The Annals of Probability
2010, Vol. 38, No. 3, 1180–1220
DOI: 10.1214/09-AOP501

OPTIMAL LOCAL HÖLDER INDEX FOR DENSITY STATES OF SUPERPROCESSES WITH $(1+\beta)$-BRANCHING MECHANISM[1]

BY KLAUS FLEISCHMANN, LEONID MYTNIK AND VITALI WACHTEL

Weierstrass Institute, Technion Israel Institute of Technology and University of Munich

For $0 < \alpha \le 2$, a super-α-stable motion X in R^d with branching of index $1+\beta \in (1,2)$ is considered. Fix arbitrary $t > 0$. If $d < \alpha/\beta$, a dichotomy for the density function of the measure X_t holds: the density function is locally Hölder continuous if $d = 1$ and $\alpha > 1+\beta$ but locally unbounded otherwise. Moreover, in the case of continuity, we determine the optimal local Hölder index.

1. Introduction and statement of results.

1.1. *Background and purpose.* For $0 < \alpha \le 2$, a super-α-stable motion $X = \{X_t : t \ge 0\}$ in R^d with branching of index $1+\beta \in (1,2]$ is a finite measure-valued process related to the log-Laplace equation

$$\frac{d}{dt} u = \mathbf{\Delta}_\alpha u + au - bu^{1+\beta}, \tag{1.1}$$

where $a \in \mathsf{R}$ and $b > 0$ are any fixed constants. Its underlying motion is described by the fractional Laplacian $\mathbf{\Delta}_\alpha := -(-\mathbf{\Delta})^{\alpha/2}$ determining a symmetric α-stable motion in R^d of index $\alpha \in (0,2]$ (Brownian motion if $\alpha = 2$) whereas its continuous-state branching mechanism described by

$$v \mapsto -av + bv^{1+\beta} =: \Psi(v), \qquad v \ge 0, \tag{1.2}$$

belongs to the domain of attraction of a stable law of index $1+\beta \in (1,2]$ (the branching is critical if $a = 0$).

It is well known that in dimensions $d < \frac{\alpha}{\beta}$ at any fixed time $t > 0$ the measure $X_t = X_t(dx)$ is absolutely continuous with probability one (cf. Fleischmann [3] where $a = 0$; the noncritical case requires the obvious changes). By an abuse of notation, we sometimes denote a version of the density function of the measure $X_t = X_t(dx)$ by the same symbol, $X_t(dx) = X_t(x)\,dx$, that is, $X_t = \{X_t(x) : x \in$

Received May 2008; revised April 2009.

[1]Supported by the German Israeli Foundation for Scientific Research and Development, Grant G-807-227.6/2003.

AMS 2000 subject classifications. Primary 60J80; secondary 60G57.

Key words and phrases. Dichotomy for density of superprocess states, Hölder continuity, optimal exponent, critical index, local unboundedness, multifractal spectrum, Hausdorff dimension.

$\mathsf{R}^d\}$. In the case of one-dimensional continuous super-Brownian motion ($\alpha = 2$, $\beta = 1$), even a joint-continuous density field $\{X_t(x) : t > 0, x \in \mathsf{R}\}$ exists, satisfying a stochastic equation (Konno and Shiga [12] as well as Reimers [16]).

From now on we assume that $d < \frac{\alpha}{\beta}$ and $\beta \in (0, 1)$. For the Brownian case $\alpha = 2$ and if $a = 0$ (critical branching), Mytnik [14] proved that a version of the density $\{X_t(x) : t > 0, x \in \mathsf{R}^d\}$ of the measure $X_t(dx)\,dt$ exists that satisfies, in a weak sense, the following stochastic partial differential equation (SPDE):

$$\frac{\partial}{\partial t} X_t(x) = \mathbf{\Delta} X_t(x) + (bX_{t-}(x))^{1/(1+\beta)} \dot{L}(t, x), \tag{1.3}$$

where $\dot{L}$ is a $(1 + \beta)$-stable noise without negative jumps.

CONVENTION 1.1. From now on, (if it is not stated otherwise explicitly) we use the term *density* to denote the density function of the measure $X_t(dx)$ with respect to the Lebesgue measure.

For the same model (as in the paragraph before Convention 1.1), in Mytnik and Perkins [15] regularity and irregularity properties of the density at fixed times had been revealed. More precisely, these densities have continuous versions if $d = 1$, whereas they are locally unbounded on open sets of positive $X_t(dx)$-measure in all higher dimensions ($d < \frac{2}{\beta}$).

The first *purpose* in the present paper is to allow also discontinuous underlying motions, that is to consider also all $\alpha \in (0, 2)$. Then actually the same type of *fixed time dichotomy* holds (recall that $d < \frac{\alpha}{\beta}$): continuity of densities if $d = 1$ and $\alpha > 1 + \beta$ whereas local unboundedness is true if $d > 1$ or $\alpha \le 1 + \beta$.

However, the *main purpose* of the paper is to address the following question: what is the optimal local Hölder index in the first case of existence of a continuous density? Here by optimality we mean that there is a critical index η_c such that for any fixed $t > 0$ there is a version of the density which is locally Hölder continuous of any index $\eta < \eta_c$ whereas there is no locally Hölder continuous version with index $\eta \ge \eta_c$.

In [15] continuity of the density at fixed times is proved by some moment methods, although moments of order larger than $1 + \beta$ are in general infinite in the $1 + \beta < 2$ case. A standard procedure to get local Hölder continuity is the Kolmogorov criterion by using "high" moments. This, for instance, can be done in the $\beta = 1$ case ($\alpha = 2$, $d = 1$) to show local Hölder continuity of any index smaller than $\frac{1}{2}$ (see the estimates in the proof of Corollary 3.4 in Walsh [19]).

Due to the lack of "high" moments in our $\beta < 1$ case we cannot use moments to get the optimal local Hölder index. Therefore we have to get deeply into the jump structure of the superprocess to obtain the needed estimates. As a result we are able to show the *local Hölder continuity* of all orders $\eta < \eta_c := \frac{\alpha}{1+\beta} - 1$, provided that $d = 1$ and $\alpha > 1 + \beta$. We also verify that the bound η_c for the local Hölder index is in fact *optimal* in the sense that there are points x_1, x_2 such that the density increments $|X_t(x_1) - X_t(x_2)|$ are of a larger order than $|x_1 - x_2|^\eta$ as $x_1 - x_2 \to 0$ for every $\eta \geq \eta_c$. For precise formulations, see Theorem 1.2 below.

1.2. *Statement of results.* Write $\mathcal{M}_\mathrm{f}$ for the set of all finite measures μ defined on R^d and $|\mu|$ for its total mass $\mu(\mathsf{R}^d)$. Let $\|f\|_U$ denote the essential supremum (with respect to Lebesgue measure) of a function $f : \mathsf{R}^d \to \mathsf{R}_+ := [0, \infty)$ over a nonempty open set $U \subseteq \mathsf{R}^d$.

Let p^α denote the continuous α*-stable transition kernel* related to the fractional Laplacian $\mathbf{\Delta}_\alpha = -(-\mathbf{\Delta})^{\alpha/2}$, and S^α the related *semigroup*.

Recall that $0 < \alpha \leq 2$, $1 + \beta \in (1, 2)$ and $d < \frac{\alpha}{\beta}$, and consider again the (α, d, β)-superprocess $X = \{X_t : t \geq 0\}$ in R^d related to (1.1). Recall also that for fixed $t > 0$, with probability one, the measure state X_t is absolutely continuous (see [3]). The following theorem is our *main result*:

THEOREM 1.2 (Dichotomy for densities). *Fix $t > 0$ and $X_0 = \mu \in \mathcal{M}_\mathrm{f}$.*

(a) (Local Hölder continuity). *If $d = 1$ and $\alpha > 1 + \beta$, then with probability one, there is a continuous version $\tilde{X}_t$ of the density function of the measure $X_t(dx)$. Moreover, for each $\eta < \eta_c := \frac{\alpha}{1+\beta} - 1$, this version $\tilde{X}_t$ is locally Hölder continuous of index η*

$$\sup_{x_1, x_2 \in K, x_1 \neq x_2} \frac{|\tilde{X}_t(x_1) - \tilde{X}_t(x_2)|}{|x_1 - x_2|^\eta} < \infty \qquad \textit{compact } K \subset \mathsf{R}.$$

(b) (Optimal local Hölder index). *Under conditions as in the beginning of part* (a), *for every $\eta \geq \eta_c$ with probability one, for any open $U \subseteq \mathsf{R}$,*

$$\sup_{x_1, x_2 \in U, x_1 \neq x_2} \frac{|\tilde{X}_t(x_1) - \tilde{X}_t(x_2)|}{|x_1 - x_2|^\eta} = \infty \qquad \textit{whenever } X_t(U) > 0.$$

(c) (Local unboundedness). *If $d > 1$ or $\alpha \leq 1 + \beta$, then with probability one, for all open $U \subseteq \mathsf{R}^d$,*

$$\|X_t\|_U = \infty \qquad \textit{whenever } X_t(U) > 0.$$

REMARK 1.3 (Any version). As in part (c), the statement in part (b) is valid also for any version X_t of the density function.

1.3. *Some discussion.* At first sight, the result of Theorem 1.2(a), (b) is a bit surprising. Let us recall again what is known about regularity properties of densities of (α, d, β)-superprocesses. The case of continuous super-Brownian motion ($\alpha = 2$, $\beta = 1$, $d = 1$) is very well studied. As already mentioned, densities exist at all times simultaneously, and they are locally Hölder continuous (in the spatial variable) for any index $\eta < \frac{1}{2}$. Moreover, it is known that $\frac{1}{2}$ is optimal in this case.

Now let us consider our result in Theorem 1.2(a), (b), specialized to $\alpha = 2$. Then we have $\eta_c = \frac{2}{1+\beta} - 1 \downarrow 0$ as $\beta \uparrow 1$ where the limit 0 is different from the optimal local Hölder index $\frac{1}{2}$ of continuous super-Brownian motion. This may confuse a reader and even raise a suspicion that something is wrong. However there is an intuitive explanation for this discontinuity as we would like to explain now.

Recall the notion of Hölder continuity *at a point*. A function f is Hölder continuous with index $\eta \in (0, 1)$ at a point x_0 if there is a neighborhood $U(x_0)$ such that

$$|f(x) - f(x_0)| \le C|x - x_0|^{\eta} \qquad \text{for all } x \in U(x_0). \tag{1.4}$$

The *optimal* Hölder index $H(x_0)$ of f at the point x_0 is defined as the supremum of all such η. Clearly, there are functions where $H(x_0)$ may vary with x_0, and the index of a local Hölder continuity in a domain cannot be larger than the smallest optimal Hölder index at the points of the domain. The densities of continuous super-Brownian motion are such that almost surely $H(x_0) = \frac{1}{2}$ for all x_0 whereas in our $\beta < 1$ case of discontinuous superprocesses the situation is quite different. The critical local Hölder index $\eta_c = \frac{\alpha}{1+\beta} - 1$ in our case is a result of the influence of relatively high jumps of the superprocess that occur close to time t. So there are (random) points x_0 with $H(x_0) = \eta_c$. But these points are *exceptional* points; loosely speaking, there are not too many of them. We conjecture[1] that at any *given* point x_0 the optimal Hölder index $H(x_0)$ equals $(\frac{1+\alpha}{1+\beta} - 1) \wedge 1 =: \bar{\eta}_c > \eta_c$. Now if $\alpha = 2$, as $\beta \uparrow 1$ one gets the index $\frac{1}{2}$ corresponding to the case of continuous super-Brownian motion.

This observation raises in fact a number of very interesting *open problems*:

CONJECTURE 1.4 (Multifractal spectrum). We conjecture that for any $\eta \in (\eta_c, \bar{\eta}_c)$ there are (random) points x_0 where the density X_t at the point x_0 is Hölder continuous with index η. What is the *Hausdorff dimension*, say $D(\eta)$, of the (random) set $\{x_0 : H(x_0) = \eta\}$? We conjecture that

$$\lim_{\eta \downarrow \eta_c} D(\eta) = 0 \quad \text{and} \quad \lim_{\eta \uparrow \bar{\eta}_c} D(\eta) = 1. \tag{1.5}$$

This function $\eta \mapsto D(\eta)$ reveals the so-called *multifractal* structure concerning the optimal Hölder index in points for the densities of superprocesses with branching of index $1 + \beta < \alpha$ and is definitely worth studying. In this connection, we refer to Jaffard [10] where multifractal properties of one-dimensional Lévy processes are studied.

Another interesting direction would be a generalization of our results to the case of SPDEs driven by Levy noises. In recent years there has been increasing

[1] We will verify this conjecture in an outcoming extended version of [4].

interest in such SPDEs. Here we may mention the papers Saint Laubert Bié [18], Mytnik [14], Mueller, Mytnik and Stan [13] as well as Hausenblas [9]. Note that in these papers properties of solutions are described in some $\mathcal{L}^p$-sense. To the best of our knowledge not too many things are known about local Hölder continuity of solutions (in case of continuity). The only result we know in this direction is [15] where some local Hölder continuity of the fixed time density of super-Brownian motion ($\alpha = 2$, $\beta < 1$, $d < \frac{2}{\beta}$, $a = 0$) was established. However, the result there was far away from being optimal. With Theorem 1.2(a), (b) we fill this gap. Our result also allows the following conjecture:

CONJECTURE 1.5 (Regularity in case of SPDE with stable noise). Consider the SPDE,

$$\frac{\partial}{\partial t} X_t(x) = \mathbf{\Delta}_\alpha X_t(x) + g(X_{t-}(x))\dot{L}(t,x), \tag{1.6}$$

where $\dot{L}$ is a $(1+\beta)$-stable noise without negative jumps, and g is such that solutions exist. Then there should exist versions of solutions such that at fixed times regularity holds just as described in Theorem 1.2(a), (b) with the same parameter classification, in particular, with the same η_{c}.

1.4. *Martingale decomposition of X.* As in the $\alpha = 2$ case of [15], for the proof we need the martingale decomposition of X. For this purpose, we will work with the following *alternative description* of the continuous-state branching mechanism Ψ from (1.2):

$$\Psi(v) = -av + \varrho \int_0^\infty dr\, r^{-2-\beta}(e^{-vr} - 1 + vr), \qquad v \ge 0, \tag{1.7}$$

where

$$\varrho := b\frac{(1+\beta)\beta}{\Gamma(1-\beta)} \tag{1.8}$$

with Γ denoting the famous Gamma function. The martingale decomposition of X in the following lemma is basically proven in Dawson [1], Section 6.1.

Denote by $\mathcal{C}_{\mathrm{b}}$ the set of all bounded and continuous functions on R^d. We add the sign $+$ if the functions are additionally nonnegative. $\mathcal{C}_{\mathrm{b}}^{(k),+}$ with $k \ge 1$ refers to the subset of functions which are k times differentiable and that all derivatives up to the order k belong to $\mathcal{C}_{\mathrm{b}}^{+}$, too.

LEMMA 1.6 (Martingale decomposition of X). *Fix $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$.*

(a) (Discontinuities). *All discontinuities of the process X are jumps upward of the form $r\delta_x$. More precisely, there exists a random measure $N(d(s,x,r))$ on $\mathsf{R}_+ \times \mathsf{R}^d \times \mathsf{R}_+$ describing the jumps $r\delta_x$ of X at times s at sites x of size r.*

(b) (Jump intensities). *The compensator* $\hat{N}$ *of* N *is given by*

$$\hat{N}(d(s,x,r)) = \varrho\, ds\, X_s(dx) r^{-2-\beta}\, dr;$$

that is, $\tilde{N} := N - \hat{N}$ *is a martingale measure on* $\mathsf{R}_+ \times \mathsf{R}^d \times \mathsf{R}_+$.

(c) (Martingale decomposition). *For all* $\varphi \in \mathcal{C}_{\mathrm{b}}^{(2),+}$ *and* $t \geq 0$,

$$\langle X_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t ds \langle X_s, \mathbf{\Delta}_\alpha \varphi \rangle + M_t(\varphi) + a I_t(\varphi)$$

with the discontinuous martingale

$$t \mapsto M_t(\varphi) := \int_{(0,t] \times \mathsf{R}^d \times \mathsf{R}_+} \tilde{N}(d(s,x,r)) r \varphi(x)$$

and the increasing process

$$t \mapsto I_t(\varphi) := \int_0^t ds \langle X_s, \varphi \rangle.$$

From Lemma 1.6 we get the related *Green's function representation*,

$$\begin{aligned}\langle X_t, \varphi \rangle = {} & \langle \mu, S_t^\alpha \varphi \rangle + \int_{(0,t] \times \mathsf{R}^d} M(d(s,x)) S_{t-s}^\alpha \varphi(x) \\ & + a \int_{(0,t] \times \mathsf{R}^d} I(d(s,x)) S_{t-s}^\alpha \varphi(x), \qquad t \geq 0, \varphi \in \mathcal{C}_{\mathrm{b}}^+,\end{aligned} \tag{1.9}$$

with M the martingale measure related to the martingale in part (c) and I the measure related to the increasing process there.

