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APPLICATIONS OF WEAK CONVERGENCE FOR HEDGING OF GAME OPTIONS[1]

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In this paper we consider Dynkin's games with payoffs which are functions of an underlying process. Assuming extended weak convergence of underlying processes $\{S^{(n)}\}_{n=0}^{\infty}$ to a limit process S we prove convergence Dynkin's games values corresponding to $\{S^{(n)}\}_{n=0}^{\infty}$ to the Dynkin's game value corresponding to S. We use these results to approximate game options prices with path dependent payoffs in continuous time models by a sequence of game options prices in discrete time models which can be calculated by dynamical programming algorithms. In comparison to previous papers we work under more general convergence of underlying processes, as well as weaker conditions on the payoffs.

1. Introduction. Consider a *càdlàg* stochastic process $\{S_t\}_{t=0}^{T}$ $(T < \infty)$ which takes on values in $\mathbb{R}_+^d$. For two given functions $f \le g$ let $X_t = g(t, S)$ and $Y_t = f(t, S)$. Define $\Gamma(S) = \inf_\sigma \sup_\tau E(X_\sigma \mathbb{I}_{\sigma<\tau} + Y_\tau \mathbb{I}_{\tau \le \sigma})$ which is the Dynkin game value for the above processes where $\mathbb{I}_A = 1$ if an event A occurs and $=0$ if not. The inf and the sup are taken over the set of stopping times no bigger than T, with respect to the usual filtration generated by the process S. Our goal is to prove (under some additional assumptions) that if a sequence of stochastic processes $\{S_t^{(n)}\}_{t=0}^{T}$, $n \ge 1$ converges in law to S then $\Gamma(S) = \lim_{n\to\infty} \Gamma(S^{(n)})$.

Although several papers dealt with stability of optimal stopping values under weak convergence of the underlying processes (see [2, 3, 5, 6, 14, 16, 17] and [19]) stability of Dynkin's games values under weak convergence of the underlying processes was not studied before. In his unpublished paper [2] Aldous represented the notion of extended weak convergence and proved the stability of optimal stopping values under extended weak convergence of the underlying processes. In this paper we extend these results for Dynkin's games. The main tools that we use for proving the result is the Skorohod representation theorem (see [7]) and the theory of extended weak convergence that Aldous developed in [2].

In [5] and [14] the authors studied binomial approximations of American put options in the Black–Scholes (BS) model. In both of these papers it was proved by using different methods, that the option price in the BS model is a limit of option prices for an appropriate sequence of Cox–Ross–Rubinstein (CRR) models.

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Furthermore they proved stability of the critical prices and the optimal stopping times. In [6] the authors studied stability of optimal stopping values and optimal stopping times under convergence in probability of the underlying processes (though, it seems that the part related to convergence of optimal stopping values contains some gaps). The most general result was obtained in [19] where the authors considered a general framework and proved Snell envelopes stability under weak convergence of the underlying processes. The authors used the fact that the Snell envelope of a positive process is a positive supermartingale, and so it is a quasi martingale and the corresponding tightness theorems (see [20]) can be employed. For Dynkin's games the value process should not, in general, be a quasi martingale and so the above method is not applicable here, and so Dynkin's games value process stability under weak convergence remains an open question, but we are able to prove in this paper the stability of Dynkin's game values under the weak convergence of the underlying processes.

One of the motivations to study Dynkin's games values stability under weak convergence of the underlying processes is applications to game options approximations. Recall, that a game contingent claim (GCC) or a game option was defined in [10] as a contract between the seller and the buyer of the option such that both have the right to exercise it any time up to a maturity date (horizon) T. If the buyer exercises the contract at time t then he gets the payment Y_t, but if the seller cancels before the buyer then the latter gets the payment X_t. The difference $\delta_t = X_t - Y_t$ is the penalty which the seller pays to the buyer for the contract cancellation. Thus the process S can be considered as a discounted risky asset, and the processes $X \geq Y$ are considered as the discounted payoff processes. In [10] it was proved that pricing a GCC in a complete market leads to the value of a Dynkin game with the payoffs X, Y under the unique martingale measure, namely if the process S is a martingale then $\Gamma(S)$ is the option price of the above game option. In [13] it was proved that for a general incomplete market, if the process S is a martingale then $\Gamma(S)$ is an arbitrage-free price.

Convergence results for Dynkin's games will allow us to approximate options prices in continuous time markets by a sequence of game options prices in discrete time markets which are defined on a discrete probability space. In addition to the theoretical interest such results have a practical value for calculations of options prices, since it is well known (see [21]) that for a discrete probability space Dynkin's games values can be calculated by dynamical programming algorithm. In this paper we give an example for approximations of game options with Russian (path dependent) type of payoffs in the Merton model.

Several papers (see [8, 11, 12]) dealt with approximations of option prices for game options. These papers used strong approximation theorems in order to obtain error estimates for discrete time approximations of game options in the BS model with Lipschitz conditions on the payoffs. The weak convergence approach does not allow one to obtain estimates of the error, but it works under weaker assumptions on the payoffs and can be applied for jump diffusion models.

The main results of this paper are formulated in the next section where we also introduce the notation that will be used. In Section 3 we derive auxiliary lemmas that we use. In Section 4 we complete the proof of the main results of the paper. In Section 5 we provide an application for approximations of game options in Merton's model with path dependent payoffs.

2. Preliminaries and main results. First we introduce some definitions and notation that will be used in this paper. Let $d \in \mathbb{N}$. Given a probability space $(\Omega, \mathcal{F}, P)$ consider a *càdlàg* stochastic process $S = \{S_t : \Omega \to \mathbb{R}^d\}_{t=0}^T$. Denote by $\mathcal{F}^S = \{\mathcal{F}_t^S\}_{t=0}^T$ the usual filtration of S, that is, the smallest right continuous filtration with respect to which S is adapted, and such that the σ algebras contain the null sets. Let $\mathcal{T}_{[0,T]}^S$ be the set of all stopping times with respect to $\mathcal{F}^S$ which take values in $[0, T]$. Denote by $\mathbb{P}^S$ the distribution of S on the canonical space $\mathbb{D}([0, T]; \mathbb{R}^d)$ equipped with the Skorohod topology, that is, for any Borel set $A \subset \mathbb{D}([0, T]; \mathbb{R}^d)$, $\mathbb{P}^S(A) = P\{S \in A\}$. For a sequence of stochastic processes $S^{(n)} : \Omega_n \to \mathbb{D}([0, T]; \mathbb{R}^d)$ we will use the notation $S^{(n)} \Rightarrow S$ to indicate that the probability measures $\mathbb{P}^{S^{(n)}}$, $n \geq 1$ converge weakly to $\mathbb{P}^S$ (where the space $\mathbb{D}([0, T]; \mathbb{R}^d)$ is equipped with the Skorohod topology).

