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IS BROWNIAN MOTION NECESSARY TO MODEL HIGH-FREQUENCY DATA?

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This paper considers the problem of testing for the presence of a continuous part in a semimartingale sampled at high frequency. We provide two tests, one where the null hypothesis is that a continuous component is present, the other where the continuous component is absent, and the model is then driven by a pure jump process. When applied to high-frequency individual stock data, both tests point toward the need to include a continuous component in the model.

1. Introduction. This paper continues our development of statistical methods designed to assess the specification of continuous-time models sampled at high frequency. The basic framework, inherited from theoretical models in mathematical finance but also common in other fields such as physics or biology, is one where the variable of interest X, in financial examples often the log of an asset price, is assumed to follow an Itô semimartingale. That semimartingale is observed on some fixed time interval $[0, T]$ at discrete regularly spaced times $i\Delta_n$, with a time lag Δ_n which is small.

A semimartingale can be decomposed into the sum of a drift, a continuous Brownian-driven part and a discontinuous, or jump, part. The jump part can in turn be decomposed into a sum of "small jumps" and "big jumps." Such a process will always generate a finite number of big jumps, but it may give rise to either a finite or infinite number of small jumps, corresponding to the finite and infinite jump activity situations, respectively. In earlier work, we developed tests to determine on the basis of the observed sampled path on $[0, T]$ whether a jump part was present, whether the jumps had finite or infinite activity, and in the latter situation proposed a definition and an estimator of a "degree of jump activity" parameter.

In this paper, we tackle the last remaining question: Does the semimartingale need to have a continuous part? In other words, is the Brownian motion present at all? From a model specification standpoint, there is a natural statistical interest in distinguishing the two situations where a continuous part is included or not, on the basis of an observed sample path. When there are no jumps, or finitely many

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jumps, and no Brownian motion, X reduces to a pure drift plus occasional jumps, and such a model is fairly unrealistic in the context of most financial data series, although it may be realistic in some other contexts. But for financial applications one can certainly consider models that consist only of a jump component, plus perhaps a drift, if that jump component is allowed to be infinitely active.

Many models in mathematical finance do not include jumps. But among those that do, the framework most often adopted consists of a jump-diffusion: these models include a drift term, a Brownian-driven continuous part and a finite activity jump part (see, e.g., [6, 7] and [16]). When infinitely many jumps are included, however, there are a number of models in the literature which dispense with the Brownian motion altogether. The log-price process is then a purely discontinuous Lévy process with infinite activity jumps or, more generally, is driven by such a process (see, e.g., [9, 10] and [14]).

The mathematical treatment of models relying on pure jump processes is quite different from the treatment of models where a Brownian motion is present. For instance, risk management procedures, derivative pricing and portfolio optimization are all significantly altered, so there is interest from the mathematical finance side in finding out which model is more likely to have generated the data.

For all these reasons, it is of importance to construct procedures which allow us to decide whether the Brownian motion is really here, or if it can be forgone in favor of a pure jump process. This is the aim of this paper: we will provide two tests allowing for a symmetric treatment of the two situations where the null hypothesis is that the Brownian motion is present, and where the null is that the Brownian motion is absent.

In the context of a specific parametric model, allowing for jump components of finite or infinite activity on top of a Brownian component, [8] find that the time series of index returns are likely to be devoid of a continuous component. An alternative but related approach to testing for the presence of a Brownian motion component to the one we propose here is due to [17]. They employ the test statistic for jumps of [5], plot its logarithm for different values of the power argument and contrast the behavior of the plot above two and below two in order to identify the presence of a Brownian component. A formal test is constructed under the null hypothesis where a continuous component is present.

The methodology that both [17] and we employ to design our respective test statistics is based on tried-and-true principles that originate in our earlier work on testing whether jumps are present [5], whether they have finite or infinite activity [3] and on estimating the index of jump activity [4], although, of course, exploited in a manner specific to the problem at hand. We compute power variations of the increments, suitably truncated and/or sampled at different frequencies. Exploiting the different asymptotic behavior of the variations as we vary these parameters gives us enough flexibility to accomplish our objectives. As is well known, powers below two will emphasize the continuous component of the underlying sampled process. Powers above two will conversely accentuate its jump component. The

power two puts them on an equal footing. Truncating the large increments at a suitably selected cutoff level can eliminate the big jumps when needed, as was shown by [15]. Finally, sampling at different frequencies can let us distinguish between situations where the variations converge to a finite limit, in which case the ratio of two variation measures constructed at different frequencies will converge to one, from situations where the variations converge to either zero or diverge to infinity, in which case the ratio will typically converge to a different constant. Since these various limiting behaviors are indicative of which component of the model dominates at a particular power, they effectively allow us to distinguish between all manners of null and alternative hypotheses.

This said, the commonality of approach should not mask the fact that each situation is, in reality, mathematically quite different. By nature, certain components of the model are turned off under particular null hypotheses. For instance, when the null hypothesis is that no Brownian motion is present, as will be the case for our first test here, then jumps drive the asymptotics. As a result, the driving component of the model that matters for the asymptotic behavior of the statistic will vary with the situation and consequently the methods employed behind the scenes to obtain the desired asymptotics will vary accordingly.

The paper is organized as follows. Section 2 describes our model and the statistical problem. Our testing procedure is described in Section 3, and the next two Sections, 4 and 5, are devoted to a simulation study of the tests and an empirical implementation of our tests on high-frequency stock returns. Section 6 is devoted to technical results and to the proof of the main theorems.

2. The model. The underlying process X which we observe at discrete times is a 1-dimensional Itô semimartingale defined on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which means that its characteristics (B, C, ν) are absolutely continuous with respect to Lebesgue measure. B is the drift, C is the quadratic variation of the continuous martingale part and ν is the compensator of the jump measure μ of X. In other words, we have

$$\begin{aligned} B_t(\omega) &= \int_0^t b_s(\omega)\,ds, \qquad C_t(\omega) = \int_0^t \sigma_s(\omega)^2\,ds, \\ \nu(\omega, dt, dx) &= dt\, F_t(\omega, dx). \end{aligned} \tag{1}$$

Here b and σ are optional process, and $F = F_t(\omega, dx)$ is a transition measure from $\Omega \times \mathbb{R}_+$ endowed with the predictable σ-field into $\mathbb{R} \setminus \{0\}$. More customarily, one may write X as

$$\begin{aligned} X_t = X_0 &+ \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s \\ &+ \int_0^t \int x 1_{\{|x|\leq 1\}}(\mu - \nu)(ds, dx) + \int_0^t \int x 1_{\{|x|>1\}}\mu(ds, dx), \end{aligned} \tag{2}$$

where W is a standard Brownian motion. It is also possible to write the last two terms above as integrals with respect to a Poisson measure and its compensator, but we do not need this here. This is a standard setup and we refer the reader to [13] for details.

We have referred above to "small jumps" and "big jumps." In the context of (2), they are represented, respectively, by the last two integrals. The size cutoff 1 adopted here is arbitrary and could be replaced by any fixed $\varepsilon > 0$, a change which amounts merely to an adjustment to the drift term B_t. Note that the small jumps integral needs to be compensated by ν since there are potentially an infinite number of such small jumps. The large jump integral is always a finite sum; it may be compensated if desired but this is not necessary. Any compensation or lack thereof is then again absorbed by an adjustment to the drift.

We now turn to the assumptions. As usual for tests, the assumptions essentially ensure that one can compute and then estimate a significance level under the null hypothesis. So here, we need some structure for the jumps of X, namely that the small jumps essentially behave like the small jumps of a stable process with some index β, up to a random intensity. As noted above, when no Brownian is present, we view the realistic situation as one where there are infinitely many small jumps. When the null is that there is a Brownian motion, we need the additional assumption that the volatility process σ_t is itself an Itô semimartingale.

We would like to give tests with a prescribed asymptotic level, as $n \to \infty$, and, of course, this is more difficult when β increases because then the process resembles more and more a continuous process plus a few big jumps: The qualitative behavior of the paths can become quite similar whether the Brownian motion is present or not. So, unsurprisingly, we can exhibit a test with prescribed level, for the null hypothesis where the Brownian motion is present, only when $\beta < 1$. The parameter β is typically unknown (although a method for estimating β in this setting is given in [4]). On the other hand, for the null hypothesis where the Brownian motion is absent we provide a test which works under no assumption on β.

With this context in mind, here is the first assumption which will be assumed throughout:

ASSUMPTION 1. (i) *The drift process b_t is locally bounded and the volatility process σ_t is càdlàg.*

(ii) *There are three constants $0 \le \beta'' \le \beta' < \beta < 2$ and a locally bounded process $L_t \ge 1$, such that the Lévy measure F_t is of the form $F_t = F'_t + F''_t$, where*

$$F'_t(dx) = \frac{\beta(1 + |x|^{\beta-\beta'} f(t,x))}{|x|^{1+\beta}} \big(a_t^{(+)} 1_{\{0<x\le z_t^{(+)}\}} + a_t^{(-)} 1_{\{-z_t^{(-)}\le x<0\}}\big)\, dx, \tag{3}$$

where $a_t^+, a_t^-, z_t^+, z_t^-$ are nonnegative predictable processes and $f = f(\omega, t, x)$ is predictable function (meaning $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable, where $\mathcal{P}$ is the predictable

σ-field on $[0,\infty)\times\Omega$), *satisfying*

$$\begin{aligned} &\frac{1}{L_t} \le z_t^{(+)} \le 1, \qquad \frac{1}{L_t} \le z_t^{(-)} \le 1, \\ &A_t := a_t^{(+)} + a_t^{(-)} \le L_t, \qquad |f(t,x)| \le L_t, \end{aligned} \tag{4}$$

and where F_t'' is a measure which is singular with respect to F_t' and satisfies

$$\int (|x|^{\beta''} \wedge 1) F_t''(dx) \le L_t. \tag{5}$$

This assumption is identical to Assumptions 1 and 2 of [4] [with some notational changes: $(\gamma, \beta', a_t^+, a_t^-)$ in that paper are called here $(\beta-\beta', \beta'', \beta a_t^+, \beta a_t^-)$, and the condition $\beta'' \le \beta'$ is not a restriction and is put here only for convenience].

For example, take a process solution of the stochastic differential equation

$$dX_t = b_t\,dt + \sigma_t\,dW_t + \delta_{t-}\,dY_t + \delta'_{t-}\,dY'_t, \tag{6}$$

where δ and δ' are càdlàg adapted processes, Y is β-stable or tempered β-stable and Y' is any other Lévy process whose Lévy measure integrates $|x|^{\beta''}$ near the origin and has an absolutely continuous part whose density is smaller than $K|x|^{-(1+\beta')}$ on $[-1,1]$ for some $K>0$ (e.g., a stable process with index strictly smaller than β'). Then X will satisfy Assumption 1.

If this assumption is satisfied with $\beta<1$, then almost surely the jumps have finite variation $\sum_{s\le t}|\Delta X_s|<\infty$ for all t or equivalently, $\int_0^t\int |x|\mu(ds,dx)<\infty$. This allows us to decompose X into the sum $X=X'+X''$, where

$$X'_t = X_0 + \int_0^t b'_s\,ds + \int_0^t \sigma_s\,dW_s, \qquad X''_t = \sum_{s\le t}\Delta X_s, \tag{7}$$

and where $b'_t = b_t - \int x 1_{\{|x|\le 1\}} F_t(dx)$ is a locally bounded process.

For clarity, we will derive the properties of both tests under the same generic Assumption 1 even though the properties of the test for the null of a Brownian present remain valid under weaker assumptions. When the null hypothesis to be tested is that the Brownian motion is present, it becomes the driving process for our test statistic and as is customary for tests or estimation problems involving a stochastic volatility, we then need an additional regularity assumption on the σ process:

ASSUMPTION 2. *We have Assumption 1 with $\beta<1$. Moreover the volatility process σ_t is an Itô semimartingale, that is, it can be written (necessarily in a unique way) as*

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s\,ds + \int_0^t \widetilde{\sigma}_s\,dW_s + N_t + \sum_{s\le t}\Delta\sigma_s 1_{\{|\Delta\sigma_s|>1\}}, \tag{8}$$

where N is a local martingale which is orthogonal to the Brownian motion W, and further the compensator of the process $[N,N]_t + \sum_{s\le t} 1_{\{|\Delta\sigma_s|>1\}}$ is of the form $\int_0^t n_s\,ds$. Moreover we suppose that:

(i) *the processes $\widetilde{b}_t$ and n_t are locally bounded*;
(ii) *the processes $\widetilde{\sigma}_t$ and b'_t defined above are càdlàg*.

3. The two tests.

3.1. *The hypotheses to be tested.* In a semimartingale model like (2), saying that the Brownian motion W is absent on the interval $[0,T]$ does not mean that there is no Brownian motion on the probability space (something which cannot be tested at all, obviously) but it means that the Brownian motion does not impact the observed process X, in the sense that the corresponding stochastic integral vanishes on this interval, or equivalently $\sigma_s = 0$ for Lebesgue-almost all s in $[0,T]$, and it would be more appropriate to say that we are testing whether "the continuous martingale part of X vanishes on $[0,T]$, or not." This is typically an ω-wise property: we can divide the set Ω into two complementary subsets

$$\Omega_T^W = \left\{\int_0^T \sigma_s^2\,ds > 0\right\}, \qquad \Omega_T^{noW} = \left\{\int_0^T \sigma_s^2\,ds = 0\right\}. \tag{9}$$

Then almost surely on the set Ω_T^{noW} the integral process $X_t^c = \int_0^t \sigma_s\,dW_s$ vanishes on $[0,T]$, whereas it does not vanish on the complement Ω_T^W. In what follows, we take Ω_T^W to represent the hypothesis that the Brownian motion is present and Ω_T^{noW} to represent the hypothesis that the Brownian motion is not present.