We add also the following lemma which can be proved as Lemma 3.1 in Le Gall and Mytnik [6]. For $p \geq 1$, let $\mathcal{L}_{\mathrm{loc}}^p(\mu) = \mathcal{L}_{\mathrm{loc}}^p(\mathsf{R}_+ \times \mathsf{R}^d, S_s^\alpha \mu(x)\, ds\, dx)$ denote the space of equivalence classes of measurable functions ψ such that

$$\int_0^T ds \int_{\mathsf{R}^d} dx\, S_s^\alpha \mu(x) |\psi(s,x)|^p < \infty, \qquad T > 0. \tag{1.10}$$

LEMMA 1.7 (L^p-space with martingale measure). *Let* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$ *and* $\psi \in \mathcal{L}_{\mathrm{loc}}^p(\mu)$ *for some* $p \in (1+\beta, 2)$. *Then the martingale*

$$t \mapsto \int_{(0,t] \times \mathsf{R}^d} M(d(s,x)) \psi(s,x) \tag{1.11}$$

is well defined.

Fix $t > 0$, $\mu \in \mathcal{M}_{\mathrm{f}}$. Suppose $d < \frac{\alpha}{\beta}$. Then the random measure X_t is a.s. absolutely continuous. From (1.9) we get the following representation of a version of its *density function* (cf. [6, 15]):

$$\begin{aligned} X_t(x) &= \mu * p_t^\alpha(x) + \int_{(0,t]\times\mathsf{R}^d} M(d(s,y))\,p_{t-s}^\alpha(x-y) \\ &\quad + a\int_{(0,t]\times\mathsf{R}^d} I(d(s,y))\,p_{t-s}^\alpha(x-y) \qquad (1.12) \\ &=: Z_t^1(x) + Z_t^2(x) + Z_t^3(x), \qquad x \in \mathsf{R}^d, \end{aligned}$$

with notation in the obvious correspondence (and kernels p^α introduced in the beginning of Section 1.2).

This representation is the starting point for the proof of the local Hölder continuity as claimed in Theorem 1.2(a). Main work has to be done to deal with Z_t^2.

1.5. *Organization of the paper.* In Section 2 we develop some tools that will be used in the following sections for the proof of Theorem 1.2. Also on the way, in Section 2.3, we are able to verify partially Theorem 1.2(a) for some range of parameters α, β using simple moment estimates. The proof of Theorem 1.2(a) is completed in Section 3 using a more delicate analysis of the jump structure of the process. Section 4 is devoted to the proof of part (c) of Theorem 1.2. In Section 5, which is the most technically involved section, we verify Theorem 1.2(b).

2. Auxiliary tools.

In this section we always assume that $d = 1$.

2.1. *On the transition kernel of α-stable motion.* The symbol C will always denote a generic positive constant, which might change from place to place. On the other hand, $c_{(\#)}$ denotes a constant appearing in formula line (or array) (#).

We start with two estimates concerning the α-stable transition kernel p^α.

LEMMA 2.1 (α-stable density increment). *For every $\delta \in [0, 1]$,*

$$|p_t^\alpha(x) - p_t^\alpha(y)| \le C\frac{|x-y|^\delta}{t^{\delta/\alpha}}\big(p_t^\alpha(x/2) + p_t^\alpha(y/2)\big), \qquad t > 0,\ x, y \in \mathsf{R}. \tag{2.1}$$

PROOF. For the case $\alpha = 2$, see, for example, Rosen [17], (2.4e). Suppose $\alpha < 2$. It suffices to assume that $t = 1$. In fact, multiply x, y by $t^{-1/\alpha}$ in the formula for the $t = 1$ case, and use that by self-similarity, $p_1^\alpha(t^{-1/a}x) = t^{1/\alpha}p_t^\alpha(x)$.

Now we use the well-known subordination formula

$$p_1^\alpha(z) = \int_0^\infty ds\, q_1^{\alpha/2}(s)\,p_s^{(2)}(z), \qquad z \in \mathsf{R}, \tag{2.2}$$

where $q^{\alpha/2}$ denotes the continuous transition kernel of a stable process on R_+ of index $\alpha/2$, and by an abuse of notation, $p^{(2)}$ refers to p^α in case $\alpha = 2$. Consequently,

$$|p_1^\alpha(x) - p_1^\alpha(y)| \le \int_0^\infty ds\, q_1^{\alpha/2}(s)\big|p_s^{(2)}(x) - p_s^{(2)}(y)\big|. \tag{2.3}$$

Hence, from the $\alpha = 2$ case,

$$
\begin{aligned}
&|p_1^\alpha(x) - p_1^\alpha(y)| \\
&\quad \le C|x-y|^\delta \int_0^\infty ds\, q_1^{\alpha/2}(s) s^{-\delta/2}\big(p_s^{(2)}(x/2) + p_s^{(2)}(y/2)\big).
\end{aligned}
\tag{2.4}
$$

The lemma will be proved if we show that

$$
\int_0^\infty ds\, q_1^{\alpha/2}(s) s^{-\delta/2} p_s^{(2)}(x/2) \le C p_1^\alpha(x/2), \qquad x \in \mathsf{R}. \tag{2.5}
$$

First, in view of (2.2),

$$
\int_1^\infty ds\, q_1^{\alpha/2}(s) s^{-\delta/2} p_s^{(2)}(x/2) \le \int_1^\infty ds\, q_1^{\alpha/2}(s) p_s^{(2)}(x/2) \le p_1^\alpha(x/2). \tag{2.6}
$$

Second, by Brownian scaling,

$$
\begin{aligned}
\int_0^1 ds\, q_1^{\alpha/2}(s) s^{-\delta/2} p_s^{(2)}(x/2) &= \int_0^1 du\, q_1^{\alpha/2}(u) u^{-(\delta+1)/2} p_1^{(2)}\left(\frac{x/2}{u^{1/2}}\right) \\
&\le p_1^{(2)}(x/2) \int_0^1 du\, q_1^{\alpha/2}(u) u^{-(\delta+1)/2} \\
&\le C p_1^{(2)}(x/2),
\end{aligned}
\tag{2.7}
$$

where in the last step we have used the fact that $q_1^{\alpha/2}(u)$ decreases, as $u \downarrow 0$, exponentially fast (cf. [2], Theorem 13.6.1). Since $p_1^{(2)}(x/2) = \mathrm{o}(p_1^\alpha(x/2))$ as $x \uparrow \infty$, we have $p_1^{(2)}(x/2) \le C p_1^\alpha(x/2)$, $x \in \mathsf{R}$. Hence,

$$
\int_0^1 ds\, q_1^{\alpha/2}(s) s^{-\delta/2} p_s^{(2)}(x/2) \le C p_1^\alpha(x/2). \tag{2.8}
$$

Combining (2.6) and (2.8) gives (2.5), completing the proof. $\square$

LEMMA 2.2 (Integrals of α-stable density increment). *If* $\theta \in [1, 1+\alpha)$ *and* $\delta \in [0, 1]$ *satisfy* $\delta < (1+\alpha-\theta)/\theta$ *then*

$$
\begin{aligned}
&\int_0^t ds \int_{\mathsf{R}} dy\, p_s^\alpha(y) |p_{t-s}^\alpha(x_1 - y) - p_{t-s}^\alpha(x_2 - y)|^\theta \\
&\quad \le C(1+t)|x_1 - x_2|^{\delta\theta}\big(p_t^\alpha(x_1/2) + p_t^\alpha(x_2/2)\big), \qquad t > 0, x_1, x_2 \in \mathsf{R}.
\end{aligned}
\tag{2.9}
$$

PROOF. By Lemma 2.1, for every $\delta \in [0, 1]$,

$$
\begin{aligned}
&|p_{t-s}^\alpha(x_1 - y) - p_{t-s}^\alpha(x_2 - y)|^\theta \\
&\quad \le C \frac{|x_1 - x_2|^{\delta\theta}}{(t-s)^{\delta\theta/\alpha}} \big(p_{t-s}^\alpha\big((x_1 - y)/2\big) + p_{t-s}^\alpha\big((x_2 - y)/2\big)\big)^\theta,
\end{aligned}
\tag{2.10}
$$

$t > s \geq 0$, $x_1, x_2, y \in \mathsf{R}$. Noting that $p^\alpha_{t-s}(\cdot) \leq C(t-s)^{-1/\alpha}$, we obtain

$$
\begin{aligned}
&|p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)|^\theta \\
&\qquad \leq C \frac{|x_1-x_2|^{\delta\theta}}{(t-s)^{(\delta\theta+\theta-1)/\alpha}} \big(p^\alpha_{t-s}\big((x_1-y)/2\big) + p^\alpha_{t-s}\big((x_2-y)/2\big)\big),
\end{aligned}
\tag{2.11}
$$

$t > s \geq 0$, $x_1, x_2, y \in \mathsf{R}$. Therefore,

$$
\begin{aligned}
&\int_0^t ds \int_{\mathsf{R}} dy\, p^\alpha_s(y) |p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)|^\theta \\
&\qquad \leq C|x_1-x_2|^{\delta\theta} \int_0^t ds (t-s)^{-(\delta\theta+\theta-1)/\alpha} \\
&\qquad\qquad \times \int_{\mathsf{R}} dy\, p^\alpha_s(y) \big(p^\alpha_{t-s}\big((x_1-y)/2\big) + p^\alpha_{t-s}\big((x_2-y)/2\big)\big).
\end{aligned}
$$

By scaling of p^α,

$$
\begin{aligned}
&\int_{\mathsf{R}} dy\, p^\alpha_s(y) p^\alpha_{t-s}\big((x-y)/2\big) \\
&\qquad = \frac{1}{2} \int_{\mathsf{R}} dy\, p^\alpha_{2^{-\alpha}s}(y/2) p^\alpha_{t-s}\big((x_2-y)/2\big) \\
&\qquad = \frac{1}{2} p^\alpha_{2^{-\alpha}s+t-s}(x/2) \\
&\qquad = \frac{1}{2} (2^{-\alpha}s+t-s)^{-1/\alpha} p^\alpha_1\big((2^{-\alpha}s+t-s)^{-1/\alpha} x/2\big) \\
&\qquad \leq t^{-1/\alpha} p^\alpha_1(t^{-1/\alpha}x/2) = p^\alpha_t(x/2),
\end{aligned}
\tag{2.12}
$$

since $2^{-\alpha}t \leq 2^{-\alpha}s + t - s \leq t$. As a result we have the inequality

$$
\begin{aligned}
&\int_0^t ds \int_{\mathsf{R}} dy\, p^\alpha_s(y) |p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)|^\theta \\
&\qquad \leq C|x_1-x_2|^{\delta\theta} \big(p^\alpha_t(x_1/2) + p^\alpha_t(x_2/2)\big) \int_0^t ds\, s^{-(\delta\theta+\theta-1)/\alpha}.
\end{aligned}
\tag{2.13}
$$

Noting that the latter integral is bounded by $C(1+t)$, since $(\delta\theta+\theta-1)/\alpha < 1$, we get the desired inequality. $\square$

2.2. *An upper bound for a spectrally positive stable process.* Let $L = \{L_t : t \geq 0\}$ denote a spectrally positive stable process of index $\kappa \in (1,2)$. Per definition, L is an R-valued time-homogeneous process with independent increments and with Laplace transform given by

$$
\mathbf{E} e^{-\lambda L_t} = e^{t\lambda^\kappa}, \qquad \lambda, t \geq 0.
\tag{2.14}
$$

Note that L is the unique (in law) solution to the following martingale problem:

$$t \mapsto e^{-\lambda L_t} - \int_0^t ds\, e^{-\lambda L_s} \lambda^\kappa \text{ is a martingale for any } \lambda > 0. \tag{2.15}$$

Let $\Delta L_s := L_s - L_{s-} > 0$ denote the jumps of L.

LEMMA 2.3 (Big values of the process in case of bounded jumps). *We have*

$$\mathbf{P}\Big(\sup_{0\le u\le t} L_u \mathbb{1}\Big\{\sup_{0\le v\le u} \Delta L_v \le y\Big\} \ge x\Big) \le \left(\frac{Ct}{xy^{\kappa-1}}\right)^{x/y}, \qquad t > 0, x, y > 0. \tag{2.16}$$

PROOF. Since for $\tau > 0$ fixed, $\{L_{\tau t} : t \ge 0\}$ is equal to $\tau^{1/\kappa} L$ in law, for the proof we may assume that $t = 1$. Let $\{\xi_i : i \ge 1\}$ denote a family of independent copies of L_1. Set

$$W_{ns} := \sum_{1\le k\le ns} \xi_k, \qquad L_s^{(n)} := n^{-1/\kappa} W_{ns}, \qquad 0 \le s \le 1, n \ge 1. \tag{2.17}$$

Denote by $D_{[0,1]}$ the Skorohod space of càdlàg functions $f : [0,1] \to \mathsf{R}$. For fixed $y > 0$, let $H : D_{[0,1]} \mapsto \mathsf{R}$ be defined by

$$H(f) = \sup_{0\le u\le 1} f(u) \mathbb{1}\Big\{\sup_{0\le v\le u} \Delta f(v) \le y\Big\}, \qquad f \in D_{[0,1]}. \tag{2.18}$$

It is easy to verify that H is continuous on the set $D_{[0,1]} \setminus J_y$ where $J_y := \{f \in D_{[0,1]} : \Delta f(v) = y \text{ for some } v \in [0,1]\}$. Since $\mathbf{P}(L \in J_y) = 0$, from the invariance principle (see, e.g., Gikhman and Skorokhod [7], Theorem 9.6.2) for $L^{(n)}$ we conclude that

$$\mathbf{P}\big(H(L) \ge x\big) = \lim_{n\uparrow\infty} \mathbf{P}\big(H\big(L^{(n)}\big) \ge x\big), \qquad x > 0. \tag{2.19}$$

Consequently, the lemma will be proved if we show that

$$\mathbf{P}\Big(\sup_{0\le u\le 1} W_{nu} \mathbb{1}\Big\{\max_{1\le k\le nu} \xi_k \le y n^{1/\kappa}\Big\} \ge x n^{1/\kappa}\Big) \le \left(\frac{C}{xy^{\kappa-1}}\right)^{x/y}, \qquad x, y > 0, n \ge 1. \tag{2.20}$$

To this end, for fixed $y', h \ge 0$, we consider the sequence,

$$\Lambda_0 := 1, \qquad \Lambda_n := e^{hW_n} \mathbb{1}\Big\{\max_{1\le k\le n} \xi_k \le y'\Big\}, \qquad n \ge 1. \tag{2.21}$$

It is easy to see that

$$\mathbf{E}\{\Lambda_{n+1} | \Lambda_n = e^{hu}\} = e^{hu} \mathbf{E}\{e^{hL_1}; L_1 \le y'\} \qquad \text{for all } u \in \mathsf{R}, \tag{2.22}$$

and that

$$\mathbf{E}\{\Lambda_{n+1}|\Lambda_n = 0\} = 0. \tag{2.23}$$

In other words,

$$\mathbf{E}\{\Lambda_{n+1}|\Lambda_n\} = \Lambda_n \mathbf{E}\{e^{hL_1}; L_1 \le y'\}. \tag{2.24}$$

This means that $\{\Lambda_n : n \ge 1\}$ is a supermartingale (submartingale) if h satisfies $\mathbf{E}\{e^{hL_1}; L_1 \le y'\} \le 1$ (respectively, $\mathbf{E}\{e^{hL_1}; L_1 \le y'\} \ge 1$). If Λ_n is a submartingale, then by Doob's inequality,

$$\mathbf{P}\Big(\max_{1\le k\le n} \Lambda_k \ge e^{hx'}\Big) \le e^{-hx'} \mathbf{E}\Lambda_n, \qquad x' > 0. \tag{2.25}$$

But if Λ_n is a supermartingale, then

$$\mathbf{P}\Big(\max_{1\le k\le n} \Lambda_k \ge e^{hx'}\Big) \le e^{-hx'} \mathbf{E}\Lambda_0 = e^{-hx'}, \qquad x' > 0. \tag{2.26}$$

