big(\Lambda^{(1)}, \Lambda^{(2)}\big) \geq e^{\delta\beta} \max\big(\sqrt{T}, \sqrt{\rho}T\big) \tag{1.23}$$

and uniformly in $(A^{(1)}, A^{(2)}) \in \mathcal{F}_T^{(1)} \times \mathcal{F}_T^{(2)}$,

$$\big|P\big(A^{(1)} \cap A^{(2)}\big) - P\big(A^{(1)}\big)P\big(A^{(2)}\big)\big| \leq SES. \tag{1.24}$$

In the study of the low temperature metastable Kawasaki dynamics (the case $U < \Delta < 2U$; see [3]) we will need such a long range decorrelation property (see [5]). This was the original motivation of this paper.

1.3 How good are our bounds?

In this paper we will not give any lower bound on the propagation velocity. But we give here some heuristic that indicates that $\max(\rho, \sqrt{\rho})$ should be the right order of the velocity propagation in different situations. This heuristic is in important part due to Francesco Manzo.

Consider for now the KS process in dimension $d = 2$ with $\rho < 1$ and in the special case $D_B = D_R = 1$. $\mathcal{R}(t)$ should then look like a kind of ball that contains all the red particles and very few blue particles. In addition, $D_B = D_R$ implies that, except for the color propagation, the particle system starts and remains at equilibrium. Let us call $n(t)$ the number of red particles at time t. Since only the particles at the border of $\mathcal{R}(t)$ should contribute to the propagation of the rumor, and since a particle typically waits for a time $1/\rho$ before meeting another particle, we should have

$$dn \simeq \mathrm{cst}\sqrt{n}\rho dt, \tag{1.25}$$

where “cst” stands for a positive constant the value of which can change from line to line. As a consequence

$$\sqrt{n} \simeq \mathrm{cst}\rho t. \tag{1.26}$$

If $r(t)$ stand for the radius of the smallest Euclidean ball that contains $\mathcal{R}(t)$, we should have

$$n \simeq \mathrm{cst} r^2 \rho, \tag{1.27}$$

so that

$$r \simeq \mathrm{cst}\frac{\sqrt{n}}{\sqrt{\rho}} \simeq \mathrm{cst}\sqrt{\rho}t. \tag{1.28}$$

If $\rho \geq 1$, we will typically have ρ particles per site and (1.25) turns into

$$dn \simeq \mathrm{cst}\rho\sqrt{\frac{n}{\rho}}\rho dt, \tag{1.29}$$

so that

$$r \simeq \mathrm{cst}\sqrt{\frac{n}{\rho}} \simeq \mathrm{cst}\rho t. \tag{1.30}$$

If $d \geq 3$ or $D_R \neq D_B$ we do not have such kind of heuristic. In the former case indeed $\mathcal{R}(t)$ should be a more complex fractal object, in the latter case the system does not stay at equilibrium. However Theorem 1 says that an upper bound of order $\max(\rho, \sqrt{\rho})$ holds independently of D_B and independently of the dimension.

For $d = 1$, $D_R = D_B$ and $\rho < 1$ the previous heuristic has to be modified. In this case the typical interparticle distance is $1/\rho$ and a particle typically waits for a time $1/\rho^2$ before meeting another particle. Then (1.25) and (1.27) turn into

$$dn \simeq \mathrm{cst}\rho^2 dt, \tag{1.31}$$

$$n \simeq \mathrm{cst} r\rho, \tag{1.32}$$

and we get

$$r \simeq \mathrm{cst}\rho t, \tag{1.33}$$

while Theorem 1 gives only an upper bound on the velocity of order $\sqrt{\rho} > \rho$. We will prove an upper bound of order ρ for the simplest case of the KS process, that is $D_B = 0$, also known as *frog model*:

Proposition 1.1. *For the KS process in dimension* 1, *with* $\rho < 1$ *and* $D_B = 0$, *there is a positive constant* δ *such that, for all* $t \geq 0$,

$$P\left(\exists z \in \mathcal{R}(t) \setminus B\left(0, \frac{\rho t}{\delta}\right)\right) \leq \frac{e^{-\delta\rho^2 t}}{\delta\rho^2}. \tag{1.34}$$

As previously we then get with the Borel–Cantelli lemma:

Corollary 1.3. *For the KS process in dimension* 1 *and with* $D_B = 0$ *there is a positive constant* δ *such that, with probability* 1

$$\mathcal{R}(t) \subset B\left(0, \frac{\rho t}{\delta}\right) \tag{1.35}$$

holds for all t *larger than some finite random time* T_0.

We will give in Section 4 some indications on how one can extend the simple proof of Proposition 1.1 to the general case of the KS processes. This is rather technical and we will not go beyond these indications.

1.4 Notation and outline of the paper

We will write "cst" for a finite and positive constant that depends only on the dimension d and the value of which can change from line to line. Given $d \geq 1$ we will write $|\cdot|$ for the d-dimensional Euclidean norm. Given a Markov process X and x in its state space, we will write P_x for the law of the process that starts from x.

In Section 2, we prove simple random walk and large deviations estimates and we recall some definitions and properties regarding the QRW approximation for the Kawasaki dynamics. In Section 3, we prove Theorem 1 for the frog model as well as Proposition 1.1. In Section 4, we prove Theorem 1 in the general case as well as Theorem 2, Corollary 1.2 and Theorem 3.

