

## MULTIFRACTAL ANALYSIS IN A MIXED ASYMPTOTIC FRAMEWORK

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Multifractal analysis of multiplicative random cascades is revisited within the framework of *mixed asymptotics*. In this new framework, the observed process can be modeled by a concatenation of independent binary cascades and statistics are estimated over a sample whose size increases as the resolution scale (or the sampling period) becomes finer. This allows one to continuously interpolate between the situation where one studies a single cascade sample at arbitrary fine scales and where, at fixed scale, the sample length (number of cascades realizations) becomes infinite. We show that scaling exponents of “mixed” partitions functions, that is, the estimator of the cumulant generating function of the cascade generator distribution depends on some “mixed asymptotic” exponent  $\chi$ , respectively, above and below two critical value  $p_{\chi}^{-}$  and  $p_{\chi}^{+}$ . We study the convergence properties of partition functions in mixed asymptotics regime and establish a central limit theorem. Moreover, within the mixed asymptotic framework, we establish a “box-counting” multifractal formalism that can be seen as a rigorous formulation of Mandelbrot’s negative dimension theory. Numerical illustrations of our results on specific examples are also provided. A possible application of these results is to distinguish data generated by log-Normal or log-Poisson models.

**1. Introduction.** Multifractal processes have been used successfully in many applications which involve series with invariance scaling properties. Well-known examples are fully developed turbulence where such processes are used to model the velocity or the dissipation energy fields [14] or finance, where they have been shown to reproduce the major “stylized facts” of return time-series [5, 7, 31]. Since pioneering works of Mandelbrot [23, 24], Kahane and Peyrière [20], a lot of mathematical studies have been devoted to multiplicative cascades, denoted in sequel as  $\mathcal{M}$ -cascades (see, e.g., [1, 3, 6, 9, 10, 12, 13, 16, 17, 22, 28, 32, 36, 37]). One of the central issues of these studies was to understand how the partition function scaling exponents [hereafter denoted as  $\tau_0(q)$ ], are related, on one hand, to the cumulant generating function of cascade weight distribution and, on the other hand, to the regularity properties of cascade samples. Actually, the goal of the multifractal formalism is to directly relate the function  $\tau_0(q)$  to the so-called singularity

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spectrum, that is, the Hausdorff dimension of the set of all the points corresponding to a given Hölder exponent. Let us mention that recently continuous versions of multiplicative cascades have been introduced [2, 4, 8]; they share most of properties with discrete cascades but do not involve any preferential scale ratio and remain invariant under time translation. In these constructions, the analog of the integral scale  $T$ , that is, the coarsest scale where the cascade iteration begins, is a correlation time.

In all the above cited references, the main results concern one single cascade over one integral scale  $T$  in the limit of arbitrary small sampling scale. However, in many applications (e.g., the above turbulence experiments) there is no reason a priori that the length of the experimental series corresponds to one (or few) integral scale(s). From a general point of view, as long as modeling a discrete (time or space) series with a cascade process is concerned, three scales are involved: (i) the resolution scale  $l$  which corresponds to the sampling period of the series, (ii) the *integral (or correlation) scale*  $T$  and (iii) the size  $L$  of the whole series. Using these notation, the total number of samples of the series is

$$N = \frac{L}{l}.$$

Therefore, when modeling a discrete series with a multifractal process, various types of asymptotics for  $N \rightarrow +\infty$  can be defined. The “high resolution asymptotics” considered in the literature, corresponds to  $l \rightarrow 0$  whereas  $L$  is fixed. On the other side, one could also consider the “infinite historic asymptotics” that corresponds to  $L \rightarrow +\infty$  whereas  $l$  is fixed. If we define  $N_T$  to be the number of integral scales involved in the series

$$N_T = \frac{L}{T}$$

and  $N_l$  the number of samples per integral scale

$$(1) \quad N_l = \frac{T}{l},$$

then we have

$$N = N_T N_l.$$

Thus, the high-resolution asymptotics corresponds to  $N_T$  fixed and  $N_l \rightarrow +\infty$  whereas the infinite historic asymptotics corresponds to  $N_l$  fixed and  $N_T \rightarrow +\infty$ . But in many applications, it is clear that since the relative values of  $N_T$  and  $N_l$  can be arbitrary, it is not obvious that one of the two mentioned asymptotics can account suitably for situation. This leads us to consider an asymptotics according to which  $N_T$  and  $N_l$  go to infinity (and therefore  $N$  goes to infinity) and at the same time preserve their relative “velocities,” that is, the ratio of their logarithm.

Some authors have already suggested the following “mixed asymptotics” [21, 29, 30]:

$$N_T = N_l^\chi,$$

where  $\chi \in \mathbb{R}^+$  is a fixed number that quantifies the relative velocities of  $N_T$  and  $N_l$ . Thus:

- $\chi = 0$  corresponds to the high resolution asymptotics,
- $\chi \rightarrow +\infty$  corresponds to the infinite historic asymptotics

and all other values are truly “mixed” asymptotics. Successful applications of the mixed asymptotics have already been performed [29, 30]. In this paper we revisit the standard problems of (i) the estimation of cascade generator cumulant generating function in the mixed asymptotic framework and of (ii) the multifractal formalism or of how to relate this function to a dimension-like quantity.

The paper is organized as follows: in Section 2 we recall basic definitions and properties of  $\mathcal{M}$ -cascades. Section 3 contains the main results of this paper. If we define a multifractal measure  $\tilde{\mu}$  as the concatenation of  $N_T$  independent  $\mathcal{M}$ -cascades of length  $T$ , with common generator law  $W$ , then we show in Theorem 2

$$\frac{1}{\log(1/l)} \log \left( N^{-1} \sum_{k=0}^{N-1} \tilde{\mu}([kl, (k+1)l])^p \right) \rightarrow p - \log_2 E[W^p] := \tau(p) + 1$$

for  $p$  in some range  $(p_\chi^-, p_\chi^+)$ . These critical exponents  $p_\chi^-, p_\chi^+$  are related to the two solutions,  $h_\chi^-, h_\chi^+$  of the equation  $D(h) = -\chi$  where

$$D(h) = \inf_p \{ph - \tau(p)\}$$

is the Legendre transform of  $\tau$ . The convergence rate is studied in Section 3.5. Let us stress that the range of validity on  $p$  of this convergence is wider in the mixed asymptotic framework ( $\chi > 0$ ) than in the high resolution asymptotic ( $\chi = 0$ ). As a consequence we can relate  $D(h)$  to a “box-counting dimension” (sometimes referred to as a box dimension [19] or a coarse-grain spectrum [34]), and derive, as stated in Theorem 3, a “box-counting multifractal formalism” for  $\tilde{\mu}$

$$\frac{1}{N_T} \#\{k \in \{0, \dots, N\} \mid \tilde{\mu}([kl, (k+1)l]) \in [l^{h-\varepsilon}, l^{h+\varepsilon}]\} \simeq l^{-D(h)}$$

in the range of  $[h_\chi^+, h_\chi^-]$ . Since for  $\chi > 0$ ,  $D(h)$  can take negative values in previous equation, this can be seen as a rigorous formulation of Mandelbrot’s negative dimension theory [25–27]. In Section 4, we extend previous results to partition functions relying on some arbitrary wavelet decomposition of the process. In Section 5 we give an interpretation of the results connected with the Besov frontier associated with our multifractal measure. Finally, in Section 6 we discuss some specific examples where the law of the cascade generator is, respectively, log-Normal, log-Poisson and log-Gamma. For illustration purpose, we also report, in each case, estimations performed from numerical simulations. Auxiliary lemmas are moved to [Appendices](#).

**2.  $\mathcal{M}$ -cascades: Definitions and properties.**

2.1. *Definition of the  $\mathcal{M}$ -cascades.* Let us first introduce some notation. Given a  $j$ -uplet  $r = (r_1, \dots, r_j)$ , for all strictly positive integers  $i \leq j$ , we note  $r|i$  the restriction of the  $j$ -uplet to its first  $i$  components, that is,

$$r|i = (r_1, \dots, r_i) \quad \forall i \in \{1, \dots, j\}.$$

By convention, if  $j = 0$ , we consider that  $r = \emptyset$  and in the sequel, we denote by  $rr'$  the  $(j + j')$ -uplet obtained by concatenation of  $r \in \{0, 1\}^j$  and  $r' \in \{0, 1\}^{j'}$ . Moreover, we note

$$\bar{r} = \begin{cases} 2^j \sum_{i=1}^j r_i 2^{-i}, & \text{if } r \neq \emptyset, \\ 0, & \text{if } r = \emptyset. \end{cases}$$

Let fix  $T \in (0, \infty)$  and  $k \in \mathbb{N}$ . We define  $I_{j,k}$  as the interval

$$(2) \quad I_{j,k} = [k2^{-j}T, (k + 1)2^{-j}T].$$

Thus, for any integer  $j \geq 1$ , the interval  $[0, T]$  can be decomposed as  $2^j$  dyadic intervals:

$$[0, T] = \bigcup_{r \in \{0,1\}^j} I_{j,\bar{r}}.$$

Let us now build the so-called  $\mathcal{M}$ -cascade measures introduced by Mandelbrot in 1974 [24]. Let  $\{W_r\}_{r \in \{0,1\}^j, j \geq 1}$  be a set of i.i.d. nonnegative random variables of mean  $\mathbb{E}[W_r] = 1$ . Given  $j \geq 1$ , we define the random measure  $\mu_j$  on  $[0, T]$  such that, for all  $r \in \{0, 1\}^j$ , the Radon–Nikodym derivative with respect to the Lebesgue measure  $\frac{d\mu_j}{dx}$  is constant on  $I_{j,\bar{r}}$  with

$$(3) \quad \frac{d\mu_j}{dx} = \prod_{i=1}^j W_{r|i} \text{ on } I_{j,\bar{r}} \quad \text{for } r \in \{0, 1\}^j.$$

As it is well known [20], the measures  $\mu_j$  have a nontrivial limit measure  $\mu_\infty$ , when  $j$  goes to  $\infty$ , as soon as  $\mathbb{E}[W \log_2 W] < 1$ . Moreover, the total mass

$$\mu_\infty([0, T]) = \lim_{j \rightarrow \infty} T2^{-j} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}$$

satisfies  $\mathbb{E}[\mu_\infty([0, T])] = T$ . Let us remark that if  $r \in \{0, 1\}^j$  then by construction we have

$$(4) \quad \begin{aligned} \mu_\infty(I_{j,\bar{r}}) &= \lim_{n \rightarrow \infty} T2^{-j} \prod_{i=1}^j W_{r|i} \left( \sum_{r' \in \{0,1\}^n} 2^{-n} \prod_{i=1}^n W_{rr'|(j+i)} \right) \\ &= 2^{-j} \left( \prod_{i=1}^j W_{r|i} \right) \bar{\mu}_\infty^{(r)}([0, T]), \end{aligned}$$

where  $\overline{\mu}_\infty^{(r')}$  is a  $\mathcal{M}$ -cascade measure on  $[0, T]$  based on the random variables  $W_{rr'}$  for  $r' \in \bigcup_{j \geq 1} \{0, 1\}^j$ . This equality is usually referred to as ‘‘Mandelbrot star equation.’’