From these inequalities and (2.24) we get

$$\mathbf{P}\Big(\max_{1\le k\le n} \Lambda_k \ge e^{hx'}\Big) \le e^{-hx'} \max\big\{1, (\mathbf{E}\{e^{hL_1}; L_1 \le y'\})^n\big\}. \tag{2.27}$$

It was proved by Fuk and Nagaev ([5] see the first formula in the proof of Theorem 4 there) that

$$\mathbf{E}\{e^{hL_1}; L_1 \le y'\} \le 1 + h\mathbf{E}\{L_1; L_1 \le y'\} + \frac{e^{hy'} - 1 - hy'}{(y')^2} V(y'), \qquad h, y' > 0,$$

where $V(y') := \int_{-\infty}^{y'} \mathbf{P}(L_1 \in du)u^2 > 0$. Noting that the assumption $\mathbf{E}L_1 = 0$ yields that $\mathbf{E}\{L_1; L_1 \le y'\} \le 0$, we obtain

$$\mathbf{E}\{e^{hL_1}; L_1 \le y'\} \le 1 + \frac{e^{hy'} - 1 - hy'}{(y')^2} V(y'), \qquad h, y' > 0. \tag{2.28}$$

Now note that

$$\begin{aligned}
&\Big\{\max_{1\le k\le n} W_k \mathbf{1}\Big\{\max_{1\le i\le k} \xi_i \le y'\Big\} \ge x'\Big\}\\
&\qquad = \Big\{\max_{1\le k\le n} e^{hW_k} \mathbf{1}\Big\{\max_{1\le i\le k} \xi_i \le y'\Big\} \ge e^{hx'}\Big\}\\
&\qquad = \Big\{\max_{1\le k\le n} \Lambda_k \ge e^{hx'}\Big\}.
\end{aligned} \tag{2.29}$$

Thus, combining (2.29), (2.28) and (2.27), we get

$$\begin{aligned}
&\mathbf{P}\Big(\max_{1\le k\le n} W_k \mathbf{1}\Big\{\max_{1\le i\le k} \xi_i \le y'\Big\} \ge x'\Big)\\
&\qquad \le \mathbf{P}\Big(\max_{1\le k\le n} \Lambda_k \ge e^{hx'}\Big)\\
&\qquad \le \exp\Big\{-hx' + \frac{e^{hy'} - 1 - hy'}{(y')^2} nV(y')\Big\}.
\end{aligned} \tag{2.30}$$

Choosing $h := (y')^{-1}\log(1 + x'y'/nV(y'))$, we arrive, after some elementary calculations, at the bound,

$$\mathbf{P}\Big(\max_{1\le k\le n} W_k \mathbf{1}\Big\{\max_{1\le i\le k} \xi_i \le y'\Big\} \ge x'\Big) \le \left(\frac{enV(y')}{x'y'}\right)^{x'/y'}, \qquad x', y' > 0. \tag{2.31}$$

Since $\mathbf{P}(L_1 > u) \sim Cu^{-\kappa}$ as $u \uparrow \infty$, we have $V(y') \le C(y')^{2-\kappa}$ for all $y' > 0$. Therefore,

$$\mathbf{P}\Big(\max_{1\le k\le n} W_k \mathbf{1}\Big\{\max_{1\le i\le k} \xi_i \le y'\Big\} \ge x'\Big) \le \left(\frac{Cn}{x'(y')^{\kappa-1}}\right)^{x'/y'}, \qquad x', y' > 0. \tag{2.32}$$

Choosing finally $x' = xn^{1/\kappa}$, $y' = yn^{1/\kappa}$, we get (2.20) from (2.32). Thus, the proof of the lemma is complete. □

LEMMA 2.4 (Small process values). *There is a constant c_κ such that*

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) \le \exp\Big\{-c_\kappa \frac{x^{\kappa/(\kappa-1)}}{t^{1/(\kappa-1)}}\Big\}, \qquad x, t > 0. \tag{2.33}$$

PROOF. It is easy to see that for all $h > 0$,

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) = \mathbf{P}\Big(\sup_{s\le t} e^{-hL_u} > e^{hx}\Big). \tag{2.34}$$

Applying Doob's inequality to the submartingale $t \mapsto e^{-hL_t}$, we obtain

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) \le e^{-hx}\mathbf{E}e^{-hL_t}. \tag{2.35}$$

Taking into account definition (2.14), we have

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) \le \exp\{-hx + th^\kappa\}. \tag{2.36}$$

Minimizing the function $h \mapsto -hx + th^\kappa$, we get the inequality in the lemma with $c_\kappa = (\kappa - 1)/(\kappa)^{\kappa/(\kappa-1)}$. □

2.3. *Local Hölder continuity with some index.* In this subsection we prove Theorem 1.2(a) for parameters $\beta \ge \frac{\alpha-1}{2}$ (see Remark 2.10), whereas for parameters $\beta < \frac{\alpha-1}{2}$ we obtain local Hölder continuity only with nonoptimal bound on indexes. We use the Kolmogorov criterion for local Hölder continuity to get these results. The proof of Theorem 1.2(a) for parameters $\beta < \frac{\alpha-1}{2}$ will be finished in Section 3.

Fix $t > 0$, $\mu \in \mathcal{M}_\mathrm{f}$, and suppose $\alpha > 1 + \beta$. Since our theorem is trivially valid for $\mu = 0$, from now on we everywhere suppose that $\mu \ne 0$. Since we are dealing with the case $d = 1$, the random measure X_t is a.s. absolutely continuous. Recall decomposition (1.12).

Clearly, the deterministic function Z_t^1 is Lipschitz continuous by Lemma 2.1. Next we turn to the random function Z_t^3.

LEMMA 2.5 (Hölder continuity of Z_t^3). *With probability one, Z_t^3 is Hölder continuous of each index $\eta < \alpha - 1$.*

PROOF. From Lemma 2.1 we get for fixed $\delta \in (0, \alpha - 1)$,

$$|p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)| \leq C \frac{|x_1 - x_2|^{\delta}}{(t-s)^{(\delta+1)/\alpha}}, \qquad t > s > 0, x_1, x_2, y \in \mathsf{R}.$$

Therefore,

$$\begin{aligned} &|Z_t^3(x_1) - Z_t^3(x_2)| \\ &\quad \leq |a| \int_0^t ds \int_{\mathsf{R}} X_s(dy) |p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)| \\ &\quad \leq C\Big(\sup_{s \leq t} X_s(\mathsf{R})\Big) |x_1 - x_2|^{\delta} \int_0^t ds (t-s)^{-(\delta+1)/\alpha} \\ &\quad \leq C \frac{\alpha}{\alpha - 1 - \delta} \Big(\sup_{s \leq t} X_s(\mathsf{R})\Big) |x_1 - x_2|^{\delta}, \qquad x_1, x_2 \in \mathsf{R}. \end{aligned} \tag{2.37}$$

Consequently,

$$\sup_{x_1 \neq x_2} \frac{|Z_t^3(x_1) - Z_t^3(x_2)|}{|x_1 - x_2|^{\delta}} < \infty \qquad \text{a.s.,} \tag{2.38}$$

and the proof is complete. $\square$

Our main work concerns Z_t^2.

LEMMA 2.6 (q-norm). *For each $\theta \in (1 + \beta, 2)$ and $q \in (1, 1 + \beta)$,*

$$\begin{aligned} &\mathbf{E}|Z_t^2(x_1) - Z_t^2(x_2)|^q \\ &\quad \leq C\bigg[\bigg(\int_0^t ds \int_{\mathsf{R}} S_s^{\alpha}\mu(dy) |p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)|^{\theta}\bigg)^{q/\theta} \\ &\qquad + \int_0^t ds \int_{\mathsf{R}} S_s^{\alpha}\mu(dy) |p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)|^q\bigg], \\ &\qquad\qquad x_1, x_2 \in \mathsf{R}. \end{aligned} \tag{2.39}$$

The proof can be done similarly to the proof of inequality (3.1) in [6].

COROLLARY 2.7 (q-norm). *For each $\theta \in (1 + \beta, 2)$, $q \in (1, 1 + \beta)$ and $\delta > 0$ satisfying $\delta < \min\{1, (1 + \alpha - \theta)/\theta, (1 + \alpha - q)/q\}$,*

$$\mathbf{E}|Z_t^2(x_1) - Z_t^2(x_2)|^q \leq C|x_1 - x_2|^{\delta q}, \qquad x_1, x_2 \in \mathsf{R}. \tag{2.40}$$

PROOF. For every $\varepsilon \in (1, 1+\alpha)$,

$$\begin{aligned}
&\int_0^t ds \int_{\mathsf{R}} S_s^\alpha \mu(dy) |p_{t-s}^\alpha(x_1 - y) - p_{t-s}^\alpha(x_2 - y)|^\varepsilon \\
&\quad = \int_{\mathsf{R}} \mu(dz) \int_0^t ds \int_{\mathsf{R}} dy\, p_s^\alpha(y - z) |p_{t-s}^\alpha(x_1 - z) - p_{t-s}^\alpha(x_2 - z)|^\varepsilon \qquad (2.41)\\
&\quad = \int_{\mathsf{R}} \mu(dz) \int_0^t ds \int_{\mathsf{R}} dy\, p_s^\alpha(y) |p_{t-s}^\alpha(x_1 - z - y) - p_{t-s}^\alpha(x_2 - z - y)|^\varepsilon.
\end{aligned}$$

Using Lemma 2.2, we get for every positive $\delta < \min\{1, (1+\alpha-\varepsilon)/\varepsilon\}$,

$$\begin{aligned}
&\int_0^t ds \int_{\mathsf{R}} S_s^\alpha \mu(dy) |p_{t-s}^\alpha(x_1 - y) - p_{t-s}^\alpha(x_2 - y)|^\varepsilon \\
&\qquad \le C|x_1 - x_2|^{\delta\varepsilon} \int_{\mathsf{R}} \mu(dz) \bigl(p_t^\alpha\bigl((x_1 - z)/2\bigr) + p_t^\alpha\bigl((x_2 - z)/2\bigr)\bigr) \\
&\qquad \le C|x_1 - x_2|^{\delta\varepsilon},
\end{aligned}$$

since μ, t are fixed. Applying this bound to both summands at the right-hand side of (2.39) finishes the proof of the lemma. $\square$

COROLLARY 2.8 (Finite q-norm of density). *If $K \subset \mathsf{R}$ is a compact and $1 \le q < 1+\beta$, then*

$$\mathbf{E}\Bigl(\sup_{x \in K} X_t(x)\Bigr)^q < \infty. \qquad (2.42)$$

PROOF. By Jensen's inequality, we may additionally assume that $q > 1$. It follows from (1.12) that

$$\Bigl(\sup_{x\in K} X_t(x)\Bigr)^q \le 4\Bigl(\Bigl(\sup_{x\in K} \mu * p_t^\alpha(x)\Bigr)^q + \sup_{x\in K} |Z_t^2(x)|^q + \sup_{x\in K} |Z_t^3(x)|^q\Bigr). \qquad (2.43)$$

Clearly, the first term at the right-hand side is finite. Furthermore, according to Corollary 1.2 of Walsh [19], inequality (2.40) implies that

$$\mathbf{E} \sup_{x\in K} |Z_t^2(x)|^q < \infty. \qquad (2.44)$$

Finally, proceeding as with the derivation of (2.37), we obtain

$$\sup_{x\in K} |Z_t^3(x)| \le C \sup_{s\le t} X_s(\mathsf{R}) \le C e^{|a|t} \sup_{s\le t} e^{-as} X_s(\mathsf{R}). \qquad (2.45)$$

Noting that $s \mapsto e^{-as} X_s(\mathsf{R})$ is a martingale, and using Doob's inequality, we conclude that

$$\mathbf{E} \sup_{x\in K} |Z_t^2(x)|^q \le C\mathbf{E}(e^{-at} X_t(\mathsf{R}))^q < \infty. \qquad (2.46)$$

This completes the proof. $\square$

Furthermore, Corollary 2.7 allows us to prove the following result:

PROPOSITION 2.9 (Local Hölder continuity of Z_t^2). *With probability one*, Z_t^2 *has a version which is locally Hölder continuous of all orders* $\eta > 0$ *satisfying*

$$\eta < \eta_{\mathrm{c}}' := \begin{cases} \dfrac{\alpha}{1+\beta} - 1, & \text{if } \beta \geq (\alpha-1)/2, \\ \dfrac{\beta}{1+\beta}, & \text{if } \beta \leq (\alpha-1)/2. \end{cases} \tag{2.47}$$

PROOF. Let θ, q and δ satisfy the conditions in Corollary 2.7. Then almost surely Z_t^2 has a version which is locally Hölder continuous of all orders smaller than $\delta - 1/q$, (cf. [19], Corollary 1.2).

Let $\varepsilon > 0$ satisfy $\varepsilon < 1 - \beta$ and $\varepsilon < \beta$. Then $\theta = \theta_\varepsilon := 1 + \beta + \varepsilon$ and $q = q_\varepsilon := 1 + \beta - \varepsilon$ are in the range of parameters we are just considering. Moreover, the condition $\delta < \min\{1, (1+\alpha-\theta)/\theta, (1+\alpha-q)/q\}$ reads as

$$\delta < \min\left\{1, \frac{\alpha-\beta-\varepsilon}{1+\beta+\varepsilon}, \frac{\alpha-\beta+\varepsilon}{1+\beta-\varepsilon}\right\} =: f(\varepsilon). \tag{2.48}$$

Hence, for all sufficiently small $\varepsilon > 0$ we can choose $\delta = \delta_\varepsilon := f(\varepsilon) - \varepsilon$. Thus, Z_t^2 has a version which is locally Hölder continuous of all orders smaller than $\delta_\varepsilon - 1/q_\varepsilon$ for this choice of $\theta_\varepsilon, q_\varepsilon, \delta_\varepsilon$. Now

$$\delta_\varepsilon - \frac{1}{q_\varepsilon} \underset{\varepsilon\downarrow 0}{\longrightarrow} \min\left\{1, \frac{\alpha-\beta}{1+\beta}, \frac{\alpha-\beta}{1+\beta}\right\} - \frac{1}{1+\beta} = \min\left\{1, \frac{\beta}{1+\beta}, \frac{\alpha-\beta-1}{1+\beta}\right\},$$

where this limit coincides with the claimed value of η_{c}', completing the proof. $\square$

REMARK 2.10 [Proof of Theorem 1.2(a) for $\beta \geq \frac{\alpha-1}{2}$]. By Lemma 2.5 and Proposition 2.9, the proof of Theorem 1.2(a) is finished for $\beta \geq \frac{\alpha-1}{2}$.

2.4. *Further estimates*. We continue to fix $t > 0$, $\mu \in \mathcal{M}_{\mathrm{f}} \setminus \{0\}$, and to suppose $\alpha > 1 + \beta$.