Next, let $f, g : [0, T] \times \mathbb{D}([0, T]; \mathbb{R}^d) \to \mathbb{R}_+$ be two measurable functions such that $f \leq g$. We will assume the following.

ASSUMPTION 2.1. For any $t \in [0, T]$ and $x, y \in \mathbb{D}([0, T]; \mathbb{R}^d)$ $f(t, x) = f(t, y)$ and $g(t, x) = g(t, y)$ if $x(s) = y(s)$ for any $s \leq t$. The functions $f(\cdot, x)$, $g(\cdot, x)$ are right-continuous functions with left-hand limits. Furthermore, let $\{x_n\}_{n=1}^\infty \subset \mathbb{D}([0, T]; \mathbb{R}^d)$ and $\{t_n\}_{n=1}^\infty \subset [0, T]$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} x_n(t_n) = x(t)$ for some $x \in \mathbb{D}([0, T]; \mathbb{R}^d)$ and $t \in [0, T]$. Then

$$\lim_{n\to\infty} f(t_n, x_n) = f(t, x) \quad \text{and} \quad \lim_{n\to\infty} g(t_n, x_n) = g(t, x). \tag{2.1}$$

For any *càdlàg* stochastic process $S = \{S_t\}_{t=0}^T$ set the *càdlàg* adapted processes

$$X_t^S = g(t, S), \qquad Y_t^S = f(t, S) \tag{2.2}$$

and consider the payoff function

$$H^S(t, s) = X_t^S \mathbb{I}_{t<s} + Y_s^S \mathbb{I}_{s\leq t}, \qquad t, s \leq T. \tag{2.3}$$

Assume that $\sup_{\tau \in \mathcal{T}_{[0,T]}^S} E X_\tau^S < \infty$ where E denotes the expectation with respect to the probability measure of the space on which the process S is defined. Let $\Gamma(S)$ be the Dynkin's game value of the payoff given by (2.3). Namely,

$$\Gamma(S) = \inf_{\sigma \in \mathcal{T}_{[0,T]}^S} \sup_{\tau \in \mathcal{T}_{[0,T]}^S} E H^S(\sigma, \tau) = \sup_{\tau \in \mathcal{T}_{[0,T]}^S} \inf_{\sigma \in \mathcal{T}_{[0,T]}^S} E H^S(\sigma, \tau). \tag{2.4}$$

The second equality follows from Corollary 12 in [15]. Furthermore from Lemma 5 in [15] it follows that

$$\begin{aligned}\Gamma(S) &= \inf_{\sigma\in\mathcal{T}^S_{[0,T]}} \sup_{\tau\in\mathcal{T}^S_{[0,T]}} EJ^S(\sigma,\tau)\\ &= \sup_{\tau\in\mathcal{T}^S_{[0,T]}} \inf_{\sigma\in\mathcal{T}^S_{[0,T]}} EJ^S(\sigma,\tau)\end{aligned} \tag{2.5}$$

$$\text{where } J^S(t,s) = \mathbb{I}_{t\wedge s=T} Y^S_T + \mathbb{I}_{t\wedge s<T}\mathbb{I}_{t\le s} X^S_t + Y^S_s \mathbb{I}_{s<t}, \quad t,s\le T.$$

First we show that $\Gamma(S)$ depends only on the distribution of S. Consider the probability space $(\mathbb{D}([0,T];\mathbb{R}^d), P^S)$. Let $\{U_t\}_{t=0}^T$ be the canonical process of coordinate projection, namely $U_t:\mathbb{D}([0,T];\mathbb{R}^d)\to\mathbb{R}^d$ is given by $U_t(x)=x(t)$ and let $\{\mathcal{G}_t\}_{t=0}^T$ be the usual filtration which is generated by the above process. Introduce the set Φ of all functions $\phi:\mathbb{D}([0,T];\mathbb{R}^d)\to[0,T]$ which satisfy $\{\phi\le t\}\in\mathcal{G}_t$ for any $t\le T$. Observe that $\sigma\in\mathcal{T}^S_{[0,T]}$ if and only if there exists a function $\phi\in\Phi$ such that $\sigma=\phi(S)$ a.s. Thus from (2.4) we obtain

$$\Gamma(S) = \inf_{\phi\in\Phi}\sup_{\psi\in\Phi} EH^S(\phi(S),\psi(S)) = \inf_{\phi\in\Phi}\sup_{\psi\in\Phi} \mathbb{E}^S H^U(\phi(U),\psi(U)), \tag{2.6}$$

where $\mathbb{E}^S$ is the expectation with respect to the probability measure $\mathbb{P}^S$. From (2.6) it follows that $\Gamma(S)$ depends only on the distribution of S.

In [2] Aldous introduced the notion of "extended weak convergence" via prediction processes. For the case where the stochastic processes are considered with respect to their natural filtration (with the usual assumptions) he proved that extended weak convergence is equivalent to a more elementary condition which does not require the use of prediction processes (see [2], Proposition 16.15). Following [6] we will use the above condition as the definition of extended weak convergence.

DEFINITION 2.2. A sequence $S^{(n)}:\Omega_n\to\mathbb{D}([0,T];\mathbb{R}^d)$, $n\ge1$ extended weak converges to a stochastic process $S:\Omega\to\mathbb{D}([0,T];\mathbb{R}^d)$ if for any k and continuous bounded functions $\psi_1,\dots,\psi_k\in C(\mathbb{D}([0,T];\mathbb{R}^d))$

$$\left(S^{(n)}, Z^{(n,1)},\dots,Z^{(n,k)}\right)\Rightarrow\left(S,Z^{(1)},\dots,Z^{(k)}\right) \qquad \text{in } \mathbb{D}([0,T];\mathbb{R}^{d+k}), \tag{2.7}$$

where for any $t\le T$, $1\le i\le k$, and $n\in\mathbb{N}$

$$Z^{(n,i)}_t = E_n\left(\psi_i\left(S^{(n)}\right)|\mathcal{F}^{S^{(n)}}_t\right), \qquad n\in\mathbb{N} \quad\text{and}\quad Z^{(i)} = E(\psi_i(S)|\mathcal{F}^S_t), \tag{2.8}$$

E_n denotes the expectation with respect to the probability measure on Ω_n and E denotes the expectation with respect to the probability measure on Ω. We will denote extended weak convergence by $S^{(n)}\Rrightarrow S$.

Next, we introduce two additional assumptions that we will work with. Let $S^{(n)}:\Omega_n \to \mathbb{D}([0,T];\mathbb{R}^d)$, $n \geq 1$ be a sequence of stochastic processes which satisfies the following assumptions.

ASSUMPTION 2.3. The random variables $g(\tau, S^{(n)})$, for $n \geq 1$ and $\tau \in \mathcal{T}^{S^{(n)}}_{[0,T]}$ are uniformly integrable.