In connection with Assumption 1 we consider the following set representing paths that have infinite jump activity of some index $\beta \in (0,2)$:

$$\Omega_T^{i\beta} = \{\overline{A}_T > 0\} \qquad \text{where } \overline{A}_t = \int_0^t A_s\,ds. \tag{10}$$

One knows that on the set $\Omega_T^{i\beta}$ the path of X over $[0,T]$ has almost surely infinitely many jumps.

We are interested in testing the following two situations:

$$\begin{cases} H_0 : \Omega_T^W & \text{vs.} \quad H_1 : \Omega_T^{noW}, \\ H_0 : \Omega_T^{noW} & \text{vs.} \quad H_1 : \Omega_T^W. \end{cases} \tag{11}$$

As discussed above, the realistic situation supposes that infinite activity jumps are present when under Ω_T^{noW} and so we will in fact provide testing procedures for the following two situations:

$$\begin{cases} H_0 : \Omega_T^W & \text{vs.} \quad H_1 : \Omega_T^{noW} \cap \Omega_T^{i\beta}, \\ H_0 : \Omega_T^{noW} \cap \Omega_T^{i\beta} & \text{vs.} \quad H_1 : \Omega_T^W \cap \Omega_T^{i\beta}. \end{cases} \tag{12}$$

In the second test, requiring $\Omega_T^{i\beta}$ with Ω_T^W under H_1 allows us to characterize precisely the properties of the statistic under this alternative (as opposed to just Ω_T^W). But it is not necessary for the actual implementation of the test which relies on its behavior under the null.

Finally, we recall that testing a null hypothesis "we are in a subset Ω_0" of Ω, against the alternative "we are in a subset Ω_1," with, of course, $\Omega_0 \cap \Omega_1 = \varnothing$, amounts to finding a critical (rejection) region $C_n \subset \Omega$ at stage n. The asymptotic size and asymptotic power for this sequence (C_n) of critical regions are the following numbers:

$$\begin{cases} a = \sup\Big(\limsup_n \mathbb{P}(C_n \mid A) : A \in \mathcal{F},\, A \subset \Omega_0, \mathbb{P}(A) > 0\Big), \\ P = \inf\Big(\liminf_n \mathbb{P}(C_n \mid A) : A \in \mathcal{F},\, A \subset \Omega_1, \mathbb{P}(A) > 0\Big). \end{cases} \tag{13}$$

3.2. *The building blocks.* Before stating the results, we introduce some notation to be used throughout. We observe the increments of X

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}, \tag{14}$$

to be distinguished from the (unobservable) jumps of the process, $\Delta X_s = X_s - X_{s-}$. In a typical application, X is a log-asset price, so $\Delta_i^n X$ is the recorded log-return over Δ_n units of time.

For any given cutoff level $u > 0$ we count the number of increments of X with size bigger than u, that is,

$$U(u, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} 1_{\{|\Delta_i^n X| > u\}}. \tag{15}$$

If $p > 0$ we also sum the pth absolute power of the increments of X, truncated at level u, that is,

$$B(p, u, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p 1_{\{|\Delta_i^n X| \le u\}}. \tag{16}$$

B is what we call a "truncated power variation." Note that in B we are retaining all increments smaller than u, whereas in U we are retaining those larger than u.

We take a sequence u_n of positive numbers, which will serve as our thresholds or cutoffs for truncating the increments, and will go to 0 as the sampling frequency increase. There will be restrictions on the rate of convergence of this sequence, expressed in the form

$$u_n/\Delta_n^{\rho-} \to 0, \qquad u_n/\Delta_n^{\rho+} \to \infty \qquad \text{for some } 0 \le \rho_- < \rho+ < \tfrac{1}{2}. \tag{17}$$

This condition becomes weaker when ρ_+ increases and when ρ_- decreases.

In practice, when a Brownian motion is present, we will often translate values of the cutoff level u_n in terms of a number of standard deviations of the continuous part of the semimartingale. That is, we express values of u_n in terms of

α_n where $u_n = \alpha_n (t^{-1} \int_0^t \sigma_s^2 \, ds)^{1/2} \Delta_n^{1/2}$. Despite the presence of jumps, the integrated volatility in that expression can be estimated using the small increments of the process, since

$$\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 1_{\{|\Delta_i^n X| \le \alpha \Delta_n^{\varpi}\}} \xrightarrow{\mathbb{P}} \int_0^t \sigma_s^2 \, ds \tag{18}$$

for any $\alpha > 0$ and $\varpi \in (0, 1/2)$. We can then vary the cutoff level α_n to yield a number of (estimated) standard deviations of the continuous part of the semimartingale. This data-driven choice can help determine a range of reasonable values for the cutoff level and provide on a path-by-path basis an equivalent, but perhaps more intuitive, scale with which to measure the magnitude of the cutoff level u_n.

When there is no Brownian motion under the null, a different scale needs to be used to assess the size of u_n. For example, we can translate u_n into the percentage of the sample that is greater than the cutoff level, and therefore not included in the computation of the truncated power variations.

3.3. *Testing for the presence of Brownian motion under the null.* In a first case, we set the null hypothesis to be "the Brownian motion is present," that is Ω_T^W, against the alternative $\Omega_T^{noW} \cap \Omega_T^{i\beta}$.

In order to construct a test, we seek a statistic with markedly different behavior under the null and alternative. One fairly natural idea is to consider powers less than 2 since in the presence of Brownian motion they would be dominated by it, while in its absence they would behave quite differently. Specifically, the large number of small increments generated by a continuous component would cause a power variation of order less than 2 to diverge to infinity. Without the Brownian motion, however, and when $p > \beta$, the power variation converges to 0 at exactly the same rate for the two sampling frequencies Δ_n and $k\Delta_n$, whereas in the former case the choice of sampling frequency will influence the magnitude of the divergence. Taking a ratio will eliminate all unnecessary aspects of the problem and focus on the key aspect, that of distinguishing between the presence and absence of the Brownian motion.

Specifically, we fix a power $p \in (0, 2)$ and an integer $k \ge 2$, and we consider the test statistics, which depend on p and on the terminal time T and on the sequence u_n subject to (17), as follows:

$$S_n = \frac{B(p, u_n, \Delta_n)_T}{B(p, u_n, k\Delta_n)_T}. \tag{19}$$

As will become clear below, taking ratios of power variations has the advantage of making the test statistic model-free. That is, its distribution under the null hypothesis can be assessed without the need for the extraneous estimation of the dynamics of the process in (2). Obviously, these dynamics can be quite complex with

potentially jumps of various activity levels, stochastic volatility, jumps in volatility, etc. So the fact that the standardized test statistic can be computed without the need to estimate the various parts of (2) is a desirable feature. In fact, implementing the test—that is, computing the statistic in (19) and estimating its asymptotic variance—will require nothing more than the computation of various truncated power variations.

The first result is a law of large numbers (LLN) giving the probability limit of the statistic S_n.

THEOREM 1. *Under Assumption 1 and if $p \in (1,2)$, we have*

$$S_n \xrightarrow{\mathbb{P}} \begin{cases} k^{1-p/2}, & \text{on the set } \Omega_T^W, \\ 1, & \text{on the set } \Omega_T^{noW} \cap \Omega_T^{i\beta}, \text{ if } p > \beta \vee 1, \rho_+ \le \dfrac{p-1}{p}. \end{cases} \tag{20}$$

This result shows that, since $k^{1-p/2} > 1$, for the test at hand an a priori reasonable critical region is $C_n = \{S_n < c_n\}$, for a sequence c_n increasing strictly to $k^{1-p/2}$: in this case the asymptotic power is 1 in restriction to the set described in the second alternative above, whereas the asymptotic level depends on how fast c_n converges to $k^{1-p/2}$.

For a more refined version of this test, with a prescribed level $a \in (0,1)$, we need a central limit theorem (CLT) associated with the convergence in (20). For this we need some notation: letting Z and Z' be two independent $\mathcal{N}(0,1)$ variables, we set

$$\begin{cases} m_p = \mathbb{E}(|Z|^p), \\ m_{k,p} = \mathbb{E}\big(|Z|^p \big| Z + \sqrt{k-1}Z'|^p\big), \\ N(p,k) = \dfrac{1}{m_{2p}}\big(k^{2-p}(1+k)m_{2p} + k^{2-p}(k-1)m_p^2 - 2k^{3-3p/2}m_{k,p}\big). \end{cases} \tag{21}$$

In terms of known functions, we have

$$\begin{cases} m_p = \dfrac{2^{p/2}}{\sqrt{\pi}}\Gamma\Big(\dfrac{p+1}{2}\Big), \\ m_{k,p} = \dfrac{2p}{\pi}(k-1)^{p/2}\Gamma\Big(\dfrac{1+r}{2}\Big)^2 F_{2,1}\Big(-\dfrac{p}{2}; \dfrac{p+1}{2}; \dfrac{1}{2}; \dfrac{-1}{k-1}\Big), \end{cases} \tag{22}$$

where $F_{2,1}$ is Gauss's hypergeometric function (see, e.g., Section 15.1 of [1]).

Then the standardized version of the CLT goes as follows (we use $\xrightarrow{\mathcal{L}-(s)}$ to denote the stable convergence in law (see, e.g., [13] for this notion); to explain the following statement, we recall that the convergence in law "in restriction to a subset Ω_0" is meaningless, but the stable convergence in law in restriction to Ω_0 makes sense):

THEOREM 2. *Suppose that Assumption 2 holds, take* $p \in (1,2)$ *and let the sequence* u_n *satisfy* (17) *with* $\rho_- > \frac{p-1}{2p-2\beta}$. *Then we have the following convergence in law*:

$$(S_n - k^{1-p/2})/\sqrt{V_n} \xrightarrow{\mathcal{L}-(s)} \mathcal{N}(0,1) \qquad \text{in restriction to } \Omega_T^W, \tag{23}$$

where

$$V_n = N(p,k)\frac{B(2p,u_n,\Delta_n)_T}{(B(p,u_n,\Delta_n)_T)^2}. \tag{24}$$

We are now ready to exhibit a critical region for testing $H_0 : \Omega_T^W$ vs. $H_1 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$ using S_n with a prescribed asymptotic level $a \in (0,1)$. Denoting by z_a the a-quantile of $N(0,1)$, that is, $\mathbb{P}(Z > z_a) = a$ where Z is $N(0,1)$, we set

$$C_n = \{S_n < k^{1-p/2} - z_a\sqrt{V_n}\}. \tag{25}$$

THEOREM 3. *Suppose that Assumption 2 holds. Let* $p \in (1,2)$ *and let the sequence* u_n *satisfy* (17) *with*

$$\frac{p-1}{2p-2\beta} = \rho_- < \rho_+ = \frac{p-1}{p} \qquad (\text{hence } p > 2\beta). \tag{26}$$

Then the asymptotic level of the critical region defined by (25) *for testing the null hypothesis "the Brownian motion is present"* (*i.e.*, Ω_T^W *against* $\Omega_T^{noW} \cap \Omega_T^{i\beta}$) *equals* a, *and the asymptotic power equals* 1.

To perform the test we need to choose p and the sequence u_n. In practice one does not know β, although it should be smaller than 1 by Assumption 2. Hence if we are willing to assume that β, although unknown, is not bigger than some prescribed $\beta_0 < 1$, one should choose $p \in (2\beta_0, 2)$, and one may take $u_n = \alpha\Delta_n^{\varpi}$ for some $\alpha > 0$ and some $\varpi \in (0, 1/2)$, and the test can be done as soon as

$$\frac{p-1}{2p-2\beta_0} < \varpi < \frac{p-1}{p}. \tag{27}$$

To properly separate the two hypotheses it is probably wise to choose p closer to $2\beta_0$ than to 2.

REMARK 1. The first part of the consistency result (20) holds also for $p \in (0,1]$ on Ω_T^W (with basically the same proof). The second part also holds for $\beta < p \le 1$ on the set on which $X_t = X_0 + \sum_{s\le t} \Delta X_s$ for all $t \le T$, that is, when there is no drift, whereas when there is a drift, S_n converges to k^{1-p} for all $p \in (0,1]$. When $0 < p \le \beta$ the limit of S_n is $k^{1-p/\beta}$ on $\Omega_T^{noW} \cap \Omega_T^{i\beta}$ when $p > 1$, and also when $p \le 1$ when again there is no drift (and the proof is more involved). Figure 1 illustrates these various limits in the case X is the sum of a Brownian martingale plus possibly a Cauchy process (with no drift).

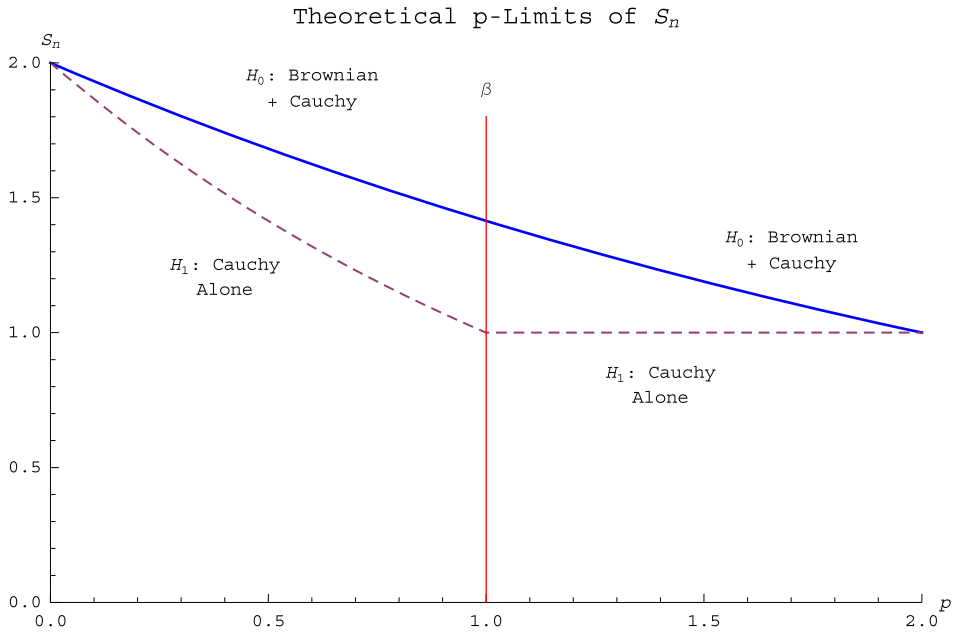

FIG. 1. *Probability limits as a function of p of the test statistic S_n with $k = 2$ in the case of a Cauchy process ($\beta = 1$, H_0) and a Brownian plus Cauchy processes (H_1).*

REMARK 2. The CLT necessitates $p \in (1, 2)$. However, more sophisticated techniques would allow us to prove the same result for all $p \in (0, 2)$, under the additional assumption that σ_t does not vanish for $t \in [0, T]$, on the set Ω_T^W (we still need $\beta < 1$, however).