LEMMA 2.11 (Local boundedness of uniformly smeared out density). *Fix a nonempty compact* $K \subset \mathsf{R}$ *and a constant* $c \geq 1$. *Then*

$$V := V_t^c(K) := \sup_{0 \leq s \leq t, x \in K} S^\alpha_{c(t-s)} X_s(x) < \infty \qquad \textit{almost surely}. \tag{2.49}$$

PROOF. Assume that the statement of the lemma does not hold, that is, there exists an event A of positive probability such that $\sup_{0 \leq s \leq t, x \in K} S^\alpha_{c(t-s)} X_s(x) = \infty$ for every $\omega \in A$. Let $n \geq 1$. Put

$$\tau_n := \begin{cases} \inf\{s < t : \text{there exists } x \in K \text{ such that } S^\alpha_{c(t-s)} X_s(x) > n\}, & \omega \in A, \\ t, & \omega \in A^{\mathrm{c}}. \end{cases}$$

If $\omega \in A$, choose $x_n = x_n(\omega) \in K$ such that $S^{\alpha}_{c(t-\tau_n)} X_{\tau_n}(x_n) > n$ whereas if $\omega \in A^{\mathrm{c}}$, take any $x_n = x_n(\omega) \in K$. Using the strong Markov property gives

$$\begin{aligned}\mathbf{E} S^{\alpha}_{(c-1)(t-\tau_n)} X_t(x_n) &= \mathbf{E}\mathbf{E}\big[S^{\alpha}_{(c-1)(t-\tau_n)} X_t(x_n) | \mathcal{F}_{\tau_n}\big] \\ &= \mathbf{E} e^{a(t-\tau_n)} S^{\alpha}_{(c-1)(t-\tau_n)} S^{\alpha}_{(t-\tau_n)} X_{\tau_n}(x_n) \\ &\geq e^{-|a|t} \mathbf{E} S^{\alpha}_{c(t-\tau_n)} X_{\tau_n}(x_n)\end{aligned} \tag{2.50}$$

[with $e^{a(t-\tau_n)}$ coming from the noncriticality of branching in (1.2)]. From the definition of (τ_n, x_n), we get

$$\mathbf{E} S^{\alpha}_{c(t-\tau_n)} X_{\tau_n}(x_n) \geq n \mathbf{P}(A) \to \infty \qquad \text{as } n \uparrow \infty. \tag{2.51}$$

In order to get a contradiction, we want to prove boundedness in n of the expectation in (2.50). If $c = 1$, then

$$\mathbf{E} X_t(x_n) \leq \mathbf{E} \sup_{x \in K} X_t(x) < \infty, \tag{2.52}$$

the last step by Corollary 2.8. Now suppose $c > 1$. Choosing a compact $K_1 \supset K$ satisfying $\operatorname{dist}(K, (K_1)^{\mathrm{c}}) \geq 1$, we have

$$\begin{aligned}&\mathbf{E} S^{\alpha}_{(c-1)(t-\tau_n)} X_t(x_n) \\ &\quad = \mathbf{E} \int_{K_1} dy\, X_t(y) p^{\alpha}_{(c-1)(t-\tau_n)}(x_n - y) \\ &\qquad + \mathbf{E} \int_{(K_1)^{\mathrm{c}}} dy\, X_t(y) p^{\alpha}_{(c-1)(t-\tau_n)}(x_n - y) \\ &\quad \leq \mathbf{E} \sup_{y \in K_1} X_t(y) + \mathbf{E} X_t(\mathsf{R}) \sup_{y \in (K_1)^{\mathrm{c}}, x \in K, 0 \leq s \leq t} p^{\alpha}_{(c-1)s}(x - y).\end{aligned}$$

By our choice of K_1 we obtain the bound,

$$\mathbf{E} S^{\alpha}_{(c-1)(t-\tau_n)} X_t(x_n) \leq \mathbf{E} \sup_{y \in K_1} X_t(y) + C = C, \tag{2.53}$$

the last step by Corollary 2.8. Altogether, (2.50) is bounded in n, and the proof is finished. □

LEMMA 2.12 (Randomly weighted kernel increments). *Fix $\theta \in [1, 1 + \alpha)$, $\delta \in [0, 1]$ with $\delta < (1 + \alpha - \theta)/\theta$, and a nonempty compact $K \subset \mathsf{R}$. Then*

$$\begin{aligned}&\int_0^t ds \int_{\mathsf{R}} X_s(dy) |p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)|^{\theta} \\ &\qquad \leq C V |x_1 - x_2|^{\delta\theta}, \qquad x_1, x_2 \in K,\ \text{a.s.},\end{aligned} \tag{2.54}$$

with $V = V_t^{2^{\alpha}}(K)$ from Lemma 2.11.

PROOF. Using (2.11) gives

$$\int_0^t ds \int_{\mathsf{R}} X_s(dy) |p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)|^{\theta}$$
$$\leq C|x_1 - x_2|^{\delta\theta} \int_0^t ds (t-s)^{-(\delta\theta+\theta-1)/\alpha}$$
$$\times \int_{\mathsf{R}} X_s(dy) \big(p_{t-s}^{\alpha}\big((x_1 - y)/2\big) + p_{t-s}^{\alpha}\big((x_2 - y)/2\big)\big),$$

uniformly in $x_1, x_2 \in \mathsf{R}$. Recalling the scaling property of p^{α}, we get

$$\int_0^t ds \int_{\mathsf{R}} X_s(dy) |p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)|^{\theta}$$
$$\leq C|x_1 - x_2|^{\delta\theta} \int_0^t ds (t-s)^{-(\delta\theta+\theta-1)/\alpha} \big(S_{2^{\alpha}(t-s)}^{\alpha} X_s(x_1) + S_{2^{\alpha}(t-s)}^{\alpha} X_s(x_2)\big).$$

We complete the proof by applying Lemma 2.11. □

REMARK 2.13 (Lipschitz continuity of Z_t^3). Using Lemma 2.12 with $\theta = 1 = \delta$, we see that Z_t^3 is in fact a.s. Lipschitz continuous.

Let $\Delta X_s := X_s - X_{s-}$ denote the jumps of the measure-valued process X.

LEMMA 2.14 (Total jump mass). *Let $\varepsilon > 0$ and $\gamma \in (0, (1+\beta)^{-1})$. There exists a constant $c_{(2.55)} = c_{(2.55)}(\varepsilon, \gamma)$ such that*

$$\mathbf{P}\big(|\Delta X_s| > c_{(2.55)}(t-s)^{(1+\beta)^{-1}-\gamma} \text{ for some } s < t\big) \leq \varepsilon. \tag{2.55}$$

PROOF. Recall the random measure N from Lemma 1.6(a). For any $c > 0$, set

$$Y_0 := N\big([0, 2^{-1}t) \times \mathsf{R} \times (c2^{-\lambda}t^{\lambda}, \infty)\big), \tag{2.56}$$

$$\begin{aligned} Y_n := N\big(&\big[(1-2^{-n})t, (1-2^{-n-1})t\big) \\ &\times \mathsf{R} \times \big(c2^{-\lambda(n+1)}t^{\lambda}, \infty\big)\big), \qquad n \geq 1, \end{aligned} \tag{2.57}$$

where $\lambda := (1+\beta)^{-1} - \gamma$. It is easy to see that

$$\mathbf{P}\big(|\Delta X_s| > c(t-s)^{\lambda} \text{ for some } s < t\big) \leq \mathbf{P}\left(\sum_{n=0}^{\infty} Y_n \geq 1\right) \leq \sum_{n=0}^{\infty} \mathbf{E}Y_n, \tag{2.58}$$

where in the last step we have used the classical Markov inequality. From the formula for the compensator $\hat{N}$ of N in Lemma 1.6(b),

$$\mathbf{E}Y_n = \varrho \int_{(1-2^{-n})t}^{(1-2^{-n-1})t} ds\, \mathbf{E}X_s(\mathsf{R}) \int_{c2^{-\lambda(n+1)}t^{\lambda}}^{\infty} dr\, r^{-2-\beta}, \qquad n \geq 1. \tag{2.59}$$

Now

$$\mathbf{E}X_s(\mathsf{R}) = X_0(\mathsf{R})e^{as} \le |\mu|e^{|a|t} =: c_{(2.60)}. \tag{2.60}$$

Consequently,

$$\mathbf{E}Y_n \le \frac{\varrho}{1+\beta} c_{(2.60)} c^{-1-\beta} 2^{-(n+1)\gamma(1+\beta)} t^{\gamma(1+\beta)}. \tag{2.61}$$

Analogous calculations show that (2.61) remains valid also in the case $n = 0$. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{E}Y_n &\le \frac{\varrho}{1+\beta} c_{(2.60)} c^{-1-\beta} t^{\gamma(1+\beta)} \sum_{n=0}^{\infty} 2^{-(n+1)\gamma(1+\beta)} \\ &= \frac{\varrho}{1+\beta} c_{(2.60)} c^{-1-\beta} t^{\gamma(1+\beta)} \frac{2^{-\gamma(1+\beta)}}{1-2^{-\gamma(1+\beta)}}.\end{aligned} \tag{2.62}$$

Choosing $c = c_{(2.55)}$ such that the expression in (2.62) equals ε, and combining with (2.58), the proof is complete. □

2.5. *Representation as time-changed stable process.* We return to general $t > 0$. Recall the martingale measure M related to the martingale in Lemmas 1.6(c) and 1.7.

LEMMA 2.15 (Representation as time-changed stable process). *Suppose $p \in (1+\beta, 2)$ and let $\psi \in \mathcal{L}^p_{\mathrm{loc}}(\mu)$ with $\psi \ge 0$. Then there exists a spectrally positive $(1+\beta)$-stable process $\{L_t : t \ge 0\}$ such that*

$$Z_t(\psi) := \int_{(0,t]\times\mathsf{R}} M(d(s,y))\psi(s,y) = L_{T(t)}, \qquad t \ge 0, \tag{2.63}$$

where $T(t) := \int_0^t ds \int_{\mathsf{R}} X_s(dy)(\psi(s,y))^{1+\beta}$.

PROOF. Let us write Itô's formula for $e^{-Z_t(\psi)}$

$$\begin{aligned}e^{-Z_t(\psi)} - 1 &= \text{local martingale} \\ &\quad + \varrho \int_0^t ds\, e^{-Z_s(\psi)} \int_{\mathsf{R}} X_s(dy) \\ &\quad\quad \times \int_0^{\infty} dr\big(e^{-r\psi(s,y)} - 1 + r\psi(s,y)\big) r^{-2-\beta}.\end{aligned} \tag{2.64}$$

Define $\tau(t) := T^{-1}(t)$, and put $t^* := \inf\{t : \tau(t) = \infty\}$. Then it is easy to get for every $v > 0$,

$$\begin{aligned}e^{-vZ_{\tau(t)}(\psi)} &= 1 + \int_0^t ds\, e^{-vZ_{\tau(s)}(\psi)} \frac{X_{\tau(s)}(v^{1+\beta}\psi^{1+\beta}(s,\cdot))}{X_{\tau(s)}(\psi^{1+\beta}(s,\cdot))} + \text{loc. mart.} \\ &= 1 + \int_0^t ds\, e^{-vZ_{\tau(s)}(\psi)} v^{1+\beta} + \text{loc. mart.}, \qquad t \le t^*.\end{aligned} \tag{2.65}$$

Since the local martingale is bounded, it is in fact a martingale. Let $\tilde{L}$ denote a spectrally positive process of index $1+\beta$, independent of X. Define

$$L_t := \begin{cases} Z_{\tau(t)}(\psi), & t \le t^*, \\ Z_{\tau(t^*)}(\psi) + \tilde{L}_{t-t^*}, & t > t^* \text{ (if } t^* < \infty). \end{cases} \tag{2.66}$$

Then we can easily get that L satisfies the martingale problem (2.15) with κ replaced by $1+\beta$. Now by time change back we obtain

$$Z_t(\psi) = \tilde{L}_{T(t)} = L_{T(t)}, \tag{2.67}$$

completing the proof. $\square$

3. Local Hölder continuity.

PROOF OF THEOREM 1.2(a). We continue to assume that $d = 1$, and that $t > 0$ and $\mu \in \mathcal{M}_{\mathrm{f}} \setminus \{0\}$ are fixed. For $\beta \ge (\alpha - 1)/2$ the desired existence of a locally Hölder continuous version of Z_t^2 of required orders is already proved in Proposition 2.9. Therefore, in what follows we shall consider the complementary case $\beta < (\alpha - 1)/2$. Fix any compact set K and $x_1 < x_2$ belonging to it. By definition (1.12) of Z_t^2,

$$\begin{aligned} Z_t^2(x_1) - Z_t^2(x_2) &= \int_{(0,t]\times\mathsf{R}} M(d(s,y))\big(p_{t-s}^\alpha(x_1 - y) - p_{t-s}^\alpha(x_2 - y)\big) \\ &= \int_{(0,t]\times\mathsf{R}} M(d(s,y))\varphi_+(s,y) \\ &\quad - \int_{(0,t]\times\mathsf{R}} M(d(s,y))\varphi_-(s,y), \end{aligned} \tag{3.1}$$

where $\varphi_+(s, y)$ and $\varphi_-(s, y)$ are the positive and negative parts of $p_{t-s}^\alpha(x_1 - y) - p_{t-s}^\alpha(x_2 - y)$. It is easy to check that φ_+ and φ_- satisfy the assumptions in Lemma 2.15. Thus, there exist stable processes L^1 and L^2 such that

$$Z_t^2(x_1) - Z_t^2(x_2) = L^1_{T_+} - L^2_{T_-}, \tag{3.2}$$

where $T_\pm := \int_0^t ds \int_{\mathsf{R}} X_s(dy)(\varphi_\pm(s,y))^{1+\beta}$.

The idea behind the proof of the existence of the required version of Z_t^2 is as follows. We first control the jumps of L^1 and L^2 for $t \le T_\pm$ and then use Lemma 2.3 to get the necessary bounds on $L^1_{T_+}, L^2_{T_-}$ themselves.

Fix any $\varepsilon \in (0, 1)$. According to Lemma 2.11, there exists a constant c_ε such that

$$\mathbf{P}(V \le c_\varepsilon) \ge 1 - \varepsilon, \tag{3.3}$$

where $V = V_t^{2^\alpha}(K)$. Consider again $\gamma \in (0, (1+\beta)^{-1})$ and set

$$A^\varepsilon := \big\{|\Delta X_s| \le c_{(2.55)}(t-s)^{(1+\beta)^{-1}-\gamma} \text{ for all } s < t\big\} \cap \{V \le c_\varepsilon\}. \tag{3.4}$$

By Lemma 2.14 and by (3.3),

$$\mathbf{P}(A^{\varepsilon}) \geq 1 - 2\varepsilon. \tag{3.5}$$

Define $Z_t^{2,\varepsilon}(x) := Z_t^2(x)\mathsf{1}(A^{\varepsilon})$. We first show that $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders η smaller than η_{c}. It follows from (3.2) that

$$\begin{aligned} &\mathbf{P}\big(|Z_t^{2,\varepsilon}(x_1) - Z_t^{2,\varepsilon}(x_2)| \geq 2r|x_1 - x_2|^{\eta}\big) \\ &\quad \leq \mathbf{P}(L^1_{T_+} \geq r|x_1 - x_2|^{\eta}, A^{\varepsilon}) \\ &\qquad + \mathbf{P}(L^2_{T_-} \geq r|x_1 - x_2|^{\eta}, A^{\varepsilon}), \qquad r > 0. \end{aligned} \tag{3.6}$$

Note that on A^{ε} the jumps of $M(d(s,y))$ do not exceed $c_{(2.55)}(t-s)^{(1+\beta)^{-1}-\gamma}$ since the jumps of X are bounded by the same values on A^{ε}. Hence the jumps of the process $u \mapsto \int_{(0,u]\times\mathsf{R}} M(d(s,y))\varphi_{\pm}(s,y)$ are bounded by

$$c_{(2.55)} \sup_{s<t}(t-s)^{(1+\beta)^{-1}-\gamma} \sup_{y\in\mathsf{R}} \varphi_{\pm}(s,y). \tag{3.7}$$

Obviously,

$$\sup_{y\in\mathsf{R}} \varphi_{\pm}(s,y) \leq \sup_{y\in\mathsf{R}} |p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)|. \tag{3.8}$$

Assume additionally that $\gamma < \eta_{\mathrm{c}}/\alpha$. Using Lemma 2.1 with $\delta = \eta_{\mathrm{c}} - \alpha\gamma$ gives

$$\begin{aligned} &\sup_{y\in\mathsf{R}} |p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)| \\ &\quad \leq C|x_1 - x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}(t-s)^{-\eta_{\mathrm{c}}/\alpha+\gamma} \sup_{z\in\mathsf{R}} p^{\alpha}_{t-s}(z) \\ &\quad \leq C|x_1 - x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}(t-s)^{-\eta_{\mathrm{c}}/\alpha+\gamma}(t-s)^{-1/\alpha} \\ &\quad = C|x_1 - x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}(t-s)^{-1/(1+\beta)+\gamma}. \end{aligned} \tag{3.9}$$

Combining (3.7)–(3.9), we see that all jumps of $u \mapsto \int_{(0,u]\times\mathsf{R}} M(d(s,y))\varphi_{\pm}(s,y)$ on the set A^{ε} are bounded by

$$c_{(3.10)}|x_1 - x_2|^{\eta_{\mathrm{c}}-\alpha\gamma} \tag{3.10}$$

for some constant $c_{(3.10)} = c_{(3.10)}(\varepsilon)$. Therefore, by an abuse of notation writing $L_{T_{\pm}}$ for $L^1_{T_+}$ and $L^2_{T_-}$,

$$\begin{aligned} &\mathbf{P}(L_{T_{\pm}} \geq r|x_1 - x_2|^{\eta}, A^{\varepsilon}) \\ &\quad = \mathbf{P}\Big(L_{T_{\pm}} \geq r|x_1 - x_2|^{\eta}, \sup_{u<T_{\pm}} \Delta L_u \leq c_{(3.10)}|x_1 - x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}, A^{\varepsilon}\Big) \\ &\quad \leq \mathbf{P}\Big(\sup_{v\leq T_{\pm}} L_v \mathsf{1}\Big\{\sup_{u<v} \Delta L_u \leq c_{(3.10)}|x_1 - x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}\Big\} \geq r|x_1 - x_2|^{\eta}, A^{\varepsilon}\Big). \end{aligned} \tag{3.11}$$