2 Preliminaries

2.1 Random walk and large deviation estimates

Lemma 2.1. *Let N and N' be two independent Poisson variables and $\gamma > 1$ such that $E[N'] \geq \gamma E[N]$. Then*

$$\text{(i)} \quad P(N \geq \gamma E[N]) \leq \exp\{-E[N](\gamma \ln \gamma - (\gamma - 1))\}, \tag{2.1}$$

$$\text{(ii)} \quad P\left(N \leq \frac{E[N]}{\gamma}\right) \leq \exp\left\{-E[N]\left(\left(1 - \frac{1}{\gamma}\right) - \frac{\ln \gamma}{\gamma}\right)\right\}, \tag{2.2}$$

$$\text{(iii)} \quad P\left(\frac{N}{E[N]} \geq \gamma \frac{N'}{E[N']}\right) \leq 2 \exp\{-E[N](t \ln t - (t - 1))\} \quad \textit{with } t := \frac{\gamma - 1}{\ln \gamma} \in]1, \gamma[. \tag{2.3}$$

Proof. We just use the Chebyshev exponential inequality. With $\lambda = E[N]$ we have, for any $t \geq 0$,

$$P(N \geq \gamma\lambda) \leq e^{-t\gamma\lambda} E[e^{tN}] = \exp\{-\lambda(t\gamma - (e^t - 1))\}. \tag{2.4}$$

Optimizing in t we find (2.1) with $t = \ln \gamma$. Similarly, for any $t \geq 0$,

$$P(N \leq \lambda/\gamma) \leq e^{t\lambda/\gamma} E[e^{-tN}] = \exp\{-\lambda((1 - e^{-t}) - t/\gamma)\}. \tag{2.5}$$

Optimizing in t we find (2.2) with $t = \ln \gamma$. Finally we have, for any $t \geq 0$,

$$P\left(\frac{N}{E[N]} \geq \gamma \frac{N'}{E[N']}\right) \leq P(N \geq t E[N]) + P\left(N' \leq \frac{t}{\gamma} E[N']\right). \tag{2.6}$$

By (2.1) and (2.2) this gives, if $t > 1$ and $t < \gamma$,

$$P\left(\frac{N}{E[N]} \geq \gamma \frac{N'}{E[N']}\right) \leq \exp\{-\lambda(t \ln t - (t-1))\} + \exp\left\{-\lambda\gamma\left(\left(1 - \frac{t}{\gamma}\right) + \frac{t}{\gamma} \ln \frac{t}{\gamma}\right)\right\}. \tag{2.7}$$

The two terms of this sum are equal when

$$t = \frac{\gamma - 1}{\ln \gamma}. \tag{2.8}$$

The concavity of the logarithm ensures

$$1 - \frac{1}{\gamma} \leq -\ln \frac{1}{\gamma} = \ln \gamma \leq \gamma - 1 \tag{2.9}$$

so that $1 < t < \gamma$ when t is defined by (2.8). This gives (2.3). □

Lemma 2.2. *Let ζ be a d-dimensional continuous-time simple random walk with jump rate* 1. *For all $t \geq 0$ and $z \in \mathbb{Z}^d$:*

- *If $|z| \leq t$ then*

$$P_0(\zeta(t) = z) \leq \frac{\text{cst}}{t^{d/2}} \exp\left\{-\frac{\text{cst}|z|^2}{t}\right\}. \tag{2.10}$$

- *If $|z| \geq t$ then*

$$P_0(\zeta(t) = z) \leq \text{cst} \exp\{-\text{cst}|z|\}. \tag{2.11}$$

Remark. Since we just need an upper bound on these probabilities we do not need the usual condition $|z| = o(t^{2/3})$ of the local central limit theorem. However, working with continuous-time random walks, we have to treat separately the case $|z| > t$.

Proof of Lemma 2.2. We will prove slightly different but clearly equivalent estimates: (2.10) when $|z| \leq 2t$ and (2.11) when $|z| \geq 2t$.

For the case $|z| \geq 2t$ we apply the previous lemma. If ζ reaches z in time t then the number of its clock rings up to time t is larger than or equal to $|z|$. Since this number has a Poissonian distribution of mean t, this occurs, by (2.1), with a probability smaller than

$$\exp\left\{-t\left(\frac{|z|}{t} \ln \frac{|z|}{t} - \left(\frac{|z|}{t} - 1\right)\right)\right\} \leq \exp\left\{-t \frac{\text{cst}|z|}{t}\right\} = e^{-\text{cst}|z|} \tag{2.12}$$

(for the last inequality we used that $|z|/t$ was bounded away from 1).

We now treat the case $|z| \le 2t$. First observe that, working with a continuous-time process with independent coordinates, it is enough to prove the result for $d = 1$. In a second step, we prove the estimate for $\tilde{\zeta}$ the discrete time version of such a one-dimensional process. Without loss of generality we can assume that $z \in \mathbb{Z}$ is nonnegative. If $z \le n/2$, then, by the Stirling formula,

$$\begin{aligned} P_0\bigl(\tilde{\zeta}(n) = z\bigr) &\le \frac{\mathrm{cst}}{\sqrt{n}} \frac{2}{\sqrt{1+z/n}\sqrt{1-z/n}} \\ &\quad \times \left[\left(1 + \frac{z}{n}\right)^{(1+z/n)/2}\left(1 - \frac{z}{n}\right)^{(1-z/n)/2}\right]^{-n} \\ &\le \frac{\mathrm{cst}}{\sqrt{n}} \exp\{-nI(z/n)\} \end{aligned} \tag{2.13}$$

with

$$I(x) := \frac{1+x}{2}\ln(1+x) + \frac{1-x}{2}\ln(1-x), \qquad x \in [-1,1]. \tag{2.14}$$

It is immediate to check that

$$\begin{cases} I(0) = I'(0) = 0, \\ \forall x \in]-1,1[, \qquad I''(x) = \dfrac{1}{1-x^2} \ge 1. \end{cases} \tag{2.15}$$

As a consequence, for all $x \in [-1,1]$,

$$I(x) \ge \frac{x^2}{2} \tag{2.16}$$

and this gives, for $z \le n/2$,

$$P_0\bigl(\tilde{\zeta}(n) = z\bigr) \le \frac{\mathrm{cst}}{\sqrt{n}} \exp\left\{-\frac{z^2}{2n}\right\}. \tag{2.17}$$