REMARK 3. Despite the fact that using powers less than 2 is the most natural way to isolate the contribution of the Brownian motion to the overall increments of the process, it is possible to design an alternative test that relies on powers greater than 2. Instead of the statistic S_n above, we could use the following statistic: pick $\gamma > 1$ and $p' > p > 2$, and set

$$\overline{S}_n = \frac{B(p', \gamma u_n, \Delta_n)_T B(p, u_n, \Delta_n)_T B(2, \gamma u_n, \Delta_n)_T}{B(p', u_n, \Delta_n)_T B(p, \gamma u_n, \Delta_n)_T B(2, u_n, \Delta_n)_T}. \tag{28}$$

Under Assumption 1, $\overline{S}_n$ converges in probability to $\gamma^{p'-p}$ on the set $\Omega_T^W \cap \Omega_T^{i\beta}$, and to $\gamma^{p'-p+2-\beta}$ on the set $\Omega_T^{noW} \cap \Omega_T^{i\beta}$, as soon as $\rho_+ < \frac{p-2}{2p-2\beta}$. We also have a CLT under Assumption 2 and if $\rho_+ < \frac{2p-4}{11p-10}$. Under H_0, $\overline{S}_n$ is model-free, just like S_n is. So one can, in an obvious way, construct a test based on $\overline{S_n}$ and which satisfies the claims of Theorem 3, under suitable conditions on the cutoff levels u_n. However, simulations studies suggest that the statistic $\overline{S_n}$ is not as well behaved as S_n, and so we do not pursue its study further.

3.4. *Testing for the absence of Brownian motion under the null.* In a second case, we set the null hypothesis to be "the Brownian motion is absent," that is, $\Omega_T^{noW} \cap \Omega_T^{i\beta}$. Designing a test under this null is trickier because the model becomes a pure jump (plus perhaps a drift) process, and we are aiming for a test that remains model-free even for this model. That is, we are looking for a statistic whose limiting behavior under the null, despite being driven by what is now a pure jump process, does not depend on the characteristics of the pure jump process, such as its degree of activity β, since those characteristics are a priori unknown.

This can be achieved as follows. We choose a real $\gamma > 1$ and a sequence u_n satisfying (17) and define the test statistic

$$S'_n = \frac{B(2, \gamma u_n, \Delta_n)_T U(u_n, \Delta_n)_T}{B(2, u_n, \Delta_n)_T U(\gamma u_n, \Delta_n)_T}. \tag{29}$$

To understand the construction of this test statistic, recall that in a power variation of order 2 the contributions from the Brownian and jump components are of the same order. But once the power variation is properly truncated, the Brownian motion will dominate it if it is present. And the truncation can be chosen to be sufficiently loose that it retains essentially all the increments of the Brownian motion at cutoff level u_n and a fortiori γu_n, thereby making the ratio of the two truncated quadratic variations converge to 1 under the alternative hypothesis. On the other hand, if the Brownian motion is not present, then the nature of the tail of jump distributions is such that the difference in cutoff levels between u_n and γu_n remains material no matter how far we go in the tail, and the limit of that same ratio will reflect it: it will now be $\gamma^{2-\beta}$ under assumptions made specific in the formal theorems below. Since absence of a Brownian motion is now the null hypothesis, the issue is then that this limit depends on the unknown β.

Canceling out that dependence is the role devoted to the ratio of the number of large increments, the U's, in (29). The U's are always dominated by the jump components of the model whether the Brownian motion is present or not. Their inclusion in the statistic is merely to ensure that the statistic is model-free, by effectively canceling out the dependence on the jump characteristics that emerges from the ratio of the truncated quadratic variations. Indeed, the limit of the ratio of the U's is γ^β under both the null and alternative hypotheses. As a result, the probability limit of S'_n will be γ^2 under the null, independent of β.

Our first result states this precisely, establishing the limiting behavior of the statistic in terms of convergence in probability:

THEOREM 4. *Let the sequence u_n satisfy* (17), *and suppose that Assumption* 1 *holds. Then*

$$S'_n \xrightarrow{\mathbb{P}} \begin{cases} \gamma^2, & \text{on the set } \Omega_T^{noW} \cap \Omega_T^{i\beta}, \\ \gamma^\beta, & \text{on the set } \Omega_T^{W} \cap \Omega_T^{i\beta}. \end{cases} \tag{30}$$

For a test with a prescribed level we need a standardized CLT.

THEOREM 5. *Suppose that Assumption 1 holds with* $\beta'' < \frac{\beta}{2+\beta}$ *and* $\beta' < \frac{\beta}{2}$, *and* (17) *holds with* $\rho_+ < \frac{1}{2+\beta} \wedge \frac{2}{5\beta} \wedge \frac{2-\beta}{3\beta}$. *Then we have*

$$(S'_n - \gamma^2)/\sqrt{V'_n} \xrightarrow{\mathcal{L}-(s)} \mathcal{N}(0,1) \qquad \text{in restriction to } \Omega_T^{noW} \cap \Omega_T^{i\beta}, \tag{31}$$

where V'_n *is given by the following formula*:

$$\begin{aligned} V'_n = \gamma^4 \Bigg(& \frac{B(4,u_n,\Delta_n)_T}{(B(2,u_n,\Delta_n)_T)^2} + \frac{1}{U(u_n,\Delta_n)} \\ & + \left(1 - \frac{2}{\gamma^2}\right)\left(\frac{B(4,\gamma u_n,\Delta_n)_T}{(B(2,\gamma u_n,\Delta_n)_T)^2} + \frac{1}{U(\gamma u_n,\Delta_n)}\right)\Bigg). \end{aligned} \tag{32}$$

Hence a critical region for testing $H_0 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$ vs. $H_1 : \Omega_T^{W} \cap \Omega_T^{i\beta}$ is

$$C'_n = \{S'_n < \gamma^2 - z_a\sqrt{V'_n}\}. \tag{33}$$

THEOREM 6. *Suppose that Assumption 1 holds with* $\beta'' < \frac{\beta}{2+\beta}$ *and* $\beta' < \frac{\beta}{2}$, *and* (17) *holds with* $\rho_+ < \frac{1}{2+\beta} \wedge \frac{2}{5\beta} \wedge \frac{2-\beta}{3\beta}$. *Then the asymptotic level of the critical region* C'_n *defined by* (33) *for testing the null hypothesis "the Brownian motion is absent"* (*i.e.*, $\Omega_T^{noW} \cap \Omega_T^{i\beta}$ *against* $\Omega_T^{W} \cap \Omega_T^{i\beta}$) *equals* a, *and the asymptotic power equals* 1.

If we take again $u_n = \alpha\Delta_n^{\varpi}$, the test can be performed if $\alpha > 0$ and

$$0 < \varpi < \frac{1}{2+\beta} \wedge \frac{2}{5\beta} \wedge \frac{2-\beta}{3\beta} \tag{34}$$

(always smaller than $1/2$). This requirement is constraining, because β is unknown, and may typically be close to 2 if we believe in the null hypothesis. Therefore in practice we must assume that β does not exceed a given $\beta_0 \in [1,2)$. This means that this limiting index β_0 is given a priori, and we do the test under the Assumption 1 with $2\beta' < \beta \le \beta_0$ and $\beta'' < \frac{\beta}{2+\beta}$, with ϖ subject to the (feasible) condition

$$0 < \varpi < \frac{2-\beta_0}{3\beta_0}. \tag{35}$$

These facts are not really surprising: first, by (30) we know that the statistic S'_n properly separates the two hypotheses only when β is not too close to 2. And, second, when β becomes very close to 2, the paths of X have big jumps but also the compensated sum of small jumps looks more and more like a Brownian path, even on the set Ω_T^{noW}.

REMARK 4. It is possible to design an alternative statistic with similar properties but make no use of the U's. Instead of the statistic S'_n in (29), we could use the following statistic: pick $\gamma > 1$, $\kappa \geq 1$ and $p > 2$, and set

$$\overline{S}'_n = \frac{B(2, u_n, \Delta_n)_T B(p, \kappa\gamma u_n, \Delta_n)_T}{B(2, \gamma u_n, \Delta_n)_T B(p, \kappa u_n, \Delta_n)_T}. \tag{36}$$

Under Assumption 1, $\overline{S}'_n$ converges in probability to γ^{p-2} on the set $\Omega_T^{noW} \cap \Omega_T^{i\beta}$, and to $\gamma^{p-\beta}$ on the set $\Omega_T^W \cap \Omega_T^{i\beta}$, as soon as $\rho_+ < \frac{p-2}{2p}$. The ratio of pth power variations plays a similar role to that of the U's, namely to cancel out the dependence of the p-lim of $\overline{S}'_n$ on β under the null. The fixed scaling factor κ allows us to use different cutoff levels for the two powers p and 2 without affecting the probability limit of the statistic. We also have a CLT if $\rho_+ < \frac{2-\beta}{3\beta}$. Under H_0, $\overline{S}'_n$ is model-free, just like S'_n is, and so a test follows. But as was the case for the statistic $\overline{S}_n$ proposed in (28), simulations studies suggest that $\overline{S}'_n$ is not as well behaved as S_n.

REMARK 5. In Theorems 2 and 3 the rate of convergence is hidden because of the standardization, but it is $1/\sqrt{\Delta_n}$, clearly optimal since there are $1 + [T/\Delta_n]$ observation altogether. In Theorems 5 and 6 the rate is $1/u_n^{\beta/2}$, which is again "optimal" when we only use the increments bigger than u_n [more precisely, if we were able to observe exactly all jumps of X with size bigger than u_n, this rate would be the optimal one, up to a $\log(1/u_n)$ term]. However, for those theorems we also have to choose u_n: the smallest u_n is, compared to Δ_n, the biggest the actual rate is, but we are limited in this choice by the upper bound on ρ_+. For example if we take $u_n = \alpha\Delta_n^{\varpi}$, and due to (35), the best rate is "almost" $1/\Delta_n^{\beta(2-\beta_0)/6\beta_0}$.

4. Simulation results. We now report simulation results documenting the finite sample performance of the test statistics S_n and S'_n. We calibrate the values to be realistic for a liquid stock trading on the NYSE, and we consider an observation length of $T = 21$ days (one month) sampled every five seconds.

We conduct simulations to determine the small sample behavior of the two statistics S_n and S'_n under their respective null and alternative hypotheses. The tables and graphs that follow report the results of 5000 simulations. The data generating process is the stochastic volatility model $dX_t = \sigma_t\, dW_t + \theta\, dY_t$, with $\sigma_t = v_t^{1/2}$, $dv_t = \xi(\eta - v_t)\, dt + \phi v_t^{1/2}\, dB_t + dJ_t$, $\mathbb{E}[dW_t\, dB_t] = \rho\, dt$, $\eta^{1/2} = 0.25$, $\phi = 0.5$, $\xi = 5$, $\rho = -0.5$, J is a compound Poisson jump process with jumps that are uniformly distributed on $[-30\%, 30\%]$ and $X_0 = 1$. The jump process Y is a β-stable process with $\beta = 1$, that is, a Cauchy process (which has infinite activity, and will be our model under $\Omega_T^{i\beta}$; this is a borderline case for the statistics S_n under the null, nevertheless we will see that this statistic behaves well). Given η, the scale parameter θ (or equivalently A) of the stable process in simulations is calibrated to deliver different various values of the tail probability $P = \mathbb{P}(|\Delta Y_t| \geq 4\eta^{1/2}\Delta_n^{1/2})$. In the

various simulations' design, we hold η fixed. Therefore the tail probability parameter P controls the relative scale of the jump component of the semimartingale relative to its continuous counterpart. We set θ such that neither of the two components of the model, $\sigma_t\,dW_t$ and θY_t, is negligible compared to the other when the hypothesis states that they should both be present. We achieve this by computing the expected percentage of the total quadratic variation attributable to jumps on a given path from the model, and set it to values that range from 5% and 95%.

4.1. *The first test.* The statistic S_n is implemented with $k = 2$ and values of p that range from 0 to 2 (recall Remark 2). Figure 2 compares the theoretical and Monte Carlo behavior of S_n as a function of the power p under the null hypothesis where a Brownian motion is present, in addition to a Cauchy pure jump process. Figure 3 shows the corresponding results under the alternative hypothesis, where there is no Brownian motion. The theoretical curves are computed from the expected values of the truncated power variations using the exact density of the increments at the sampling interval $\Delta_n = 5$ seconds, rather than their asymptotic limits for $\Delta_n \to 0$. This introduces a slight Jensen's inequality effect in the figure but appears to capture well the small sample behavior of the statistic.