Since

$$T_{\pm} \le \int_0^t ds \int_{\mathsf{R}} X_s(dy) |p_{t-s}^{\alpha}(x_1 - y) - p_{t-s}^{\alpha}(x_2 - y)|^{1+\beta}, \tag{3.12}$$

applying Lemma 2.12 with $\theta = 1 + \beta$ and $\delta = 1$ (since $\beta < (\alpha - 1)/2$), we get the bound

$$T_{\pm} \le c_{(3.13)} |x_1 - x_2|^{1+\beta} \qquad \text{on } \{V \le c_{\varepsilon}\}, \tag{3.13}$$

for some $c_{(3.13)} = c_{(3.13)}(\varepsilon)$. Consequently,

$$\begin{aligned} &\mathbf{P}(L_{T_{\pm}} \ge r|x_1 - x_2|^{\eta}, A^{\varepsilon}) \\ &\quad \le \mathbf{P}\Big(\sup_{v \le c_{(3.13)}|x_1 - x_2|^{1+\beta}} L_v \mathbf{1}\Big\{ \sup_{u<v} \Delta L_u \le c_{(3.10)}|x_1 - x_2|^{\eta_c - \alpha\gamma} \Big\} \\ &\qquad\qquad \ge r|x_1 - x_2|^{\eta} \Big). \end{aligned}$$

Using Lemma 2.3 with $\kappa = 1 + \beta$, $t = c_{(3.13)}|x_1 - x_2|^{1+\beta}$, $x = r|x_1 - x_2|^{\eta}$, and $y = c_{(3.10)}|x_1 - x_2|^{\eta_c - \alpha\gamma}$, and noting that

$$\begin{aligned} 1 + \beta - \eta - \beta(\eta_c - \alpha\gamma) &= 2 + 2\beta - \alpha + (\eta_c - \eta) + \beta\alpha\gamma \\ &> 2 + 2\beta - \alpha, \end{aligned} \tag{3.14}$$

we obtain

$$\begin{aligned} &\mathbf{P}(L_{T_{\pm}} \ge r|x_1 - x_2|^{\eta}, A^{\varepsilon}) \\ &\qquad \le \big(c_{(3.15)} r^{-1} |x_1 - x_2|^{(2\beta + 2 - \alpha)}\big)^{(c_{(3.10)}^{-1} r |x_1 - x_2|^{\eta - \eta_c + \alpha\gamma})} \end{aligned} \tag{3.15}$$

for some $c_{(3.15)} = c_{(3.15)}(\varepsilon)$. Applying this bound with $\gamma = (\eta_c - \eta)/2\alpha$ to the summands at the right-hand side in (3.6), and noting that $2\beta + 2 - \alpha$ is also constant here, we have

$$\begin{aligned} &\mathbf{P}\big(|Z_t^{2,\varepsilon}(x_1) - Z_t^{2,\varepsilon}(x_2)| \ge 2r|x_1 - x_2|^{\eta}\big) \\ &\qquad \le 2\big(c_{(3.15)} r^{-1} |x_1 - x_2|\big)^{(c_{(3.16)} r |x_1 - x_2|^{(\eta - \eta_c)/2})}. \end{aligned} \tag{3.16}$$

This inequality yields that all the conditions of Theorem III.5.6 of Gihman and Skorokhod [8] hold with $g(h) = 2h^{\eta}$ and $q(r, h) = 2(c_{(3.15)} r^{-1} h)^{(c_{(3.16)} r h^{(\eta - \eta_c)/2})}$, from which we conclude that almost surely $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders $\eta < \eta_c$.

By an abuse of notation, from now on the symbol $Z_t^{2,\varepsilon}$ always refers to this continuous version. Consequently,

$$\lim_{k \uparrow \infty} \mathbf{P}\Big(\sup_{x_1, x_2 \in K, x_1 \ne x_2} \frac{|Z_t^{2,\varepsilon}(x_1) - Z_t^{2,\varepsilon}(x_2)|}{|x_1 - x_2|^{\eta}} > k \Big) = 0. \tag{3.17}$$

Combining this with the bound

$$\begin{aligned}(3.18)\qquad &\mathbf{P}\Big(\sup_{x_1,x_2\in K,x_1\neq x_2}\frac{|Z_t^2(x_1)-Z_t^2(x_2)|}{|x_1-x_2|^\eta}>k\Big)\\ &\quad\le \mathbf{P}\Big(\sup_{x_1,x_2\in K,x_1\neq x_2}\frac{|Z_t^{2,\varepsilon}(x_1)-Z_t^{2,\varepsilon}(x_2)|}{|x_1-x_2|^\eta}>k,\,A^\varepsilon\Big)+\mathbf{P}(A^{\varepsilon,\mathrm{c}})\end{aligned}$$

(with $A^{\varepsilon,\mathrm{c}}$ denoting the complement of A^ε), gives

$$(3.19)\qquad \limsup_{k\uparrow\infty}\mathbf{P}\Big(\sup_{x_1,x_2\in K,x_1\neq x_2}\frac{|Z_t^2(x_1)-Z_t^2(x_2)|}{|x_1-x_2|^\eta}>k\Big)\le 2\varepsilon.$$

Since ε may be arbitrarily small, this immediately implies

$$(3.20)\qquad \sup_{x_1,x_2\in K,x_1\neq x_2}\frac{|Z_t^2(x_1)-Z_t^2(x_2)|}{|x_1-x_2|^\eta}<\infty,\qquad \text{almost surely.}$$

This is the desired local Hölder continuity of Z_t^2, for all $\eta<\eta_{\mathrm{c}}$. Because $\eta_{\mathrm{c}}<\alpha-1$, together with Lemma 2.5 the proof of Theorem 1.2(a) is complete. □

4. Local unboundedness: Proof of Theorem 1.2(c). In the proof we use ideas from the proofs of Theorems 1.1(b) and 1.2 of [15]. Throughout this section, suppose $d>1$ or $\alpha\le 1+\beta$. Recall that $t>0$ and $X_0=\mu\in\mathcal{M}_{\mathrm{f}}\setminus\{0\}$ are fixed. We want to verify that for each version of the density function X_t the property

$$(4.1)\qquad \|X_t\|_B=\infty\qquad \mathbf{P}\text{-a.s. on the event } \{X_t(B)>0\}$$

holds whenever B is a fixed open ball in R^d. Then the claim of Theorem 1.2(c) follows as in the proof of Theorem 1.1(b) in [15]. We thus fix such B.

As in [15] to get (4.1) we first show that on the event $\{X_t(B)>0\}$ there are always sufficiently "big" jumps of X that occur close to time t. This is done in Lemma 4.3 below. Then with the help of properties of the log-Laplace equation derived in Lemma 4.4 we are able to show that the "big" jumps are large enough to ensure the unboundedness of the density at time t. Loosely speaking the density is getting unbounded in the proximity of big jumps.

In order to fulfil the above program, we start with deriving the continuity of $X_\cdot(B)$ at (fixed) time t.

LEMMA 4.1 (Path continuity at fixed times). *For the fixed $t>0$,*

$$(4.2)\qquad \lim_{s\to t}X_s(B)=X_t(B)\qquad a.s.$$

PROOF. Since t is fixed, X is continuous at t with probability 1. Therefore,

$$(4.3)\qquad X_t(B)\le\liminf_{s\to t}X_s(B)\le\limsup_{s\to t}X_s(B)\le\limsup_{s\to t}X_s(\overline{B})\le X_t(\overline{B})$$

with $\overline{B}$ denoting the closure of B. But since $X_t(dx)$ is absolutely continuous with respect to Lebesgue measure, we have $X_t(B) = X_t(\overline{B})$. Thus the proof is complete. □

LEMMA 4.2 (Explosion). *Let $f:(0,t) \to (0,\infty)$ be measurable such that*

$$\int_{t-\delta}^{t} ds\, f(t-s) = \infty \qquad \text{for all sufficiently small } \delta \in (0,t). \tag{4.4}$$

Then for these δ,

$$\int_{t-\delta}^{t} ds\, X_s(B) f(t-s) = \infty, \qquad \mathbf{P}\text{-a.s. on the event } \{X_t(B) > 0\}. \tag{4.5}$$

PROOF. Fix δ as in the lemma. Fix also ω such that $X_t(B) > 0$ and $X_s(B) \to X_t(B)$ as $s \uparrow t$. For this ω, there is an $\varepsilon \in (0,\delta)$ such that $X_s(B) > \varepsilon$ for all $s \in (t-\varepsilon, t)$. Hence

$$\int_{t-\delta}^{t} ds\, X_s(B) f(t-s) \geq \varepsilon \int_{t-\varepsilon}^{t} ds\, f(t-s) = \infty \tag{4.6}$$

and we are done. □

Set

$$\vartheta := \frac{1}{1+\beta} \tag{4.7}$$

and for $\varepsilon \in (0,t)$ let $\tau_\varepsilon(B)$ denote the first moment in $(t-\varepsilon, t)$ in which a "big jump" occurs. More precisely, define

$$\tau_\varepsilon(B) := \inf\left\{ s \in (t-\varepsilon, t) : |\Delta X_s|(B) > (t-s)^\vartheta \log^\vartheta\left(\frac{1}{t-s}\right)\right\}. \tag{4.8}$$

LEMMA 4.3 (Existence of big jumps). *For $\varepsilon \in (0,t)$ and the open ball B,*

$$\mathbf{P}\big(\tau_\varepsilon(B) = \infty\big) \leq \mathbf{P}\big(X_t(B) = 0\big). \tag{4.9}$$

PROOF. For simplicity, through the proof we write τ for $\tau_\varepsilon(B)$. It suffices to show that

$$\mathbf{P}\{\tau = \infty, X_t(B) > 0\} = 0. \tag{4.10}$$

To verify (4.10) we will mainly follow the lines of the proof of Theorem 1.2(b) of [6]. For $u \in (0,\varepsilon]$, define

$$Z_u := N\left((s,x,r) : s \in (t-\varepsilon, t-\varepsilon+u), x \in B, r > (t-s)^\vartheta \log^\vartheta\left(\frac{1}{t-s}\right)\right)$$

with the random measure N introduced in Lemma 1.6(a). Then

$$\{\tau = \infty\} = \{Z_\varepsilon = 0\}. \tag{4.11}$$

Recall the formula for the compensator $\hat{N}$ of N in Lemma 1.6(b). From a classical time change result for counting processes (see, e.g., Theorem 10.33 in [11]), we get that there exists a standard Poisson process $A = \{A(v) : v \geq 0\}$ such that

$$
\begin{aligned}
Z_u &= A\biggl(\varrho \int_{t-\varepsilon}^{t-\varepsilon+u} ds\, X_s(B) \int_{(t-s)^{\vartheta}\log^{\vartheta}(1/(t-s))}^{\infty} dr\, r^{-2-\beta}\biggr) \\
&= A\biggl(\frac{\varrho}{1+\beta} \int_{t-\varepsilon}^{t-\varepsilon+u} ds\, X_s(B) \frac{1}{(t-s)\log(1/(t-s))}\biggr),
\end{aligned}
\tag{4.12}
$$

where we used notation (4.7). Then

$$
\begin{aligned}
&\mathbf{P}\bigl(Z_\varepsilon = 0, X_t(B) > 0\bigr) \\
&\quad \leq \mathbf{P}\biggl(\int_{t-\varepsilon}^{t} ds\, X_s(B) \frac{1}{(t-s)\log(1/(t-s))} < \infty, X_t(B) > 0\biggr).
\end{aligned}
\tag{4.13}
$$

It is easy to check that

$$
\int_{t-\delta}^{t} ds \frac{1}{(t-s)\log(1/(t-s))} = \infty \qquad \text{for all } \delta \in (0, \varepsilon).
\tag{4.14}
$$

Therefore, by Lemma 4.2,

$$
\int_{t-\varepsilon}^{t} ds\, X_s(B) \frac{1}{(t-s)\log(1/(t-s))} = \infty \qquad \text{on } \{X_t(B) > 0\}.
\tag{4.15}
$$

Thus, the probability in (4.13) equals 0. Hence, together with (4.11) claim (4.10) follows. $\square$

Set $\varepsilon_n := 2^{-n}$, $n \geq 1$. Then we choose open balls $B_n \uparrow B$ such that

$$
\overline{B_n} \subset B_{n+1} \subset B \quad \text{and} \quad \sup_{y \in B^c, x \in B_n, 0 < s \leq \varepsilon_n} p_s^{\alpha}(x-y) \underset{n\uparrow\infty}{\longrightarrow} 0.
\tag{4.16}
$$

Fix $n \geq 1$ such that $\varepsilon_n < t$. Define $\tau_n := \tau_{\varepsilon_n}(B_n)$.

In order to get a lower bound for $\|X_t\|_B$ we use the following inequality:

$$
\|X_t\|_B \geq \int_B dy\, X_t(y) p_r^{\alpha}(y-x), \qquad x \in B, r > 0.
\tag{4.17}
$$

On the event $\{\tau_n < t\}$, denote by ζ_n the spatial location in B_n of the jump at time τ_n, and by r_n the size of the jump, meaning that $\Delta X_{\tau_n} = r_n \delta_{\zeta_n}$. Then specializing (4.17),

$$
\|X_t\|_B \geq \int_B dy\, X_t(y) p_{t-\tau_n}^{\alpha}(y-\zeta_n) \qquad \text{on the event } \{\tau_n < t\}.
\tag{4.18}
$$

From the strong Markov property at time τ_n, together with the branching property of superprocesses, we know that conditionally on $\{\tau_n < t\}$, the process

$\{X_{\tau_n+u}: u \geq 0\}$ is bounded below in distribution by $\{\widetilde{X}_u^n: u \geq 0\}$ where $\widetilde{X}^n$ is a super-Brownian motion with initial value $r_n\delta_{\zeta_n}$. Hence, from (4.18) we get

$$\begin{aligned}
&\mathbf{E}\exp\{-\|X_t\|_B\}\\
&\quad\leq \mathbf{E}1_{\{\tau_n<t\}}\exp\biggl\{-\int_B dy\, X_t(y)p^\alpha_{t-\tau_n}(y-\zeta_n)\biggr\}+\mathbf{P}(\tau_n=\infty)\\
&\quad\leq \mathbf{E}1_{\{\tau_n<t\}}\mathbf{E}_{r_n\delta_{\zeta_n}}\exp\biggl\{-\int_B dy X_{t-\tau_n}(y)p^\alpha_{t-\tau_n}(y-\zeta_n)\biggr\}\\
&\qquad+\mathbf{P}(\tau_n=\infty).
\end{aligned}\tag{4.19}$$

Note that on the event $\{\tau_n<t\}$, we have

$$r_n \geq (t-\tau_n)^\vartheta \log^\vartheta\biggl(\frac{1}{t-\tau_n}\biggr) =: h_\beta(t-\tau_n).\tag{4.20}$$

We now claim that

$$\lim_{n\uparrow\infty}\sup_{0<s<\varepsilon_n, x\in B_n, r\geq h_\beta(s)}\mathbf{E}_{r\delta_x}\exp\biggl\{-\int_B dy\, X_s(y)p^\alpha_s(y-x)\biggr\}=0.\tag{4.21}$$

To verify (4.21), let $s\in(0,\varepsilon_n)$, $x\in B_n$ and $r\geq h_\beta(s)$. Then using the Laplace transition functional of the superprocess we get

$$\begin{aligned}
\mathbf{E}_{r\delta_x}\exp\biggl\{-\int_B dy\, X_s(y)p^\alpha_s(y-x)\biggr\} &= \exp\{-r v^n_{s,x}(s,x)\}\\
&\leq \exp\{-h_\beta(s)v^n_{s,x}(s,x)\},
\end{aligned}\tag{4.22}$$

where the nonnegative function $v^n_{s,x}=\{v^n_{s,x}(s',x'): s'>0,\ x'\in \mathsf{R}^d\}$ solves the log-Laplace integral equation

$$\begin{aligned}
v^n_{s,x}(s',x') &= \int_{\mathsf{R}^d} dy\, p^\alpha_{s'}(y-x')1_B(y)p^\alpha_s(y-x)\\
&\quad+\int_0^{s'}dr'\int_{\mathsf{R}^d}dy\, p^\alpha_{s'-r'}(y-x')[a v^n_{s,x}(r',y)\\
&\qquad\qquad - b(v^n_{s,x}(r',y))^{1+\beta}]
\end{aligned}\tag{4.23}$$

related to (1.1).

LEMMA 4.4 (Another explosion). *Under the conditions $d>1$ or $\alpha\leq 1+\beta$, we have*

$$\lim_{n\uparrow\infty}\Bigl(\inf_{0<s<\varepsilon_n, x\in B_n} h_\beta(s)v^n_{s,x}(s,x)\Bigr)=+\infty.\tag{4.24}$$

Let us postpone the proof of Lemma 4.4.