In the sequel we need the following set of assumptions:

- (5)  $\mathbb{E}[W \log_2 W] < 1, \quad \mathbb{P}(W = 1) < 1,$
- (6)  $\mathbb{P}(W > 0) = 1, \quad \mathbb{E}[W^p] < \infty \quad \text{for all } p \in \mathbb{R}.$

Let  $\tau(p)$  be the smooth and concave function defined on  $\mathbb{R}$  by

$$(7) \quad \tau(p) = p - \log_2 \mathbb{E}[W^p] - 1.$$

Let us notice that  $\log_2 \mathbb{E}[W^p]$  is nothing but the cumulant generating function (log-Laplace transform) of the logarithm of cascade generator distribution. It is shown in [20] that for  $p > 1$ , the condition  $\tau(p) > 0$  implies the finiteness of  $\mathbb{E}[\mu_\infty([0, T])^p]$ . By Theorem 4 in [28], the conditions (6) imply the existence of finite negative moments  $\mathbb{E}[\mu_\infty([0, T])^p]$ , for all  $p < 0$ .

2.2. *Multifractal properties of  $\mathcal{M}$ -cascades.* A  $\mathcal{M}$ -cascade is a multifractal measure and the study of its multifractal properties reduces to the study of the *partition function*

$$(8) \quad \mathcal{S}_\mu(j, p) = \sum_{k=0}^{2^j-1} \mu_\infty(I_{j,k})^p.$$

Basically, one can show [28, 32] that, for fixed  $p$ , this partition function behaves, when  $j$  goes to  $\infty$ , as a power law function of the scale  $|I_{j,k}| = T2^{-j}$ . More precisely, let us introduce the two following critical exponents:

$$p_0^+ = \inf\{p \geq 1 \mid p\tau'(p) - \tau(p) \leq 0\} \in (1, \infty],$$

$$p_0^- = \sup\{p \leq 0 \mid p\tau'(p) - \tau(p) \leq 0\} \in [-\infty, 0).$$

If  $p_0^+$  (resp.,  $p_0^-$ ) is finite we set  $h_0^+ = \tau'(p_0^+)$  [resp.,  $h_0^- = \tau'(p_0^-)$ ].

**THEOREM 1** (Scaling of the partition function [32]). *Let  $p \in \mathbb{R}$ , the power law scaling exponent of  $\mathcal{S}_\mu(j, p)$  is given by*

$$(9) \quad \lim_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_\mu(j, p)}{-j} \xrightarrow{a.s.} \tau_0(p),$$

where  $\tau_0(p)$  is defined by

$$(10) \quad \tau_0(p) = \begin{cases} \tau(p), & \forall p \in (p_0^-, p_0^+), \\ h_0^+ p, & \forall p \geq p_0^+, \\ h_0^- p, & \forall p \leq p_0^-. \end{cases}$$

The proof can be found in [32] (see also related results in [9, 28]). This theorem basically states that  $\mathcal{S}_\mu(j, p)$  behaves like

$$\mathcal{S}_\mu(j, p) \simeq 2^{-j\tau_0(p)}.$$

Let us note that the partition function (8) can be rewritten in the following way:

$$(11) \quad \mathcal{S}_\mu(j, p) = \sum_{r \in \{0,1\}^j} \mu_\infty(I_{j,\bar{r}})^p,$$

and using (4), one gets

$$(12) \quad \mathcal{S}_\mu(j, p) = 2^{-jp} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}^p (\bar{\mu}_\infty^{(r)}([0, T]))^p,$$

where the  $\{\bar{\mu}_\infty^{(r)}([0, T])\}_{r \in \{0,1\}^j}$  are i.i.d. random variables with the same law as  $\mu_\infty([0, T])$ . Thus, a simple computation shows that

$$\mathbb{E}[\mathcal{S}_\mu(j, p)] = 2^{-jp} 2^j \mathbb{E}[W^p]^j \mathbb{E}[\mu_\infty([0, T])^p] = 2^{-j\tau(p)} \mathbb{E}[\mu_\infty([0, T])^p].$$

One sees that the last theorem states that, in the case  $p \in [p_0^-, p_0^+]$ ,  $\mathcal{S}_\mu(j, p)$  scales as its mean value.

On the other hand, the fact that for  $p \notin [p_0^-, p_0^+]$  the partition function scales as given in (10) instead of scaling as its mean value, is referred to as the ‘‘linearization effect.’’ A possible explanation of this effect is that for  $p$  larger than the critical exponents  $p_0^+$  (resp., smaller than  $p_0^-$ ), the sum involved in the partition function (11) is dominated by its supremum (resp., infimum) term. Thus one should not expect a law of large numbers to hold for the behavior of this sum. Another possible interpretation of this theorem in the case  $p > p_0^+$  is given in Section 5.

### 3. Mixed asymptotics for $\mathcal{M}$ -cascades.

3.1. *Mixed asymptotics: Definitions and notation.* A convenient way to construct a multifractal measure on  $\mathbb{R}_+$ , with an integral scale equal to  $T$ , is to patch independent realizations of  $\mathcal{M}$ -cascades measures. More precisely, consider  $\{\mu_\infty^{(m)}\}_{m \in \mathbb{N}}$  a sequence of i.i.d.  $\mathcal{M}$ -cascades on  $[0, T]$  as defined in Section 2.1, and define the stochastic measure on  $[0, \infty)$  by

$$(13) \quad \tilde{\mu}([t_1, t_2]) = \sum_{m=0}^{+\infty} \mu_\infty^{(m)}([t_1 - mT, t_2 - mT]) \quad \text{for all } 0 \leq t_1 \leq t_2.$$

This model is entirely defined as soon as both  $T$  and the law of  $W$  are fixed. The discretized time model for the  $N$  samples of the series is  $\{\tilde{\mu}[kl, (k + 1)l]\}_{0 \leq k < N-1}$ .

3.2. *Scaling properties.* In this section, we study the partition function for the measure  $\tilde{\mu}$  as defined in (13) in the mixed asymptotic limit.  $T$  is fixed, we choose the sampling step

$$l = T2^{-j}, \quad N_l = 2^j,$$

and the number of integral scales is related with the sampling step as

$$N_T = \lfloor N_l^\chi \rfloor \sim 2^{j\chi}$$

with  $\chi > 0$  fixed. According to (1), one gets for the total number of data

$$N = N_T 2^j \sim 2^{j(1+\chi)}.$$

The mixed asymptotics corresponds to the limit  $j \rightarrow +\infty$ . The partition function of  $\tilde{\mu}$  can be written as [recall (2)]

$$(14) \quad \mathcal{S}_{\tilde{\mu}}(j, p) = \sum_{k=0}^{N-1} \tilde{\mu}(I_{j,k})^p$$

$$(15) \quad = \sum_{m=0}^{N_T-1} \mathcal{S}_{\mu}^{(m)}(j, p),$$

where  $\mathcal{S}_{\mu}^{(m)}(j, p)$  is the partition function of  $\mu_{\infty}^{(m)}$ , that is,

$$(16) \quad \mathcal{S}_{\mu}^{(m)}(j, p) = \sum_{k=0}^{2^j-1} \mu_{\infty}^{(m)}(I_{j,k})^p.$$

Let us state the results of this section. We introduce the two critical exponents in the mixed asymptotic framework

$$(17) \quad p_{\chi}^{+} = \inf\{p \geq 1 \mid p\tau'(p) - \tau(p) \leq -\chi\} \in (1, \infty],$$

$$(18) \quad p_{\chi}^{-} = \sup\{p \leq 0 \mid p\tau'(p) - \tau(p) \leq -\chi\} \in [-\infty, 0),$$

and when these critical exponents are finite we set  $h_{\chi}^{+} = \tau'(p_{\chi}^{+})$ ,  $h_{\chi}^{-} = \tau'(p_{\chi}^{-})$ .

**THEOREM 2** (Scaling of the partition function in a mixed asymptotics). *Let  $\tilde{\mu}$  be the random measure defined by (13) where the law of  $W$  satisfies (5) and (6). We assume that, either  $p_{\chi}^{+} < \infty$  with  $\tau(p_{\chi}^{+}) > 0$ , or  $p_{\chi}^{+} = \infty$  with  $\tau(p) > 0$  for all  $p > 1$ .*

(1) *For all  $p \in \mathbb{R}$ , the power law scaling of  $\mathcal{S}_{\tilde{\mu}}(j, p)$  is given by*

$$(19) \quad \lim_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_{\tilde{\mu}}(j, p)}{-j} \xrightarrow{a.s.} \tau_{\chi}(p),$$

where  $\tau_\chi(p)$  is defined by

$$\tau_\chi(p) = \begin{cases} \tau(p) - \chi, & \forall p \in (p_\chi^-, p_\chi^+), \\ h_\chi^+ p, & \forall p \geq p_\chi^+, \\ h_\chi^- p, & \forall p \leq p_\chi^-. \end{cases}$$

Moreover the convergence (19) is uniform with respect to values of  $p$  restricted to any compact subset of  $\mathbb{R}$ .

(2) If  $p_\chi^+ < \infty$ , one has

$$(20) \quad \lim_{j \rightarrow +\infty} \frac{\log_2 \sup_{k \in [0, N-1]} \tilde{\mu}(I_{j,k})}{-j} = h_\chi^+ \quad \text{almost surely,}$$

and if  $p_\chi^- > -\infty$ , one has

$$(21) \quad \lim_{j \rightarrow +\infty} \frac{\log_2 \inf_{k \in [0, N-1]} \tilde{\mu}(I_{j,k})}{-j} = h_\chi^- \quad \text{almost surely.}$$

REMARK 1. In the case  $p_\chi^+ < \infty$ , the assumption  $\tau(p_\chi^+) > 0$  is stated in Theorem 2 to insure  $\mathbb{E}[\mu_\infty([0, T]^p)] < \infty$  for  $p \in [0, p_\chi^+)$ . If  $p_\chi^+ = \infty$  then we ask  $\tau(p) > 0$  for all  $p > 1$ , and hence the cascade measure has finite moments of any positive orders. Actually, this can only happens if  $W$  is bounded by 2. Such assumption was not needed in Theorem 1, since one can check that necessarily  $\tau(p_0^+) > 0$ .

REMARK 2. Let us stress that the behavior of the partition function is largely affected by the choice of a mixed asymptotic: the ‘‘linearization effect’’ now occurs for  $p$  in the set  $(-\infty, p_\chi^-) \cup (p_\chi^+, \infty)$ , which is smaller when  $\chi$  increases.

Equations (20) and (21) show that when the ‘‘linearization effect’’ occurs, the scaling of the partition function (14) is governed by its supremum and infimum terms for, respectively, large positive and negative  $p$  values.

This theorem will be proved in three parts. In Section 3.3.2, we will prove equation (19) of Theorem 2 only for  $p \in (p_\chi^-, p_\chi^+)$ . In Section 3.3.3, we will prove the case  $p \notin (p_\chi^-, p_\chi^+)$  and (20) and (21) is shown in Section 3.3.4 to be a simple corollary of this last case.