Recall that for concreteness α is expressed as a number of standard deviations of the Brownian part of X: that is, the level of truncation u_n is expressed in terms of the number α of standard deviations of the continuous martingale part of the process, defined in multiples of the long-term volatility parameter $\eta^{1/2}$: α is defined by $u_n = \alpha\eta^{1/2}\Delta_n^{1/2}$. Our view of the joint choice of (ϖ, α) is that they are not independent parameters in finite sample: they are different parameters for asymptotic purposes but in finite samples the only relevant quantity is the actual resulting cutoff size u_n. This is why we are reporting the values of the cutoffs u_n in the form of the α that would correspond to $\varpi = 1/2$. This has the advantage

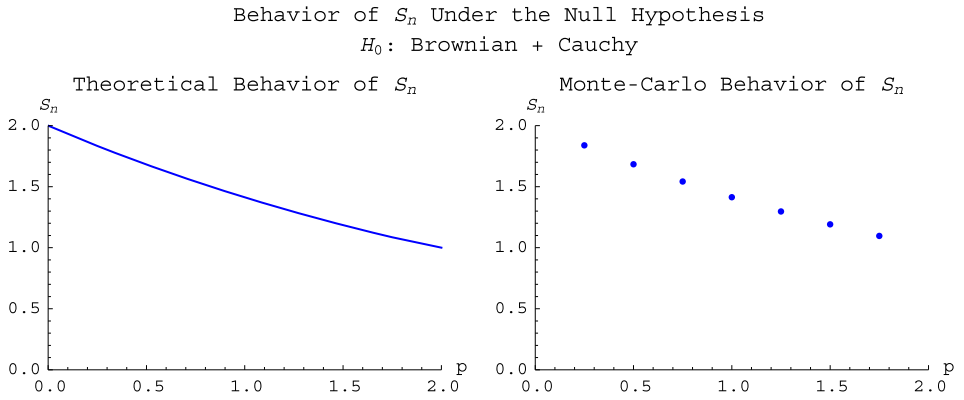

FIG. 2. *Theoretical and Monte Carlo behavior of S_n as a function of the power p under the null hypothesis where a Brownian motion is present, in addition to a pure jump (Cauchy) process.*

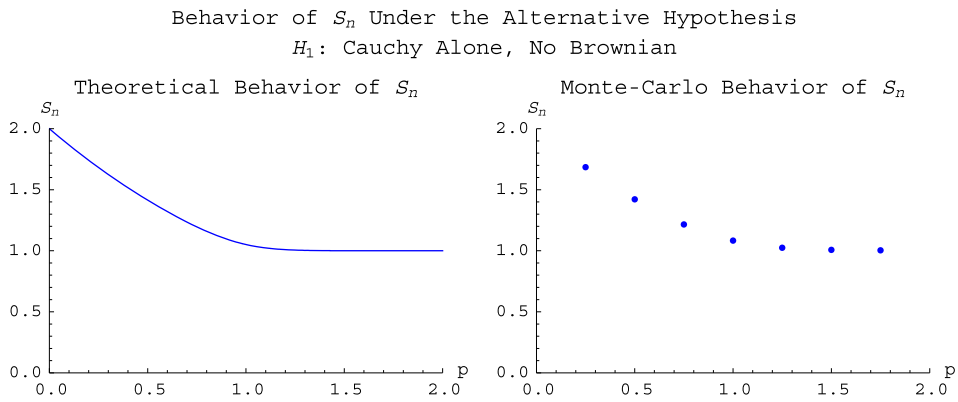

FIG. 3. *Theoretical and Monte Carlo behavior of S_n as a function of the power p under the alternative hypothesis where a Brownian is absent.*

of providing an easily interpretable size of the cutoff compared to the size of the increments that would be expected from the Brownian component of the process: we can then think in terms of truncating at a level that corresponds to $\alpha = 4$, 6, etc., standard deviations of the continuous part of the model. Since the ultimate purpose of the truncation is either to eliminate or conserve that part, it provides an immediate and intuitively clear reference point. Of course, given u_n and this α, it is possible to back this into the value of the α corresponding to any ϖ, for that given sample size, including the value(s) of ϖ that satisfy the required inequalities imposed by the asymptotic results. This approach would lose its effectiveness if we were primarily interested in testing the validity of the asymptotic approximation as the sample size varies, but for applications, by definition on a finite sample, it seems to us that the interpretative advantage outweighs this disadvantage.

The statistic in the plots is computed with a truncation level corresponding to $\alpha = 7$. Table 1 looks at the dependence of the results on the choice of α.

TABLE 1

Testing $H_0 : \Omega_T^W$ vs. $H_1 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$: Monte Carlo rejection rate for the test for the presence of a Brownian motion using the statistic S_n

Degree of truncation α	**Test theoretical level**	**Sample rejection rate (%) for power p**						
		0.25	**0.5**	**0.75**	**1.0**	**1.25**	**1.5**	**1.75**
6	10%	9.1	9.4	9.4	9.2	9.1	9.3	8.9
	5%	4.6	4.7	4.8	4.7	4.5	4.3	4.1
7	10%	9.7	9.7	9.8	9.8	9.9	10.2	9.8
	5%	5.0	5.0	5.1	5.0	4.9	4.7	4.4
8	10%	9.7	9.9	9.9	10.0	9.9	10.1	9.9
	5%	5.0	5.1	5.1	5.1	4.9	4.8	4.5

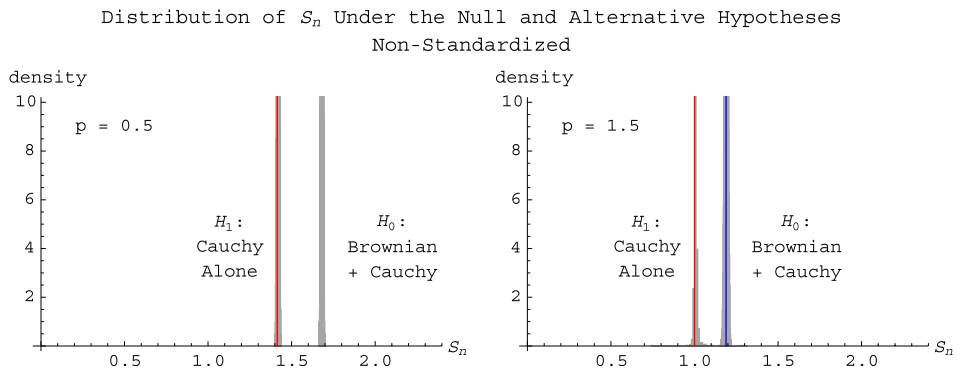

FIG. 4. *Nonstandardized distribution of S_n under the null and alternative hypotheses for two values of p.*

Next, we report in Figure 4 histograms of the values of the unstandardized S_n computed under $H_0 : \Omega_T^W$ and $H_1 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$, respectively, and with the same level of truncation $\alpha = 7$. The vertical lines represent the anticipated limits of the statistic in the two situations, $k^{1-p/2}$ under H_0 and either 1 when $p > \beta$ or $k^{1-p/\beta}$ when $0 < p \leq \beta$ under H_1, based on Theorem 1 and Remark 1. Since here $\beta = 1$, the two graphs with $p = 0.5$ and $p = 1.5$ illustrate the two situations where $p < \beta$ and $p \geq \beta$.

Figure 5 reports the Monte Carlo distribution of the statistic S_n, standardized according to Theorem 2, compared to the limiting $\mathcal{N}(0, 1)$ distribution. Table 1 reports the Monte Carlo rejection rates of the test of $H_0 : \Omega_T^W$ vs. $H_1 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$ at the 10% and 5% level, using the test statistic S_n, for various levels of trunca-

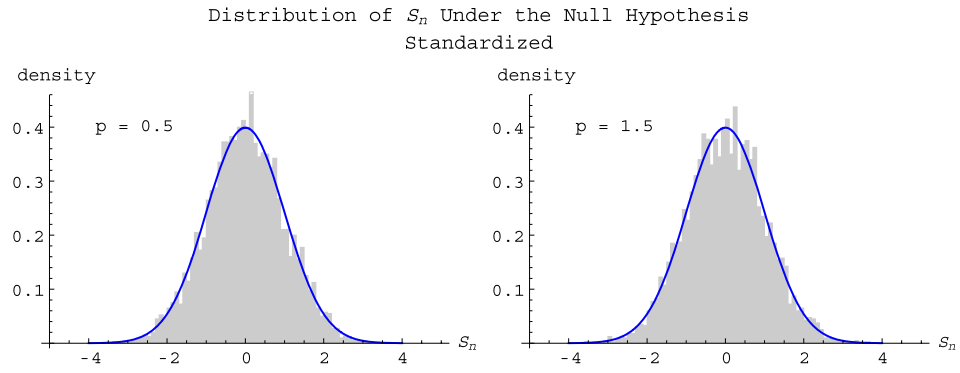

FIG. 5. *Standardized distribution of S_n under the null hypothesis of a Brownian motion present for two values of p. The histogram represents the small sample distribution while the solid curve is the asymptotic $\mathcal{N}(0, 1)$ density.*

tion α. We find that the test behaves well, with empirical test levels close to their theoretical counterparts.

4.2. *The second test.* We now turn to the second problem, that of testing $H_0 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$ vs. $H_1 : \Omega_T^{W} \cap \Omega_T^{i\beta}$. For this test, S_n' is implemented with a second truncation level twice as large as the first, that is, $\gamma = 2$. The simulation evidence suggests that the results are largely similar for values of γ within a range of 1.5 to 2.5. Parameter values are identical to those employed for the first test. Since there is no Brownian motion under the null, the truncation level u_n is set in terms of the percentage of observations that are excluded by the truncation. For comparison with the truncation levels employed in the first test, we report it here again in terms of α, a number of standard deviations for the Brownian motion using the same parameter values as under the first test's null, or this test's alternative hypothesis.

Under the null, the model is driven exclusively by the Cauchy process. Figure 6 shows the limiting value of S_n' under H_0, as a function of the truncation level α, comparing the theoretical limit of $\gamma^2 = 4$ given in Theorem 4 (left graph) and the corresponding average value of S_n' from the Monte Carlo simulations (right graph). Figure 7 shows the corresponding values under the alternative hypothesis, where the increments of X are now generated by a Brownian motion plus a Cauchy process. The theoretical limit on the left graph is computed from the expected values under the exact distribution of the increments at the sampling frequency Δ_n rather than the p-lim $\gamma^{2-\beta} = 2$ obtained in the limit where $\Delta_n \to 0$, with the same remark about Jensen's inequality applying here. We note that for small truncation levels ($\alpha = 4$) the interaction of the Brownian and the stable processes is material, driving the actual limit above 2. If desired, small sample corrections for this interaction can be implemented along the same lines as in Section 5 of [4].

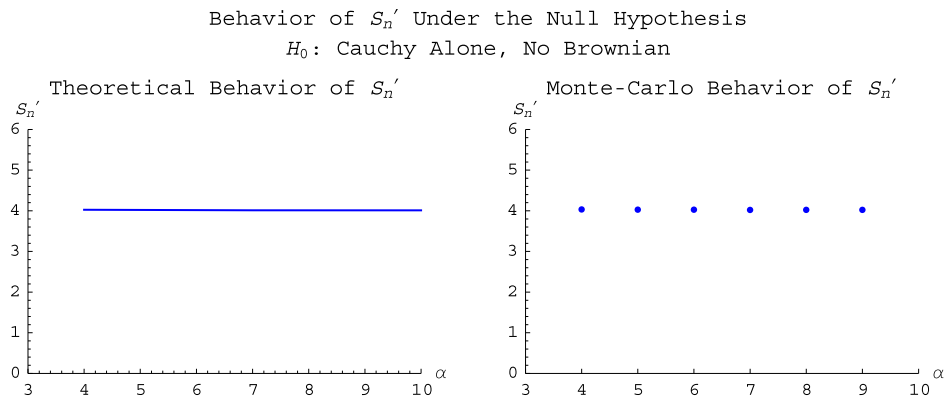

FIG. 6. *Theoretical and Monte Carlo behavior of S_n' as a function of the truncation level α under the null hypothesis where a Brownian motion is absent. The model is a pure jump (Cauchy) process.*

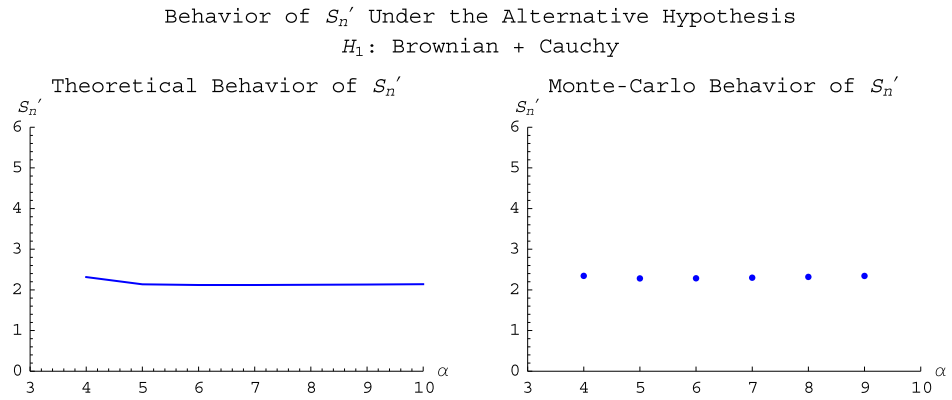

FIG. 7. *Theoretical and Monte Carlo behavior of S_n' as a function of the truncation level α under the alternative hypothesis where a Brownian motion is present, in addition to a pure jump (Cauchy) process.*

The test statistic in simulations under the alternative appears to be slightly biased upwards. Quite naturally, this effect worsens as the pure jump process gets closer to a Brownian motion (for instance if $\beta = 1.5$ instead of 0.5 or 1), and/or when the scale parameter θ of the jump process increases since that makes isolating the effect of the Brownian motion component of the model relatively more difficult.

Generally speaking, S_n' is, under its alternative, more finicky than S_n is under either its null or alternative. The reason for this is that S_n' requires under H_1 a Goldilocks-like conjunction of factors whereby the Brownian motion component of the model is sufficiently large to drive the behavior of the ratio of truncated quadratic variations, while the jump component of the model cannot be so small as to render inaccurate the ratio of the number of increments larger than the truncation level.

Figure 8 reports the Monte Carlo distributions of S_n' under H_0 and H_1; the vertical lines represent the theoretical limits. Under H_1, we note again that S_n' is slightly biased upwards. Fortunately, this bias is limited to H_1 so it does not adversely affect the implementation of the test per se, which is based on the behavior of S_n' under H_0. But it can affect the interpretation of the results of the test implemented on real data, since, as we will see below, we will find empirical values of S_n' below 4. Figure 9 reports the Monte Carlo and asymptotic distribution of the statistic S_n' standardized under H_0 as prescribed by Theorem 5.