ASSUMPTION 2.4. For any $\varepsilon > 0$

$$\lim_{\delta \downarrow 0} \lim_{n\to\infty} \sup_{0<u<\delta} \sup_{\tau \in \mathcal{F}^{S^{(n)}}_{[0,T]}} P\big(\big|S^{(n)}_{(\tau+u)\wedge T} - S^{(n)}_{\tau}\big| > \varepsilon\big) = 0.$$

The above assumption is called the "Aldous tightness criterion" and was introduced in [1]. The following theorem is the main result of the paper.

THEOREM 2.5. *If $S^{(n)} \Rightarrow S$, then $\Gamma(S) = \lim_{n\to\infty} \Gamma(S^{(n)})$.*

3. Auxiliary lemmas. Let $I \subset [0,T]$. For any stochastic process $S:\Omega \to \mathbb{D}([0,T];\mathbb{R}^d)$ denote by Δ^S_I the set of all stopping times τ which take on a finite number of values in I and such that for any $t \in I$, $\{\tau = t\} \in \sigma\{S_u | u \leq t\}$.

LEMMA 3.1. *Let $I \subset [0,T]$ be a dense set which contains the point T. Then*

$$\sup_{\tau \in \Delta^S_I} \inf_{\sigma \in \mathcal{F}^S_{[0,T]}} EH^S(\sigma,\tau) = \Gamma(S) = \inf_{\sigma \in \Delta^S_I} \sup_{\tau \in \mathcal{F}^S_{[0,T]}} EJ^S(\sigma,\tau). \tag{3.1}$$

PROOF. We start with the proof of the first equality. Choose $\varepsilon > 0$ and $\tau \in \mathcal{T}^S_{[0,T]}$ which satisfies

$$\inf_{\sigma \in \mathcal{F}^S_{[0,T]}} EH^S(\sigma,\tau) > \Gamma(S) - \varepsilon. \tag{3.2}$$

For any n let $E_n \subset I$ be a finite set which contains T and satisfies $\bigcup_{e\in E_n}(e - \frac{1}{n}, e] \supseteq [0,T]$. Define

$$\tau_n = \min\left\{e \in E_n \middle| e \geq T \wedge \left(\frac{1}{n} + \tau\right)\right\}, \qquad n \in \mathbb{N}. \tag{3.3}$$

Fix n and $t \in I \setminus \{T\}$. Clearly,

$$\{\tau_n = t\} = \left\{\min\left\{e \in E_n \middle| e - \frac{1}{n} \geq \tau\right\} = t\right\} \in \mathcal{F}^S_{t-1/n} \subset \sigma\{S_u | u \leq t\},$$

which means that for any n, $\tau_n \in \Delta^S_I$. Since $\tau \leq \tau_n \leq \tau + \frac{2}{n}$, we have $\tau_n \downarrow \tau$. Set, $\Phi_n = \sup_{\sigma \in \mathcal{F}^S_{[0,T]}} (H^S(\sigma,\tau) - H^S(\sigma,\tau_n))^+$, $n \in \mathbb{N}$. Observe that

$$\limsup_{n\to\infty} \Phi_n \leq \limsup_{n\to\infty} \sup_{\tau \leq t \leq \tau_n} (Y^S_\tau - Y^S_t)^+ = 0 \qquad \text{a.s.}$$

Since the sequence Φ_n, $n \in \mathbb{N}$ is uniformly integrable then $\lim_{n\to\infty} E\Phi_n = 0$. From (3.2) we obtain

$$\begin{aligned}\Gamma - \sup_{\tau\in\Delta_I^S} \inf_{\sigma\in\mathcal{F}^S_{[0,T]}} & EH^S(\sigma,\tau) \\ &\le \varepsilon + \limsup_{n\to\infty} \sup_{\sigma\in\mathcal{F}^S_{[0,T]}} E\big(H^S(\sigma,\tau) - H^S(\sigma,\tau_n)\big)^+ \\ &\le \limsup_{n\to\infty} E\Big[\sup_{\sigma\in\mathcal{F}^S_{[0,T]}} \big(H^S(\sigma,\tau) - H^S(\sigma,\tau_n)\big)^+\Big] \\ &\le \varepsilon + \lim_{n\to\infty} E\Phi_n = \varepsilon,\end{aligned}$$

and the first equality in (3.1) follows. The proof of the second equality is similar. □

Let $S^{(n)} \Rightarrow S$ and assume that the sequence $S^{(n)}$, $n \in \mathbb{N}$, satisfies the assumptions from Section 2. The following two lemmas are a small modification of similar results that were obtained in [2]. For the reader's convenience we provide a self-contained proof for Lemmas 3.2 and 3.3 which follows the ideas that were used in [2].

LEMMA 3.2. *Let $E \subset [0,T]$ be a finite set such that any $t \in E \setminus \{T\}$ is a continuity point of the process S a.s. Then for any $\tau \in \Delta^S_E$ there exists a sequence of stopping times $\tau_n \in \mathcal{T}^{S^{(n)}}_{[0,T]}$ with values in E such that $(S^{(n)},\tau_n) \Rightarrow (S,\tau)$ on the space $\mathbb{D}([0,T];\mathbb{R}^d) \times [0,T]$.*

PROOF. By using the Skorohod representation theorem (see [7]) it follows that without loss of generality we can assume that there exists a probability space $(\Omega,\mathcal{F},P)$ on which the process S and the sequence $S^{(n)}$ are defined and $S^{(n)} \to S$ a.s. on $\mathbb{D}([0,T];\mathbb{R}^d)$. In order to prove the lemma it is sufficient to show that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and a sequence of stopping times $\tau_n \in \mathcal{T}^{S^{(n)}}_{[0,T]}$ with values in E such that

$$P\left(\bigcup_{n=N}^{\infty} \{\tau_n \neq \tau\}\right) < \varepsilon. \tag{3.4}$$

Choose $\varepsilon > 0$. Let $E \setminus \{T\} = \{t_1 < t_2 < \cdots < t_k\}$. Denote $A_i = \{\tau = t_i\} \in \sigma\{S_u | u \le t_i\}$, $i \le k$. Since t_i is a continuity point of the process S there exist continuous functions $\phi_i : \mathbb{D}([0,T];\mathbb{R}^d) \to [0,1]$, $i \le k$, such that

$$E|\mathbb{I}_{A_i} - \phi_i(S)| < \frac{\varepsilon}{2^{(i+1)}}, \tag{3.5}$$

and the function $\phi_i(x)$ depends only on the restriction of x to the interval $[0, t_i]$. For any $n \in \mathbb{N}$ define the stopping time $\tau_n \in \mathcal{T}^{S_n}_{[0,T]}$ by

$$\tau_n = T \wedge \min\{t_i | \phi_i(S^{(n)}) > \tfrac{1}{2}\}, \tag{3.6}$$

where $\min\{t_i | \phi_i(S^{(n)}) > \frac{1}{2}\} = \infty$ if for any i, $\phi_i(S^{(n)}) \le \frac{1}{2}$. Observe that $\phi_i^{(n)}(S^{(n)})$ is a $\mathcal{F}^{S_n}_{t_i}$ measurable random variable, thus τ_n is indeed a stopping time with respect to the filtration $\mathcal{F}^{S^{(n)}}$. Set