This is easily extended to the case $z \ge n/2$, that is, $z/n \ge 1/2$. Since the inequality in (2.15) is a strict inequality as soon as $x > 0$, the function $x \in [\frac{1}{2}, 1] \to \frac{2I(x)}{x^2}$ is increasing, $2I(x) \ge 8I(1/2)x^2$ for $x \ge 1/2$ and $8I(1/2) > 1$, we have

$$\begin{aligned} P_0\bigl(\tilde{\zeta}(n) = z\bigr) &\le \mathrm{cst}\exp\{-nI(z/n)\} \le \mathrm{cst}\exp\left\{-n \cdot 8I(1/2)\frac{z^2}{2n^2}\right\} \\ &\le \mathrm{cst}\sqrt{\frac{n}{z^2}}\exp\left\{-\frac{z^2}{2n}\right\} \le \frac{\mathrm{cst}}{\sqrt{n}}\exp\left\{-\frac{z^2}{2n}\right\}. \end{aligned} \tag{2.18}$$

Finally, we use the previous lemma to conclude. From the estimates on $\tilde{\zeta}$ we deduce

$$P_0\bigl(\zeta(n) = z\bigr) \le E\left[\frac{\mathrm{cst}}{\sqrt{N}}\exp\left\{-\frac{z^2}{2N}\right\}\right], \tag{2.19}$$

where N is a Poisson variable of mean t. Intersecting with the event N in or out of the interval $[\frac{t}{\gamma}, \gamma t]$, we get

$$\begin{aligned} P_0\big(\zeta(n) \le z\big) \le E\bigg[\frac{\text{cst}}{\sqrt{N}} \exp\Big\{-\frac{z^2}{2N}\Big\}\bigg|\frac{t}{\gamma} \le N \le \gamma t\bigg] \\ + P\bigg(N \notin \Big[\frac{t}{\gamma}, \gamma t\Big]\bigg). \end{aligned} \tag{2.20}$$

By (2.1), (2.2) applied with a large enough γ we can find two positive constants c_1, c_2 with $4c_1 < c_2$ such that

$$P_0\big(\zeta(n) = z\big) \le \frac{\text{cst}}{\sqrt{t}} \exp\Big\{-c_1 \frac{z^2}{t}\Big\} + \exp\{-2c_2 t\} \tag{2.21}$$

$$\le \frac{\text{cst}}{\sqrt{t}} \bigg(\exp\Big\{-c_1 \frac{z^2}{t}\Big\} + \exp\{-c_2 t\}\bigg) \tag{2.22}$$

and we get (2.10) using $z \le 2t$, i.e., $4t \ge z^2/t$. □

2.2 Quasi random walks

With the notation we introduced in Section 1.2 for the Kawasaki dynamics and given an arbitrarily small parameter $\alpha > 0$ as well as an unbounded slowly increasing function λ satisfying (1.18), we recall in this section a few definitions and results from [4].

Definition 2.1. A process $Z = (Z_1, \dots, Z_N)$ on Λ_β^N is called a random walk with pauses (RWP) associated with the stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \cdots, \qquad i \in \{1, \dots, N\}, \tag{2.23}$$

if for any i in $\{1, \dots, N\}$, Z_i is constant on all time intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \ge 0$, and if the process $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_N)$ obtained from Z by cutting off these pauses intervals, that is, with

$$\tilde{Z}_i(s) := Z_i\bigg(s + \sum_{k < j_i(s)} \tau_{i,k} - \sigma_{i,k}\bigg), \qquad s \ge 0, \tag{2.24}$$

where

$$j_i(s) := \inf\bigg\{j \ge 0 : s + \sum_{k<j} \tau_{i,k} - \sigma_{i,k} \le \sigma_{i,j}\bigg\}. \tag{2.25}$$

$\tilde{Z}_i(s)$, $i \in \{1, \dots, N\}$, are independent random walks in law.

Now with

$$T_\alpha := e^{(\Delta-\alpha)\beta} \tag{2.26}$$

QRW processes are defined as follows.

Definition 2.2. We say that a process $\xi = (\xi_1, \dots, \xi_N)$ on Λ_β^N is a QRW process with parameter $\alpha > 0$ up to a stopping time $\mathcal{T}$, written QRW$(\alpha, \mathcal{T})$, if there exists a coupling between ξ and a RWP process Z associated with stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \cdots, \qquad i \in \{1, \dots, N\}, \tag{2.27}$$

such that

(i) $\xi(0) = Z(0)$,
(ii) for any i in $\{1, \dots, N\}$ ξ_i and Z_i evolve jointly ($\xi_i - Z_i$ is constant) outside the pause intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \ge 0$, and
(iii) for any $t_0 \ge 0$,

the following events occur with probability $1 - SES$ uniformly in i and t_0:

$$F_i(t_0) := \big\{\sharp\{k \ge 0 : \sigma_{i,k} \in [t_0 \wedge \mathcal{T}, (t_0 + T_\alpha) \wedge \mathcal{T}]\} \le l(\beta)\big\}, \tag{2.28}$$

$$\begin{aligned} G_i(t_0) := \big\{&\forall k \ge 0, \forall t \ge t_0, \sigma_{i,k} \in [t_0 \wedge \mathcal{T}, (t_0 + T_\alpha) \wedge \mathcal{T}] \\ &\Rightarrow |\xi(t \wedge \tau_{i,k} \wedge \tau) - \xi(t \wedge \sigma_{i,k} \wedge \tau)| \le l(\beta)\big\} \end{aligned} \tag{2.29}$$

for some $\beta \mapsto l(\beta)$ that satisfies

$$\lim_{\beta \to +\infty} \frac{1}{\beta} \ln l(\beta) = 0. \tag{2.30}$$

In words, the fact that for each i the events $F_i(t_0)$ and $G_i(t_0)$ occur for all $t_0 \ge 0$ means, on the one hand, that in each time interval before time $\mathcal{T}$ and of length $1/\rho$ almost, there are few pauses for the associated RWP Z_i (a nonexponentially large number) and, on the other hand, that ξ_i stays close to Z_i. The two processes are close in the sense that during each of these few pause intervals the distance between the two processes cannot increase of more than the same nonexponentially large quantity l. Recalling the definition of $\mathcal{T}_{\alpha,\lambda}$ before Theorem 2:

Proposition 2.1. *For any unbounded and slowly increasing function λ that satisfies (1.18) and any positive $\alpha < \Delta$, $\hat\eta$ is a* QRW$(\alpha, \mathcal{T}_{\alpha,\lambda})$ *process.*

We refer to [4] for the proof. In that paper we proved a "nonsuperdiffusivity property" as consequence of the QRW property: for all $\delta > 0$, uniformly in the initial configuration and uniformly in $T = T(\beta) \in [2, T_\alpha^2]$,

$$P\big(\mathcal{T}_{\alpha,\lambda} > T, \exists t \in [0, T], \exists i \in \{1, \dots, N\}, |\hat\eta_i(t) - \hat\eta_i(0)| > e^{\delta\beta} T\big) \le SES. \tag{2.31}$$

In [4] we also introduced at any time $t_0 \ge 0$ a partition of $\{1, \dots, N\}$ in *clouds of potentially interacting particles on time scale T_α*: we associate with each particle i

a ball centered at its position at time t_0 with radius

$$r := e^{\alpha\beta/4}\sqrt{T_\alpha}, \tag{2.32}$$

we call B_0 their union

$$B_0 := \bigcup_i B(\hat{\eta}_i(t_0), r), \tag{2.33}$$

and we say that two particles are in the same cloud if they are, at time t_0, in the same connected component of B_0. It is easy to check that if $t_0 < \mathcal{T}_{\alpha,\lambda}$, then no cloud contains more than λ particles. And, as a consequence of (2.31), with probability $1 - SES$, interactions between particles during the time interval $[t_0, (t_0 + T_\alpha) \wedge \mathcal{T}_{\alpha,\lambda}[$ will only take place *inside* the different clouds (and not between particles of different clouds).

3 The frog model

3.1 Proof of Theorem 1 for the KS process with $D_B = 0$

There is a natural notion of generation in the model. We say that the first particle at the origin is of first generation and that a particle that turns red when it encounters a particle of kth generation is of $(k+1)$th generation. (If a blue particle moves on a site with more than one red particles then its generation number is determined by the lowest generation number of the red particles.) Now, to drive the red color outside an Euclidean ball $B(0, r)$ by time t, the first particle initially in $z_1 = 0$ has to activate at some time t_1 a second generation particle in some site z_2, and this particle has to activate at some time $t_1 + t_2$ a third generation particle in some site $z_3, \ldots$ and, for some n, an nth generation particle will have to reach some site z_{n+1} outside $B(0, r)$ at some time $t_1 + \cdots + t_n \le t$. Taking into account the fact that more than one blue particle can stand in a site reached by a red particle and using Lemma 2.2 we get, for all r and t,

$$P\big(\exists z \in \mathcal{R}(t), |z| > r\big) \le Q(r, t) \tag{3.1}$$

with

$$\begin{aligned} Q(r, t) := \sum_{n\ge 1} \sum_{\substack{z_1,\ldots,z_{n+1} \\ z_1 = 0 \\ z_{n+1}\notin B(0,r)}} \int_{t_1+\cdots+t_n\le t} \sum_{j_2,\ldots,j_n\ge 0} \prod_{k=2}^{n} e^{-\rho}\frac{\rho^{j_k}}{j_k!} j_k \\ \times \prod_{k=1}^{n} \left(\left(\frac{\mathrm{cst}}{t_k^{d/2}} e^{-\mathrm{cst}|z_{k+1}-z_k|^2/t_k}\right) \vee \left(\mathrm{cst} e^{-\mathrm{cst}|z_{k+1}-z_k|}\right)\right) dt_k, \end{aligned} \tag{3.2}$$

where here, like in the sequel, we did not write to alleviate the notation, that the integral is restricted to positive variables only.

Permuting the last sum with the product, making a spherical change of variable and using the triangular inequality, we get

$$Q(r,t) \le \sum_{n\ge 1} \int_{\substack{r_1+\cdots+r_n\ge r\\ t_1+\cdots+t_n\le t}} \rho^{n-1} \prod_{k=1}^{n} \big(q_1(r_k,t_k) \vee q_2(r_k)\big) r_k^{d-1}\, dr_k\, dt_k \tag{3.3}$$

with

$$q_1(r_k,t_k) := \frac{\mathrm{cst}}{t_k^{d/2}} e^{-\mathrm{cst} r_k^2/t_k}, \tag{3.4}$$

$$q_2(r_k,t_k) = q_2(r_k) := \mathrm{cst} e^{-\mathrm{cst} r_k}. \tag{3.5}$$

Grouping together the different terms according to the respective values of q_1 and q_2 and using, for all $0 \le j \le n$,

$$\binom{n}{j} \le 2^j 2^{n-j} \tag{3.6}$$

we get

$$\begin{aligned} Q(r,t) &\le \frac{1}{\rho} \int_{\substack{R_1+R_2\ge r\\ T_1+T_2\le t}} \sum_{n\ge 1}\sum_{j=0}^{n} \binom{n}{j} \left(\int_{\substack{r_1+\cdots+r_j\ge R_1\\ t_1+\cdots+t_j\le T_1}} \rho^j \prod_{k=1}^{j} q_1(r_k,t_k) r_k^{d-1}\, dr_k\, dt_k \right) \\ &\quad \times \left(\int_{\substack{r_1+\cdots+r_{n-j}\ge R_2\\ t_1+\cdots+t_{n-j}\le T_2}} \rho^{n-j} \prod_{k=1}^{n-j} q_2(r_k) r_k^{d-1}\, dr_k\, dt_k \right) dR_1\, dR_2\, dT_1\, dT_2 \\ &\le \frac{1}{\rho} \int_{\substack{R_1+R_2\ge r\\ T_1+T_2\le t}} \sum_{n\ge 1}\sum_{j=0}^{n} Q_1^{(j)}(R_1,T_1) Q_2^{(n-j)}(R_2,T_2)\, dR_1\, dR_2\, dT_1\, dT_2 \\ &= \frac{1}{\rho} \int_{\substack{R_1+R_2\ge r\\ T_1+T_2\le t}} Q_1(R_1,T_1) Q_2(R_2,T_2)\, dR_1\, dR_2\, dT_1\, dT_2 \end{aligned} \tag{3.7}$$