As said above, the histograms are computed using $T = 21$ days (one month) sampled every five seconds. With this length of the series, the empirical distribution of the statistic is very well approximated by its asymptotic $\mathcal{N}(0, 1)$ limit. Shorter time periods (such as $T = 1$ day) tend to result in right-skewness of the Monte Carlo distribution of S_n'. We do not view the need for a longer series as a

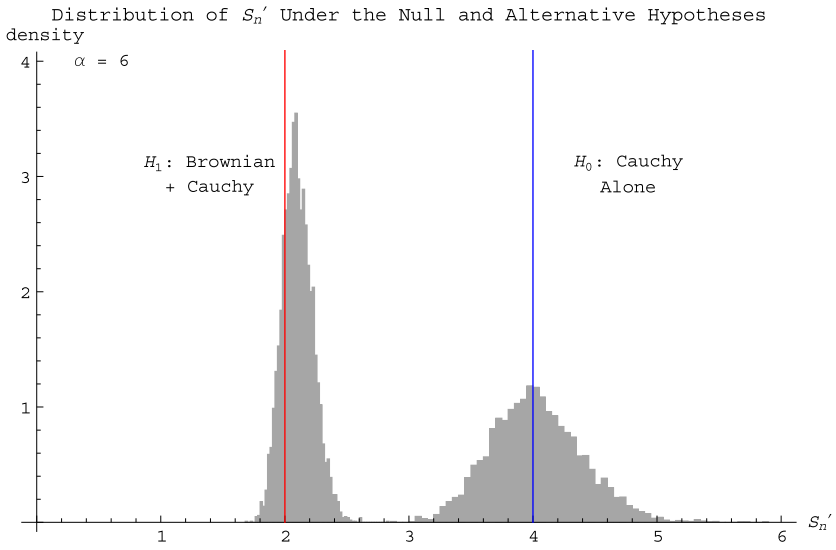

FIG. 8. *Nonstandardized distribution of S_n' under the null and alternative hypotheses.*

serious obstacle to the empirical implementation of the test since one would not typically expect the Brownian motion component of the model to be turned on or off on a daily basis: one would expect the market to operate in such a way that the Brownian component is either there all the time or not there at all. But if an answer is nevertheless desired on a day-by-day basis, then the first test can always be implemented, as it requires substantially shorter time spans.

Finally, the test's rejection rate under the null hypothesis is reported in Table 2. Since the test is one-sided (we reject H_0 when the standardized S_n' is too low), the

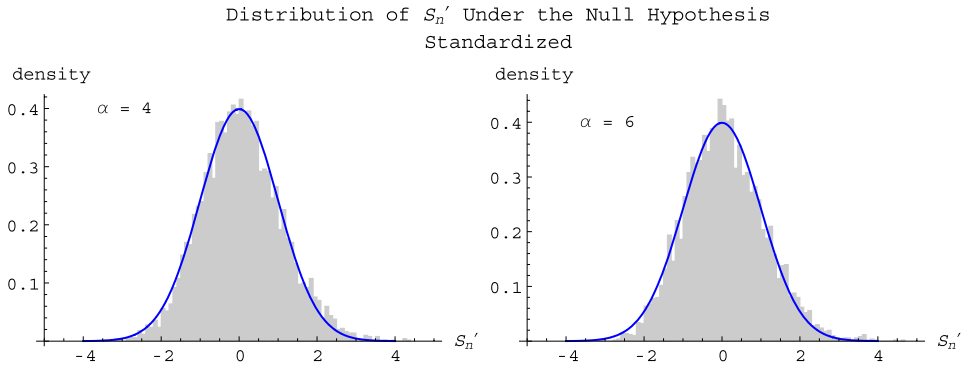

FIG. 9. *Standardized distribution of S_n' under the null hypothesis of a Brownian motion absent for two values of the truncation level. The histogram represents the small sample distribution while the solid curve is the asymptotic $\mathcal{N}(0, 1)$ density.*

TABLE 2
Testing $H_1 : \Omega_T^{noW} \cap \Omega_T^{i\beta}$ vs. $H_0 : \Omega_T^W$: Monte Carlo rejection rate for the test for the absence of a Brownian motion using the statistic S_n'

	Sample rejection rate (%) for truncation level α				
Test theoretical level	**4**	**5**	**6**	**7**	**8**
10%	9.0	9.2	9.1	9.0	9.2
5%	4.0	4.2	4.1	4.2	4.2

right-skewness of the statistic visible in Figure 9 results in a slight under-rejection by the test.

5. Empirical results. In this section, we apply the two test statistics to real data, consisting of all transactions recorded during the year 2006 on two of the most actively traded stocks, Intel (INTC) and Microsoft (MSFT). The data source is the TAQ database. Using the correction variables in the dataset, we retain only transactions that are labeled "good trades" by the exchanges: regular trades that were not corrected, changed, or signified as cancelled or in error; and original trades which were later corrected, in which case the trade record contains the corrected data for the trade. Beyond that, no further adjustment to the raw data are made.

We first consider the test where the null hypothesis consists of a continuous component being present. Figures 10 and 11 show the values of the test statistic S_n, plotted for a range of values of the power p, for the two data series. The empirical values of S_n are labeled on the plots with numbers representing the sampling interval employed, in seconds, with values ranging from $\Delta_n = 5$ seconds to $\Delta_n = 30$ minutes. In addition to the empirical estimates, the figures display the two limits of S_n under the null where a Brownian is present and the alternative hypothesis where it is absent. The theoretical limits correspond to those given in Figures 2 and 3, except that the theoretical limit under H_1 (no Brownian present) is plotted for a value of $\beta = 1.6$, in line with the estimates of β given in [4] for these data series. Quite naturally, the closer β is to 2, the closer the jump component can mimic the behavior of a Brownian motion and the harder it becomes to tell the two hypotheses apart. The limit under H_0 is independent of β. Also on the figures are the two limits corresponding to the situation where market microstructure noise dominates. We include the two polar cases where the noise is either of a pure additive form or of a pure rounding form.

When the observations are blurred with either an additive white noise or with noise due to rounding, the respective limits are then 2 and $\sqrt{2}$. Indeed, suppose that instead of observing the exact value of X we have on top of it an additive white noise, that is we observe $X_{i\Delta_n} + Z_i$ (at stage n, where the Z_i's are i.i.d.,

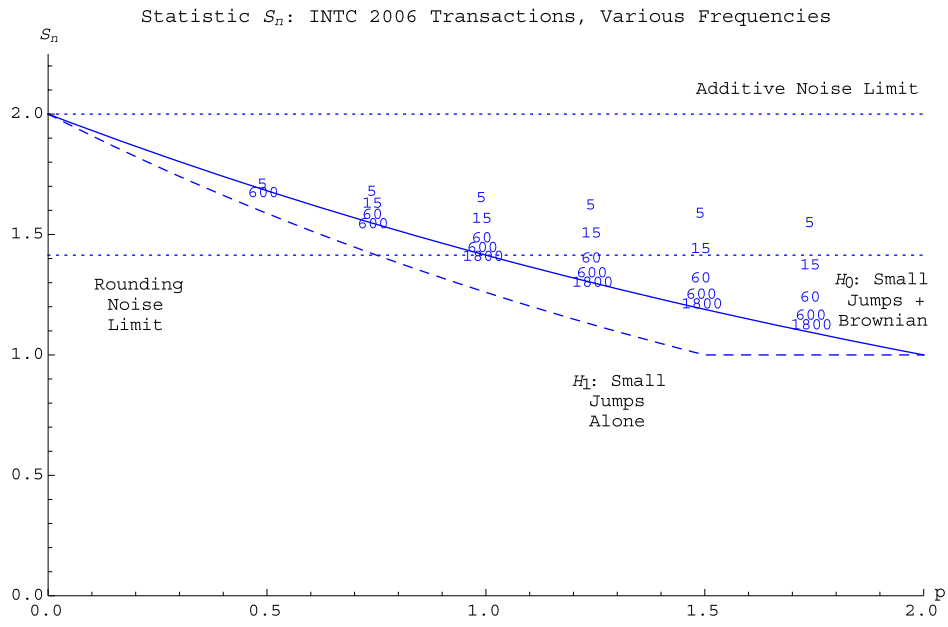

FIG. 10. *Empirical estimates of S_n at various values of p and sampling frequencies from all Intel transactions during* 2006.

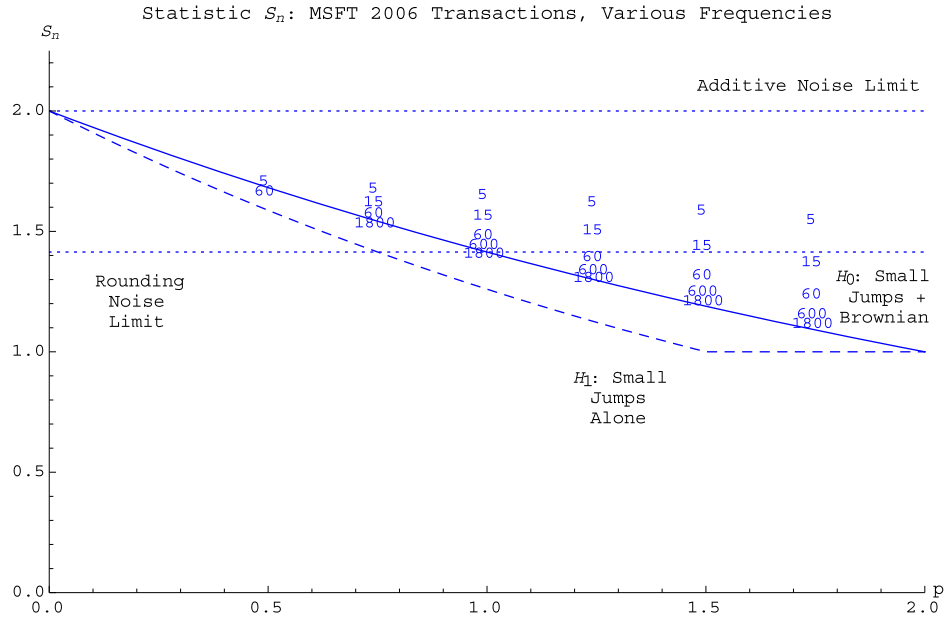

FIG. 11. *Empirical estimates of S_n at various values of p and sampling frequencies from all Microsoft transactions during* 2006.

and independent of the process X). If we suppose that Z_i has a density which is continuous and nonvanishing at 0, then the noise is the leading factor in the behavior of $B(p, u_n, \Delta_n)_T$ as soon as $p/(2(p+1)) \geq \rho_+$ [recall (17)]. In this case, the variables $(\Delta_n/u_n^{p+1})B(p, u_n, \Delta_n)_T$ converge in probability to TC_p for some constant C_p, and thus S_n converges in probability to the sampling frequency ratio k, which is 2 here. When the noise is pure rounding at some level α_n, then again it is the leading factor and $\sqrt{\Delta_n}\alpha_n^{1-p}B(p, u_n, \Delta_n)_T$ converges in probability to some positive limiting variable, as soon as $\alpha_n/u_n \to 0$ and $\alpha_n^2\Delta_n \to \infty$. Thus S_n converges in probability to $\sqrt{k}$ [when $\alpha_n > u_n$ we have $B(p, u_n, \Delta_n)_T = 0$ and then S_n is not even well defined; however, here the truncation level u_n used in practice is quite bigger than the rounding level of 1 cent].

The values of α are similar to those employed in simulations, and indexed in terms of standard deviations of the continuous martingale part of the log-price: we first estimate the volatility of the continuous part of X using the small increments, those of order $\Delta_n^{1/2}$, and then use that estimate to form the cutoff level used in the construction of the test statistic. To account for potential time series variation in the volatility process σ_t, that procedure is implemented separately for each day and we compute the sum, for that day, of the absolute value of the increments that are smaller than the cutoff, to the appropriate power p. For the full year, we then add the truncated power variations computed for each day.

The results in both Figures 10 and 11 tell a similar story. First, the empirical estimates are always on the side away from the limit under H_1, indicating that the null hypothesis of a Brownian motion present will not be rejected. Second, as the sampling frequency decreases, the empirical values get closer to the theoretical limit under H_0. For very high sampling frequencies, the results are consistent with some mixture of the noise driving the asymptotics. They then slowly settle down toward the limit corresponding to a null hypothesis of a Brownian present as the sampling frequency decreases, and the noise presumably becomes less of a factor.

Next, we turn to the results of the second test on the same data series in Figures 12 and 13. The test statistic S_n' is implemented with $\gamma = 2$, with data sampled every $\Delta_n = 5$ seconds. The empirical estimates are represented by a star, with the vertical dashes representing a 95% confidence interval. Also represented on the plots are the limits corresponding to H_0 (no Brownian) and H_1 (Brownian present). The theoretical limits correspond to those given in Figures 6 and 7, except that the theoretical limit under H_1 is plotted for a value of $\beta = 1.5$ for the same reason as above. We find that the empirical estimates tend to be lower than the value specified by H_0, which leads to a rejection of the null hypothesis of no Brownian motion. The estimates are, however, generally higher than their expected value under H_1, consistent with the upward bias identified in simulations, the bias being more pronounced when β gets closer to 2.

To summarize, the answer from both tests appears consistent with the presence of a continuous component in the data: using S_n, we do not reject the null of a Brownian motion present, while using S_n' we reject the null of its absence.