$$C_i = (A_i \cap \{\phi_i(S) > 1/2\}) \cup (A_i^c \cap \{\phi_i(S) < 1/2\}), \qquad i \le k,$$

and

$$C = \bigcap_{i=1}^{k} C_i.$$

Since for any i, ϕ_i is a continuous function we obtain that for any $\omega \in C$ there exists $N(\omega)$ such that for any $n \ge N(\omega)$, $\tau(\omega) = \tau_n(\omega)$. From (3.5) and the Markov inequality we obtain

$$\begin{aligned} P(C) = 1 - P(\Omega \setminus C) &\ge 1 - \sum_{i=1}^{k} P\left(|\phi_i(S) - \mathbb{I}_{A_i}| \ge \frac{1}{2}\right) \\ &\ge 1 - \sum_{i=1}^{k} \frac{\varepsilon}{2^i} > 1 - \varepsilon. \end{aligned} \tag{3.7}$$

Set $E_n = \bigcap_{m=n}^{\infty} \{\tau_m = \tau\}$, $n \in \mathbb{N}$. Observe that the sequence E_n, $n \ge 1$ is an increasing sequence of events and $\bigcup_{n=1}^{\infty} E_n \supset C$. From (3.7) it follows that there exists $N \in \mathbb{N}$ such that $P(E_N) > 1 - \varepsilon$ and (3.4) follows. $\square$

LEMMA 3.3. *Assume that $\tau_n \in \mathcal{T}^{S^{(n)}}_{[0,T]}$, $n \ge 1$ is a sequence of stopping times which satisfies $(S^{(n)}, \tau_n) \Rightarrow (S, \nu)$ on the space $\mathbb{D}([0,T]; \mathbb{R}^d) \times [0,T]$ for some random variable ν. Then*

$$(S^{(n)}, S^{(n)}_{\tau_n}, \tau_n) \Rightarrow (S, S_\nu, \nu) \tag{3.8}$$

on the space $\mathbb{D}([0,T]; \mathbb{R}^d) \times \mathbb{R}^d \times [0,T]$. In addition, if $S^{(n)} \Rrightarrow S$, then for any t, $\{\nu \le t\}$ and $\mathcal{F}^S_T$ are conditionally independent given $\mathcal{F}^S_t$, and so for any uniformly integrable càdlàg stochastic process $\{V_t\}_{t=0}^T$ adapted to the filtration $\mathcal{F}^S_{[0,T]}$

$$\inf_{\tau \in \mathcal{T}^S_{[0,T]}} EV_\tau \le EV_\nu \le \sup_{\tau \in \mathcal{T}^S_{[0,T]}} EV_\tau. \tag{3.9}$$

PROOF. By using the Skorohod representation theorem it follows that without loss of generality we can assume that there exists a probability space $(\Omega, \mathcal{F}, P)$

on which the process S, ν and the sequence $S^{(n)}, \tau_n$ are defined and $(S^{(n)}, \tau_n) \to (S, \nu)$ a.s. on $\mathbb{D}([0,T];\mathbb{R}^d) \times [0,T]$. Thus in order to prove (3.8) it is sufficient to show that $S^{(n)}_{\tau_n} \to S_\nu$ in probability. Choose $\varepsilon > 0$. The process $Z_t := S_{(\nu+t)\wedge T}$, $t \ge 0$ is a *càdlàg* process. It is well known (see, e.g., [4], Chapter 3) that for a *càdlàg* process the set of points for which the process is not continuous (with positive probability) is at most countable, thus there exists a sequence $u_n \downarrow 0$ such that for any n the process Z is continuous at u_n, which means that for any $\omega \in \Omega$, $\nu(\omega) + u_n$ is a continuity point of the function $S(\omega)$ provided that $\nu(\omega) + u_n < T$. Since the map $(f, t) \to f(t)$ from $\mathbb{D}([0,T];\mathbb{R}^d) \times [0,T]$ to $\mathbb{R}^d$ is continuous at (f_0, t_0) if t_0 is a continuity point of f_0 (see [4], Chapter 3) we obtain that for any $\omega \in E_1 := \{\nu < T\}$, $\lim_{n\to\infty} S^{(n)}_{(\tau_n+u_n)\wedge T} = S_\nu$. This together with Assumption 2.4 gives

$$\begin{aligned} &\lim_{n\to\infty} P\big(E_1 \cap \{|S_\nu - S^{(n)}_{\tau_n}| > 2\varepsilon\}\big) \\ &\quad \le \lim_{n\to\infty} P\big(E_1 \cap \{|S_\nu - S^{(n)}_{(\tau_n+u_n)\wedge T}| > \varepsilon\}\big) \\ &\qquad + \lim_{n\to\infty} P\{|S^{(n)}_{\tau_n} - S^{(n)}_{(\tau_n+u_n)\wedge T}| > \varepsilon\} = 0. \end{aligned} \tag{3.10}$$

Next, we deal with the event $E_2 := \{\nu = T\}$. For any $\delta > 0$ and $n \in \mathbb{N}$ set $\tau_n^{(\delta)} = \mathbb{I}_{\tau_n < T-\delta}\tau_n + \mathbb{I}_{\tau_n \ge T-\delta} T \in \mathcal{T}^{S^{(n)}}_{[0,T]}$. Observe that for any $\omega \in E_2$ there exists $N(\omega) \in \mathbb{N}$ such that for any $n > N(\omega)$, $\tau_n^{(\delta)} = T$. Since the map $f \to f(T)$ from $\mathbb{D}([0,T];\mathbb{R}^d)$ to $\mathbb{R}^d$ is continuous we obtain that for any $\delta > 0$ and $\omega \in E_2$, $\lim_{n\to\infty} S^{(n)}_{\tau_n^{(\delta)}} = S_T = S_\nu$. Thus from Assumption 2.4 we obtain

$$\begin{aligned} &\lim_{n\to\infty} P\big(B \cap \{|S_\nu - S^{(n)}_{\tau_n}| > 2\varepsilon\}\big) \\ &\quad \le \limsup_{\delta\downarrow 0} \limsup_{n\to\infty} P\big(B \cap \{|S_\nu - S^{(n)}_{\tau_n^{(\delta)}}| > \varepsilon\}\big) \\ &\qquad + \limsup_{\delta\downarrow 0} \limsup_{n\to\infty} P\{|S^{(n)}_{\tau_n} - S^{(n)}_{\tau_n^{(\delta)}}| > \varepsilon\} = 0. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11) it follows that $\lim_{n\to\infty} P\{|S_\nu - S^{(n)}_{\tau_n}| > 2\varepsilon\} = 0$ and (3.8) follows. Next, let $S^{(n)} \Rrightarrow S$ and assume without loss of generality that $(\Omega, \mathcal{F}, P)$ is large enough such that there exists a random variable H distributed uniformly on the interval $[0,1]$ and independent of $\mathcal{F}^S_T$. First we show that for any $t < T$, $\{\nu \le t\}$ and $\mathcal{F}^S_T$ are conditionally independent given $\mathcal{F}^S_t$, that is,

$$E(\mathbb{I}_{\nu\le t}|\mathcal{F}^S_t) = E(\mathbb{I}_{\nu\le t}|\mathcal{F}^S_T). \tag{3.12}$$