COMPLETION OF PROOF OF THEOREM 1.2(c). Our claim (4.21) readily follows from estimate (4.22) and (4.24). Moreover, according to (4.21), by passing to the limit $n \uparrow \infty$ in the right-hand side of (4.19), and then using Lemma 4.3, we arrive at

$$\mathbf{E}\exp\{-\|X_t\|_B\} \le \limsup_{n\uparrow\infty} \mathbf{P}(\tau_n = \infty) \le \limsup_{n\uparrow\infty} \mathbf{P}\big(X_t(B_n) = 0\big). \tag{4.25}$$

Since the event $\{X_t(B) = 0\}$ is the nonincreasing limit as $n \uparrow \infty$ of the events $\{X_t(B_n) = 0\}$ we get

$$\mathbf{E}\exp\{-\|X_t\|_B\} \le \mathbf{P}\big(X_t(B) = 0\big). \tag{4.26}$$

Since obviously $\|X_t\|_B = 0$ if and only if $X_t(B) = 0$, we see that (4.1) follows from this last bound. The proof of Theorem 1(c) is finished for $U = B$. $\square$

PROOF OF LEMMA 4.4. We start with a determination of the asymptotics of the first term at the right-hand side of the log-Laplace equation (4.23) at $(s', x') = (s, x)$. Note that

$$\begin{aligned}&\int_{\mathsf{R}^d} dy\, p_s^\alpha(y-x)1_B(y)p_s^\alpha(y-x)\\ &\quad= \int_{\mathsf{R}^d} dy\, p_s^\alpha(y-x)p_s^\alpha(y-x) - \int_{B^{\mathrm{c}}} dy\, p_s^\alpha(y-x)p_s^\alpha(y-x).\end{aligned} \tag{4.27}$$

In the latter formula line, the first term equals $p_{2s}^\alpha(0) = Cs^{-d/\alpha}$ whereas the second one is bounded from above by

$$\sup_{0<s<\varepsilon_n, x\in B_n, y\in B^{\mathrm{c}}} p_s^\alpha(y-x) \underset{n\uparrow\infty}{\longrightarrow} 0, \tag{4.28}$$

where the last convergence follows by assumption (4.16) on B_n. Hence from (4.27) and (4.28) we obtain

$$\int_{\mathsf{R}^d} dy\, p_s^\alpha(y-x)1_B(y)p_s^\alpha(y-x) = Cs^{-d/\alpha} + \mathrm{o}(1) \qquad \text{as } n \uparrow \infty, \tag{4.29}$$

uniformly in $s \in (0, \varepsilon_n)$ and $x \in B_n$.

To simplify notation, we write $v^n := v_{s,x}^n$. Next, from (4.23) we can easily get the upper bound

$$\begin{aligned}v^n(s', x') &\le e^{|a|s'} \int_{\mathsf{R}^d} dy\, p_{s'}^\alpha(y-x')p_s^\alpha(y-x)\\ &= e^{|a|s'} p_{s'+s}^\alpha(x-x').\end{aligned} \tag{4.30}$$

Then we have

$$
\begin{aligned}
&\int_0^s dr' \int_{\mathsf{R}^d} dy\, p^\alpha_{s-r'}(y-x)(v^n(r',y))^{1+\beta} \\
&\qquad \le e^{|a|(1+\beta)s} \int_0^s dr' \int_{\mathsf{R}^d} dy\, p^\alpha_{s-r'}(y-x)\big(p^\alpha_{r'+s}(x-y)\big)^{1+\beta} \\
&\qquad \le e^{|a|(1+\beta)s} (p^\alpha_s(0))^\beta \int_0^s dr' \int_{\mathsf{R}^d} dy\, p^\alpha_{s-r'}(y-x) p^\alpha_{r'+s}(x-y) \\
&\qquad = e^{|a|(1+\beta)s} (p^\alpha_s(0))^\beta \int_0^s dr'\, p^\alpha_{2s}(0) = Ce^{|a|(1+\beta)s} s^{1-d(1+\beta)/\alpha}
\end{aligned}
\tag{4.31}
$$

and, similarly,

$$
\int_0^s dr' \int_{\mathsf{R}^d} dy\, p^\alpha_{s-r'}(y-x) a v^n(r',y) \ge -C|a|e^{|a|s} s^{1-d/\alpha}. \tag{4.32}
$$

Summarizing, by (4.23), (4.29), (4.31) and (4.32),

$$
v^n(s,x) \ge Cs^{-d/\alpha} + \mathrm{o}(1) - Ce^{|a|(1+\beta)s} s^{1-d(1+\beta)/\alpha} - C|a|e^{|a|s} s^{1-d/\alpha} \tag{4.33}
$$

uniformly in $s \in (0, \varepsilon_n)$ and $x \in B_n$. According to our general assumption $d < \alpha/\beta$, we conclude that the right-hand side of (4.33) behaves like $Cs^{-d/\alpha}$ as $s \downarrow 0$ uniformly in $s \in (0, \varepsilon_n)$. Now recalling definitions (4.20) and (4.7) as well as our assumption that $d > 1$ or $\alpha \le 1 + \beta$, we immediately get

$$
\lim_{n\uparrow\infty} \inf_{0<s<\varepsilon_n} h_\beta(s) s^{-d/\alpha} = +\infty. \tag{4.34}
$$

By (4.33), this implies (4.24), and the proof of the lemma is complete. □

5. Optimal local Hölder index: Proof of Theorem 1.2(b). We return to $d = 1$ and continue to assume that $t > 0$ and $\mu \in \mathcal{M}_\mathrm{f} \setminus \{0\}$ are fixed. In the proof of Theorem 1.2(b) we implement the following idea. We show that there exists a sequence of "big" jumps of X that occur close to time t and these jumps in fact destroy the local Hölder continuity of any index greater or equal than η_c.

As in the proof of Theorem 1.2(c) in the previous section, we may work with a fixed open interval U. For simplicity we consider $U = (0, 1)$. Put

$$
I^{(n)}_k := \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), \qquad n \ge 1, 0 \le k \le 2^n - 1. \tag{5.1}
$$

Choose n_0 such that $2^{-\alpha n_0} < t$. For $n \ge n_0$ and $2 \le k \le 2^n + 1$, denote by $A_{n,k}$ the following event:

$$
\left\{\Delta X_s\big(I^{(n)}_{k-2}\big) \ge \frac{c_{(5.2)}}{2^{\alpha/(1+\beta)n}} n^{1/(1+\beta)} \text{ for some } s \in \big[t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)}\big)\right\} \tag{5.2}
$$

with $c_{(5.2)} := (\alpha 2^{-\alpha} \log 2)^{1/(1+\beta)}$, and for $N \ge n_0$ write

$$
\widetilde{A}_N := \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} A_{n,k}. \tag{5.3}
$$

LEMMA 5.1 (Again existence of big jumps). *For any* $N \geq n_0$,

$$\mathbf{P}\{\widetilde{A}_N | X_t(U) > 0\} = 1. \tag{5.4}$$

PROOF. For $s \in [t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)})$ we have

$$\begin{aligned}\left((t-s)\log\left(\frac{1}{t-s}\right)\right)^{1/(1+\beta)} &\geq \left(2^{-\alpha(n+1)}\log 2^{\alpha n}\right)^{1/(1+\beta)} \\ &= c_{(5.2)} 2^{-\alpha/(1+\beta)n} n^{1/(1+\beta)}.\end{aligned} \tag{5.5}$$

Therefore,

$$\begin{aligned}\bigcup_{k=2}^{2^n+1} A_{n,k} \supseteq \Bigg\{ \Delta X_s(U) \geq \left((t-s)\log\left(\frac{1}{t-s}\right)\right)^{1/(1+\beta)} \\ \text{for some } s \in \left[t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)}\right) \Bigg\}\end{aligned}$$

and, consequently,

$$\begin{aligned}\widetilde{A}_N &= \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} A_{n,k} \\ &\supseteq \left\{ \Delta X_s(U) \geq \left((t-s)\log\left(\frac{1}{t-s}\right)\right)^{1/(1+\beta)} \text{ for some } s \geq t - 2^{-N} \right\}\end{aligned} \tag{5.6}$$

and we are done by Lemma 4.3. $\square$

Now we are going to define increments of Z_t^2 on the dyadic sets $\{\frac{k}{2^n} : k = 0, \dots, 2^n\}$. By definition (1.12),

$$\begin{aligned}&Z_t^2\left(\frac{k}{2^n}\right) - Z_t^2\left(\frac{k+1}{2^n}\right) \\ &\quad= \int_{(0,t]\times\mathsf{R}} M(d(s,y))\left(p_{t-s}^{\alpha}\left(\frac{k}{2^n} - y\right) - p_{t-s}^{\alpha}\left(\frac{k+1}{2^n} - y\right)\right) \\ &\quad= \int_{(0,t]\times\mathsf{R}} M(d(s,y))\left(p_{t-s}^{\alpha}\left(\frac{k}{2^n} - y\right) - p_{t-s}^{\alpha}\left(\frac{k+1}{2^n} - y\right)\right)_+ \\ &\qquad+ \int_{(0,t]\times\mathsf{R}} M(d(s,y))\left(p_{t-s}^{\alpha}\left(\frac{k}{2^n} - y\right) - p_{t-s}^{\alpha}\left(\frac{k+1}{2^n} - y\right)\right)_- .\end{aligned} \tag{5.7}$$

Then according to Lemma 2.15 there exist spectrally positive stable processes $L_{n,k}^+$ and $L_{n,k}^-$ of index $1 + \beta$ such that

$$Z_t^2\left(\frac{k}{2^n}\right) - Z_t^2\left(\frac{k+1}{2^n}\right) = L_{n,k}^+(T_+) - L_{n,k}^-(T_-), \tag{5.8}$$

where

$$T_{\pm} := \int_0^t ds \int_{\mathsf{R}} X_s(dy)\Big(p^{\alpha}_{t-s}\Big(\frac{k}{2^n} - y\Big) - p^{\alpha}_{t-s}\Big(\frac{k+1}{2^n} - y\Big)\Big)^{1+\beta}_{\pm}. \tag{5.9}$$

Fix $\varepsilon \in (0, \frac{1}{1+\beta})$ for a while. Let us define the following events:

$$\begin{aligned} B_{n,k} &:= \{L^+_{n,k}(T_+) \geq 2^{-\eta_c n} n^{1/(1+\beta)-\varepsilon}\} \cap \{L^-_{n,k}(T_-) \leq 2^{-\eta_c n - \varepsilon n}\} \\ &=: B^+_{n,k} \cap B^-_{n,k} \end{aligned} \tag{5.10}$$

(with notation in the obvious correspondence). Define the following event:

$$\begin{aligned} D_N &:= \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A_{n,k} \cap B_{n,k}) \\ &\supseteq \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} A_{n,k} \Big\backslash \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A_{n,k} \cap B^{\mathrm{c}}_{n,k}). \end{aligned} \tag{5.11}$$

An estimation of the probability of D_N is crucial for the proof of Theorem 1.2(b). In fact we are going to show that conditionally on $\{X_t(U) > 0\}$, the event D_N happens with probability one for any N. This in turn implies that for any N one can find $n \geq N$ sufficiently large such that there exists an interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ on which the increment $Z^2_t(\frac{k}{2^n}) - Z^2_t(\frac{k+1}{2^n})$ is of order $L^+_{n,k}(T_+) \geq 2^{-\eta_c n} n^{1/(1+\beta)-\varepsilon}$ [since the other term $L^-_{n,k}(T_-)$ is much smaller on that interval]. This implies the statement of Theorem 1.2(b). Detailed arguments follow.

By Lemma 5.1 we get

$$\mathbf{P}\{D_N | X_t(U) > 0\} \geq 1 - \mathbf{P}\Big\{\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A_{n,k} \cap B^{\mathrm{c}}_{n,k}) \Big| X_t(U) > 0\Big\}. \tag{5.12}$$

Recall A^{ε} defined in (3.4). Note that

$$\begin{aligned} &\mathbf{P}\Big(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A_{n,k} \cap B^{\mathrm{c}}_{n,k})\Big) \\ &\quad \leq \mathbf{P}(A^{\varepsilon,\mathrm{c}}) + \mathbf{P}\Big(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A^{\varepsilon} \cap A_{n,k} \cap B^{\mathrm{c}}_{n,k})\Big) \\ &\quad \leq 2\varepsilon + \mathbf{P}\Big(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A^{\varepsilon} \cap A_{n,k} \cap B^{\mathrm{c}}_{n,k})\Big). \end{aligned} \tag{5.13}$$

LEMMA 5.2 (Probability of small increments). *For all $\varepsilon > 0$ sufficiently small*,

$$\lim_{N\uparrow\infty} \mathbf{P}\Big(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A^{\varepsilon} \cap A_{n,k} \cap B^{\mathrm{c}}_{n,k})\Big) = 0. \tag{5.14}$$

We postpone the proof of this lemma to the end of this section. Instead we will show now, how it implies Theorem 1.2(b).

COMPLETION OF PROOF OF THEOREM 1.2(b). From Lemma 5.2 and (5.13) it follows that

$$\limsup_{N\uparrow\infty} \mathbf{P}\left\{\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}(A_{n,k}\cap B_{n,k}^{\mathrm{c}})\,\middle|\,X_t(U)>0\right\} \le \frac{2\varepsilon}{\mathbf{P}(X_t(U)>0)}. \tag{5.15}$$

Since ε can be arbitrarily small, the latter lim sup expression equals 0. Combining this with estimate (5.12), we get

$$\lim_{N\uparrow\infty} \mathbf{P}\{D_N|X_t(U)>0\}=1. \tag{5.16}$$

Since $D_N \downarrow \bigcap_{N=n_0}^{\infty} D_N =: D_\infty$ as $N\uparrow\infty$, we conclude that

$$\mathbf{P}\{D_\infty|X_t(U)>0\}=1. \tag{5.17}$$

This means that, almost surely on $\{X_t(U)>0\}$, there is a sequence (n_j,k_j) such that

$$Z_t^2\left(\frac{k_j}{2^{n_j}}\right)-Z_t^2\left(\frac{k_j+1}{2^{n_j}}\right)\ge 2^{-\eta_{\mathrm{c}} n_j} n_j^{1/(1+\beta)-\varepsilon}. \tag{5.18}$$

This inequality implies the claim in Theorem 1.2(b). □

We now prepare for the proof of Lemma 5.2. Actually by using (5.10), we represent the probability in (5.14) as a sum of the two following probabilities:

$$\begin{aligned}
&\mathbf{P}\left(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{\mathrm{c}})\right)\\
&\quad=\mathbf{P}\left(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{+,\mathrm{c}})\right)\\
&\qquad+\mathbf{P}\left(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}})\right).
\end{aligned} \tag{5.19}$$

Now we will handle each term on the right-hand side of (5.19) separately.