with for $m = 1, 2$ and all $j \ge 1$

$$Q_m^{(j)}(R_m,T_m) := \int_{\substack{r_1+\cdots+r_j\ge R_m\\ t_1+\cdots+t_j\le T_m}} (2\rho)^j \prod_{k=1}^{j} q_m(r_k,t_k) r_k^{d-1}\, dr_k\, dt_k, \tag{3.8}$$

$$Q_m(R_m,T_m) := \sum_{n\ge 1} Q_m^{(n)}(R_m,T_m). \tag{3.9}$$

For any $R, T \ge 0$ we will estimate separately $Q_1(R,T)$ and $Q_2(R,T)$.

We have

$$Q_1(R,T) \le \sum_{n\ge 1} (\mathrm{cst}\rho)^n \int_{\substack{r_1+\cdots+r_n\ge R\\ t_1+\cdots+t_n\le T}} \prod_{k=1}^{n} e^{-\mathrm{cst} r_k^2/t_k} \left(\frac{r_k}{\sqrt{t_k}}\right)^{d-1} \frac{dr_k\, dt_k}{\sqrt{t_k}}. \tag{3.10}$$

Making a change of variable $x_k = \mathrm{cst} r_k^2/t_k$ and observing that, by the Cauchy–Schwartz inequality,

$$\begin{cases} \sum_k \sqrt{t_k}\sqrt{x_k} \ge \mathrm{cst} R, \\ \sum_k t_k \le T, \end{cases} \quad \Rightarrow \quad \begin{cases} \sum_k x_k \ge \mathrm{cst} R^2/T, \\ \sum_k t_k \le T, \end{cases} \tag{3.11}$$

we get, with Γ the Euler function,

$$Q_1(R,T) \le \sum_{n\ge 1} (\mathrm{cst}\rho)^n \int_{\substack{x_1+\cdots+x_n\ge \mathrm{cst} R^2/T\\ t_1+\cdots+t_n\le T}} \prod_{k=1}^{n} e^{-x_k} x_k^{(d-1)/2} \frac{dx_k\, dt_k}{x_k^{1/2}} \tag{3.12}$$

$$\le \sum_{n\ge 1} (\mathrm{cst}\rho)^n \int_{\substack{x_1+\cdots+x_n\ge \mathrm{cst} R^2/T\\ t_1+\cdots+t_n\le T}} \prod_{k=1}^{n} e^{-x_k} x_k^{d/2-1} \frac{dx_k\, dt_k}{\Gamma(d/2)}. \tag{3.13}$$

Since the volume of the n-dimensional simplex of side-length T is $T^n/n!$ and the sum of independent variables with a Γ distribution follows a Γ law,

$$\begin{aligned} Q_1(R,T) &\le \sum_{n\ge 1} \frac{(\mathrm{cst}\rho T)^n}{n!} \int_{x\ge \mathrm{cst} R^2/T} e^{-x} x^{nd/2-1} \frac{dx}{\Gamma(nd/2)} \\ &\le \sum_{n\ge 1} \frac{(\mathrm{cst}\rho T)^n}{n!} P\left(N' \le \left\lceil \frac{nd}{2} \right\rceil\right) \\ &\le e^{\mathrm{cst}\rho T} P(N' \le \mathrm{cst} N), \end{aligned}$$

where N and N' are independent Poisson variables of mean $\mathrm{cst}\cdot \rho T$ and $\mathrm{cst}\cdot R^2/T$, respectively. Now, for any large enough γ, if $R \ge \gamma\sqrt{\rho}T$, then by (2.3)

$$\begin{aligned} Q_1(R,T) &\le e^{\mathrm{cst}\rho T} P\left(\frac{N}{E[N]} \ge \mathrm{cst} \frac{R^2/T}{\rho T} \frac{N'}{E[N']}\right) \\ &\le e^{\mathrm{cst}\rho T} e^{-\mathrm{cst} R^2/T} \le e^{\mathrm{cst}\rho T} e^{-\mathrm{cst}\sqrt{\rho} R} \end{aligned} \tag{3.14}$$

so that, for any large enough γ,

$$Q_1(R,T) \le e^{\mathrm{cst}\rho T} \exp\{-\mathrm{cst}\sqrt{\rho} R \mathbb{1}_{[\gamma\sqrt{\rho}T, +\infty[}(R)\}. \tag{3.15}$$

Turning to $Q_2(R,T)$ we have

$$\begin{aligned}
Q_2(R,T) &\le \sum_{n\ge 1}(\mathrm{cst}\rho)^n \int_{\substack{r_1+\cdots+r_n\ge R\\ t_1+\cdots+t_n\le T}} \prod_{k=1}^{n} e^{-\mathrm{cst} r_k} r_k^{d-1}\, dr_k\, dt_k \\
&\le \sum_{n\ge 1}(\mathrm{cst}\rho)^n \int_{\substack{x_1+\cdots+x_n\ge \mathrm{cst}R\\ t_1+\cdots+t_n\le T}} \prod_{k=1}^{n} e^{-x_k} x_k^{d-1}\, dx_k\, dt_k \\
&\le \sum_{n\ge 1}\frac{(\mathrm{cst}\rho T)^n}{n!}\int_{x\ge \mathrm{cst}R} e^{-x} x^{nd-1}\frac{dx}{\Gamma(nd)} \\
&\le e^{\mathrm{cst}\rho T} P(N'\le \mathrm{cst}N),
\end{aligned} \tag{3.16}$$

where N and N' are independent Poissonian variables of mean $\mathrm{cst}\cdot\rho T$ and $\mathrm{cst}\cdot R$, respectively. Then, for any large enough γ, if $R\ge\gamma\rho T$, we get by (2.3)

$$Q_2(R,T)\le e^{\mathrm{cst}\rho T} P\left(\frac{N}{E[N]}\ge\frac{\mathrm{cst}R}{\rho T}\frac{N'}{E[N']}\right)\le e^{\mathrm{cst}\rho T}e^{-\mathrm{cst}R} \tag{3.17}$$

so that, for any large enough γ,

$$Q_2(R,T)\le e^{\mathrm{cst}\rho T}\exp\{-\mathrm{cst}R\mathbb{1}_{[\gamma\rho T,+\infty[}(R)\}. \tag{3.18}$$