Fix $t < T$, a $\psi \in C(\mathbb{D}([0,T];\mathbb{R}^d))$ and $\phi \in C[0,T]$. Define the (*càdlàg*) stochastic processes $Z_u = E(\psi(S)|\mathcal{F}^S_u)$ and $Z^{(n)}_u = E(\psi(S^{(n)})|\mathcal{F}^{S^{(n)}}_u)$. Let $u_n \downarrow t$ be a sequence such that for any n the process Z is continuous at u_n. Clearly,

$$E\big[\phi(\tau_m \wedge u_n)\big(\psi(S^{(m)}) - Z^{(m)}_{u_n}\big)\big] = 0 \qquad \forall n, m \in \mathbb{N}. \tag{3.13}$$

Since $S^{(m)} \Rrightarrow S$ we obtain that for any n, $Z^{(m)}_{u_n} \Rightarrow Z_{u_n}$. Fix n. The sequence $(S^{(m)}, Z^{(m)}_{u_n}, \tau_m)$, $m \ge 1$ is tight and so from Prohorov's theorem (see [4]) it follows that there exists a subsequence $(S^{(m_k)}, Z^{(m_k)}_{u_n}, \tau_{m_k})$ which converges in law to (S, Z_{u_n}, ν). This together with (3.13) gives $E[\phi(\nu \wedge u_n)(\psi(S) - Z(u_n))] = 0$. The function $\psi \in C(\mathbb{D}([0,T];\mathbb{R}^d))$ is arbitrary, and so from density arguments it follows that for any $B \in \mathcal{F}^S_{[0,T]}$ and $n \in \mathbb{N}$, $E[\phi(\nu \wedge u_n)(\mathbb{I}_B - E(\mathbb{I}_B|\mathcal{F}^S_{u_n}))] = 0$. Since $\{\nu \le t\} = \{\nu \wedge u_n \le t\}$ and ϕ is arbitrary then by using density arguments it follows that $E[\mathbb{I}_{\nu \le t}(\mathbb{I}_B - E(\mathbb{I}_B|\mathcal{F}^S_{u_n}))] = 0$, and by letting $n \to \infty$ we obtain that for any $B \in \mathcal{F}^S_{[0,T]}$, $E[\mathbb{I}_{\nu \le t}(\mathbb{I}_B - E(\mathbb{I}_B|\mathcal{F}^S_t))] = 0$. Thus for any $B \in \mathcal{F}^S_{[0,T]}$,

$$\begin{aligned} E[(\mathbb{I}_{\nu\le t}|\mathcal{F}^S_T)\mathbb{I}_B] &= E(\mathbb{I}_{\nu \le t}\mathbb{I}_B) = E[\mathbb{I}_{\nu\le t}E(\mathbb{I}_B|\mathcal{F}^S_t)] \\ &= E[E(\mathbb{I}_{\nu\le t}|\mathcal{F}^S_t)E(\mathbb{I}_B|\mathcal{F}^S_t)] \\ &= E[E(\mathbb{I}_{\nu\le t}|\mathcal{F}^S_t)\mathbb{I}_B] \end{aligned}$$

and (3.12) follows. Next, define the stochastic process $Q_t = E(\mathbb{I}_{\nu\le t}|\mathcal{F}^S_T)$, $t \le T$. Clearly Q is a positive increasing (adapted to $\mathcal{F}^S$) *càdlàg* process and $Q_T = 1$ a.s. Set $\sigma = \inf\{t|Q_t \ge H\}$. From the definition of H we obtain

$$E(\mathbb{I}_{\sigma\le t}|\mathcal{F}^S_T) = E(\mathbb{I}_{Q_t \ge H|\mathcal{F}^S_t}) = Q_t = (\mathbb{I}_{\nu\le t}|\mathcal{F}^S_T),$$

and since V is $\mathcal{F}^S_T$ measurable, then $EV_\sigma = EV_\nu$. Finally, for any $0 \le u \le 1$ define $\sigma_u = \inf\{t|Q_t \ge u\} \in \mathcal{T}^S_{[0,T]}$. Since H is independent of V and Q, then $EV_\nu = EV_\sigma = \int_0^1 (EV_{\sigma_u})\,du$ and (3.9) follows. $\square$

4. Proof of main results. In this section we complete the proof of Theorem 2.5. Denote $\Gamma = \Gamma(S)$ and $\Gamma_n = \Gamma(S^{(n)})$, $n \ge 1$. First we prove that $\Gamma \le \lim_{n\to\infty}\Gamma_n$. Here and in the sequel, for the sake of simplicity we will assume that indices have been renamed so that the whole sequence converges. Choose $\varepsilon > 0$. Denote by $I \subset [0,T]$ the union of the point $\{T\}$ together with all continuity points of the process S. From Lemma 3.1 it follows that there exists $\tau \in \Delta^S_I$ such that

$$\Gamma(S) < \varepsilon + \inf_{\sigma \in \mathcal{F}^S_{[0,T]}} EH^S(\sigma, \tau). \tag{4.1}$$

From Lemma 3.2 we can choose a sequence of stopping times $\tau_n \in \mathcal{T}^{S^{(n)}}_{[0,T]}$, $n \ge 1$, such that $(S^{(n)}, \tau_n) \Rightarrow (S, \tau)$ on the space $\mathbb{D}([0,T];\mathbb{R}^d) \times [0,T]$. From Lemma 3.3 we obtain

$$\big(S^{(n)}, S^{(n)}_{\tau_n}, \tau_n\big) \Rightarrow (S, S_\tau, \tau) \tag{4.2}$$

on the space $\mathbb{D}([0,T];\mathbb{R}^d)\times\mathbb{R}^d\times[0,T]$. From (2.4) it follows that for any $n\in\mathbb{N}$ there exists a stopping time $\sigma_n\in\mathcal{T}^{S^{(n)}}_{[0,T]}$ such that

$$\Gamma_n > E_n H^{S^{(n)}}(\sigma_n,\tau_n)-\varepsilon. \tag{4.3}$$

The sequence $(S^{(n)},\sigma_n)$ is tight in $\mathbb{D}([0,T];\mathbb{R}^d)\times[0,T]$ and so $(S^{(n)},\sigma_n)\Rightarrow(S,\nu)$ for some random variable $\nu\le T$. From Lemma 3.3

$$\big(S^{(n)},S^{(n)}_{\sigma_n},\sigma_n\big)\Rightarrow(S,S_\nu,\nu) \tag{4.4}$$