LEMMA 5.3 [First term in (5.19)]. *For* $\varepsilon\in(0,\frac{1}{1+\beta})$,

$$\lim_{N\uparrow\infty}\mathbf{P}\left(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{+,\mathrm{c}})\right)=0. \tag{5.20}$$

PROOF. Consider the process $L^+_{n,k}(s)$, $s \le T_+$. On $A_{n,k}$ there exists a jump of the martingale measure M of the form $r^*\delta_{s^*,y^*}$ for some

$$\begin{aligned} &r^* \ge c_{(5.2)} 2^{-\alpha/(1+\beta)n} n^{1/(1+\beta)}, \\ &s^* \in [t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)}], \qquad y^* \in I^{(n)}_{k-2}. \end{aligned} \tag{5.21}$$

Hence

$$\begin{aligned} \Delta L^+_{n,k}(s^*) \ge \inf_{y \in I^{(n)}_{k-2}, s \in [2^{-\alpha(n+1)}, 2^{-\alpha n}]} \biggl(p^\alpha_s \Bigl(\frac{k}{2^n} - y \Bigr) - p^\alpha_s \Bigl(\frac{k+1}{2^n} - y \Bigr) \biggr)_+ \\ \times c_{(5.2)} 2^{-\alpha/(1+\beta)n} n^{1/(1+\beta)}. \end{aligned} \tag{5.22}$$

It is easy to get

$$\begin{aligned} &\inf_{y \in I^{(n)}_{k-2}, s \in [2^{-\alpha(n+1)}, 2^{-\alpha n}]} \biggl(p^\alpha_s \Bigl(\frac{k}{2^n} - y \Bigr) - p^\alpha_s \Bigl(\frac{k+1}{2^n} - y \Bigr) \biggr)_+ \\ &\quad = \inf_{\substack{2^{-n} \le z \le 2^{-n+1}, \\ s \in [2^{-\alpha(n+1)}, 2^{-\alpha n}]}} \bigl(p^\alpha_s(z) - p^\alpha_s(z + 2^{-n}) \bigr)_+ \\ &\quad = \inf_{\substack{2^{-n} \le z \le 2^{-n+1}, \\ s \in [2^{-\alpha(n+1)}, 2^{-\alpha n}]}} s^{-1/\alpha} \bigl(p^\alpha_1(z s^{-1/\alpha}) - p^\alpha_1\bigl((z + 2^{-n}) s^{-1/\alpha}\bigr) \bigr)_+ \\ &\quad \ge 2^n \inf_{\substack{2^{-n} \le z \le 3 \cdot 2^{-n}, \\ s \in [2^{-\alpha(n+1)}, 2^{-\alpha n}]}} |(p^\alpha_1)'(z s^{-1/\alpha})| 2^{-n} s^{-1/\alpha} \\ &\quad \ge 2^n \inf_{1 \le x \le 6} |(p^\alpha_1)'(x)| =: c_{(5.23)} 2^n, \end{aligned} \tag{5.23}$$

where $c_{(5.23)} > 0$. In fact, from (2.2),

$$\frac{d}{dz} p^\alpha_1(z) = -\int_0^\infty ds\, q^{\alpha/2}_1(s) \frac{z}{2s} p^{(2)}_s(z) \ne 0, \qquad z \ne 0, \tag{5.24}$$

and $(p^{(2)}_\alpha)'(x) \ne 0$ for any $x \ne 0$. Apply (5.23) in (5.22) to arrive at

$$\Delta L^+_{n,k}(s^*) \ge c_{(5.25)} 2^{(1-\alpha/(1+\beta))n} n^{1/(1+\beta)} = c_{(5.25)} 2^{-\eta_c n} n^{1/(1+\beta)}. \tag{5.25}$$

Using Lemma 2.12 with $\theta = 1 + \beta$ and

$$\delta = (1+\beta) \mathsf{1}_{\{2\beta < \alpha - 1\}} + (\alpha - \beta - \varepsilon) \mathsf{1}_{\{2\beta \ge \alpha - 1\}}, \tag{5.26}$$

we get, with c_ε appearing in definition (3.4) of A^ε,

$$T_\pm \le c_\varepsilon \bigl(2^{-n(1+\beta)} \mathsf{1}_{\{2\beta < \alpha - 1\}} + 2^{-n(\alpha - \beta - \varepsilon)} \mathsf{1}_{\{2\beta \ge \alpha - 1\}} \bigr) =: t_n \qquad \text{on } A^\varepsilon. \tag{5.27}$$

Hence for all n sufficiently large we obtain

$$\begin{aligned}
&\mathbf{P}\big(L^{+}_{n,k}(T_+) < 2^{-\eta_c n} n^{1/(1+\beta)-\varepsilon}, A^{\varepsilon} \cap A_{n,k}\big)\\
&\quad \le \mathbf{P}\big(L^{+}_{n,k}(T_+) < 2^{-\eta_c n} n^{1/(1+\beta)-\varepsilon},\\
&\qquad\quad \Delta L^{+}_{n,k}(s^*) \ge c_{(5.25)} 2^{-\eta_c n} n^{1/(1+\beta)}, A^{\varepsilon}\big)\\
&\quad \le \mathbf{P}\Big(\inf_{s \le T^+} L^{+}_{n,k}(s) < -\frac{1}{2} c_{(5.25)} 2^{-\eta_c n} n^{1/(1+\beta)}, A^{\varepsilon}\Big)\\
&\quad \le \mathbf{P}\Big(\inf_{s \le t_n} L^{+}_{n,k}(s) < -\frac{1}{2} c_{(5.25)} 2^{-\eta_c n} n^{1/(1+\beta)}\Big) \qquad (5.28)\\
&\quad \le \exp\big\{-c_{\beta}(t_n)^{-1/\beta}\big(c_{(5.25)} 2^{-\eta_c n} n^{1/(1+\beta)}\big)^{(1+\beta)/\beta}\big\}\\
&\quad \le \exp\big\{-c_{\varepsilon} n^{1/\beta}\big(t_n^{-1} 2^{-\eta_c(1+\beta)n}\big)^{1/\beta}\big\}\\
&\quad \le \exp\big\{-c_{\varepsilon} n^{1/\beta} 2^{(1-\varepsilon)n}\big\},
\end{aligned}$$

where (5.28) follows by Lemma 2.4, and the rest is simple algebra. From this we get that for N sufficiently large,

$$\begin{aligned}
\mathbf{P}\Bigg(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A^{\varepsilon} \cap A_{n,k} \cap B^{+,c}_{n,k}\big)\Bigg) &\le \sum_{n=N}^{\infty} \sum_{k=2}^{2^n+1} \mathbf{P}\big(A^{\varepsilon} \cap A_{n,k} \cap B^{+,c}_{n,k}\big)\\
&\le \sum_{n=N}^{\infty} \sum_{k=2}^{2^n+1} \exp\big\{-c_{\varepsilon} n^{1/\beta} 2^{(1-\varepsilon)n}\big\} \qquad (5.29)\\
&= \sum_{n=N}^{\infty} 2^n \exp\big\{-c_{\varepsilon} n^{1/\beta} 2^{(1-\varepsilon)n}\big\},
\end{aligned}$$

which converges to 0 as $N \uparrow \infty$, and we are done with the proof of Lemma 5.3. □

LEMMA 5.4 [Second term in (5.19)]. *For all $\varepsilon > 0$ sufficiently small*,

$$\lim_{N \uparrow \infty} \mathbf{P}\Bigg(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A^{\varepsilon} \cap A_{n,k} \cap B^{-,c}_{n,k}\big)\Bigg) = 0. \qquad (5.30)$$

The proof of this lemma will be postponed almost to the end of the section. For its preparation, fix $\rho \in (0, \frac{1}{2})$. Define

$$A^{\rho}_n := \Big\{\omega : \text{there exists } I^{(n)}_k \text{ with } \sup_{s \in [t-2^{-\alpha(1-\rho)n}, t)} X_s\big(I^{(n)}_k\big) \ge 2^{-n(1-2\rho)}\Big\}.$$

Note that

$$\begin{aligned}
&\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}})\right)\\
&\qquad\le \mathbf{P}(A_n^{\rho})+\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}(A_n^{\rho,\mathrm{c}}\cap A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}})\right)\\
&\qquad\le \mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}\mathbf{P}(A_n^{\rho,\mathrm{c}}\cap A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}).
\end{aligned}\tag{5.31}$$

Now let us introduce the notation

$$B_{n,k}^{-,1}:=\Big\{\sup_{s\le T_-}\Delta L_{n,k}^{-}(s)\le 2^{-\eta_{\mathrm{c}}n-\varepsilon n}\Big\}.\tag{5.32}$$

Then we have

$$\begin{aligned}
&\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}})\right)\\
&\qquad\le \mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}\mathbf{P}(A_n^{\rho,\mathrm{c}}\cap A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}})\\
&\qquad\le \mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}\mathbf{P}(A^{\varepsilon}\cap B_{n,k}^{-,\mathrm{c}}\cap B_{n,k}^{-,1})\\
&\qquad\quad+\sum_{k=2}^{2^n+1}\mathbf{P}(A^{\varepsilon}\cap A_n^{\rho,\mathrm{c}}\cap A_{n,k}\cap B_{n,k}^{-,1,\mathrm{c}})\\
&\qquad=:\mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}P_{n,k}^{\varepsilon}+\sum_{k=2}^{2^n+1}P_{n,k}^{\varepsilon,\varrho}.
\end{aligned}\tag{5.33}$$

In the following lemmas we consider the three terms in (5.33) separately.

LEMMA 5.5 [First term in (5.33)]. *There exists a constant $c_{(5.34)}$ independent of $\rho\in(0,\frac{1}{2})$ such that*

$$\mathbf{P}(A_n^{\rho})\le c_{(5.34)}2^{-\rho n},\qquad n\ge n_0.\tag{5.34}$$

PROOF. Fix $n\ge n_0$. Define the stopping time $\tau_n=\tau_n(\rho)$ as

$$\inf\big\{s\in\big[t-2^{-\alpha(1-\rho)n},t\big):X_s\big(I_k^{(n)}\big)\ge 2^{-n(1-2\rho)}\text{ for some }I_k^{(n)}\big\},\tag{5.35}$$

if $\omega \in A_n^\rho$, and as t if $\omega \in A_n^{\rho,\mathrm{c}}$. Fix any $\omega \in A_n^\rho$. By definition of τ_n there exists a sequence $\{(s_j, I_{k_j}^{(n)}) : j \geq 1\}$ such that

$$s_j \downarrow \tau_n \qquad \text{as } j \uparrow \infty \quad \text{and} \quad X_{s_j}\bigl(I_{k_j}^{(n)}\bigr) \geq 2^{-n(1-2\rho)}, \qquad j \geq 1. \tag{5.36}$$

There exists a subsequence $\{j_r : r \geq 1\}$ such that $I_{k_{j_r}}^{(n)} = I_{\tilde{k}}^{(n)}$ for some $\tilde{k} \in \mathsf{Z}$. Hence, for the fixed $\omega \in A_n^\rho$,

$$X_{\tau_n}\bigl(I_{\tilde{k}}^{(n)}\bigr) = \lim_{r\to\infty} X_{s_{j_r}}\bigl(I_{\tilde{k}}^{(n)}\bigr) \geq 2^{-n(1-2\rho)}. \tag{5.37}$$

Put $\tilde{B} := [\tilde{k}2^{-n} - 2^{-n(1-\rho)}, (\tilde{k}+1)2^{-n} + 2^{-n(1-\rho)}]$. Then there is a constant $c_{(5.38)}$ independent of ρ such that

$$\begin{aligned}\int_{\tilde{B}} dy\, p_{t-s}^\alpha(y-z) \geq c_{(5.38)}&\\ \text{for all } z \in I_{\tilde{k}}^{(n)} \text{ and } s \in \bigl[t - 2^{-\alpha(1-\rho)n}, t\bigr).&\end{aligned} \tag{5.38}$$

Now, by the strong Markov property,

$$\begin{aligned}\mathbf{E}X_t(\tilde{B}) &= \mathbf{E}e^{a(t-\tau_n)} S_{t-\tau_n}^\alpha X_{\tau_n}(\tilde{B})\\ &\geq e^{-|a|t}\mathbf{E}\biggl\{\int_{\tilde{B}} dy \int_{\mathsf{R}} X_{\tau_n}(dz) p_{t-\tau_n}^\alpha(y-z);\, A_n^\rho\biggr\}\\ &\geq e^{-|a|t}\mathbf{E}\biggl\{\int_{I_{\tilde{k}}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}} dy\, p_{t-\tau_n}^\alpha(y-z);\, A_n^\rho\biggr\}\\ &\geq c_{(5.38)}\mathbf{E}\bigl\{X_{\tau_n}\bigl(I_{\tilde{k}}^{(n)}\bigr);\, A_n^\rho\bigr\}.\end{aligned}$$

Taking into account (5.37) and (5.38) then gives

$$\mathbf{E}X_t(\tilde{B}) \geq c_{(5.38)} 2^{-n(1-2\rho)} \mathbf{P}(A_n^\rho). \tag{5.39}$$

On the other hand, in view of Corollary 2.8,

$$\begin{aligned}\mathbf{E}X_t(\tilde{B}) &\leq |\tilde{B}|\mathbf{E} \sup_{0\leq x\leq 1} X_t(x)\\ &\leq 2\bigl(2^{-n} + 2^{-n(1-\rho)}\bigr)\mathbf{E} \sup_{0\leq x\leq 1} X_t(x) \leq C2^{-n(1-\rho)},\end{aligned} \tag{5.40}$$

where we wrote $|\tilde{B}|$ for the length of the interval $\tilde{B}$. Combining (5.39) and (5.40) completes the proof. $\square$

LEMMA 5.6 [Second term in (5.33)]. *For fixed* $\varepsilon \in (0, \frac{1}{1+\beta})$ *and all* n *large enough*,

$$P_{n,k}^\varepsilon \leq 2^{-3n/2}, \qquad 2 \leq k \leq 2^n + 1. \tag{5.41}$$

PROOF. Since $T_- \le t_n$ on A^ε [recall notation (5.27)],

$$P^\varepsilon_{n,k} \le \mathbf{P}\Bigl(\sup_{v\le t_n} L_v 1\Bigl\{\sup_{u\le v} \Delta L_u \le 2^{-n(\eta_c+\varepsilon)}\Bigr\} \ge 2^{-n\eta_c}\Bigr). \tag{5.42}$$

Applying now Lemma 2.3, with notation of t_n from (5.27) we obtain

$$P^\varepsilon_{n,k} \le \bigl(c_\varepsilon 2^{\varepsilon\beta n-(1-\eta_c)(1+\beta)n} + c_\varepsilon 2^{\eta_c(1+\beta)n+\varepsilon\beta n-(\alpha-\beta-\varepsilon)n}\bigr)^{(2^{n\varepsilon})}. \tag{5.43}$$

Inserting the definition of η_c and making n sufficiently large, the estimate in the lemma follows. $\square$

In order to deal with the third term $P^{\varepsilon,\varrho}_{n,k}$, we need to define additional events

$$\begin{aligned} A^{\varepsilon,\rho,1}_{n,k} := \Bigl\{&\text{There exists a jump of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \\ &\text{for some } (r^*,s^*,y^*) \text{ such that } r^* \ge (t-s)^{1/(1+\beta)+2\varepsilon/\alpha}, \\ &\quad\Bigl|\frac{k+1}{2^n} - y^*\Bigr| \le (t-s)^{1/\alpha-2\varepsilon}, s^* \ge t-2^{-\alpha(1+\rho)n}\Bigr\} \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} A^{\varepsilon,\rho,2}_{n,k} &:= A^{\rho,c}_n \cap A_{n,k} \\ &\quad\cap \Bigl\{\text{There exists a jump of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \\ &\qquad \text{for some } (r^*,s^*,y^*) \text{ such that } r^* \ge (t-s)^{1/(1+\beta)+2\varepsilon/\alpha}, \\ &\qquad y^* \in \Bigl[\frac{k+1/2}{2^n}, \frac{k+1+2^{\rho n+\alpha 2\varepsilon(1-\rho)n}}{2^n}\Bigr], \\ &\qquad\qquad s^* \in \bigl[t-2^{-\alpha(1-\rho)n}, t-2^{-\alpha(1+\rho)n}\bigr]\Bigr\}. \end{aligned}$$

So far we assumed that $\varepsilon \in (0, \frac{1}{1+\beta})$ and $\rho \in (0, \frac{1}{2})$. Suppose additionally that

$$\frac{\alpha(\alpha+1)2\varepsilon}{1-\eta_c+2\varepsilon(\alpha^2+\alpha-1)} \le \rho. \tag{5.45}$$

LEMMA 5.7 [Splitting of the third term in (5.33)]. *For $\rho, \varepsilon > 0$ sufficiently small and satisfying (5.45) we have*

$$P^{\varepsilon,\varrho}_{n,k} \le \mathbf{P}(A^{\varepsilon,\rho,1}_{n,k}) + \mathbf{P}(A^{\varepsilon,\rho,2}_{n,k}) \tag{5.46}$$

for all $0 \le k \le 2^n - 1$ and $n \ge n_\varepsilon$.

PROOF. First let us describe the strategy of the proof. We are going to show that whenever a jump of $L^-_{n,k}(s)$, $s \le T_-$, of size at least $2^{-n(\eta_c+\varepsilon)}$ occurs, then it may happen only in the points indicated in the definition of $A^{\varepsilon,\rho,1}_{n,k}$ and $A^{\varepsilon,\rho,2}_{n,k}$. To show this we will in fact show that outside the sets mentioned in $A^{\varepsilon,\rho,1}_{n,k}$ and $A^{\varepsilon,\rho,2}_{n,k}$ the jumps of $L^-_{n,k}(s)$, $s \le T_-$, are less than $2^{-n(\eta_c+\varepsilon)}$.

To implement this strategy, first let us recall that all the jumps of $L^-_{n,k}(s)$, $s \le T_-$, equal to

$$\Delta X_{s*}(y^*)\left(p^\alpha_{t-s}\left(\frac{k+1}{2^n} - y^*\right) - p^\alpha_{t-s}\left(\frac{k}{2^n} - y^*\right)\right)_+ \tag{5.47}$$

for some $(s^*, y^*) \in [0, t) \times \mathsf{R}$.