Turning back to $Q(r,t)$, we get, for any large enough γ,

$$\begin{aligned}
Q(r,t) \le \frac{1}{\rho}\int_{\substack{R_1+R_2\ge r\\ T_1+T_2\le t}} & e^{\mathrm{cst}\rho(T_1+T_2)} \\
&\times\exp\bigl\{-\mathrm{cst}\bigl(\sqrt{\rho}R_1\mathbb{1}_{[\gamma\sqrt{\rho}T_1,+\infty[}(R_1) \\
&\qquad + R_2\mathbb{1}_{[\gamma\rho T_2,+\infty[}(R_2)\bigr)\bigr\}\, dR_1\, dR_2\, dT_1\, dT_2
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\le \frac{1}{\rho}\int_{\substack{R_1+R_2\ge r\\ T_1+T_2\le t}} & e^{\mathrm{cst}\rho t} \\
&\times\exp\bigl\{-\mathrm{cst}\bigl(\sqrt{\rho}R_1\mathbb{1}_{[\gamma\rho T_1,+\infty[}\bigl(\sqrt{\rho}R_1\bigr) \\
&\qquad + R_2\mathbb{1}_{[\gamma\rho T_2,+\infty[}(R_2)\bigr)\bigr\}\, dR_1\, dR_2\, dT_1\, dT_2.
\end{aligned} \tag{3.20}$$

Now if $\rho\le 1$, then

$$R_2\mathbb{1}_{[\gamma\rho T_2,+\infty[}(R_2)\ge\sqrt{\rho}R_2\mathbb{1}_{[\gamma\rho T_2,+\infty[}\bigl(\sqrt{\rho}R_2\bigr), \tag{3.21}$$

and if $\rho\ge 1$, then

$$\sqrt{\rho}R_1\mathbb{1}_{[\gamma\rho T_1,+\infty[}\bigl(\sqrt{\rho}R_1\bigr)\ge R_1\mathbb{1}_{[\gamma\rho T_1,+\infty[}(R_1). \tag{3.22}$$

As a consequence, with

$$\bar\rho := \max\bigl(\rho,\sqrt{\rho}\bigr)\quad\text{and}\quad X_m=\frac{\rho}{\bar\rho}R_m,\qquad m=1,2, \tag{3.23}$$

we have

$$Q(r,t) \leq \frac{\bar{\rho}^2}{\rho \cdot \rho^2} \int_{\substack{X_1+X_2 \geq \rho r/\bar{\rho} \\ T_1+T_2 \leq t}} e^{\text{cst}\rho t} \times \exp\big\{-\text{cst}\big(X_1 \mathbb{1}_{[\gamma\rho T_1,+\infty[}(X_1) + X_2 \mathbb{1}_{[\gamma\rho T_2,+\infty[}(X_2)\big)\big\}\, dX_1\, dX_2\, dT_1\, dT_2 \tag{3.24}$$

$$\leq \frac{\bar{\rho}^2 e^{\text{cst}\rho t}}{\rho^3} \times \int_{\substack{X_1+X_2 \geq \rho r/\bar{\rho} \\ T_1+T_2 \leq t}} e^{-\text{cst}(X_1+X_2-\gamma\rho(T_1+T_2))}\, dX_1\, dX_2\, dT_1\, dT_2. \tag{3.25}$$

If $r \geq 2\gamma\bar{\rho}t$, that is,

$$\frac{\rho r}{2\bar{\rho}} \geq \gamma\rho t \tag{3.26}$$

then

$$\begin{aligned} Q(r,t) &\leq \frac{\bar{\rho}^2 e^{\text{cst}\rho t}}{\rho^3} \int_{\substack{X_1+X_2 \geq \rho r/\bar{\rho} \\ T_1+T_2 \leq t}} e^{-\text{cst}(X_1+X_2)/2}\, dX_1\, dX_2\, dT_1\, dT_2 \\ &\leq \text{cst}\frac{\bar{\rho}^2 e^{\text{cst}\rho t}}{\rho^3} t^2 \frac{\rho r}{\bar{\rho}} e^{-\text{cst}\rho r/(2\bar{\rho})} \leq \text{cst}\frac{\bar{\rho}^2 e^{\text{cst}\rho t}}{\rho^5} e^{-\text{cst}\rho r/(2\bar{\rho})} \\ &\leq \frac{\text{cst}\bar{\rho}^2}{\rho^5} e^{\text{cst}\rho t} e^{-\text{cst}\gamma\rho t} \end{aligned} \tag{3.27}$$

and, with a large enough γ, we get

$$Q(r,t) \leq \frac{\text{cst}\bar{\rho}^2}{\rho^5} e^{-\text{cst}\rho t}. \tag{3.28}$$