on the space $\mathbb{D}([0,T];\mathbb{R}^d)\times\mathbb{R}^d\times[0,T]$. From (4.2) and (4.4) it follows that the sequence $(S^{(n)},S^{(n)}_{\tau_n},S^{(n)}_{\sigma_n},\tau_n,\sigma_n)$, $n\ge 1$, is tight on the space $\mathbb{D}([0,T];\mathbb{R}^d)\times\mathbb{R}^{2d}\times[0,T]^2$. Thus $(S^{(n)},S^{(n)}_{\tau_n},S^{(n)}_{\sigma_n},\tau_n,\sigma_n)\Rightarrow(S,S_\tau,S_\nu,\tau,\nu)$. By using the Skorohod representation theorem it follows that without loss of generality we can assume that there exists a probability space $(\Omega,\mathcal{F},P)$ on which $(S^{(n)},S^{(n)}_{\tau_n},S^{(n)}_{\sigma_n},\tau_n,\sigma_n)\to(S,S_\tau,S_\nu,\tau,\nu)$ a.s. on the space $\mathbb{D}([0,T];\mathbb{R}^d)\times\mathbb{R}^{2d}\times[0,T]^2$. This together with Assumption 2.1 gives

$$H^S(\nu,\tau)\le\liminf_{n\to\infty}H^{S^{(n)}}(\sigma_n,\tau_n). \tag{4.5}$$

From (4.3) and (4.5)

$$EH^S(\nu,\tau)\le\liminf_{n\to\infty}EH^{S^{(n)}}(\sigma_n,\tau_n)\le\lim_{n\to\infty}\Gamma_n+\varepsilon. \tag{4.6}$$

By applying Lemma 3.3 for the process $Q_t:=H^S(t,\tau)$ it follows

$$\inf_{\sigma\in\mathcal{F}^S_{[0,T]}}EH^S(\sigma,\tau)\le EH^S(\nu,\tau)\le\lim_{n\to\infty}\Gamma_n+\varepsilon. \tag{4.7}$$

From (4.1) and (4.7) we obtain $\Gamma\le\lim_{n\to\infty}\Gamma_n$. In order to complete the proof we prove that $\Gamma\ge\lim_{n\to\infty}\Gamma_n$. Choose $\varepsilon\ge 0$. From Lemma 3.1 there exists a stopping time $\sigma\in\Delta^S_I$ which takes values on a finite set E and satisfies

$$\Gamma(S)>\sup_{\tau\in\mathcal{F}^S_{[0,T]}}EJ^S(\sigma,\tau)-\varepsilon. \tag{4.8}$$

From Lemma 3.2 and 3.3 it follows that we can choose a sequence of stopping times $\sigma_n\in\mathcal{T}^{S^{(n)}}_{[0,T]}$, $n\ge 1$, with values in E such that

$$\big(S^{(n)},S^{(n)}_{\sigma_n},\sigma_n\big)\Rightarrow(S,S_\sigma,\sigma) \tag{4.9}$$

in law on the space $\mathbb{D}([0,T];\mathbb{R}^d)\times\mathbb{R}^d\times[0,T]$. From (2.5) it follows that for any $n\in\mathbb{N}$ there exists a stopping time $\tau_n\in\mathcal{T}^{S^{(n)}}_{[0,T]}$ such that

$$\Gamma_n<\inf_{\sigma_n\in\mathcal{T}^{S^{(n)}}_{[0,T]}}E_nJ^{S^{(n)}}(\sigma_n,\tau_n)+\varepsilon. \tag{4.10}$$

The sequence $(S^{(n)}, \tau_n)$ is tight, and thus from Lemma 3.3,

$$\big(S^{(n)}, S^{(n)}_{\tau_n}, \tau_n\big) \Rightarrow (S, S_\nu, \nu) \tag{4.11}$$

on the space $\mathbb{D}([0,T];\mathbb{R}^d) \times \mathbb{R}^d \times [0,T]$, for some $\nu \le T$. As before, by using the Skorohod representation theorem it follows that without loss of generality we can assume that there exists a probability space $(\Omega, \mathcal{F}, P)$ on which $(S^{(n)}, S^{(n)}_{\tau_n}, S^{(n)}_{\sigma_n}, \tau_n, \sigma_n) \to (S, S_\nu, S_\sigma, \nu, \sigma)$ a.s. on the space $\mathbb{D}([0,T];\mathbb{R}^d) \times \mathbb{R}^{2d} \times [0,T]^2$. Observe that if $\sigma = T$, then $\sigma_n = T$ for sufficiently large n. Thus from Assumption 2.1

$$J^S(\sigma, \nu) \ge \limsup_{n\to\infty} J^{S^{(n)}}(\sigma_n, \tau_n).$$

This together with (4.10) and Assumption 2.3 gives

$$EJ^S(\sigma, \nu) \ge \limsup_{n\to\infty} EJ^{S^{(n)}}(\sigma_n, \tau_n) \ge \lim_{n\to\infty} \Gamma_n - \varepsilon. \tag{4.12}$$

By applying Lemma 3.3 for the process $Q_t := J^S(\sigma, t)$ we obtain

$$\sup_{\tau \in \mathcal{T}^S_{[0,T]}} EJ^S(\sigma, \tau) \ge EJ^S(\sigma, \nu). \tag{4.13}$$

From (4.8), (4.12) and (4.13), $\Gamma \ge \lim_{n\to\infty} \Gamma_n$.

5. Applications to game options. In this section we give an example for an application of Theorem 2.5. We will consider discrete time approximations of game options prices in the Merton (one-dimensional) model. Approximation of American options in the Merton model were considered in [18]. Let $(\Omega, \mathcal{F}, P)$ be a probability space together with a standard Brownian motion $\{W_t\}_{t=0}^T$, a Poisson process $\{N_t\}_{t=0}^T$ with intensity λ and independent of W and a sequence of i.i.d. random variables $\{U_i\}_{i=1}^\infty$ with values in $(-1, \infty)$, independent of N and W. We assume that $EU_1 < \infty$. A Merton model with horizon $T < \infty$ consists of a savings account with an interest rate $r > 0$, and of a risky asset (stock). Assume that the discounted stock price $\{S_t\}_{t=0}^T$ [i.e., a ratio of the original stock price and $\exp(rt)$] is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{j=1}^{N_t} \ln(1 + U_j)\right), \qquad S_0, \sigma > 0, \mu = -\lambda EU_1. \tag{5.1}$$

The equality $\mu = -\lambda EU_1$ guarantees that S is a martingale with respect to P and the usual filtration $\mathcal{F}^S$. Introduce a game option with Russian payoff functions. Namely the discounted payoffs are $Y_t^S = f(t,S)$ and $X_t^S = g(t,S)$ where $f, g : [0,T] \times \mathbb{D}([0,T];\mathbb{R}) \to \mathbb{R}_+$ are given by

$$f(t,x) = e^{-rt}\max\Big(M, \sup_{0\le u\le t} e^{ru}x_u\Big) \quad \text{and}$$
$$g(t,x) = f(t,x) + \delta x_t, \qquad \delta, r, M > 0. \tag{5.2}$$