3.3. Proof of Theorem 2. Although Theorem 2 generalizes the results of [32] to the case  $\chi \neq 0$ , our method differs with the one used in this paper. First we need an auxiliary result which is helpful in the sequel.

3.3.1. *Limit theorem for a rescaled cascade.* For each  $m$  we denote as  $(W_r^{(m)})_{r \in \cup_j \{0,1\}^j}$  the set of i.i.d. random variables used for the construction of the measure  $\mu_\infty^{(m)}$ . Moreover we assume that for each  $m \geq 0, j \geq 0, r \in \{0, 1\}^j$  we are given a random variable  $Z^{(m,r)}$ , measurable with respect to the sigma-field  $\sigma(W_{r'r'}^{(m)} \mid r' \in \cup_j \{0, 1\}^j)$ . We make the assumption that the law of  $Z^{(m,r)}$  does not depend on  $(m, r)$ , and denote by  $Z$  a variable with this law.

Let us consider the quantities, for  $p \in \mathbb{R}$

$$(22) \quad \mathcal{M}_j^{(m)}(p) = 2^{-jp} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j (W_{r|i}^{(m)})^p Z^{(m,r)}$$

and

$$(23) \quad \mathcal{N}_j(p) = \sum_{m=0}^{N_T-1} \mathcal{M}_j^{(m)}(p).$$

PROPOSITION 1. Assume that for some  $\epsilon > 0, \mathbb{E}[|Z|^{1+\epsilon}] < \infty$  and  $-p \times \tau'(p) + \tau(p) < \chi$ , then

$$2^{j(\tau(p)-\chi)} \mathcal{N}_j(p) \xrightarrow{j \rightarrow \infty} \mathbb{E}[Z] \quad \text{almost surely.}$$

PROOF. From (22) and (23) and definition (7) we get

$$\begin{aligned} \mathbb{E}[\mathcal{N}_j(p)] &= N_T 2^j 2^{-jp} \mathbb{E}[W^p]^j \mathbb{E}[Z] \\ &\stackrel{j \rightarrow \infty}{\sim} 2^{j\chi} 2^{-j\tau(p)} \mathbb{E}[Z]. \end{aligned}$$

Hence the proposition will be proved if we show

$$(24) \quad 2^{j(\tau(p)-\chi)} (\mathcal{N}_j(p) - \mathbb{E}[\mathcal{N}_j(p)]) \xrightarrow{j \rightarrow \infty} 0 \quad \text{almost surely.}$$

For an arbitrary small  $\epsilon > 0$ , we study the  $L^{1+\epsilon}(\mathbb{P})$  norm of the difference. Set

$$(25) \quad L_N^{1+\epsilon} = \mathbb{E}[|\mathcal{N}_j(p) - \mathbb{E}[\mathcal{N}_j(p)]|^{1+\epsilon}].$$

Applying successively Lemmas 1 and 2 of Appendix A, we get

$$\begin{aligned} L_N^{1+\epsilon} &\leq C 2^{j\chi} \mathbb{E}[|\mathcal{M}_j^{(0)}(p)|^{1+\epsilon}] \\ &\leq C 2^{-j[(1+\epsilon)\tau(p)-\chi]} \sum_{k=0}^j 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)}. \end{aligned}$$

We deduce that  $2^{j(\tau(p)-\chi)(1+\epsilon)} L_N^{1+\epsilon}$  is bounded by the quantity

$$C 2^{-j\chi\epsilon} \sum_{k=0}^j 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)}.$$

Clearly, as soon as  $2^{-\chi\epsilon}2^{-\tau(p(1+\epsilon))}2^{(1+\epsilon)\tau(p)} < 1$ , this quantity is, in turn, bounded by  $C2^{-j\chi\epsilon'}$  for some  $\epsilon' > 0$ . Taking the log, a sufficient condition is

$$\frac{\tau(p)(1 + \epsilon) - \tau(p(1 + \epsilon))}{\epsilon} < \chi,$$

which is implied for  $\epsilon$  small enough by

$$-p\tau'(p) + \tau(p) < \chi.$$

Thus we have shown that  $2^{j(\tau(p)-\chi)(1+\epsilon)}L_N^{1+\epsilon}$  is asymptotically smaller than  $2^{-j\epsilon'}$  with some  $\epsilon' > 0$ . Using the Bienaymé–Chebyshev inequality leads to

$$\mathbb{P}\{2^{j(\tau(p)-\chi)}|\mathcal{N}_j(p) - \mathbb{E}[\mathcal{N}_j(p)]| \geq \eta\} \leq \frac{2^{j(\tau(p)-\chi)(1+\epsilon)}L_N^{1+\epsilon}}{\eta^{1+\epsilon}} \leq \frac{C2^{-j\epsilon'}}{\eta^{1+\epsilon}}$$

for any  $\eta > 0$ . A simple use of the Borel–Cantelli lemma shows (24).  $\square$

3.3.2. *Proof of Theorem 2 for  $p \in (p_\chi^-, p_\chi^+)$ .* From (15) and (16) and the representation (12) for the partition function of a single cascade, we see that that  $\mathcal{S}_{\tilde{\mu}}(j, p)$  exactly has the same structure as the quantity  $\mathcal{N}_j(p)$  of Section 3.3.1 where  $Z^{(m,r)} = \bar{\mu}_\infty^{(m,r)}([0, T])^p$  are random variables distributed as  $Z = \mu_\infty([0, T])^p$ .

By definition [recall (17) and (18)], the condition  $-p\tau'(p) + \tau(p) < \chi$  holds for any  $p \in (p_\chi^-, p_\chi^+)$ , and by Remark 1,  $\mathbb{E}[|Z|^{1+\epsilon}] < \infty$  for  $\epsilon$  small enough.

Thus, an application of Proposition 1 yields the almost sure convergence

$$(26) \quad 2^{j\tau_\chi(p)}\mathcal{S}_{\tilde{\mu}}(j, p) = 2^{j(\tau(p)-\chi)}\mathcal{S}_{\tilde{\mu}}(j, p) \xrightarrow{j \rightarrow \infty} \mathbb{E}[\mu_\infty([0, T])^p].$$

This proves Theorem 2 for the case  $p \in (p_\chi^-, p_\chi^+)$ .

3.3.3. *Proof of Theorem 2 for  $p \notin (p_\chi^-, p_\chi^+)$ .* The following proof is an adaptation of the corresponding proof in [33]. We need the following notation:

$$\begin{aligned} \mathcal{S}_{\tilde{\mu}}^*(j) &= \sup_{k \in [0, N-1]} \tilde{\mu}_\infty([k2^{-j}T, (k+1)2^{-j}T]), \\ m_{\text{sup}}(p) &= \limsup_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_{\tilde{\mu}}(j, p)}{-j}, & m_{\text{inf}}(p) &= \liminf_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_{\tilde{\mu}}(j, p)}{-j}, \\ m_{\text{sup}}^* &= \limsup_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_{\tilde{\mu}}(j)^*}{-j}, & m_{\text{inf}}^* &= \liminf_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_{\tilde{\mu}}(j)^*}{-j}. \end{aligned}$$

In Section 3.3.2 we proved that for all  $p \in (p_\chi^-, p_\chi^+)$  the following holds almost surely:

$$m_{\text{sup}}(p) = m_{\text{inf}}(p) = \tau_\chi(p).$$

We may assume that in an event which has probability one, this equality holds for all  $p$  in a countable and dense subset of  $(p_\chi^-, p_\chi^+)$ .

From the sub-additivity of  $x \mapsto x^\rho$ ,

$$\forall \rho \in ]0, 1[, \forall p \in \mathbb{R} \quad \mathcal{S}_{\tilde{\mu}}(p, j)^\rho \leq \mathcal{S}_{\tilde{\mu}}(\rho p, j),$$

and thus

$$m_{\text{inf}}(p) \geq \frac{m_{\text{inf}}(\rho p)}{\rho}.$$

But we have seen that  $m_{\text{inf}}(\rho p) = \tau_\chi(\rho p)$ , for a dense subset of  $\rho p \in (p_\chi^-, p_\chi^+)$ . Assume now for simplicity that  $p \geq p_\chi^+$  and let  $\rho \rightarrow (p_\chi^+/p)$ , we get

$$(27) \quad \forall p \geq p_\chi^+ \quad \frac{m_{\text{inf}}(p)}{p} \geq \frac{\tau_\chi(p_\chi^+)}{p_\chi^+} = \frac{\tau(p_\chi^+) - \chi}{p_\chi^+} = h_\chi^+,$$

where we have used (17).

On the other hand, let  $p > 0, q \in [0, p_\chi^+)$ , and  $q' \in [0, q)$ , we have

$$\begin{aligned} \mathcal{S}_{\tilde{\mu}}(j, q) &= \sum_{k=0}^{N-1} \tilde{\mu}_\infty([k2^{-j}T, (k+1)2^{-j}T])^q \\ &\leq \mathcal{S}_{\tilde{\mu}}^*(j)^{q-q'} \mathcal{S}_{\tilde{\mu}}(j, q') \\ &\leq \mathcal{S}_{\tilde{\mu}}(j, p)^{(q-q')/p} \mathcal{S}_{\tilde{\mu}}(j, q'). \end{aligned}$$

Thus

$$m_{\text{sup}}(q) \geq (q - q') \frac{m_{\text{sup}}(p)}{p} + m_{\text{inf}}(q'),$$

then

$$\frac{m_{\text{sup}}(p)}{p} \leq \frac{m_{\text{sup}}(q) - m_{\text{inf}}(q')}{q - q'} = \frac{\tau_\chi(q) - \tau_\chi(q')}{q - q'}.$$

Taking the limit  $q' \rightarrow q^-$

$$\frac{m_{\text{sup}}(p)}{p} \leq \inf_{q \in [0, p_\chi^+)} \tau'_\chi(q) \leq \tau'_\chi(p_\chi) = h_\chi^+.$$

Merging this last relation with (27) leads to

$$(28) \quad \forall p \geq p_\chi^+ \quad h_\chi^+ \leq \frac{m_{\text{inf}}(p)}{p} \leq \frac{m_{\text{sup}}(p)}{p} \leq h_\chi^+,$$

which proves Theorem 2 for  $p \in [p_\chi^+, +\infty[$ . The proof for  $p \leq p_\chi^-$  is similar. Finally, it remains to show that the convergence is uniform for  $p$  in a compact set. This is obtained by remarking that if a sequence of concave functions converges pointwise (on a dense set) to a continuous concave function, then the convergence is necessarily uniform on compact sets.

3.3.4. *Proof of (20) and (21).* The following proof is an adaptation of the corresponding proof in [33]. We have for  $p > 0$ ,

$$\mathcal{S}_{\tilde{\mu}}^*(j)^p \leq \mathcal{S}_{\tilde{\mu}}(j, p) \leq N \mathcal{S}_{\tilde{\mu}}^*(j)^p = \lfloor 2^{j\chi} \rfloor 2^j \mathcal{S}_{\tilde{\mu}}^*(j)^p,$$

thus

$$pm_{\text{inf,sup}}^* \geq m_{\text{inf,sup}}(p) \geq 1 + \chi + pm_{\text{inf,sup}}^*,$$

which means that

$$\frac{m_{\text{inf,sup}}(p)}{p} - \frac{1 + \chi}{p} \geq m_{\text{inf,sup}}^* \geq \frac{m_{\text{inf,sup}}(p)}{p}$$

and taking the limit  $p \rightarrow +\infty$  and using (28) proves that

$$m_{\text{sup}}^* = m_{\text{inf}}^* = h_{\chi}^+,$$

which proves (20). The proof of (21) is obtained analogously by considering  $p < 0$ .