3.2 Proof of Proposition 1.1

In the previous proof we could have used, instead of the estimates from Lemma 2.2 on $P_0(\zeta(t) = z)\,dt$, an estimate on

$$dP_0\big(\tau_z(\zeta) \leq t\big) = P_0\big(\tau_z(\zeta) \in [t, t+dt]\big) \tag{3.29}$$

with

$$\tau_z(\zeta) := \inf\{t \geq 0 : \zeta(t) = z\}. \tag{3.30}$$

While in dimension $d \geq 2$ the two quantities are quite close, in dimension $d = 1$ they are substantially different. In addition, using $\tau_z(\zeta)$ in dimension 1 allows to give a simpler proof of a stronger result when ρ is small enough. Indeed, for all r and t,

$$\begin{aligned} &P\big(\exists z\in\mathcal{R}(t), |z|>r\big) \\ &\quad\le \sum_{n\ge1}\rho^{n-1}\sum_{r_1+\cdots+r_n\ge r}\int_{t_1+\cdots+t_n\le t}\prod_{k=1}^{n} dP_0\big(\tau_{r_k}(\zeta)\le t_k\big) && (3.31)\\ &\quad\le \sum_{n\ge1}\rho^{n-1}\sum_{R\ge r}\sum_{r_1+\cdots+r_n=R} P_0\big(\tau_R(\zeta)\le t\big). && (3.32)\end{aligned}$$

Then, by the reflexion principle and Lemma 2.2

$$\begin{aligned} P\big(\exists z\in\mathcal{R}(t), |z|>r\big) &\le \frac{\text{cst}}{\rho}\sum_{R\ge r}\sum_{n\ge1}\frac{\rho^n R^n}{n!}\big(e^{-\text{cst}R^2/t}\vee e^{-\text{cst}R}\big)\\ &\le \frac{\text{cst}}{\rho}\sum_{R\ge r}e^{\rho R}\big(e^{-\text{cst}R^2/t}\vee e^{-\text{cst}R}\big). \end{aligned} \tag{3.33}$$

Now if $r\ge\gamma\rho t$ for some large enough γ we get, for ρ small enough,

$$P\big(\exists z\in\mathcal{R}(t), |z|>r\big)\le\frac{\text{cst}}{\rho}\sum_{R\ge r}e^{-\text{cst}\rho R}\le\frac{\text{cst}}{\rho^2}e^{-\text{cst}\rho r}\le\frac{\text{cst}}{\rho^2}e^{-\text{cst}\rho^2 t}. \tag{3.34}$$

This proves Proposition 1.1 for small ρ's.

When ρ is bounded away from 0, the estimate in Proposition 1.1 is just a consequence of Theorem 1 for the frog model.

4 RB and RBK processes

4.1 Proof of Theorem 1

We can proceed like in the case of the frog model except for the fact that a particle does not anymore turn red at the same point where it started. We have then to sum over the possible starting points. With the notation

$$s_k = t_1+\cdots+t_{k-1},\qquad k\ge 2, \tag{4.1}$$

and for any $i\ge1$ we have

$$\begin{aligned} &P\big(\exists z\in\mathcal{R}(t), |z|>r\big)\\ &\quad\le\sum_{n\ge1}\sum_{\substack{z_1,\dots,z_{n+1}\\ z_1=0\\ z_{n+1}\notin B(0,r)}}\int_{t_1+\cdots+t_n\le t}\sum_{\substack{z'_2,\dots,z'_n\\ j_2,\dots,j_n\ge0}}\prod_{k=2}^{n}e^{-\rho}\frac{\rho^{j_k}}{j_k!}j_k\\ &\qquad\times P\big(z'_k+Z_i^B(s_k)=z_k\big) && (4.2)\\ &\qquad\times\prod_{k=1}^{n}\Big(\frac{\text{cst}}{t_k^{d/2}}e^{-\text{cst}|z_{k+1}-z_k|^2/t_k}\vee\text{cst}e^{-\text{cst}|z_{k+1}-z_k|}\Big)dt_k. && (4.3)\end{aligned}$$

Now permuting the last sum with the product and using (1.3) we get

$$P\big(\exists z \in \mathcal{R}(t), |z| > r\big) \le Q(r,t) \tag{4.4}$$

with $Q(r,t)$ defined in (3.2) and estimated in the previous section.

Remark. Unfortunately the proof of Proposition 1.1 cannot be extended so simply to the general case, even if we restrict ourselves to KS processes. To do so we would have to link the differential

$$dP_0\big(\tau_{z_R}(\zeta_R) \le t\big) = P_0\big(\tau_{z_R}(\zeta_R) \in [t, t+dt]\big) \tag{4.5}$$

with the sum

$$\sum_{z_B>0} P_{(0,z_B)}\big(\tau_0(\zeta_B - \zeta_R) \in [t, t+dt], \zeta_R(t) = z_R\big) \tag{4.6}$$

with ζ_R and ζ_B independent continuous-time random walks with jump rates $D_R = 1$ and $D_B > 0$. In the case $D_B = 1$ this can be done using the independence between $\zeta_B - \zeta_R$ and $\zeta_B + \zeta_R$. In the case $D_B \neq 1$ we can only use an "asymptotic independence" between $\zeta_B - \zeta_R$ and $\zeta_B + D_B\zeta_R$. In both cases this is a quite technical task: we will not go in this paper beyond the result for the frog model.

4.2 Proof of Theorem 2 and Corollary 1.2

Proof of Theorem 2. We first note that, for β large enough, the right-hand side of (1.19) is larger than 1 if $\rho T \le 1$. Without lost of generality we can then assume $T \ge \rho^{-1}$. Now we can adapt the proof for the frog model using the QRW property and the last observations of Section 2.2:

$$\begin{aligned} &P\big(\exists z \in \mathcal{R}(T), |z| > R, \mathcal{T}_{\alpha,\lambda} > T\big) \\ &\quad \le \sum_{n=1}^{\lceil \lambda l T/T_\alpha \rceil} \sum_{\substack{z_1,\dots,z_{n+1} \in \Lambda_\beta \\ z_1 = 0 \\ z_{n+1} \notin B(0,R)}} \int_{t_1+\dots+t_n \le T} \prod_{k=1}^{n} \mathrm{cst}\lambda^3 l^2 \\ &\qquad \times \left(\frac{\mathrm{cst}}{t_k^{d/2}} e^{-\mathrm{cst}|z_{k+1}-z_k|^2/t_k} \vee \mathrm{cst} e^{-\mathrm{cst}|z_{k+1}-z_k|}\right) dt_k + SES. \end{aligned} \tag{4.7}$$

In this formula the first sum is limited to $\lceil \lambda l T/T_\alpha \rceil$ since, on the one hand, T is at most exponential in β and in each interval of length T_α, with probability $1 - SES$, interactions are limited to clouds that contains λ particles at most and, on the other hand, particles are coupled with random walks with l pauses at most. The factor l^2 is due to the fact that, with probability $1 - SES$, in each pause interval the distance between a particle and its associated random walk with pauses increases of at most l. One factor λ is due to the fact that at most λ red particles can leave a given cluster before $\mathcal{T}_{\alpha,\lambda}$: no cluster can contain more than λ particles before the first “anomalous concentration.” The last factor λ^2 is due to the fact that at each time $t < \mathcal{T}_{\alpha,\lambda}$ a given particle can turn red other particles inside a radius λ at most.