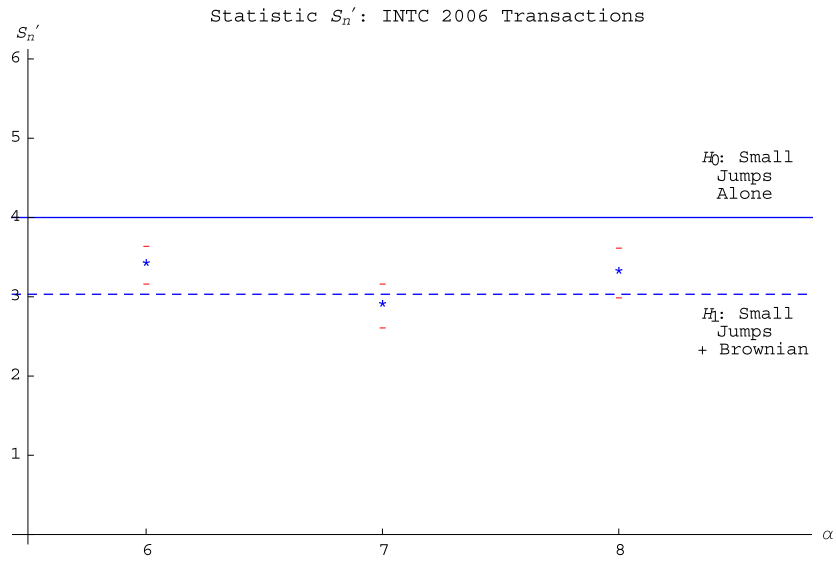

FIG. 12. *Empirical estimates of S_n' for various truncation levels α from all Intel transactions during* 2006.

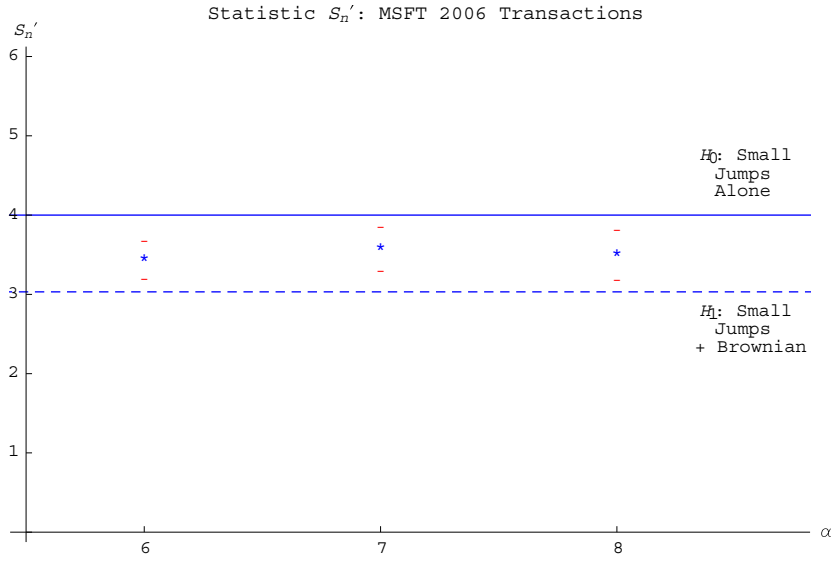

FIG. 13. *Empirical estimates of S_n' for various truncation levels α from all Microsoft transactions during* 2006.

6. Technical results. By a standard localization procedure, we can replace the local boundedness hypotheses in our assumptions by a boundedness assumption, and also assume that the process X itself, and thus the jump process ΔX_t, are bounded as well. That is, for all results which need Assumption 1 we may assume further that, for some constant $C > 0$,

$$|b_t|, |\sigma_t|, L_t, |\Delta X_t| \le C, \qquad \text{hence also } F_t([-C, C]^c) = 0. \tag{37}$$

When we need Assumption 2 we may assume the above, together with

$$|\widetilde{b}_t|, n_t, |\widetilde{\sigma}_t|, \int |x| F_t(dx) \le C. \tag{38}$$

We call these *reinforced Assumptions* 1 or 2, and they are assumed in all the sequels instead of mere Assumptions 1 or 2, according to the case.

Recall that if $\beta < 1$, we have (7) with b'_t bounded as well. Otherwise the decomposition (7) is no longer valid, but under reinforced Assumption 1 we can always write

$$Y'_t = X_0 + \int_0^t b''_s \, ds + \int_0^t \sigma_s \, dW_s, \qquad Y''_t = X_t - Y'_t, \tag{39}$$

where $b''_t = b_t + \int x 1_{\{|x|>1\}} F_t(dx)$ defines a bounded process, and Y'' is a purely discontinuous martingale.

Also, K below denotes a constant which may change from line to line and may depend on C above.

The key to all results is clearly the behavior of the processes $B(p, u_n, \Delta_n)$ and $U(u_n, \Delta_n)$. For establishing this behavior, it is convenient to introduce a few auxiliary processes, for $u > 0$ an arbitrary cut-off level and Y an arbitrary process

$$\begin{cases} B'(Y, p, \Delta_n)_t = \displaystyle\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n Y|^p, \\ D(p, u)_t = \displaystyle\sum_{s \le t} |\Delta X_s|^p 1_{\{|\Delta X_s| \le u\}}, \\ D'(u)_t = \displaystyle\sum_{s \le t} 1_{\{|\Delta X_s| > u\}}. \end{cases} \tag{40}$$

6.1. *Central limit theorems for the auxiliary processes.* This subsection is devoted to recalling or proving some limit theorems for $B'(X, p, \Delta_n)$ and for the auxiliary processes introduced in (40). First, we recall from Theorem 2.4 of [12] that under Assumption 1 (and even much more generally),

$$\begin{cases} 0 < p < 2 \quad \Rightarrow \quad \Delta_n^{1-p/2} B'(X, p, \Delta_n)_t \xrightarrow{\mathbb{P}} A(p)_t = m_p \displaystyle\int_0^t |\sigma_s|^p \, ds, \\ p \ge 2, \, X \text{ continuous} \quad \Rightarrow \quad \Delta_n^{1-p/2} B'(X, p, \Delta_n)_t \xrightarrow{\mathbb{P}} A(p)_t, \\ (u_n) \text{ satisfies (17)} \quad \Rightarrow \quad B(2, u_n, \Delta_n)_t \xrightarrow{\mathbb{P}} A(2)_t \end{cases} \tag{41}$$

[the last property is proved when $u_n = \alpha \Delta_n^{\varpi}$ with $\alpha > 0$ and $\varpi \in (0, 1/2)$, but the proof works as well when (17) holds].

LEMMA 1. *Suppose that X is continuous, and let $t \geq 0$ and $p > 1$ and $k \geq 2$. Under Assumption 2 the two-dimensional variables*

$$\begin{aligned} &\frac{1}{\sqrt{\Delta_n}}\big(\Delta_n^{1-p/2} B'(X, p, \Delta_n)_t - A(p)_t, \\ &\qquad \Delta_n^{1-p/2} B'(X, p, k\Delta_n)_t - k^{p/2-1} A(p)_t\big) \end{aligned} \tag{42}$$

stably converge in law to a limit which is defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and which, conditionally on $\mathcal{F}$, is a centered Gaussian variable with variance–covariance matrix given by

$$\frac{1}{m_{2p}} \begin{pmatrix} (m_{2p} - m_p^2) A(2p)_T & (m_{k,p} - k^{p/2} m_p^2) A(2p)_T \\ (m_{k,p} - k^{p/2} m_p^2) A(2p)_T & k^{p-1}(m_{2p} - m_p^2) A(2p)_T \end{pmatrix}. \tag{43}$$

(The same would hold if $p \in (0, 1]$, under the additional assumption that σ_t is bounded away from 0.)

PROOF OF LEMMA 1. We can assume reinforced Assumption 2. The result will follow from Theorem 7.1 of [11]. Assumption (H) in that paper is slightly more restrictive than reinforced Assumption 2, but a close look at the proof yields that this theorem still holds in the present situation.

We apply the quoted Theorem 7.1 to the two-dimensional function on $\mathbb{R}^k$ whose components are $|x_1|^p + \cdots + |x_k|^p$ and $|x_1 + \cdots + x_k|^p$. This function is C^1 with derivatives having polynomial growth. With the notation of that paper, variable (42) with the nontruncated variations is equal to $Z_n + R_n$, where

$$Z_n = \frac{1}{\sqrt{\Delta_n}}\left(\Delta_n V'(f, k, \Delta_n)_t - \frac{1}{k}\int_0^t \rho_{\sigma_s}^{\otimes k}(f)\, ds\right) \tag{44}$$

and R_n is a remainder term with second component equal to 0, and with first component

$$\Delta_n^{1/2-p/2} \sum_{i=k[t/k\Delta_n]+1}^{[t/\Delta_n]} |\Delta_i^n X|^p. \tag{45}$$

By (37) we have $E(|\Delta_i^n X|^p) \leq K\Delta_n^{p/2}$, and hence, since there are at most k summands in the definition of R_n, we deduce that $R_n \xrightarrow{\mathbb{P}} 0$. On the other hand, the aforementioned result yields that Z_n converges stably in law to a limiting variable, which is exactly as described in the statement of the lemma. □

LEMMA 2. *Let $t \geq 0$, and suppose Assumption 1 and $p > \beta$ and $u_n \to 0$. Then*

$$u_n^{\beta-p} D(p, u_n)_t \xrightarrow{\mathbb{P}} \frac{\beta}{p-\beta}\overline{A}_t, \qquad u_n^{\beta} D'(u_n)_t \xrightarrow{\mathbb{P}} \overline{A}_t. \tag{46}$$

Moreover, if $\beta' < \beta/2$ the four-dimensional variables

$$\begin{pmatrix} \dfrac{1}{u_n^{\beta/2}}\Big(u_n^{\beta-p}D(p,u_n)_t - \dfrac{\beta}{p-\beta}\overline{A}_t\Big) \\ \dfrac{1}{u_n^{\beta/2}}\Big((\gamma u_n)^{\beta-p}D(p,\gamma u_n)_t - \dfrac{\beta}{p-\beta}\overline{A}_t\Big) \\ \dfrac{1}{u_n^{\beta/2}}\big(u_n^{\beta}D'(u_n)_t - \overline{A}_t\big) \\ \dfrac{1}{u_n^{\beta/2}}\big((\gamma u_n)^{\beta}D'(\gamma u_n)_t - \overline{A}_t\big) \end{pmatrix} \tag{47}$$

stably converge in law to a limit which is defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and which, conditionally on $\mathcal{F}$, is a centered Gaussian variable with variance–covariance matrix $\overline{A}_t\widetilde{C}$, where $\widetilde{C}$ is the 4×4 matrix

$$\widetilde{C}_t = \begin{pmatrix} \dfrac{\beta}{2p-\beta} & \dfrac{\beta\gamma^{\beta-p}}{2p-\beta} & 0 & 0 \\ \dfrac{\beta\gamma^{\beta-p}}{2p-\beta} & \dfrac{\beta\gamma^{\beta}}{2p-\beta} & \dfrac{\beta(1-\gamma^{\beta-p})}{p-\beta} & 0 \\ 0 & \dfrac{\beta(1-\gamma^{\beta-p})}{p-\beta} & 1 & 1 \\ 0 & 0 & 1 & \gamma^{\beta} \end{pmatrix}. \tag{48}$$

PROOF. Assumption 1 here implies Assumption 6 of [2], with the same β and $\overline{A}_t$, and with β' there substituted with any number in (β', β) here. Then all statements concerning $D(p,u_n)_t$ are in Proposition 5 of that paper. However, we must redo the proof to obtain the joint convergence for the processes $D(p,u_n)$ and $D'(u_n)$.

Let $\widetilde{D}(p,u)$ and $\widetilde{D}'(u)$ be the predictable compensators of $D(p,u)$ and $D'(u)$, and $M(u) = u^{\beta-p}(D(p,u) - \widetilde{D}(p,u))$ and $M'(u) = u^{\beta}(D'(u) - \widetilde{D}'(u))$. Observe that $\widetilde{D}'(u)_t = \int_0^t F_s([-u,u]^c)\,ds$ and $|F_t([-v,v]^c) - v^{-\beta}A_t| \leq KL_t v^{-\beta'}$ by Assumption 1. Therefore, exactly as in the paper (and (C.23) and (C.24) in it), we see that if $q > \beta$,

$$\begin{cases} \beta' < \beta \quad \Rightarrow \quad u_n^{\beta-q}\widetilde{D}(q,u_n)_t \to \dfrac{\beta\overline{A}_t}{q-\beta}, \qquad u_n^{\beta}\widetilde{D}'(u_n)_t \to \overline{A}_t, \\ \beta' < \dfrac{\beta}{2} \quad \Rightarrow \quad \dfrac{1}{u_n^{\beta/2}}\Big|u_n^{\beta-q}\widetilde{D}(q,u_n)_t - \dfrac{\beta\overline{A}_t}{q-\beta}\Big| \to 0, \\ \qquad\qquad\qquad\qquad \dfrac{1}{u_n^{\beta/2}}|u_n^{\beta}\widetilde{D}'(u_n)_t - \overline{A}_t| \to 0. \end{cases} \tag{49}$$

The processes $M(u)$ and $M'(u)$ are martingales, and if $u \leq v$ the brackets are given by the following formulas:

$$\langle M(u), M(v)\rangle = (uv)^{\beta-p}\widetilde{D}(2p,u), \qquad \langle M'(u), M'(v)\rangle = (uv)^{\beta}\widetilde{D}'(v),$$
$$\langle M(u), M'(v)\rangle = 0, \qquad \langle M'(u), M(v)\rangle = u^{\beta}v^{\beta-p}\big(\widetilde{D}(p,v) - \widetilde{D}(p,u)\big).$$

This, applied with (u,v) equal to (u_n,u_n) or $(u_n,\gamma u_n)$ or $(\gamma u_n,\gamma u_n)$, and combined with the first part of (49), yield that the bracket matrix at time t of the 4-dimensional continuous martingale $\overline{M}^n = u_n^{-\beta/2}(M(u_n), M(\gamma u_n), M'(u_n), M'(\gamma u_n))$ converges to $\overline{A}_t\widetilde{C}$ in probability, where $\widetilde{C}$ is given by (48). Then as in Proposition 5 of [2] one deduces that $\overline{M}^n_t$ converges stably in law to the limit described in the statement of the lemma. It remains to deduce from the second part of (49) that the difference between $\overline{M}^n_t$ and the variable defined by (47) goes to 0 in probability. □

6.2. *The behavior of* $B(p,u_n,\Delta_n)_T$. In this subsection we establish the behavior of $B(p,u_n,\Delta_n)$ for the relevant values of p and for the cases not covered by (41). This is done in several lemmas.