3.4. *Multifractal formalism and “negative dimensions.”* Let  $D(h)$  be the Legendre transform of  $\tau(p)$

$$D(h) = \min_p (ph - \tau(p)).$$

The *multifractal formalism* [15] gives an interesting interpretation of  $D(h)$ , as soon as  $D(h) > 0$ , in terms of dimension of set of points with the same regularity. For  $\mathcal{M}$ -cascades, this formalism holds [28], that is,  $D(h)$  corresponds to the Hausdorff dimension of the points  $t \in [0, T]$  around which  $\mu_{\infty}$  scales with the exponent  $h$

$$(29) \quad D(h) = \dim_H \left\{ t, \limsup_{\epsilon \rightarrow 0} \frac{\log_2 \mu_{\infty}([t - \epsilon, t + \epsilon])}{\log_2(\epsilon)} = h \right\}.$$

The right-hand side of (29) is usually referred to as the *singularity spectrum* and therefore the multifractal formalism simply states that  $D(h)$  can be identified with the singularity spectrum of the cascade. A consequence of the multifractal formalism is that the statistical distribution of the singularities can be predicted from  $D(h)$ . In a mixed asymptotic framework, our next result shows that this kind of multifractal formalism still holds for  $D(h) < 0$  in the sense that  $D(h)$  governs the behavior of the population histogram in proportion to the number of cascade samples. In other words,  $D(h)$  coincides with a “latent” box-counting dimension (sometimes referred to as a box dimension [19] or a coarse-grain spectrum [34]). Hence the Legendre transform of  $\tau(p)$  can be interpreted as a “population” dimension even for singularity values above and below  $h_0^+$  and  $h_0^-$ . Since for these values, one has  $D(h) < 0$ , they have been called “negative dimensions” by Mandelbrot [26]. This simply means that they cannot be observed on a single cascade sample; rather one needs at least  $2^{j\chi}$  realizations to observe them with a “cardinality” like  $2^{j(\chi + D(h))}$ . In that respect, they have also been referred to as “latent” singularities [27].

**THEOREM 3.** Assume  $p_\chi^+ < \infty$ ,  $p_\chi^- > -\infty$  and  $\tau(p_\chi^+) > 0$ . Let  $h \in (h_\chi^+, h_\chi^-)$ , then

$$(30) \quad \lim_{\varepsilon \rightarrow 0} \lim_j \frac{1}{j} \log_2 \#\{k \in \{0, \dots, N-1\} \mid 2^{-j(h+\varepsilon)} \leq \tilde{\mu}(I_{j,k}) \leq 2^{-j(h-\varepsilon)}\} \\ = \chi + D(h),$$

$$(31) \quad \lim_{\varepsilon \rightarrow 0} \overline{\lim}_j \frac{1}{j} \log_2 \#\{k \in \{0, \dots, N-1\} \mid 2^{-j(h+\varepsilon)} \leq \tilde{\mu}(I_{j,k}) \leq 2^{-j(h-\varepsilon)}\} \\ = \chi + D(h).$$

**PROOF.** The proof of the theorem relies on an application of the Gärtner–Ellis theorem to a well chosen empirical measure. Denote  $\nu_j$  the probability on  $\mathbb{R}$  defined as the sum of Dirac masses

$$\nu_j(dx) = \frac{1}{N_T 2^j} \sum_{m=0}^{N_T-1} \sum_{k=0}^{2^j-1} \delta_{\{\log(\mu_\infty^{(m)}(I_{j,k}))\}} = \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\{\log(\tilde{\mu}(I_{j,k}))\}}.$$

Clearly, the log Laplace transform of the measure  $\nu_j$  is related to the partition function of  $\tilde{\mu}$  in the following way:

$$\log_2 \int_{\mathbb{R}} e^{px} \nu_j(dx) = \log_2 \mathcal{S}_{\tilde{\mu}}(j, p) - \log_2(N_T) - j.$$

Then, Theorem 2 implies that on a set of probability one, the following convergence holds for all  $p \in \mathbb{R}$ :

$$\frac{1}{j} \log_2 \int_{\mathbb{R}} e^{px} \nu_j(dx) \xrightarrow{j \rightarrow \infty} -\tau_\chi(p) - \chi - 1 := \Lambda_\chi(p).$$

Now, on the set where this convergence holds, we apply the Gärtner–Ellis theorem [11] to the sequence of probability measures  $(\nu_j)_j$ . This yields that, almost surely, the following large deviation inequalities hold for any set  $A \subset \mathbb{R}$ :

$$(32) \quad \underline{\lim}_j \frac{1}{j} \log_2 \int_{\mathbb{R}} 1_{\{x/j \in A\}} \nu_j(dx) \geq - \inf_{a \in A} \Lambda_\chi^*(a),$$

$$(33) \quad \overline{\lim}_j \frac{1}{j} \log_2 \int_{\mathbb{R}} 1_{\{x/j \in A\}} \nu_j(dx) \leq - \inf_{a \in A} \Lambda_\chi^*(a).$$

Let  $h \in (h_\chi^+, h_\chi^-)$  and specify  $A = [-h - \varepsilon, -h + \varepsilon]$ , then the quantity

$$N \int_{\mathbb{R}} 1_{\{x/j \in A\}} \nu_j(dx)$$

is the cardinality of the set

$$\{k \in \{0, \dots, N-1\} \mid 2^{-j(h+\varepsilon)} \leq \tilde{\mu}(I_{j,k}) \leq 2^{-j(h-\varepsilon)}\}.$$

Meanwhile, if  $a \in (-h_\chi^-, -h_\chi^+)$  simple considerations on the Legendre transform show that  $\Lambda_\chi^*(a) = -D(-a) + 1$ . Hence if  $A = [-h - \varepsilon, -h + \varepsilon] \subset (-h_\chi^-, -h_\chi^+)$  the right-hand side in the Gärtner–Ellis theorem reads

$$-\inf_{a \in [-h-\varepsilon, -h+\varepsilon]} (-D(-a) + 1),$$

which converges to  $D(h) - 1$  as  $\varepsilon \rightarrow 0$ . Now, we deduce (30) and (31) from (32) and (33).  $\square$

3.5. *Central limit theorems.* In this section, we briefly study the rate of the convergence of  $\mathcal{S}_{\bar{\mu}}(j, p)$  as  $j \rightarrow \infty$  of (19) in Theorem 2. Using the same notation as in the proof of Theorem 2, we write

$$(34) \quad \mathcal{S}_{\bar{\mu}}(j, p) = \lfloor 2^{j\chi} \rfloor 2^{-j\tau(p)} \mathbb{E}[\mu_\infty([0, T])^p] + A_j + B_j,$$

where

$$A_j = \sum_{m=0}^{N_T-1} 2^{-jp} \sum_{r \in \{0,1\}^j} \left( \prod_{i=1}^j W_{r|i}^{(m)p} \right) (\bar{\mu}_\infty^{(m,r)}([0, T])^p - \mathbb{E}[\mu_\infty([0, T])^p])$$

and

$$B_j = \sum_{m=0}^{N_T-1} \left( 2^{-jp} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}^{(m)p} - 2^{-j\tau(p)} \right) \mathbb{E}[\mu_\infty([0, T])^p].$$

PROPOSITION 2. Assume (5) and (6) and that, either  $p_\chi^+ < \infty$  with  $\tau(p_\chi^+) > 0$ , or  $p_\chi^+ = \infty$  with  $\tau(p) > 0$  for all  $p > 1$ . If  $p_\chi^-/2 < p < p_\chi^+/2$ , then

$$2^{j(\tau(2p)-\chi)/2} A_j \xrightarrow{j \rightarrow \infty} \mathcal{N}(0, \text{Var}(\mu_\infty([0, T])^p)).$$

PROOF. Consistently with the notation of Section 3.3.1, we define, for every  $r \in \{0, 1\}^j$  and  $m = 0, \dots, N_T - 1$ , the random variables  $\tilde{Z}^{(m,r)} = \bar{\mu}_\infty^{(m,r)}([0, T])^p - \mathbb{E}[\mu_\infty([0, T])^p]$  and denote by  $\tilde{Z} = \mu_\infty([0, T])^p - \mathbb{E}[\mu_\infty([0, T])^p]$  their common law. Furthermore, we will need the quantity

$$(35) \quad \eta_{m,r,j}(p) = 2^{j(\tau(2p)-\chi)/2} 2^{-jp} \left( \prod_{i=1}^j W_{r|i}^{(m)p} \right) \tilde{Z}^{(m,r)},$$

and the following family of  $\sigma$ -fields: for  $j \geq 0$

$$\mathcal{F}_{-1,j} := \sigma(W_r^{(m)}, |r| \leq j, m = 0, \dots, N_T - 1)$$

and for every  $k = 0, \dots, n(j) = 2^j(N_T - 1)$

$$\mathcal{F}_{k,j} = \mathcal{F}_{-1,j} \vee \sigma(\tilde{Z}^{(m,r)}, \bar{r} + 2^j m \leq k).$$

For fixed  $j$ , we have a one-to-one correspondence between  $(m, r)$  and  $k = \bar{r} + 2^j m$ , so abusing notation slightly, we write  $\eta_{k,j}(p)$  instead of  $\eta_{m,r,j}(p)$  in (35) when no confusion is possible. With the notation

$$2^{(\tau(2p)-\chi)/2} A_j = \sum_{k=0}^{n(j)} \eta_{k,j}(p),$$

where  $\eta_{k,j}(p)$  is  $\mathcal{F}_{k,j}$ -measurable and

$$\mathbb{E}[\eta_{k,j}(p) \mid \mathcal{F}_{k-1,j}] = 0 \quad \forall k = 0, \dots, n(j).$$

Thus, we are dealing with a triangular array of martingale increments. Let us consider the sum of the conditional variances,

$$(36) \quad V_j = \sum_{k=0}^{n(j)} \mathbb{E}[\eta_{k,j}(p)^2 \mid \mathcal{F}_{k-1,j}].$$

We have

$$V_j = \text{Var}(\tilde{Z}) 2^{j(\tau(2p)-\chi)} 2^{-2jp} \sum_{m=0}^{N_T-1} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j (W_{r|i}^{(m)})^{2p};$$

thus by application of Proposition 1 (with the choice of  $Z^{(m,r)}$  equal to 1) we get

$$V_j \xrightarrow{j \rightarrow \infty} \text{Var}(\tilde{Z}).$$

Hence the proposition will be proved if we can show that the triangular array satisfies a Lindeberg condition, for some  $\epsilon > 0$ ,

$$V_j^{(\epsilon)} = \sum_{k=0}^{n(j)} \mathbb{E}[|\eta_{k,j}(p)|^{2+\epsilon} \mid \mathcal{F}_{k-1,j}] \xrightarrow{j \rightarrow \infty} 0.$$