Recall that by definition (3.4), on the event A^ε,

$$|\Delta X_s| \le c_{(2.55)}(t-s)^{(1+\beta)^{-1}-\gamma} \tag{5.48}$$

with $\gamma \in (0, (1+\beta)^{-1})$. On the other hand using Lemma 2.1 with $\delta = 1$ we obtain

$$p^\alpha_{t-s}\left(\frac{k+1}{2^n} - y\right) - p^\alpha_{t-s}\left(\frac{k}{2^n} - y\right) \le C2^{-n}(t-s)^{-2/\alpha}. \tag{5.49}$$

From (5.48) and (5.49) we infer

$$\begin{aligned} &\sup_{s \le t - 2^{-\alpha(1-\rho)n}} \Delta X_s \sup_{y \in \mathsf{R}}\left(p^\alpha_{t-s}\left(\frac{k+1}{2^n} - y\right) - p^\alpha_{t-s}\left(\frac{k}{2^n} - y\right)\right) \\ &\quad \le C c_{(2.55)} 2^{-n}\left(2^{-\alpha(1-\rho)n}\right)^{1/(1+\beta)-\gamma-2/\alpha} \\ &\quad = C2^{-n(\eta_c - \alpha\gamma + \rho(1-\eta_c+\alpha\gamma))}. \end{aligned} \tag{5.50}$$

Furthermore if the jump ΔX_s occurs at the point y^* with

$$\left|y^* - \frac{k+1}{2^n}\right| \ge (t-s)^{1/\alpha - 2\varepsilon}, \tag{5.51}$$

then again by Lemma 2.1, for any $\delta \in [0, 1]$,

$$\begin{aligned} &p^\alpha_{t-s}\left(\frac{k+1}{2^n} - y\right) - p^\alpha_{t-s}\left(\frac{k}{2^n} - y\right) \\ &\quad \le C2^{-n\delta}(t-s)^{-\delta/\alpha} p^\alpha_{t-s}\left((t-s)^{1/\alpha - 2\varepsilon}\right). \end{aligned} \tag{5.52}$$

Since

$$p^\alpha_1(x) \le Cx^{-1-\alpha}, \qquad x \in \mathsf{R}, \tag{5.53}$$

we get the bound

$$p^\alpha_{t-s}\left(\frac{k+1}{2^n} - y\right) - p^\alpha_{t-s}\left(\frac{k}{2^n} - y\right) \le C2^{-n\delta}(t-s)^{-(\delta+1)/\alpha + 2\varepsilon(\alpha+1)}. \tag{5.54}$$

Hence

$$\sup_{s<t} \sup_{y:|y-(k+1)/2^n|\geq(t-s)^{1/\alpha-2\varepsilon}} \Delta X_s(y)\biggl(p^\alpha_{t-s}\Bigl(\frac{k+1}{2^n}-y\Bigr) - p^\alpha_{t-s}\Bigl(\frac{k}{2^n}-y\Bigr)\biggr) \tag{5.55}$$
$$\leq C c_{(2.55)} 2^{-n\delta}(t-s)^{-(\delta+1)/\alpha+2\varepsilon(\alpha+1)+1/(\beta+1)-\gamma}.$$

Set

$$\delta := \eta_{\mathrm{c}} + \alpha\bigl(2\varepsilon(\alpha+1)-\gamma\bigr). \tag{5.56}$$

Note that for all ε and γ sufficiently small, we have $\delta \in [0,1]$, and we can apply the previous estimates. Thus we obtain

$$\sup_{s<t} \sup_{y:|y-(k+1)/2^n|\geq(t-s)^{1/\alpha-2\varepsilon}} \Delta X_s(y)\biggl(p^\alpha_{t-s}\Bigl(\frac{k+1}{2^n}-y\Bigr) - p^\alpha_{t-s}\Bigl(\frac{k}{2^n}-y\Bigr)\biggr) \tag{5.57}$$
$$\leq C c_{(2.55)} 2^{-n(\eta_{\mathrm{c}}+\alpha(2\varepsilon(\alpha+1)-\gamma))}.$$

Now if we take $\gamma = 2\varepsilon(\alpha+1-1/\alpha)$, which belongs to these admissible γ, and ρ as in (5.45), we conclude that the right-hand side of (5.50) and (5.57) is bounded by

$$C2^{-n(\eta_{\mathrm{c}}+2\varepsilon)}. \tag{5.58}$$

For any jump $r^*\delta_{(s^*,y^*)}$ of M such that $r^* \leq (t-s)^{1/(1+\beta)+2\varepsilon/\alpha}$ and $s^* < t$ we may apply Lemma 2.1 with $\delta = \eta_{\mathrm{c}} + 2\varepsilon$ to get that

$$\Delta X_{s*}(y^*)\biggl(p^\alpha_{t-s}\Bigl(\frac{k+1}{2^n}-y^*\Bigr) - p^\alpha_{t-s}\Bigl(\frac{k}{2^n}-y^*\Bigr)\biggr) \leq C2^{-n(\eta_{\mathrm{c}}+2\varepsilon)}. \tag{5.59}$$

Now recall (5.47). Hence combining (5.50), (5.57), (5.58) and (5.59) the conclusion of Lemma 5.7 follows. □

In the next two lemmas we will bound the two probabilities on the right-hand side of (5.46).

LEMMA 5.8 [First term in (5.46)]. *For all $\rho, \varepsilon > 0$ sufficiently small and satisfying*

$$6\varepsilon(\alpha+1+\beta) \leq \rho, \tag{5.60}$$

we have

$$\mathbf{P}(A^{\varepsilon,\rho,1}_{n,k}) \leq 2^{-n-n\rho/2} \tag{5.61}$$

for all k, n considered.

PROOF. It is easy to see that

$$\begin{aligned} A_{n,k}^{\varepsilon,\rho,1} \subseteq \bigcup_{l=(1+\rho)n}^{\infty} \Big\{ & \text{There exists a jump of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \\ & \text{for some } (r^*, s^*, y^*) \text{ such that } r^* \geq 2^{-l(\alpha/(1+\beta)+2\varepsilon)}, \\ & \left|\frac{k+1}{2^n} - y^*\right| \leq 2^{-l(1-2\varepsilon\alpha)}, s^* \in \big[t - 2^{-\alpha l}, t - 2^{-\alpha(l+1)}\big)\Big\} \\ =: \bigcup_{l=(1+\rho)n}^{\infty} & A_{n,k,l}^{\varepsilon,\rho,1}. \end{aligned}$$

Recall the random measure N describing the jumps of X. Write $Y_{n,k,l}$ for the N-measure of

$$\begin{aligned} &\big[t(1-2^{-\alpha l}), t\big(1-2^{-\alpha(l+1)}\big)\big] \times \left[\frac{k+1}{2^n} - 2^{-l(1-2\alpha\varepsilon)}, \frac{k+1}{2^n} + 2^{-l(1-2\alpha\varepsilon)}\right] \\ &\quad \times \big[2^{-l(\alpha/(1+\beta)+2\varepsilon)}, \infty\big). \end{aligned}$$

Then, by Markov's inequality,

$$\mathbf{P}(A_{n,k,l}^{\varepsilon,\rho,1}) = \mathbf{P}(Y_{n,k,l} \geq 1) \leq \mathbf{E}Y_{n,k,l}. \tag{5.62}$$

Therefore,

$$\mathbf{P}(A_{n,k}^{\varepsilon,\rho,1}) \leq \sum_{l\geq(1+\rho)n} \mathbf{P}(A_{n,k,l}^{\varepsilon,\rho,1}) \leq \sum_{l\geq(1+\rho)n} \mathbf{E}Y_{n,k,l}. \tag{5.63}$$

From the formula for the compensator of N we get

$$\begin{aligned} \mathbf{E}Y_{n,k,l} = \varrho \int_{t(1-2^{-\alpha l})}^{t(1-2^{-\alpha(l+1)})} ds\, \mathbf{E}X_s\bigg(\bigg[&\frac{k+1}{2^n} - 2^{-l(1-2\alpha\varepsilon)}, \\ &\frac{k+1}{2^n} + 2^{-l(1-2\alpha\varepsilon)}\bigg]\bigg) \\ \times \int_{2^{-l(\alpha/(1+\beta)+2\varepsilon)}}^{\infty} & dr\, r^{-2-\beta} \\ \leq C2^{-\alpha l}2^{-l(1-2\alpha\varepsilon)}&2^{l(\alpha+2\varepsilon(1+\beta))}. \end{aligned} \tag{5.64}$$

Consequently,

$$\mathbf{P}(A_{n,k,l}^{\varepsilon,\rho,1}) \leq C \sum_{l\geq(1+\rho)n} 2^{-l+2\varepsilon(\alpha+1+\beta)l} \leq C2^{-(1+\rho)n+2\varepsilon(\alpha+1+\beta)(1+\rho)n}. \tag{5.65}$$

Noting that $2\varepsilon(\alpha+1+\beta)(1+\rho) \leq \rho/2$ under the conditions in the lemma, we complete the proof. $\square$

LEMMA 5.9 [Second term in (5.46)]. *For all $\varepsilon, \rho > 0$ sufficiently small*,

$$\mathbf{P}(A_{n,k}^{\varepsilon,\rho,2}) \le 2^{-3n/2} \tag{5.66}$$

for all k, n considered.

PROOF. It is easy to see by construction that

$$A_{n,k}^{\varepsilon,\rho,2} \subseteq A_n^{\rho,\mathrm{c}} \cap \Big\{\text{There exist at least two jumps of } M$$

$$\text{of the form } r^*\delta_{(s^*,y^*)} \text{ such that}$$

$$r^* \ge 2^{-n(\alpha(1+\rho)/(1+\beta)+2\varepsilon(1+\rho))}, \tag{5.67a}$$

$$y^* \in \left[\frac{k-2}{2^n}, \frac{k+1+2^{\rho n+2\alpha\varepsilon(1-\rho)n}}{2^n}\right], \tag{5.67b}$$

$$s^* \in \left[t - 2^{-\alpha(1-\rho)n}, t - 2^{-\alpha(1+\rho)n}\right]\Big\}. \tag{5.67c}$$

On the event $A_n^{\rho,\mathrm{c}}$, for the intensity of jumps satisfying (5.67a)–(5.67c), we have

$$\begin{aligned}
&\int_{t-2^{-\alpha(1-\rho)n}}^{t-2^{-\alpha(1+\rho)n}} ds\, X_s\left(\left[\frac{k-2}{2^n}, \frac{k+1+2^{\rho n+2\alpha\varepsilon(1-\rho)n}}{2^n}\right]\right) \\
&\quad \times \int_{2^{-n(\alpha(1+\rho)/(1+\beta)+2\varepsilon(1+\rho))}}^{\infty} dr\, r^{-2-\beta} \\
&\quad \le 2^{-\alpha(1-\rho)n} 2^{-n(1-2\rho)} 2^{\rho n+2\alpha\varepsilon(1-\rho)n+2} 2^{n(\alpha(1+\rho)+2\varepsilon(1+\rho)(1+\beta))} \\
&\quad \le 2^{-n} 2^{10(\rho+2\varepsilon)n} \le 2^{-3/4n}
\end{aligned}$$

for all ε and ρ sufficiently small. Since the number of such jumps can be represented by means of a time-changed standard Poisson process, the probability to have at least two jumps is bounded by the square of the above bound and we are done. □

LEMMA 5.10 [Third term in (5.33)]. *For all $\rho, \varepsilon > 0$ sufficiently small, satisfying* (5.45) *and* (5.60), *we have*

$$P_{n,k}^{\varepsilon,\varrho} \le 2^{-3n/2} + C2^{-n-\rho n/2}, \qquad 2 \le k \le 2^n + 1, n \ge n_\varepsilon. \tag{5.68}$$

PROOF. The proof follows immediately from Lemmas 5.7, 5.8 and 5.9. □

PROOF OF LEMMA 5.4. Applying Lemmas 5.5, 5.6 and 5.10 to (5.33) we obtain

$$\mathbf{P}\left(\bigcup_{k=2}^{2^n+1} (A^\varepsilon \cap A_{n,k} \cap B_{n,k}^{-,\mathrm{c}})\right) \le c_{(5.34)} 2^{-\rho n} + 2^{-n/2} + C2^{-\rho n/2} + 2^{-n/2} \tag{5.69}$$

for all $\rho, \varepsilon > 0$ sufficiently small satisfying (5.45) and (5.60) as well as all $n \geq n_\varepsilon$. Since these terms are summable in n, the claim of the lemma follows. □

PROOF OF LEMMA 5.2. The proof follows immediately from (5.10) and Lemmas 5.3 and 5.4. □

Acknowledgments. We thank Don Dawson for very helpful discussions of the subject. We also thank the referees for their useful comments and suggestions which improved the exposition.

REFERENCES

[1] DAWSON, D. A. (1993). Measure-valued Markov processes. In *École D'Été de Probabilités de Saint-Flour XXI—1991. Lecture Notes in Math.* **1541** 1–260. Springer, Berlin. MR1242575

[2] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **II**, 2nd ed. Wiley, New York. MR0270403

[3] FLEISCHMANN, K. (1988). Critical behavior of some measure-valued processes. *Math. Nachr.* **135** 131–147. MR944225

[4] FLEISCHMANN, K., MYTNIK, L. and WACHTEL, V. (2009). Hölder index for density states of $(\alpha, 1, \beta)$-superprocesses at a given point. Preprint. Available at arXiv:0901.2315v1.

[5] FUK, D. K. and NAGAEV, S. V. (1971). Probability inequalities for sums of independent random variables. *Theory Probab. Appl.* **16** 643–660.

[6] LE GALL, J.-F. and MYTNIK, L. (2005). Stochastic integral representation and regularity of the density for the exit measure of super-Brownian motion. *Ann. Probab.* **33** 194–222. MR2118864

[7] GIKHMAN, I. I. and SKOROKHOD, A. V. (1969). *Introduction to the Theory of Random Processes*. W. B. Saunders Co., Philadelphia, PA. MR0247660

[8] GIKHMAN, I. I. and SKOROKHOD, A. V. (1980). *The Theory of Stochastic Processes. I*, English ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **210**. Springer, Berlin. MR636254

[9] HAUSENBLAS, E. (2007). SPDEs driven by Poisson random measure with non Lipschitz coefficients: Existence results. *Probab. Theory Related Fields* **137** 161–200. MR2278455

[10] JAFFARD, S. (1999). The multifractal nature of Lévy processes. *Probab. Theory Related Fields* **114** 207–227. MR1701520

[11] JAKOD, J. (1979). *Calcul Stochastique et Problèmes de Martigales. Lecture Notes in Math.* **714** Springer, Berlin.

[12] KONNO, N. and SHIGA, T. (1988). Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Related Fields* **79** 201–225. MR958288

[13] MUELLER, C., MYTNIK, L. and STAN, A. (2006). The heat equation with time-independent multiplicative stable Lévy noise. *Stochastic Process. Appl.* **116** 70–100. MR2186840

[14] MYTNIK, L. (2002). Stochastic partial differential equation driven by stable noise. *Probab. Theory Related Fields* **123** 157–201. MR1900321

[15] MYTNIK, L. and PERKINS, E. (2003). Regularity and irregularity of $(1+\beta)$-stable super-Brownian motion. *Ann. Probab.* **31** 1413–1440. MR1989438

[16] REIMERS, M. (1989). One-dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields* **81** 319–340. MR983088

[17] ROSEN, J. (1987). Joint continuity of the intersection local times of Markov processes. *Ann. Probab.* **15** 659–675. MR885136

[18] SAINT LOUBERT BIÉ, E. (1998). Étude d'une EDPS conduite par un bruit poissonnien. *Probab. Theory Related Fields* **111** 287–321. MR1633586

[19] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'Été de Probabilités de Saint-Flour, XIV*—1984. *Lecture Notes in Math*. **1180** 265–439. Springer, Berlin. MR876085

K. FLEISCHMANN
WEIERSTRASS INSTITUTE
FOR APPLIED ANALYSIS
AND STOCHASTICS
MOHRENSTRASSE 39
D-10117 BERLIN
GERMANY
E-MAIL: fleischm@wias-berlin.de

L. MYTNIK
FACULTY OF INDUSTRIAL ENGINEERING
AND MANAGEMENT
TECHNION ISRAEL INSTITUTE
OF TECHNOLOGY
HAIFA 32000
ISRAEL
E-MAIL: leonid@ie.technion.ac.il
URL: http://ie.technion.ac.il/leonid.phtml

V. WACHTEL
MATHEMATICAL INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTRASSE 39
D-80333 MUNICH
GERMANY
E-MAIL: wachtel@mathematik.uni-muenchen.de