LEMMA 3. *Under Assumption* 1, *and if* u_n *satisfies* (17), *we have*

$$B(4,u_n,\Delta_n)_t \xrightarrow{\mathbb{P}} 0. \tag{50}$$

PROOF. We first observe that $B(4,v,\Delta_n)_T$ converges in probability to $G(v)_T = \sum_{s\le T}|\Delta X_s|^4 1_{\{|\Delta X_s|\le v\}}$ for any fixed $v>0$ such that $P(\exists s\le T: |\Delta X_s| = v) = 0$. Hence there is a sequence $v_m\to 0$ such that $B(4,v_m,\Delta_n)_T$ converges in probability to $G(v_m)_T$. On the one hand $B(4,u_n,\Delta_n)_T \le B(4,v_m,\Delta_n)_T$ as soon as $u_n\le v_m$. On the other hand we have $G(v_m)_T\to 0$ as $m\to\infty$. Then the result follows. □

LEMMA 4. *Assume* (17) *and reinforced Assumption* 1, *and let* $p>0$. *If either* $p\le 2$, *or* $p>2$ *with* $\rho_- > \frac{p-2}{2p-2\beta}$, *we have*

$$\Delta_n^{1-p/2}B(p,u_n,\Delta_n)_t \xrightarrow{\mathbb{P}} A(p)_t. \tag{51}$$

PROOF. We consider decomposition (39). In view of (41), it is enough to prove that under the conditions of the lemma we have

$$\Delta_n^{1-p/2}\big(B(p,u_n,\Delta_n)_t - B'(Y',p,\Delta_n)_t\big) \xrightarrow{\mathbb{P}} 0. \tag{52}$$

The left-hand side above is $\Delta_n^{1-p/2}\sum_{i=1}^{[t/\Delta_n]}\zeta_i^n$, where

$$\zeta_i^n = |\Delta_i^n Y' + \Delta_i^n Y''|^p 1_{\{|\Delta_i^n X|\le u_n\}} - |\Delta_i^n Y'|^p.$$

With $\kappa=1$ when $p>1$ and $\kappa=0$ otherwise, we have the following inequalities, for all $m,q>0$:

$$|\Delta_i^n Y'| \ge \frac{u_n}{2} \quad\Rightarrow\quad |\zeta_i^n| \le K|\Delta_i^n Y'|^{p+q}/u_n^q,$$

$$|\Delta_i^n X| > 2u_n, \qquad |\Delta_i^n Y'| \le \frac{u_n}{2} \tag{53}$$

$$\Rightarrow \quad |\zeta_i^n| \le |\Delta_i^n Y'|^p |\Delta_i^n Y''|^m / u_n^m,$$

$$|\Delta_i^n X| \le 2u_n, \qquad |\Delta_i^n Y'| \le \frac{u_n}{2}$$

$$\Rightarrow \quad |\zeta_i^n| \le K\big((|\Delta_i^n Y''| \wedge u_n)^p + \kappa |\Delta_i^n Y'|^{p-1}(|\Delta_i^n Y''| \wedge u_n)\big),$$

where we have used the inequality $||x+y|^p - |x|^p| \le K(|y|^p + |x|^{p-1}|y|)$ when $p > 1$ and $||x+y|^p - |x|^p| \le |y|^p$ when $p \le 1$. In view of (37), we have the estimates

$$\text{(54)} \quad \begin{cases} \mathbb{E}(|\Delta_i^n Y''|^2) \le K\Delta_n, \qquad q > 0 \quad \Rightarrow \quad \mathbb{E}(|\Delta_i^n Y'|^q) \le K_q \Delta_n^{q/2}, \\ r \in (\beta, 2] \quad \Rightarrow \quad \mathbb{E}\big((|\Delta_i^n Y''| \wedge u_n)^2\big) \le K_r \Delta_n u_n^{2-r} \end{cases}$$

(the first estimate is obvious and the second one follows from Burkholder–Davis–Gundy inequality; the third one follows from (6.25) of [11] applied to the process Y'' and with $\alpha_n = u_n/\sqrt{\Delta_n}$, which goes to ∞ by (17), and with r as above). Then, using Hölder's inequality and $(|x| \wedge u_n)^p \le u_n^{p-2}(|x| \wedge u_n)^2$ when $p > 2$, we deduce from (53) applied with $q = m = 1$ and from $u_n \le K$ that

$$\Delta_n^{1-p/2}\mathbb{E}(|\zeta_i^n|) \le \begin{cases} K\Delta_n\Big(\dfrac{\Delta_n^{1/2}}{u_n} + u_n^{p(1-r/2)} + \kappa u_n^{1-r/2}\Big), & \text{if } p \le 2, \\ K\Delta_n\Big(\dfrac{\Delta_n^{1/2}}{u_n} + \Delta_n^{1-p/2} u_n^{p-r} + u_n^{1-r/2}\Big), & \text{if } p > 2. \end{cases}$$

We have $\Delta_n^{1/2}/u_n \to 0$ by (17), hence $E(\Delta_n^{1-p/2}\sum_{i=1}^{[t/\Delta_n]} |\zeta_i^n|) \to 0$, as soon as $p \le 2$, or $p > 2$ and $\rho_- \ge \frac{p-2}{2(p-r)}$. Since r is arbitrary in $(\beta, 2]$, we deduce the result. □

LEMMA 5. *Let $p \in (0, 2]$, and assume reinforced Assumption 1 with $\beta < 1$ and (17) with further $\rho_- > \frac{p-1}{2p-2\beta}$ when $p \ge 1$. Then, with X' given by (7), we have*

$$\text{(55)} \qquad \Delta_n^{1/2-p/2}\big(B(p, u_n, \Delta_n)_t - B'(X', p, \Delta_n)_t\big) \xrightarrow{\mathbb{P}} 0.$$

PROOF. The proof of this lemma is similar to that of the previous one. The left-hand side of (55) is $\Delta_n^{1/2-p/2}\sum_{i=1}^{[t/\Delta_n]} \zeta_i^n$, where

$$\zeta_i^n = |\Delta_i^n X' + \Delta_i^n X''|^p 1_{\{|\Delta_i^n X| \le u_n\}} - |\Delta_i^n X'|^p.$$

Then (53) holds with (X', X'') instead of (Y', Y''), whereas (54) is replaced by

$$\text{(56)} \quad \begin{cases} \mathbb{E}(|\Delta_i^n X''|) \le K\Delta_n, \qquad q > 0 \quad \Rightarrow \quad \mathbb{E}(|\Delta_i^n X'|^q) \le K_q \Delta_n^{q/2}, \\ r \in (\beta, 1) \quad \Rightarrow \quad \mathbb{E}(|\Delta_i^n X''| \wedge u_n) \le K_r \Delta_n u_n^{1-r} \end{cases}$$

(we now use (6.26) of [11] applied with $\alpha_n = u_n/\sqrt{\Delta_n}$ and r as above). Hence, using (53) for the pair (X', X''), plus the fact that $(|x| \wedge u_n)^p \le u_n^{p-m}(|x| \wedge u_n)^m$ for $0 < m \le p$ and Hölder's inequality, we deduce that for all $q > 0$ and $m \in (0, 1)$ and $r \in (\beta, 1)$, and with κ as in the previous proof,

$$\begin{aligned}
&\Delta_n^{1/2-p/2}\mathbb{E}(|\zeta_i^n|)\\
&\quad\le K_r\Delta_n\left(\frac{\Delta_n^{q/2-1/2}}{u_n^q} + \frac{\Delta_n^{m-1/2}}{u_n^m} + \Delta_n^{1/2-p/2}u_n^{p-r} + \kappa\Delta_n^{m-1}u_n^{1-mr}\right)\\
&\quad\le K_r\Delta_n(\Delta_n^{v_1} + \Delta_n^{v_1} + \Delta_n^{v_3} + \kappa\Delta_n^{v_4}),
\end{aligned}$$

where $v_1 = q(\frac{1}{2} - \rho_+) - \frac{1}{2}$ and $v_2 = m(1 - \rho_+ r) - \frac{1}{2}$ and $v_3 = \frac{1-p}{2} + (p - r)\rho_-$ and $v_4 = m - 1 + \rho_-(1 - mr)$. Since $\rho_+ < 1/2$ we have $v_1 > 0$ for q large enough. When $r \downarrow \beta$ and $m \uparrow 1$, we have $v_2 \to v_2' = (1 - \rho_+\beta) - \frac{1}{2}$ and $v_3' \to v_3' = \frac{1-p}{2} + (p - \beta)\rho_-$ and $v_4 \to v_4' = (1 - \beta)\rho_-$, and (55) will follow from $v_j' > 0$ for $j = 2, 3$ and also for $j = 4$ when $p > 1$. We have $v_2' > 0$ because $\beta\rho_+ < \frac{1}{2}$. When $p < 1$ we have $v_3' > 0$. When $p = 1$ then $v_3' > 0$ if $\rho_- > 0$, and when $p > 1$ we have $v_3' > 0$ and $v_4' > 0$ as soon as $\rho_- > \frac{p-1}{2p-2\beta}$. So (55) is proved. □

The previous lemma essentially gives the behavior of $B(p, u_n, \Delta_n)$ when the leading term is due to the continuous martingale part of X. When this part vanishes, we have another type of behavior, which we describe now.

LEMMA 6. *Let $p > 1$, and assume reinforced Assumption 1.*

(i) *If $p > \beta$ and (17) holds with $\rho_+ < \frac{p-1}{p}$ we have*

$$u_n^{\beta-p}\big(B(p, u_n, \Delta_n)_t - D(p, u_n)_t\big) \xrightarrow{\mathbb{P}} 0 \qquad \text{on the set } \Omega_t^{noW}. \tag{57}$$

(ii) *If $p \ge 2$ and (17) holds with $\rho_+ \le \frac{2-\beta}{3\beta}$, and if $\beta' < \beta/2$, we have*

$$u_n^{\beta/2-p}\big(B(p, u_n, \Delta_n)_t - D(p, u_n)_t\big) \xrightarrow{\mathbb{P}} 0 \qquad \text{on the set } \Omega_t^{noW}. \tag{58}$$

PROOF. Since the variables $B(p, u_n, \Delta_n)_t$ are the same on the set Ω_t^{noW} when they are computed on the basis of X or on the basis of the process $X_t - \int_0^t \sigma_s\, dW_s$, it is no restriction to assume that $\sigma_s = 0$ identically.

The proof is based on the result of [2], when $\sigma_t = 0$ identically. We have Assumption 7 of that paper with $H = \beta$ and $a = 1 - \beta'/\beta$ and thus $\phi'(x) = x^{-\beta'}$. We can then apply Lemmas 8 of that paper with the version of $\eta(p)_n$ given at the end of Lemma 7 (because $X^c = 0$ here), to obtain that for $p > 1 \vee \beta$ and if $\rho_+ < \frac{p-1}{p}$ and for any $r \in (0, \frac{2}{3\rho_+\beta} - \frac{2}{3})$,

$$\mathbb{E}\big(|B(p, u_n, \Delta_n)_t - D(p, u_n)_t|\big) \le K_r t u_n^{p-\beta}\eta(p)_n, \tag{59}$$

where

$$\eta(p)_n = \sum_{j=1}^{5} (u_n)^{x_j},$$

$$\begin{cases} x_1 = \dfrac{1}{\rho_+} - \beta(1+r), & x_2 = \dfrac{2}{\rho_+} - \beta(2+3r), \\ x_3 = r\left(1 - \dfrac{\beta}{p}\right), & x_4 = \dfrac{p-1}{p\rho_+} + \dfrac{\beta}{p} - 1, \qquad x_5 = \beta - \beta'. \end{cases}$$

Clearly, (57) follows from (59), as soon as we can choose $r \in (0, \frac{2}{3\rho_+\beta} - \frac{2}{3})$ such that $x_j > 0$ for all $j = 1, \ldots, 5$: this is obvious when $\beta < p$ and $\rho_+ \le \frac{p-1}{p}$.

As for (58), it will also follow from (59) if we can choose r as above, such that $x_j > \beta/2$ for all $j = 1, \ldots, 5$. This property holds for x_5 because $\beta' < \beta/2$ is assumed, and for x_4 because $\rho_+ < 1/2$. For $j = 1, 2, 3$, and since x_1 and x_2 do not depend on p and x_3 increases with p, it is enough to consider the case $p = 2$. Then if we let r decrease strictly to $\frac{\beta}{2-\beta}$, we see that $x_3 > \beta/2$, whereas x_1 and x_2 increase to $\frac{1}{\rho_+} - \frac{2\beta}{2-\beta}$ and to $\frac{2}{\rho_+} - \frac{\beta(4+\beta)}{2-\beta}$ respectively, and these quantities are strictly bigger than $\beta/2$ if ρ_+ is strictly smaller than $\frac{4-2\beta}{\beta(6-\beta)}$ and $\frac{8-4\beta}{\beta(10+\beta)}$. Now, recall that one should also have $\frac{\beta}{2-\beta} < r < \frac{2}{3\rho_+\beta} - \frac{2}{3}$, which is possible if and only if $\rho_+ < \frac{4-2\beta}{\beta(4+\beta)}$. All these conditions on ρ_+ are ensured if $\rho_+ \le \frac{2-\beta}{3\beta}$. □

6.3. *The behavior of $U(u_n, \Delta_n)$.* The behavior of $U(u_n, \Delta_n)$ has been exhibited in [4], including a central limit theorem. However, here we need a joint CLT, at least on the set $\Omega_T^{noW} \cap \Omega_T^{i\beta}$, for the pair $(U(u_n, \Delta_n)_T, B(2, u_n, \Delta_n))$, and even for this pair jointly with the similar pair with the truncation levels γu_n. For this we will use Lemma 2, and we thus need to show that the difference $U(u_n, \Delta_n) - D'(u_n)$ is negligible, after a suitable normalization. To this effect, we use the contorted way of using the aforementioned CLT for $U(u_n, \Delta_n)_T$, but knowing this result it seems the shortest route toward the desired joint CLT.