But, we have

$$V_j^{(\epsilon)} = \mathbb{E}[|W|^{2+\epsilon}] 2^{j(\tau(2p)-\chi)(1+\epsilon/2)} 2^{-j(2+\epsilon)p} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j (W_{r|i}^{(m)})^{2+\epsilon}$$

and by application of the Proposition 1, the order of magnitude of  $V_j^{(\epsilon)}$  is  $2^{j((\tau(2p)-\chi)(1+\epsilon/2)-(\tau(2+p)\epsilon)-\chi))}$ . Thus, it can be seen that  $V_j^{(\epsilon)}$  converges to zero, for  $\epsilon$  small enough, by the condition  $2p\tau'(2p) - \tau(2p) > -\chi$ . This completes the proof of the proposition.  $\square$

**PROPOSITION 3.** *Assume (5) and (6) and that, either  $p_\chi^+ < \infty$  with  $\tau(p_\chi^+) > 0$ , or  $p_\chi^+ = \infty$  with  $\tau(p) > 0$  for all  $p > 1$ . Then:*

1. If  $\tau(2p) - 2\tau(p) > 0$  we have

$$(37) \quad 2^{j(\tau(p)-\chi/2)} B_j \rightarrow \mathcal{N}(0, c(p)),$$

where  $c(p) = \mathbb{E}[\mu_\infty([0, T])^p]^2 \frac{2^{2\tau(p)-\tau(2p)+1}-1}{2-2^{2\tau(p)-\tau(2p)+1}}$ .

2. If  $\tau(2p) - 2\tau(p) = 0$ , we have  $\text{Var}(B_j) = O(j2^{-j(\tau(2p)-\chi)})$ .

3. If  $\tau(2p) - 2\tau(p) < 0$ , we have  $\text{Var}(B_j) = O(2^{-j(\tau(2p)-\chi)})$ .

PROOF. Denote  $v_j^{(m)}$  the measures defined at the step  $j$  of the construction of the  $\mathcal{M}$ -cascade on  $[0, 1]$  based on  $W^p / \mathbb{E}[W^p]$

$$v_j^{(m)}([0, 1]) = 2^{-j} \sum_{r \in \{0, 1\}^j} \prod_{i=1}^j (W_{r|i}^{(m,r)})^p \mathbb{E}[W^p]^{-j} \quad \text{for } m \in \{0, \dots, N_T - 1\}.$$

With this notation we have

$$(38) \quad B_j = \mathbb{E}[\mu_\infty([0, T])^p] 2^{-j\tau(p)} \sum_{m=0}^{N_T-1} (v_j^{(m)}([0, 1]) - 1)$$

and using Kahane and Peyrière’s results [20], we know that for each  $m$  the sequence  $(v_j^{(m)}([0, 1]))_j$  is bounded in  $\mathbf{L}^q$  as soon as  $\tau(pq) - q\tau(p) > 0$ .

We first focus on the case  $\tau(2p) - 2\tau(p) > 0$ . Hence the sequence  $(v_j^{(m)}([0, 1]))_j$  is bounded in  $\mathbf{L}^{2+\epsilon}$ -norm for some  $\epsilon > 0$ . Using that  $(v_j^{(m)}([0, 1]) - 1)_m$  is a centered i.i.d. sequence and classical considerations for triangular array of martingale increments, one can show that a central limit theorem holds:

$$N_T^{-1/2} \sum_{m=0}^{N_T-1} (v_j^{(m)}([0, 1]) - 1) \xrightarrow{j \rightarrow \infty} \mathcal{N}(0, \text{Var}(v_\infty^{(0)}([0, 1]))),$$

where  $v_\infty^{(0)}([0, 1]) = \lim_{j \rightarrow \infty} v_j^{(0)}([0, 1])$ . From this and (38) we deduce (37) with  $c(p) = \mathbb{E}[\mu_\infty([0, T])^p]^2 \text{Var}(v_\infty^{(0)}([0, 1]))$ . Computing the variance of a cascade measure as a function of its generator yields to the expression of  $c(p)$  given in the statement of the proposition.

In the cases  $\tau(2p) - 2\tau(p) \leq 0$ , by (38) again we have

$$\text{Var}(B_j) = [2^{j\chi}] 2^{-j2\tau(p)} \mathbb{E}[\mu_\infty([0, T])^p]^2 \text{Var}(v_j^{(0)}([0, 1])).$$

Now  $\text{Var}(v_j^{(0)}([0, 1])) = \mathbb{E}[v_j^{(0)}([0, 1])^2] - 1$  is unbounded as  $j \rightarrow \infty$ , but a careful look at the computations in Lemma 2 with  $\epsilon = 1$  yields to

$$\mathbb{E}[v_j^{(0)}([0, 1])^2] \stackrel{j \rightarrow \infty}{\sim} \sum_{l=0}^j 2^{-l} 2^{l(2\tau(p)-\tau(2p))}.$$

We deduce that  $\text{Var}(B_j) = O(2^{-j(2\tau(p)-\chi)} \sum_{l=0}^j 2^{-l} 2^{l(2\tau(p)-\tau(2p))})$ . Then, the theorem follows in the cases  $\tau(2p) - 2\tau(p) = 0$  and  $\tau(2p) - 2\tau(p) < 0$ .  $\square$

REMARK 3. By (34) the difference between  $2^{(\tau(p)-\chi)j} \mathcal{S}_{\tilde{\mu}}(j, p)$  and its limit is decomposed into two dissimilar error terms: particularly the fact that the contribution of  $B_j$  converges to zero is due to the observation of a large number of integral scales, whereas the contribution of  $A_j$  vanishes as the sampling step tends to zero. For those reasons, in the case  $\chi = 0$ , only a central limit theorem for a term similar to  $A_j$  was studied in [32] while the contribution of  $B_j$  was considered as a bias term (see also [33]).

In the case  $\tau(2p) - 2\tau(p) > 0$ , the contribution of  $B_j$  strictly dominates and  $2^{j(\tau(p)-\chi)} \mathcal{S}_{\tilde{\mu}}(j, p) - \mathbb{E}[\mu_{\infty}([0, T])^p]$  is of magnitude  $2^{-j\chi/2} \sim N_T^{-1/2}$ .

If  $\tau(2p) - 2\tau(p) < 0$ , the magnitude of  $A_j$  and  $B_j$  are the same and  $2^{j(\tau(p)-\chi)} \mathcal{S}_{\tilde{\mu}}(j, p) - \mathbb{E}[\mu_{\infty}([0, T])^p]$  is asymptotically bounded by terms of magnitude  $2^{j(-\chi+2\tau(p)-\tau(2p))/2}$ . This rate of convergence is slower than  $N_T^{-1/2}$ .

The variance terms appearing in Propositions 2 and 3 can be estimated from the data. For instance, the quantity  $\text{Var}(\mu_{\infty}([0, T])^p)$  is simply estimated by the corresponding empirical variance of the sequence  $(\tilde{\mu}([mT, (m + 1)T])^p)_m$ .

If the purpose is to estimate  $\tau(p)$  rather than study the partition function itself, then it appears that  $\frac{1}{j} \log(\mathcal{S}_{\tilde{\mu}}(j, p))$  suffers from a bias of magnitude  $1/j$  and is not a satisfactory estimator. The same problem occurs in the case of a single cascade and this bias term is removed by considering the ratio between partition functions at different scales [32, 33]. In our situation the partition function at finer scale are constructed from series with longer size. Hence a similar approach necessitates the introduction of a partition function at scale  $T2^{-(j+1)}$  based on the part of the series available at the coarser scale. We let  $\widehat{\mathcal{S}}_{\tilde{\mu}}(j + 1, p) = \sum_{m=0}^{\lfloor 2^{j\chi} \rfloor - 1} \mathcal{S}_{\tilde{\mu}}^{(m)}(j + 1, p)$ , and then it can be seen that  $\widehat{\mathcal{S}}_{\tilde{\mu}}(j + 1, p) / \mathcal{S}_{\tilde{\mu}}(j, p)$  provides us with an estimator of  $2^{-\tau(p)}$ . Applying the same algebraic manipulation as in the beginning of the proof of Proposition 8.2 of [33] it can be seen that  $\widehat{\mathcal{S}}_{\tilde{\mu}}(j + 1, p) - 2^{-\tau(p)} \mathcal{S}_{\tilde{\mu}}(j, p)$  has the same structure as the above term  $A_j$ , and thus a central limit theorem can be shown  $2^{j(\tau(2p)-\chi)} (\widehat{\mathcal{S}}_{\tilde{\mu}}(j + 1, p) - 2^{-\tau(p)} \mathcal{S}_{\tilde{\mu}}(j, p)) \xrightarrow{j \rightarrow \infty} \mathcal{N}(0, \hat{c}(p))$  for some constant  $\hat{c}(p)$ . In turn, this implies that  $2^{j(\chi-2\tau(p)+\tau(2p))/2} (\frac{\widehat{\mathcal{S}}_{\tilde{\mu}}(j+1,p)}{\mathcal{S}_{\tilde{\mu}}(j,p)} - 2^{-\tau(p)})$  is asymptotically Gaussian. Based on this result, a confidence interval could be constructed for the estimation of  $2^{-\tau(p)}$ , but we do not pursue here for the sake of shortness.

**4. Extension to wavelet based partition functions.** In most applications, one must use wavelets to analyze a signal in order to suppress eventual superimposed regular behavior (such as a trend). In this section, we assess the regularity of the path  $t \mapsto \tilde{\mu}([0, t])$  via the behavior of its wavelet coefficients. Moreover, this choice suppresses the unnatural fact that the partition function was constructed on exactly the same grid as the dyadic cascade measure.

4.1. *Notation.* In this section we assume, for notational convenience, that  $T = 1$ . Consider now  $g$  a “generalized box” function. It is a real valued function that satisfies the following assumptions:

- (H1)  $g$  has compact support included in  $[0, 2^J]$ , for some  $J \geq 0$ .
- (H2)  $g$  is piecewise continuous.
- (H3)  $g$  is at least nonzero on an interval.

Following the common wavelet notation, we define

$$g_{j,k}(t) = g(2^j t - k).$$

The support of  $g_{j,k}(t)$  is

$$(39) \quad \text{Supp } g_{j,k} = [2^{-j}k, 2^{-j}k + 2^{J-j}].$$

In the sequel, if  $\mu$  is a random measure, for any Borel function  $f$  we will use the notation

$$\langle \mu, f \rangle = \int f(t) d\mu(t).$$

4.2. *The generalized partition function: Scaling properties.* We define the generalized partition function of an  $\mathcal{M}$ -cascade  $\mu_\infty$  on  $[0, 1]$  at scale  $2^{-j}$  as

$$(40) \quad \mathcal{S}_{\mu,g}(j, p) = \sum_{k=0}^{2^j-2^{j-1}-1} |\langle \mu_\infty, g_{j,k} \rangle|^p.$$

Remark that for simplicity we removed a finite number of border terms, and that, in the case where  $g(t)$  is the “box” function  $g(t) = 1_{[0,1]}(t)$  we recover the partition function of Section 2.1.