From [13] it follows that

$$V := \Gamma(S) = \inf_{\sigma\in\mathcal{T}^S_{[0,T]}} \sup_{\tau\in\mathcal{T}^S_{[0,T]}} EH^S(\sigma,\tau) \tag{5.3}$$

is an arbitrage-free price (recall that the Merton model is incomplete). Following [18] we construct a sequence of discrete time approximations. For any $n\in\mathbb{N}$ let $(\Omega_n, \mathcal{F}_n, P_n)$ be a probability space together with three independent sequences of i.i.d. random variables $\{\xi_k^{(n)}\}_{k=1}^n$, $\{\rho_k^{(n)}\}_{k=1}^n$ and $\{u_k^{(n)}\}_{k=1}^n$. The first one is a sequence of Bernoulli random variables such that $P_n\{\xi_k^{(n)} = 1\} = \frac{\lambda T}{n}$, the second sequence satisfies $P_n\{\rho_k^{(n)} = 1\} = 1 - P_n\{\rho_k^{(n)} = -1\} = \frac{n/(n+\mu\lambda T) - \exp(-\sigma\sqrt{T/n})}{\exp(\sigma\sqrt{T/n}) - \exp(-\sigma\sqrt{T/n})}$ (we assume that n is sufficiently large such that above term is positive) and the third sequence given by $u_k^{(n)} \sim \ln(1+U_1)$. For any $0\le k\le n$ and $kT/n \le t < (k+1)T/n$ set

$$W_t^{(n)} = \sqrt{\frac{T}{n}}\sum_{i=1}^k \rho_i^{(n)}, \qquad N_t^{(n)} = \sum_{i=1}^k \xi_i^{(n)} \quad \text{and}$$
$$S_t^{(n)} = S_0 \exp\Bigg(\sigma W_t^{(n)} + \sum_{i=1}^{N_t^{(n)}} u_i^{(n)}\Bigg). \tag{5.4}$$

The n-step discrete time market is active at times $\{0, \frac{T}{n}, \frac{2T}{n}, \dots, T\}$ and consists of a savings account with an interest rate $r > 0$, and of a risky asset whose discounted stock price $S^{(n)}$ is given by (5.4). Consider a game option with the discounted payoffs $Y_t^{S^{(n)}} = f(t, S^{(n)})$ and $X_t^{S^{(n)}} = g(t, S^{(n)})$. Let Δ_n be the set of all stopping times with respect to the filtration $\mathcal{F}^{S^{(n)}}$ with values in the set $\{0, \frac{T}{n}, \frac{2T}{n}, \dots, T\}$. Since the process $\{S_{kT/n}^{(n)}\}_{k=0}^n$ is a martingale under P_n it follows that V_n which is given by

$$V_n = \inf_{\sigma\in\Delta_n}\sup_{\tau\in\Delta_n} E_n H^{S^{(n)}}(\sigma,\tau) \tag{5.5}$$

is an arbitrage-free price. Next, we describe a dynamical programming algorithm which allows us to calculate V_n. For $0\le k\le n$ define the functions

$\psi_k^{(n)}, \phi_k^{(n)} : \mathbb{R}^k \times \{-1,1\}^k \times \{0,1\}^k \to \mathbb{R}_+$ by

$$
\begin{aligned}
&\psi_k^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k) \\
&\quad = \exp\left(-\frac{rkT}{n}\right) \\
&\qquad \times \max\left(M, S_0 \max_{0\le i\le k} \exp\left(\frac{r((i+1)\wedge k)T}{n}\right.\right. \\
&\qquad\qquad \left.\left. + \sigma\sqrt{\frac{T}{n}}\sum_{j=1}^{i} y_j + \sum_{j=1}^{m_i} x_j\right)\right) \quad \text{and}
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
&\phi_k^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k) \\
&\quad = \psi_k^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k) \\
&\qquad + \delta S_0 \exp\left(\sigma\sqrt{\frac{T}{n}}\sum_{i=1}^{k} y_i + \sum_{i=1}^{m_k} x_i\right),
\end{aligned}
$$

where $m_i = m_i(z_1,\ldots,z_k) = \sum_{q=1}^{i} z_q$. Since the process $S^{(n)}$ is constant on intervals of the form $[iT/n,(i+1)T/n)$, for any $0 \le k \le n$

$$
\begin{aligned}
&\psi_k^{(n)}(u_1^{(n)},\ldots,u_k^{(n)},\rho_1^{(n)},\ldots,\rho_k^{(n)},\xi_1^{(n)},\ldots,\xi_k^{(n)}) = Y_{kT/n}^{S^{(n)}} \quad \text{and} \\
&\phi_k^{(n)}(u_1^{(n)},\ldots,u_k^{(n)},\rho_1^{(n)},\ldots,\rho_k^{(n)},\xi_1^{(n)},\ldots,\xi_k^{(n)}) = X_{kT/n}^{S^{(n)}}.
\end{aligned}
\tag{5.7}
$$

Finally, define a sequence $\{J_k^{(n)}\}_{k=0}^n$ of functions $J_k^{(n)} : \mathbb{R}^k \times \{-1,1\}^k \times \{0,1\}^k \to \mathbb{R}_+$ by the following backward recursion:

$$
\begin{aligned}
&J_n^{(n)} = \psi_n^{(n)} \quad \text{and} \\
&J_k^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k) \\
&\quad = \min\big(\phi_k^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k), \\
&\qquad\quad \max\big(\psi_k^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k), \\
&\qquad\qquad E_n[J_{k+1}^{(n)}(x_1,\ldots,x_k,u_1^{(n)}, \\
&\qquad\qquad\qquad y_1,\ldots,y_k,\rho_1^{(n)},z_1,\ldots,z_k,\xi_1^{(n)})]\big)\big) \\
&\qquad\qquad\qquad\qquad \text{for } k = n-1, n-2, \ldots, 0.
\end{aligned}
\tag{5.8}
$$

From (5.7) and by using the dynamical programming algorithm that was obtained in [21] for general Dynkin's games in discrete time, it follows that

$$
V_n = J_0^{(n)}. \tag{5.9}
$$

The following result says that the option price in the continuous time Merton model can be approximated by the sequence $\{V_n\}_{n=1}^{\infty}$ (which can be calculated explicitly as shown above).

THEOREM 5.1. $V = \lim_{n\to\infty} V_n$.