LEMMA 7. *Assume reinforced Assumption 1.*

(i) *Under (17) we have*

$$u_n^{\beta} U(u_n, \Delta_n)_t \xrightarrow{\mathbb{P}} \overline{A}_t. \tag{60}$$

(ii) *If moreover $\beta'' < \frac{\beta}{2+\beta}$ and $\beta' < \frac{\beta}{2}$ and (17) holds with $\rho_+ < \frac{1}{2+\beta} \wedge \frac{2}{5\beta}$, then*

$$u_n^{\beta/2}\big(U(u_n, \Delta_n)_t - D'(u_n)_t\big) \xrightarrow{\mathbb{P}} 0. \tag{61}$$

PROOF. In [4] the truncation level was set as $u_n = \alpha \Delta_n^{\varpi}$. However, it is obvious that it works with any truncation level u_n subject to (17), with the conditions

on ϖ replaced by exactly the same conditions on ρ_+. With this in view, (i) follows from Proposition 1 of that paper. The proof of (ii) is much more involved, and broken into several steps.

Step (1) We write $U(u_n,\Delta_n)_t - D'(u_n)_t$ as $H(1)^n_t + H(2)^n_t - H(3)^n_t$, where $H(3)^n_t = D'(u_n)_t - D'(u_n)_{\Delta_n[t/\Delta_n]}$ and $H(j)^n_t = \sum_{i=1}^{[t/\Delta_n]} \zeta(j)^n_i$ for $j=1,2$, with

$$\zeta(1)^n_i = 1_{\{\Delta^n_i D'(u_n)=0, |\Delta^n_i X|>u_n\}},$$
$$\zeta(2)^n_i = 1_{\{\Delta^n_i D'(u_n)\geq 1, |\Delta^n_i X|>u_n\}} - \Delta^n_i D'(u_n).$$

In this step we prove

$$u_n^{\beta/2} H(3)^n_t \xrightarrow{\mathbb{P}} 0. \tag{62}$$

The left-hand side above is nonnegative, with expectation $\mathbb{E}(\widetilde{D}'(u_n)_t - \widetilde{D}'(u_n)_{\Delta_n[t/\Delta_n]})$, which is smaller than $K\Delta_n/u_n^{\beta/2+\beta'}$ (see the proof of Lemma 2). Since $\rho_+(\beta/2+\beta') < 3\rho_+\beta/2 < 1$ we deduce (62).

Step (2) Let us assume for a moment that we have

$$u_n^{\beta/2} H(2)^n_t \xrightarrow{\mathbb{P}} 0. \tag{63}$$

In Proposition 2 of [4], and upon replacing $\alpha\Delta_n^{\varpi}$ by u_n, it is proved that under our assumptions on β', β'' and ρ_+, the sequence $Z_n = u_n^{-\beta/2}(u_n^{\beta} U(u_n,\Delta_n)_t - \overline{A}_t)$ converges in law to a limiting variable $\overline{W}_t$ which is centered. On the other hand, Lemma 2 yields that $Z'_n = u_n^{-\beta/2}(u_n^{\beta} D'(u_n)_t - \overline{A}_t)$ converges in law to a limiting variable $\overline{W}'_t$ which is also centered (and, indeed, has the same law as $\overline{W}_t$).

Up to taking a subsequence, assume that the pair (Z_n, Z'_n) converges in law to a pair (Z,Z') of variables which are centered, whereas $Z_n - Z'_n = u_n^{\beta/2}(H(1)^n_t + H(2)^n_t)$. In view of (63) it follows that $u_n^{\beta/2} H(1)^n_t$ converges in law to $Z - Z'$. Therefore, since by construction $H(1)^n_t \geq 0$ we must have $Z - Z' \geq 0$. Since $Z - Z'$ is centered, we must have $Z' = Z$ a.s. In other words, for any subsequence of (Z_n, Z'_n) which converges in law, the limit is a.s. 0, and by a subsequence principle it follows that the original sequence $Z_n - Z'_n$ goes to 0 in law, hence in probability; this obviously implies (61).

At this stage, we are left to prove (63) which will be implied by the following:

$$\mathbb{E}(u_n^{\beta/2} |\zeta(2)^n_i|) \leq \Delta_n v_n \tag{64}$$

for a sequence $v_n \to 0$.

We recall the property (B.12) of [2]: denoting by $R^n_1, \ldots, R^n_m, \ldots$ the successive jump times of $D'(u_n)$ occurring after $(i-1)\Delta_n$ (with any fixed i), we have $P(R^n_j \leq i\Delta_n) \leq K^j \Delta_n^j u_n^{-j\beta}$. This implies

$$\mathbb{P}\big(\Delta^n_i D'(u_n) \geq 2\big) \leq K\Delta_n^2 u_n^{-2\beta}, \qquad \mathbb{E}\big(\Delta^n_i D'(u_n) 1_{\{\Delta^n_i D'(u_n)\geq 2\}}\big) \leq K\Delta_n^2 u_n^{-2\beta}.$$

Since $\rho_+ < 2/(3\beta)$ we have $\Delta_n/u_n^{3\beta/2} \to 0$. Therefore, for proving (64) it remains to show that

$$u_n^{\beta/2}\mathbb{P}\big(\Delta_i^n D'(u_n) = 1, |\Delta_i^n X| \le u_n\big) \le \Delta_n v_n. \tag{65}$$

Set

$$X''(u_n)_t = \sum_{s\le t} \Delta X_s 1_{\{|\Delta X_s|>u_n\}}, \qquad X'(u_n) = X - X''(u_n).$$

We have estimate (B.15) of [2] again, with $H = \beta$ and $\phi'(x) = x^{-\beta'}$. Thus, since on the set $\{\Delta_i^n D'(u_n) = 1\}$ the process $X''(u_n)$ is piecewise constant and with a single jump on the interval $\{(i-1)\Delta_n, i\Delta_n]\}$, and the size of this jump is bigger than u_n, we deduce

$$\mathbb{P}\big(\Delta_i^n D'(u_n) = 1, |\Delta_i^n X''(u_n)| \le u_n(1+w_n)\big) \le K\Delta_n(u_n^{-\beta}w_n + u_n^{-\beta'}) \tag{66}$$

for any choice of the sequence w_n decreasing to 0.

Finally we use estimate (61) of [4] to obtain for all $q \ge 2$

$$\mathbb{P}\big(\Delta_i^n D'(u_n) = 1, |\Delta_i^n X'(u_n)| > u_n w_n\big) \le K\frac{\Delta_n^2}{w_n^2 u_n^{2\beta}} + K_q\frac{\Delta_n^{q/2}}{w_n^q u_n^q}. \tag{67}$$

Of course the left-hand side of (65) is smaller than $u_n^{\beta/2}$ times the sum of the left-hand sides of (66) and (67). Therefore, it remains to prove that we can choose the sequence w_n and $q \ge 2$ in such a way that $y_n(j) \to 0$ for $j = 1, 2, 3, 4$, where

$$y_n(1) = u_n^{\beta/2-\beta'}, \qquad y_n(2) = \frac{w_n}{u_n^{\beta/2}},$$

$$y_n(3) = \frac{\Delta_n}{u_n^{3\beta/2}w_n^2}, \qquad y_n(4) = \frac{\Delta_n^{q/2-1}}{u_n^{q-\beta/2}w_n^q}.$$

We have $y_n(1) \to 0$ by hypothesis. Upon taking $w_n = u_n^r$ for some r, this amounts to showing that one can find $r > 0$ and $q \ge 2$ such that $r > \frac{\beta}{2}$ and $\frac{1}{\rho_+} - 2r > \frac{3\beta}{2}$ and $q - 2 > (q(2r+2) - \beta)\rho_+$. The last condition is satisfied for q large enough as soon as $2(r+1)\rho_+ < 1$. Then it is easy to see that the choice of r is possible if and only if $\rho_+ < \frac{1}{2+\beta} \wedge \frac{2}{5\beta}$. □

6.4. *Central limit theorems for $B(p, u_n, \Delta_n)$ and $U(u_n, \Delta_n)$.* The previous results allow us to derive joint CLTs for the processes $B(p, u_n, \Delta_n)$ and $U(u_n, \Delta_n)$, as required for Theorems 2 and 5. For the first of these two theorems, we use the following proposition which follows from Lemmas 1 and 5:

PROPOSITION 1. *Let $p \in (1, 2]$ and $t \ge 0$ and $k \ge 2$. Under Assumption 2 and (17) with $\rho_- > \frac{p-1}{2(p-\beta)}$ the two-dimensional variables*

$$\frac{1}{\sqrt{\Delta_n}}\big(\Delta_n^{1-p/2}B(p, u_n, \Delta_n)_t - A(p)_t, \Delta_n^{1-p/2}B(p, u_n, k\Delta_n)_t - k^{p/2-1}A(p)_t\big)$$

stably converge in law to a limit which is defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and which, conditionally on* $\mathcal{F}$, *is a centered Gaussian variable with variance–covariance matrix given by* (43).

For the second theorem, we use the following consequence of Lemmas 2, 6 and 7:

PROPOSITION 2. *Let* $t \geq 0$ *and* $\gamma > 1$, *and suppose Assumption* 1.

(i) *If* $u_n \to 0$ *we have*

$$u_n^{\beta} U(u_n, \Delta_n)_t \xrightarrow{\mathbb{P}} \overline{A}_t. \tag{68}$$

(ii) *If* $p > \beta$ *and* (17) *holds with* $\rho_+ \leq \frac{p-1}{p}$, *we have*

$$u_n^{\beta-p} B(p, u_n, \Delta_n)_t \xrightarrow{\mathbb{P}} \frac{\beta}{p-\beta} \overline{A}_t \quad \text{in restriction to the set } \Omega_t^{noW}. \tag{69}$$

(iii) *If further* $\beta'' < \frac{\beta}{2+\beta}$ *and* $\beta' < \frac{\beta}{2}$, *and if* (17) *holds with* $\rho_+ < \frac{1}{2+\beta} \wedge \frac{2}{5\beta} \wedge \frac{2-\beta}{3\beta}$, *the four-dimensional variables*

$$\begin{pmatrix} \dfrac{1}{u_n^{\beta/2}} \left(u_n^{\beta-p} B(p, u_n, \Delta_n)_t - \dfrac{\beta}{p-\beta} \overline{A}_t \right) \\ \dfrac{1}{u_n^{\beta/2}} \left((\gamma u_n)^{\beta-p} B(p, \gamma u_n, \Delta_n)_t - \dfrac{\beta}{p-\beta} \overline{A}_t \right) \\ \dfrac{1}{u_n^{\beta/2}} \left(u_n^{\beta} U(u_n, \Delta_n)_t - \overline{A}_t \right) \\ \dfrac{1}{u_n^{\beta/2}} \left((\gamma u_n)^{\beta} U(\gamma u_n, \Delta_n)_t - \overline{A}_t \right) \end{pmatrix}$$

stably converge in law, in restriction to the set Ω_t^{noW}, *to a limit which is defined on an extension of* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *and which, conditionally on* $\mathcal{F}$, *is a centered Gaussian variable with variance–covariance matrix* $\overline{A}_t \widetilde{C}$, *with* $\widetilde{C}$ *given by* (48).

6.5. *Proof of the theorems*. It remains to prove the main theorems, for which we can assume the reinforced assumptions if necessary, without restriction.

First, the consistency results (20) and (30) are obvious consequences of (51), (69) and (68), plus the facts that $\overline{A}_T > 0$ on $\Omega_T^{i\beta}$ and $A(p)_T > 0$ on Ω_T^W.

Second, in order to prove Theorem 2 we use Proposition 1 which, upon using the "delta method," shows that under the stated assumptions the variables $\frac{1}{\sqrt{\Delta_n}}(S_n - k^{p/2-1})$ converge stably in law, in restriction to Ω_T^W, to a variable which conditionally on $\mathcal{F}$ is centered Gaussian with variance

$$V = N(p,k) \frac{A(2p)_T}{(A(p)_T)^2}.$$

With V_n given by (24), we have $\frac{1}{\Delta_n} V_n \xrightarrow{\mathbb{P}} V$ by (51), and the result readily follows.

In the same way, Proposition 2 yields that $\frac{1}{u_n^{\beta/2}}(S'_n - \gamma^2)$ converges stably in law, in restriction to $\Omega_T^{noW} \cap \Omega_T^{i\beta}$, to a variable which conditionally on $\mathcal{F}$ is centered Gaussian with variance

$$V' = \frac{\gamma^4}{\overline{A}_T}\left(\frac{\beta(2-\beta)^2}{4-\beta} + 1\right)(1 + \gamma^\beta - 2\gamma^{\beta-2}).$$

If V'_n is given by (32), then $\frac{1}{u_n^\beta} V'_n \xrightarrow{\mathbb{P}} V'$ in restriction to $\Omega_T^{noW} \cap \Omega_T^{i\beta}$ by (69) and (68). This finishes the proof of Theorem 5.

Finally, for both Theorems 3 and 6, the claims concerning the asymptotic level of the tests are trivial consequences of two central limit Theorems 2 and 5. It remains to prove that the asymptotic power is 1 in both cases. By virtue of (20) and (30), this will follow from the next two properties, under the appropriate assumptions

$$\text{(70)} \qquad \begin{cases} V_n \xrightarrow{\mathbb{P}} 0, & \text{on the set } \Omega_T^{noW} \cap \Omega_T^{i\beta}, \\ V'_n \xrightarrow{\mathbb{P}} 0, & \text{on the set } \Omega_T^{W} \cap \Omega_T^{i\beta}. \end{cases}$$

The first of these properties follows from (69), and the second one follows from (41), (50) and (68).

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