Let us study the scaling of  $\mathbb{E}[\mathcal{S}_{\mu,g}(j, p)]$ .

PROPOSITION 4. *Assume (5) and (6) and let  $p > 0$ . Then, we have  $K_1 \times 2^{-j\tau(p)} \leq \mathbb{E}[\mathcal{S}_{\mu,g}(j, p)] \leq K_2 2^{-j\tau(p)}$  for  $K_1, K_2$ , two positive constants depending on  $p, W$  and  $g$ .*

PROOF. Since  $|g(t)|$  is clearly a bounded function, we have

$$\mathbb{E}[|\langle \mu_\infty, g_{j,k} \rangle|^p] \leq C \mathbb{E}[\mu_\infty([2^{-j}k, 2^{-j}k + 2^{J-j}])^p],$$

where  $C$  is a constant. We write  $\mu_\infty([2^{-j}k, 2^{-j}k + 2^{J-j}]) = \sum_{l=0}^{2^j-1} \mu_\infty(I_{j,k+l})$ , and deduce

$$(41) \quad \mathbb{E}[|\langle \mu_\infty, g_{j,k} \rangle|^p] \leq C \mathbb{E}[\mu_\infty[0, 2^{-j}]^p] = K 2^{-j(\tau(p)+1)},$$

where  $K$  only depends on  $g$  and the law of  $W$ . By (40) we get the upper bound for  $\mathbb{E}[\mathcal{S}_{\mu,g}(j, p)]$ .

For the lower bound, let us write that  $\mathcal{S}_{\mu,g}(j, p)$  is greater than

$$\sum_{k'=0}^{2^{j-J}-1} |\langle \mu_\infty, g_{j,2^J k'} \rangle|^p.$$

But  $g_{j,2^J k'}$  is supported on  $[k'2^{J-j}, (k'+1)2^{J-j}]$ , and thus applying Lemma 3 in the Appendix B with  $a = j - J$ , we deduce

$$\langle \mu_\infty, g_{j,2^J k'} \rangle = 2^{J-j} \left( \prod_{i=1}^{j-J} W_{r|i} \right) Z,$$

where, in law,  $Z$  is equal to  $\langle \mu_\infty, g_{J,0} \rangle$ . Thus  $\mathbb{E}[|\langle \mu_\infty, g_{j,k} \rangle|^p]$  is greater than

$$2^{p(J-j)} \mathbb{E}[W^p]^{j-J} \mathbb{E}[|\langle \mu_\infty, g_{J,0} \rangle|^p] = K 2^{-j(\tau(p)+1)} \mathbb{E}[|\langle \mu_\infty, g_{J,0} \rangle|^p].$$

Applying Lemma 4 with  $f = g_{J,0}$  shows that  $\mathbb{E}[|\langle \mu_\infty, g_{J,0} \rangle|^p]$  is some positive constant. Then the lower bound for  $\mathbb{E}[\mathcal{S}_{\mu,g}(j, p)]$  easily follows.  $\square$

4.3. *The partition function in the mixed asymptotic framework.* Following (15), we define the partition function in the mixed asymptotic framework as

$$\mathcal{S}_{\tilde{\mu},g}(j, p) = \sum_{m=0}^{N_T-1} \mathcal{S}_{\mu,g}^{(m)}(j, p),$$

where  $\mathcal{S}_{\mu,g}^{(m)}(j, p)$  is the partition function of  $\mu_\infty^{(m)}$ , that is,

$$\mathcal{S}_{\mu,g}^{(m)}(j, p) = \sum_{k=0}^{2^j-2^J-1} |\langle \mu_\infty^{(m)}, g_{j,k} \rangle|^p.$$

We have the following result.

**THEOREM 4** (Scaling of the generalized partition function in a mixed asymptotic). *Let  $p > 0$ , then under the same assumptions as Theorem 2 the power law scaling of  $\mathcal{S}_{\tilde{\mu},g}(j, p)$  is given by*

$$\lim_{j \rightarrow \infty} \frac{\log_2 \mathcal{S}_{\tilde{\mu},g}(j, p)}{-j} \xrightarrow{a.s.} \tau_\chi(p).$$

**PROOF.** Using Proposition 4 we have,  $\lim_{j \rightarrow \infty} \frac{1}{-j} \log_2 \mathbb{E}[\mathcal{S}_{\tilde{\mu},g}(j, p)] = \tau_\chi(p)$ , and we just need to prove that, almost surely,

$$\mathcal{S}_{\tilde{\mu},g}(j, p) - \mathbb{E}[\mathcal{S}_{\tilde{\mu},g}(j, p)] = o(2^{-j\tau_\chi(p)}).$$

Using Lemmas 5 and 6 of Appendix B, this is done in the exact same way as the proof of (24) in Proposition 1.  $\square$

**5. Link with Besov spaces.** Following [18], one may define for a measure  $\mu$  on  $[0, T]$ , the boundary of its Besov domain as the function  $s_\mu : (0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$  given by

$$s_\mu(1/p) = \sup \left\{ \sigma \in \mathbb{R} \mid \sup_{j \geq 0} 2^{j\sigma} \left( 2^{-j} \sum_{k=0}^{2^j-1} |\mu(I_{j,k})|^p \right)^{1/p} < \infty \right\}.$$

The following proposition can be shown (see [18]).

**PROPOSITION 5.** *The function  $s_\mu$  is an increasing, concave function, with a derivative bounded by 1.*

Let us stress that the condition  $s'_\mu(1/p) \leq 1$  is a simple consequence of the Sobolev embedding for Besov spaces. Theorem 4 characterizes the Besov domain for  $\mu_\infty$  a  $\mathcal{M}$ -cascade on  $[0, T]$

$$\forall p > 0 \quad s_{\mu_\infty}(1/p) = \begin{cases} \frac{\tau(p)+1}{p}, & \text{if } \frac{1}{p} > \frac{1}{p_0}, \\ h_0^+ + \frac{1}{p}, & \text{if } \frac{1}{p} \leq \frac{1}{p_0}. \end{cases}$$

If we denote  $s(\frac{1}{p}) = \frac{\tau(p)+1}{p}$ , then it is simply checked that the condition  $1/p > 1/p_0^+$  is equivalent to  $s'(1/p) < 1$ . Hence Proposition 5 explains why for  $1/p \leq 1/p_0^+$  the boundary of the Besov domain must be linear with a slope equal to one.

In mixed asymptotic the support of the measure grows with  $j$  but we can still define, using the notation of Section 3, the index

$$s_{\tilde{\mu}}^\chi(1/p) = \sup \left\{ \sigma \in \mathbb{R} \mid \sup_{j \geq 0} 2^{j\sigma} \left( N_T^{-1} 2^{-j} \sum_{k=0}^{N_T 2^j - 1} |\tilde{\mu}(I_{j,k})|^p \right)^{1/p} < \infty \right\}.$$

Then, it is simply checked that Theorem 2 implies  $s_{\tilde{\mu}}^\chi(1/p) = s(1/p)$  when  $s'(1/p) < 1 + \chi$ , and  $s_{\tilde{\mu}}^\chi(1/p) = h_\chi^+ + \frac{1+\chi}{p}$  otherwise. This shows how the linear part in  $s_{\tilde{\mu}}^\chi$  is shifted to larger values of  $p$  under the mixed asymptotic framework.

**6. Numerical examples and applications.** Our goal in this section is not to focus on statistical issues or on precise estimates of multifractal exponents from empirical data. We rather aim at illustrating the results of Theorem 2 on precise examples, namely random cascades with, respectively, log-Normal, log-Poisson and log-Gamma statistics. For the sake of simplicity we will consider exclusively scaling of partition function for  $p \geq 0$ .<sup>1</sup> In order to facilitate the comparison of the three models,  $\lambda^2$  will represent the so-called intermittency coefficient, that is,

$$\lambda^2 = -\tau''(0),$$

---

<sup>1</sup>Numerical methods for estimating  $\tau(p)$  for  $p < 0$  are trickier to handle [38].

where  $\tau(p)$  is defined in (7). This value will be fixed for the three considered models. Let  $\{W_r\}$  be the cascade random generators as defined in (3), and let  $\omega_r = \ln W_r$ .

In the simplest, log-Normal case the  $\{\omega_r\}_r$  are normally distributed random variables of variance  $\lambda^2 \ln(2)$ . Thanks to the condition  $\mathbb{E}[W_r] = \mathbb{E}[e^{\omega_r}] = 1$ , their mean is necessarily  $-\lambda^2 \ln(2)/2$ . In that case, the cumulant generating function  $\tau(p)$  defined in (7) is simply a parabola

$$\tau^{(LN)}(p) = p\left(1 + \frac{\lambda^2}{2}\right) - \frac{\lambda^2}{2}p^2 - 1.$$

In the log-Poisson case, the variables  $\omega_r$  are written as  $\omega_r = m_0 \ln(2) + \delta n_r$  where the  $n_r$  are integers distributed according to a Poisson law of mean  $\gamma \ln(2)$ . It results that  $\tau(p) = p(1 - m_0) + \gamma(1 - e^{p\delta}) - 1$ . If one sets  $\tau(1) = 0$  and  $\tau''(0) = -\lambda^2$ , one finally gets the expression of  $\tau(p)$  of a log-Poisson cascade with intermittency coefficient  $\lambda^2$

$$(42) \quad \tau^{(LP)}(p) = p\left(1 + \frac{\lambda^2}{\delta^2}(e^\delta - 1)\right) + \frac{\lambda^2}{\delta^2}(1 - e^{p\delta}).$$

In third case the variables  $\omega_r$  are drawn from a Gamma distribution. If  $x$  is a random variable of p.d.f.  $\beta^\alpha \ln(2)^\alpha x^{\alpha \ln(2)-1} e^{-\beta x} / \Gamma(\alpha \ln(2))$ , then one chooses  $\omega_r = x + m_0 \ln(2)$  and it is easy to show that  $\tau(p)$  is defined only for  $p < \beta$ , and in this case  $\tau(p) = p(1 - m_0) + \alpha(1 - p/\beta)$ . By fixing  $\tau(1) = 1$  and  $\tau''(0) = \lambda^2$ , one obtains

$$(43) \quad \tau^{(LG)}(p) = p\left(1 - \lambda^2 \beta^2 \ln \frac{\beta - 1}{\beta}\right) + \lambda^2 \beta^2 \ln \frac{\beta - p}{\beta}.$$

Notice that one recovers the log-Normal case from both log-Poisson and log-Gamma statistics in the limits  $\delta \rightarrow 0$  and  $\beta \rightarrow +\infty$ , respectively.