PROOF. First we prove that $V = \lim_{n\to\infty} \Gamma(S^{(n)})$. Let us check that the conditions of Theorem 2.5 are satisfied. It can be easily checked that the functions f, g satisfy Assumption 2.1 and that for any $k > 1$, $\sup_{n\ge 1} E_n(S_T^{(n)})^k < \infty$. Thus from Doob's inequality we obtain that $\sup_{n\ge 1} E_n[(\sup_{0\le t\le T} S_t^{(n)})^k] < \infty$ and Assumption 2.3 follows. It is well known that $(W^{(n)}, N^{(n)}) \Rightarrow (W, N)$ on the space $\mathbb{D}([0,T];\mathbb{R}^2)$, and so by using the Skorohod representation theorem we can build a probability space on which $(W^{(n)}, N^{(n)}) \to (W, N)$ a.s. and on which there exists a sequence of i.i.d. random variables $U_1, \dots, U_n, \dots$ which is independent of $\{W^{(n)}, N^{(n)}\}_{n=1}^{\infty}, W, N$. Thus $\sigma W^{(n)} + \sum_{i=1}^{N^{(n)}} U_i \to \sigma W + \sum_{i=1}^{N} U_i$ a.s. on the space $\mathbb{D}([0,T];\mathbb{R})$ thus $\ln S^{(n)} \Rightarrow \ln S$. For any n the process $\ln S^{(n)}$ has independent increments and the process $\ln S$ is continuous in probability with independent increments. From Corollaries 1 and 2 in [9] we obtain that $\ln S^{(n)} \Rrightarrow \ln S$ and that $\ln S^{(n)}$, $n \ge 1$, satisfies Assumption 2.4, which means that $S^{(n)} \Rrightarrow S$ and $S^{(n)}$, $n \ge 1$, satisfies Assumption 2.4. We conclude that the conditions of Theorem 2.5 are satisfied, and the equality $V = \lim_{n\to\infty} \Gamma(S^{(n)})$ follows. In order to complete the proof it remains to show that

$$\lim_{n\to\infty} |V_n - \Gamma(S^{(n)})| = 0. \tag{5.10}$$

For any n define the maps $\Phi_n, \Psi_n : \mathcal{T}_{[0,T]}^{S^{(n)}} \to \Delta_n$

$$\Phi_n(\sigma) = \frac{T}{n} \max\{k | kT/n \le \sigma\} \quad \text{and} \quad \Psi_n(\sigma) = \frac{T}{n} \min\{k | kT/n \ge \sigma\}. \tag{5.11}$$

From (5.5)

$$\inf_{\sigma\in\Delta_n} \sup_{\tau\in\mathcal{T}_{[0,T]}^{S^{(n)}}} E_n H^{S^{(n)}}(\sigma, \Psi_n(\tau)) = V_n = \inf_{\sigma\in\mathcal{T}_{[0,T]}^{S^{(n)}}} \sup_{\tau\in\Delta_n} E_n H^{S^{(n)}}(\Phi_n(\sigma), \tau).$$

Thus

$$\begin{aligned}\Gamma(S^{(n)}) - V_n &\le \inf_{\sigma\in\Delta_n} \sup_{\tau\in\mathcal{T}_{[0,T]}^{S^{(n)}}} E_n H^{S^{(n)}}(\sigma, \tau) \\ &\quad - \inf_{\sigma\in\Delta_n} \sup_{\tau\in\mathcal{T}_{[0,T]}^{S^{(n)}}} E_n H^{S^{(n)}}(\sigma, \Psi_n(\tau)) \\ &\le \sup_{\tau\in\mathcal{T}_{[0,T]}^{S^{(n)}}} E_n |Y_\tau^{S^{(n)}} - Y_{\Psi_n(\tau)}^{S^{(n)}}| \quad \text{and}\end{aligned} \tag{5.12}$$

$$
\begin{aligned}
V_n - \Gamma\big(S^{(n)}\big) &\le \inf_{\sigma \in \mathcal{T}_{[0,T]}^{S^{(n)}}} \sup_{\tau \in \Delta_n} E_n H^{S^{(n)}}(\Phi_n(\sigma), \tau) \\
&\quad - \inf_{\sigma \in \mathcal{T}_{[0,T]}^{S^{(n)}}} \sup_{\tau \in \Delta_n} E_n H^{S^{(n)}}(\sigma, \tau) \\
&\le \sup_{\sigma \in \mathcal{T}_{[0,T]}^{S^{(n)}}} E_n \big| X_{\Phi_n(\sigma)}^{S^{(n)}} - X_{\sigma}^{S^{(n)}} \big|.
\end{aligned}
$$

For any $0 \le t_1, t_2 \le T$, and $u_1, u_2 \ge 0$, we have the following inequalities: $|e^{-rt_1}u_1 - e^{-rt_2}u_2| \le |u_2 - u_1| + r|t_1 - t_2| \max(u_1, u_2)$ and $|e^{rt_1}u_1 - e^{rt_2}u_2| \le e^{rT}(|u_2 - u_1| + r|t_1 - t_2| \max(u_1, u_2))$. This together with (5.12) gives

$$
\begin{aligned}
&\big|V_n - \Gamma\big(S^{(n)}\big)\big| \\
&\quad \le \sup_{\tau \in \mathcal{T}_{[0,T]}^{S^{(n)}}} E_n\big(\big|Y_{\tau}^{S^{(n)}} - Y_{\Psi_n(\tau)}^{S^{(n)}}\big| + \big|X_{\Phi_n(\tau)}^{S^{(n)}} - X_{\tau}^{S^{(n)}}\big|\big) \\
&\quad \le 2\frac{rT}{n}\Big(M + E_n \sup_{0 \le t \le T} e^{rt} S_t^{(n)}\Big) + 2e^{rT}\frac{rT}{n} E_n \sup_{0 \le t \le T} S_t^{(n)} \\
&\qquad + (\delta + 2e^{rT}) \sup_{\tau \in \mathcal{T}_{[0,T]}^{S^{(n)}}} E_n \big|S_{T \wedge (\Phi_n(\tau) + T/n)}^{(n)} - S_{\Phi_n(\tau)}^{(n)}\big|.
\end{aligned}
\tag{5.13}
$$

The sequence $S^{(n)}$, $n \ge 1$, satisfies Assumption 2.4, and so from (5.13)

$$
\begin{aligned}
&\lim_{n \to \infty} \big|V_n - \Gamma\big(S^{(n)}\big)\big| \\
&\quad \le (\delta + 2e^{rT}) \lim_{n \to \infty} \sup_{\tau \in \mathcal{T}_{[0,T]}^{S^{(n)}}} E_n \big|S_{T \wedge (\Phi_n(\tau) + T/n)}^{(n)} - S_{\Phi_n(\tau)}^{(n)}\big| \\
&\quad = 0.
\end{aligned}
\tag{5.14}
$$

□

REMARK 5.2. Similar results can be obtained for game options in the Merton model with integral payoffs (Asian options). It can be checked that integral payoffs satisfy Assumption 2.1 and by estimates in the spirit of this section we can get convergence results for this case also. Of course, put and call options can be treated even in a more simple way since their payoffs depend only on the present stock price.

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