Then we can repeat the calculation of Section 3.1 with two main differences. On the one hand we do not have the factor ρ^{n-1} anymore in our sum, on the other hand this sum is limited to $\lceil \lambda l T/T_\alpha \rceil$. Instead of (2.3) we use then (2.2) repeatedly. For example, defining Q_1 and Q_2 in an analogous way and observing that for any $\delta > 0$, λ and l are smaller than $e^{\delta\beta}$ for β large enough, we have now, choosing a small enough α and using $T \geq \rho^{-1}$,

$$Q_1(R,T) \leq \sum_{n=1}^{\lfloor e^{\delta\beta}\rho T \rfloor} \frac{(e^{\delta\beta}T)^n}{n!} P\left(N' \leq \left\lceil \frac{nd}{2} \right\rceil\right) + SES \tag{4.8}$$

$$\leq \sum_{n=1}^{\lfloor e^{\delta\beta}\rho T \rfloor} \frac{(e^{\delta\beta}T)^n}{n!} P(N' \leq e^{2\delta\beta}\rho T) + SES \tag{4.9}$$

with N' a Poisson variable of mean $\mathrm{cst} \cdot R^2/T$. For any $\delta_1 > \delta$, if $R \geq e^{\delta_1\beta}\sqrt{\rho}T$ the last probability can be estimated from above by

$$P(N' \leq e^{2\delta\beta}\rho T) \leq \exp\{-\mathrm{cst}\sqrt{\rho}R\} + SES \tag{4.10}$$

and the remaining sum can be estimated from above by

$$\sum_{n=1}^{\lfloor e^{\delta\beta}\rho T \rfloor} \frac{(e^{\delta\beta}T)^n}{n!} \leq \exp\{e^{\delta\beta}T\} P(N \leq e^{\delta\beta}\rho T) + SES \tag{4.11}$$

with N a Poisson variable of mean $e^{\delta\beta}T$, so that, by (2.2),

$$\sum_{n=1}^{\lfloor e^{\delta\beta}\rho T \rfloor} \frac{(e^{\delta\beta}T)^n}{n!} \leq \exp\{e^{\delta\beta}T\} \exp\{-e^{\delta\beta}T\big((1-\rho) + \rho\ln\rho\big)\} \tag{4.12}$$

$$= \exp\{e^{\delta\beta}T - e^{\delta\beta}T + e^{\delta\beta}T\rho + e^{\delta\beta}T\rho\Delta\beta\} \tag{4.13}$$

$$\leq \exp\{e^{2\delta\beta}\rho T\} + SES. \tag{4.14}$$

Using (4.9), (4.10) and (4.14) we get, for any R, T,

$$Q_1(R,T) \leq \exp\{e^{2\delta\beta}\rho T\} \exp\{-\mathrm{cst}\sqrt{\rho}R\mathbb{1}_{[e^{\delta_1\beta}\sqrt{\rho}T,+\infty[}(R)\} + SES. \tag{4.15}$$

We can estimate Q_2 in the same way and the rest of the calculation goes like in Section 3. □

Proof of Corollary 1.2. We distinguish between two cases: $K < \Delta$ and $K \geq \Delta$. In the former case (1.21) is a consequence of the last remarks of Section 2.2:

interactions are restricted to clouds of potentially interacting particles on time scale $T_\alpha > T$ for a small enough α, then (1.21) follows from the nonsuperdiffusivity property (2.31). In the latter case (1.21) follows from Theorem 2 applied with $\delta' := \delta/4$ and $T' := e^{\delta\beta/2}T$ instead of δ and T. □

4.3 Proof of Theorem 3

Given $\Lambda^{(1)}$ and $\Lambda^{(2)}$ with the condition (1.23) we define a new coloring process. With

$$B := \Lambda^{(1)} \cup \Lambda^{(2)} \tag{4.16}$$

and

$$W := \left\{ z \in \Lambda_\beta : \inf_{b\in B} |z - b| > e^{-\delta\beta/2} d\big(\Lambda^{(1)}, \Lambda^{(2)}\big) \right\} \tag{4.17}$$

we say that all the particles that start from B are black, all the particles that start from W are white and all the particles that start from $(B \cup W)^c$ do not have any color at time $t = 0$. Then, for $t > 0$, black particles keep their black color, white particles keep their white color, noncolored particles that enter B turn black, non-colored particles that enter W turn white, and noncolored particles that share some cluster with a colored particle turn black or white choosing randomly a colored particle inside the cluster and taking the same color. We can define a black zone and a white zone like we defined the red zone. As a consequence of Corollary 1.2, with probability $1 - SES$, the black and white zones will not intersect up to time $T \wedge \mathcal{T}_{\alpha,\lambda}$ and we will never see black and white particles in a same cluster up to time $T \wedge \mathcal{T}_{\alpha,\lambda}$.

Now we couple in the more natural way the previous process, with a process that starts from the same initial configuration, uses the same marks and clocks for the particles and evolves in the same way except for the fact that each particle in W or that enters in W disappears. For this process the restrictions of the dynamics to $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are clearly independent and the previous observation shows that, with probability $1 - SES$, these restrictions for the two processes coincide up to time $T \wedge \mathcal{T}_{\alpha,\lambda}$. This proves the theorem.

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