For the 3 cases, one can explicitly compute all the mixed asymptotic exponents as functions of  $\chi$ : in particular the values of  $p_\chi^\pm$  read

$$p_{\chi,n}^\pm = \pm \sqrt{\frac{2(1 + \chi)}{\lambda^2}},$$

$$p_{\chi,p}^\pm = \frac{W(\pm, (\delta^2(1 + \chi) - \lambda^2)/(e\lambda^2)) + 1}{\delta} = p_{\chi,n}^\pm + \frac{2(1 + \chi)}{3\lambda^2}\delta + \mathcal{O}(\delta^2),$$

$$p_{\chi,g}^\pm = \beta \left[ 1 + \frac{1 + \chi}{\lambda 2 \beta^2} - e^{1+W(\pm, -e^{-1-(1+\chi)/(\lambda^2 \beta^2)})} \right] = p_{\chi,n}^\pm - \frac{4(1 + \chi)}{3\lambda^2 \beta} + \mathcal{O}(\beta^{-2}),$$

where suffixes  $n, p, g$  stand for, respectively, log-Normal, log-Poisson and log-Gamma cascades, and  $W(\pm, z)$  represent the two branches of the Lambert  $W(z)$  function, namely the solution of  $W(z)e^{W(z)} = z$  that take (resp., positive and negative) real values for the considered arguments. For log-Poisson and log-Gamma

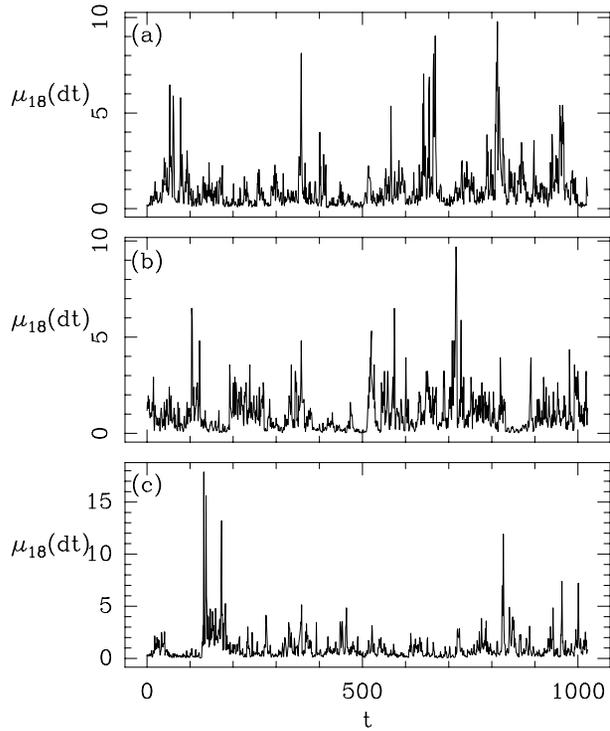


FIG. 1. Three synthetic samples of  $\mathcal{M}$ -cascades with  $T = 2^{13}$  and intermittency coefficient  $\lambda^2 = 0.2$ : (a) log-Normal sample, (b) log-Poisson sample with  $\delta = -0.1$  and (c) log-Gamma sample with  $\beta = 10$ . In fact we used  $\mu_{j_{\max}+5}[n, n+1]_{n=0, \dots, L}$  as a proxy of  $\mu_{\infty}[n, n+1]$  with  $j_{\max} = \log_2(T) = 13$  [see (3) for the definition of  $\mu_j$ ].

cases, we have also indicated the asymptotic behavior in the limits  $\delta \rightarrow 0$  and  $\beta \rightarrow \infty$ . The values  $h_{\chi}^{\pm}$  can be easily deduced from their definition:  $h_{\chi}^{\pm} = \tau'(p_{\chi}^{\pm})$ .

In Figure 1 is plotted a sample of each of the three examples of  $\mathcal{M}$ -cascades. We chose  $T = 2^{13}$  and  $\lambda^2 = 0.2$  for all models while, in the log-Poisson case we have set  $\delta = -0.1$  and  $\beta = 10$  in the log-Gamma model. In each case, an approximation of the  $\mathcal{M}$ -cascade sample is generated. We chose to generate  $\mu_{18}$  [as defined by (3)] so that the smallest scale involved is  $l_{\min} = 2^{-18}T = 2^{-5}$  (we have checked that the results reported below do not depend on  $l_{\min}$ ). An approximation of  $\tilde{\mu}$  is generated by concatenating i.i.d. realizations of  $\mu_{18}$ . Then, for each model and for each chosen value of  $\chi$ ,  $\tau_{\chi}(p)$  ( $p = 0, \dots, 6$ ) was obtained from a least square fit of the curve  $\log_2 S_{\tilde{\mu}}(j, p)$  versus  $j$  over the range  $j = 0, \dots, 6$ . Let us recall that, for each value of  $j$ , the mixed asymptotic regime corresponds to sampling  $\tilde{\mu}$  at scale  $l = 2^{-j}T$  and over an interval of size  $L = 2^{j\chi}T$ . The exponents reported in Figures 2 and 3 represent the mean values of exponents estimated in that way using  $N = 130$  experiments.

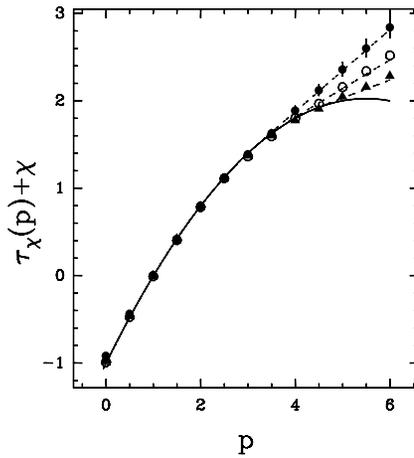


FIG. 2. Estimates of the function  $\tau_\chi(p)$  of log-Normal cascades with  $\lambda^2 = 0.2$  for  $\chi = 0$  ( $\bullet$ ),  $\chi = 0.5$  ( $\circ$ ) and  $\chi = 1$  ( $\blacktriangle$ ). Dashed lines represent the corresponding analytical expression from Theorem 2, and the solid line represents the function  $\tau(p)$  as defined in (7).  $\tau_\chi$  is estimated from the average over 130 trials of  $2^{j\chi}$  cascades samples. Error bars are reported on the  $\chi = 0$  curve as vertical solid bar. These errors are of order of symbol size.

The log-Normal mixed asymptotic scaling exponents for  $\chi = 0, 0.5, 1$  are represented in Figure 2. For illustration purpose we have plotted  $\tau_\chi(p) + \chi$  as a function of  $p$ : one clearly observes that, as the value of  $\chi$  increases, the value of  $p_\chi^+$  below which the function is linear, also increases while the value of the slope  $h_\chi^+$  decreases. As expected, when  $\chi$  increases  $\tau_\chi(p) + \chi$  matches  $\tau(p)$  over an increasing range of  $p$  values. Notice that the estimated exponents are very close the

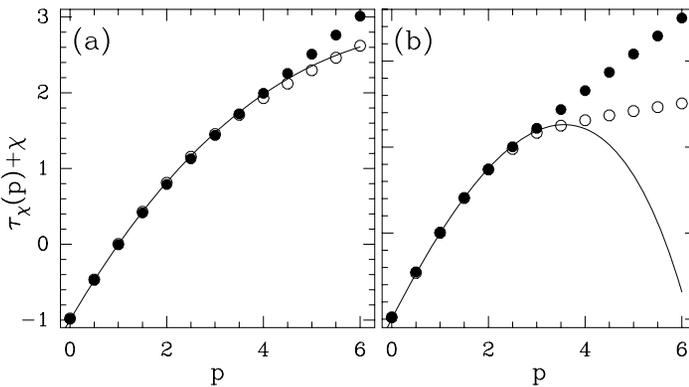


FIG. 3. Estimates of the function  $\tau_\chi(p)$  of log-Poisson and log-Gamma for  $\chi = 0$  ( $\bullet$ ) and  $\chi = 1$  ( $\circ$ ). In both models we chose  $T = 2^{13}$  and  $\lambda^2 = 0.2$ . (a) Log-Poisson case with  $\delta = -0.1$ . (b) Log-Gamma case with  $\beta = 10$ . Solid lines represent the curves  $\tau(p)$ .

analytical predictions as represented by the dashed lines. Error bars on the mean value estimates are simply computed from the estimated r.m.s. over the 130 trials and are reported only for the  $\chi = 0$  curve. We can see that these errors are smaller or close to the symbol thickness.

In Figure 3 are reported estimates of  $\tau_\chi(p)$ ,  $\chi = 0, 1$  for log-Poisson [Figure 3(a)] and log-Gamma [Figure 3(b)] samples. The solid lines represent the theoretical  $\tau(p)$  functions for both models as provided by (42) and (43). We used the same estimation procedure as for the log-Normal case. One sees that, in both cases, since the intermittency coefficient is the same for the three models, the classical  $\tau_0(p)$  curves are very similar to the log-Normal curve (Figure 2). However, these models behave very differently in mixed regime: for  $\chi = 1$ , log-Poisson and log-Gamma both estimated scaling exponents become closer to the respective values of  $\tau(p)$  and are very easy to distinguish. Let us mention that such a analysis has been recently performed by two of us in order to distinguish two popular log-Normal and log-Poisson models for spatial fluctuations of energy dissipation in fully developed turbulence [29].

APPENDIX A: LEMMA USED FOR THE PROOF OF THEOREM 2

LEMMA 1. *We have*

$$L_N^{1+\epsilon} \leq C 2^{j\chi} \mathbb{E}[|\mathcal{M}_j^{(0)}(p)|^{1+\epsilon}],$$

where  $L_N^{1+\epsilon}$  is defined by (25) and  $C$  is a constant that depends only on  $\epsilon$ .

PROOF. According to [35], if  $\epsilon \in [0, 1]$  and if  $\{X_i\}_{1 \leq i \leq P}$  are centered independent random variables one has

$$\mathbb{E}\left[\left|\sum_{i=1}^P X_i\right|^{1+\epsilon}\right] \leq C \sum_{i=1}^P \mathbb{E}[|X_i|^{1+\epsilon}],$$

where  $C$  is a constant that depends only on  $\epsilon$  (and neither on the law of  $X$  nor on  $P$ ). Applying it with  $P = N_T = \lfloor 2^{j\chi} \rfloor$  to the expression (23) of  $\mathcal{N}_j(p)$ , and using the fact that the random variables  $\{\mathcal{M}_j^{(m)}(p)\}_m$  defined by (22) are i.i.d., one gets

$$\begin{aligned} L_N^{1+\epsilon} &\leq C 2^{j\chi} \mathbb{E}[|\mathcal{M}_j^{(0)}(p) - \mathbb{E}[\mathcal{M}_j^{(0)}(p)]|^{1+\epsilon}] \\ &\leq C 2^{j\chi} (\mathbb{E}[|\mathcal{M}_j^{(0)}(p)|^{1+\epsilon}] + \mathbb{E}[|\mathbb{E}[\mathcal{M}_j^{(0)}(p)]|^{1+\epsilon}]). \end{aligned}$$

Using the Jensen’s inequality we get the result.  $\square$

LEMMA 2. *Assume that  $\mathbb{E}[|Z|^{1+\epsilon}] < \infty$ . Then we have for all  $m$ ,*

$$\mathbb{E}[|\mathcal{M}_j^{(m)}(p)|^{1+\epsilon}] \leq C 2^{-j(1+\epsilon)\tau(p)} \sum_{k=0}^j 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)},$$

where  $C$  is a constant that depends only on  $p$  and  $\epsilon$ .

PROOF. The proof of this result is very much inspired from [33]. Since the law of  $\mathcal{M}_j^{(m)}(p)$  is independent of  $m$ , we forget the superscript  $m$  throughout the proof. Using definition (22), one gets

$$(44) \quad \mathbb{E}[|\mathcal{M}_j(p)|^{1+\epsilon}] = 2^{-jp(1+\epsilon)} \mathbb{E} \left[ \left| \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}^p Z^{(r)} \right|^{1+\epsilon} \right].$$

Let

$$(45) \quad X = 2^{-jp} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}^p Z^{(r)},$$

then

$$X^2 = 2^{-2jp} \sum_{r_1 \in \{0,1\}^j} \sum_{r_2 \in \{0,1\}^j} \prod_{i=1}^j W_{r_1|i}^p W_{r_2|i}^p Z^{(r_1)} Z^{(r_2)}.$$

It can be rewritten as

$$(46) \quad X^2 = Y + D,$$

where  $Y$  corresponds to the nondiagonal terms

$$(47) \quad Y = 2^{-2jp} \sum_{\substack{r_1 \in \{0,1\}^j \\ r_2 \in \{0,1\}^j \\ r_2 \neq r_1}} \prod_{i=1}^j W_{r_1|i}^p W_{r_2|i}^p Z^{(r_1)} Z^{(r_2)}$$

and  $D$  to the diagonal terms

$$(48) \quad D = 2^{-2jp} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}^{2p} (Z^{(r)})^2.$$

The left-hand side of (44) is nothing but  $\mathbb{E}[|X|^{1+\epsilon}]$ . By writing that  $\mathbb{E}[|X|^{1+\epsilon}] = \mathbb{E}[(X^2)^{(1+\epsilon)/2}]$ , using the sub-additivity of  $x \mapsto x^{(1+\epsilon)/2}$ , we get

$$(49) \quad \mathbb{E}[|X|^{1+\epsilon}] \leq \mathbb{E}[|Y|^{(1+\epsilon)/2}] + \mathbb{E}[D^{(1+\epsilon)/2}].$$

Let us first work with the  $Y$  term. We factorize the common beginning of the words  $r_1$  and  $r_2$  in the expression (47) of  $Y$

$$Y = 2^{-2jp} \sum_{k=0}^{j-1} \sum_{r \in \{0,1\}^k} \prod_{i=1}^k W_{r|i}^{2p} \sum_{\substack{r_1, r_2 \in \{0,1\}^{j-k} \\ r_1|0 \neq r_2|0}} \prod_{i=k+1}^j W_{r_1|i}^p W_{r_2|i}^p Z^{(r_1)} Z^{(r_2)}.$$

Again by the sub-additivity of  $x \mapsto x^{(1+\epsilon)/2}$  and using the fact that the  $W_{r|i}$  are i.i.d., one gets

$$\begin{aligned} & \mathbb{E}[|Y|^{(1+\epsilon)/2}] \\ & \leq 2^{-jp(1+\epsilon)} \sum_{k=0}^{j-1} \mathbb{E}[W^{p(1+\epsilon)}]^k \\ & \quad \times \sum_{r \in \{0,1\}^k} \mathbb{E} \left[ \left( \sum_{\substack{r_1, r_2 \in \{0,1\}^{j-k} \\ r_1|0 \neq r_2|0}} \prod_{i=k+1}^j W_{rr_1|i}^p W_{rr_2|i}^p \right. \right. \\ & \quad \left. \left. \times |Z^{(r_1)} Z^{(r_2)}| \right)^{(1+\epsilon)/2} \right], \end{aligned}$$

and by using Jensen's inequality

$$\begin{aligned} & \mathbb{E}[|Y|^{(1+\epsilon)/2}] \\ & \leq 2^{-jp(1+\epsilon)} \sum_{k=0}^{j-1} \mathbb{E}[W^{p(1+\epsilon)}]^k \\ (50) \quad & \times \sum_{r \in \{0,1\}^k} \left( \sum_{\substack{r_1, r_2 \in \{0,1\}^{j-k} \\ r_1|0 \neq r_2|0}} \prod_{i=k+1}^j \mathbb{E}[W_{rr_1|i}^p W_{rr_2|i}^p] \right. \\ & \quad \left. \times \mathbb{E}[|Z^{(r_1)} Z^{(r_2)}|] \right)^{(1+\epsilon)/2}. \end{aligned}$$

The variables  $Z^{(r_1)}$  and  $Z^{(r_2)}$  are independent with finite expectation; thus the term  $\mathbb{E}[|Z^{(r_1)} Z^{(r_2)}|]$  is bounded by a constant  $C$ . Using  $\prod_{i=k+1}^j \mathbb{E}[W_{rr_1|i}^p W_{rr_2|i}^p] = \mathbb{E}[W^p]^{2(j-k)}$ , we deduce

$$\begin{aligned} \mathbb{E}[|Y|^{(1+\epsilon)/2}] & \leq C 2^{-jp(1+\epsilon)} \sum_{k=0}^{j-1} \mathbb{E}[W^{p(1+\epsilon)}]^k \mathbb{E}[W^p]^{(j-k)(1+\epsilon)} \\ & \quad \times \sum_{r \in \{0,1\}^k} \left| \sum_{\substack{r_1, r_2 \in \{0,1\}^{j-k} \\ r_1|0 \neq r_2|0}} 1 \right|^{(1+\epsilon)/2}. \end{aligned}$$

There are  $2^k$  possible values for  $r$  and less than  $2^{2(j-k)}$  values for the couple  $(r_1, r_2)$ , thus

$$\mathbb{E}[|Y|^{(1+\epsilon)/2}] \leq 2^{-jp(1+\epsilon)} K \sum_{k=0}^{j-1} \mathbb{E}[W^{p(1+\epsilon)}]^k \mathbb{E}[W^p]^{(j-k)(1+\epsilon)} 2^k 2^{(j-k)(1+\epsilon)}.$$

Since  $2^{-j\tau(p)} = 2^{-jp}2^j\mathbb{E}[W^p]^j$

$$(51) \quad \mathbb{E}[|Y|^{(1+\epsilon)/2}] \leq K 2^{-j(1+\epsilon)\tau(p)} \sum_{k=0}^{j-1} 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)}.$$

Let us now take care of the diagonal terms of  $X$  (48). First, we write

$$D^{(1+\epsilon)/2} \leq 2^{-jp(1+\epsilon)} \sum_{r \in \{0,1\}^j} \prod_{i=1}^j W_{r|i}^{p(1+\epsilon)} |Z^{(r)}|^{1+\epsilon},$$

and using the  $\mathbb{E}[|Z|^{1+\epsilon}] < \infty$  we deduce that

$$(52) \quad \mathbb{E}[D^{(1+\epsilon)/2}] \leq C 2^{-jp(1+\epsilon)} 2^j \mathbb{E}[W^{p(1+\epsilon)}]^j = C 2^{-j\tau(p(1+\epsilon))}.$$

Merging (51) and (52) into (49) leads to

$$\mathbb{E}[|X|^{1+\epsilon}] \leq K 2^{-j(1+\epsilon)\tau(p)} \sum_{k=0}^j 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)}$$

and since  $\mathbb{E}[|\mathcal{M}_j(p)|^{1+\epsilon}] = \mathbb{E}[|X|^{1+\epsilon}]$ , it completes the proof.  $\square$

APPENDIX B: LEMMA USED FOR THE PROOF OF THEOREM 4

LEMMA 3. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be some Borel function whose support is included in  $I_{a,\bar{r}} = [\frac{\bar{r}}{2^a}, \frac{\bar{r}+1}{2^a}]$  for  $r \in \{0, 1\}^a, a \geq 0$ . Then

$$(53) \quad \langle \mu_\infty, f \rangle = 2^{-a} \left( \prod_{i=1}^a W_{r|i} \right) \langle \bar{\mu}_\infty^{(r)}, \tilde{f} \rangle,$$

where  $\tilde{f}(x) = f(2^{-a}(x + \bar{r}))$  and  $\bar{\mu}_\infty^{(r)}$  is a cascade measure on  $[0, 1]$  measurable with respect to the sigma field  $\sigma\{W_{r'r'}, r' \in \{0, 1\}^{a'}, a' \geq 1\}$ .

PROOF. The scaling relation (53) is easily obtained, by the definition of the measure  $\mu_\infty$ , if  $f$  is the characteristic function of some interval  $I_{a+a',r\bar{r}'}$  where  $r' \in \{0, 1\}^{a'}, a' \geq 0$ . This relation extends to any Borel function  $f$  by standard arguments of measure theory.  $\square$

LEMMA 4. Let  $h : [0, 1] \rightarrow \mathbb{R}$  be a piecewise continuous, nonzero, function. Then  $\mathbb{E}[|\langle \mu_\infty, h \rangle|^p] > 0$  for all  $p > 0$ .

PROOF. By contradiction, assume that for some  $p > 0, \mathbb{E}[|\langle \mu_\infty, h \rangle|^p] = 0$ . Hence  $\langle \mu_\infty, h \rangle = 0, \mathbb{P}$ -almost surely. But using Lemma 3,

$$\begin{aligned} 0 &= \langle \mu_\infty, h \rangle = \langle \mu_\infty, h1_{[0,1/2]} \rangle + \langle \mu_\infty, h1_{(1/2,1]} \rangle \\ &= \frac{1}{2} W_0 \langle \bar{\mu}_\infty^{(0)}, h^{(0)} \rangle + \frac{1}{2} W_1 \langle \bar{\mu}_\infty^{(1)}, h^{(1)} \rangle, \end{aligned}$$

where  $h^{(0)}(\cdot) = h(2^{-1}\cdot)$ ,  $h^{(1)}(\cdot) = h(2^{-1}(\cdot + 1))$  and  $\bar{\mu}_\infty^{(0)}, \bar{\mu}_\infty^{(1)}$  are independent cascade measures on  $[0, 1]$ . Thus we deduce  $W_0 \langle \bar{\mu}_\infty^{(0)}, h^{(0)} \rangle = -W_1 \langle \bar{\mu}_\infty^{(1)}, h^{(1)} \rangle$  almost surely, and since  $W > 0$  this shows that with probability one the two independent variables  $\langle \bar{\mu}_\infty^{(0)}, h^{(0)} \rangle$  and  $\langle \bar{\mu}_\infty^{(1)}, h^{(1)} \rangle$  vanish simultaneously. This is only possible either, if they both vanish on a set of full probability, or if they both vanish on a negligible set. Assume the latter, then the following identity holds almost surely:

$$\frac{\langle \bar{\mu}_\infty^{(0)}, h^{(0)} \rangle}{\langle \bar{\mu}_\infty^{(1)}, h^{(1)} \rangle} = -\frac{W_1}{W_0},$$

where the variables on right- and left-hand side are independent. These variables must be constant, which is excluded by the assumption  $\mathbb{P}(W = 1) < 1$  [recall (5)].

Thus we deduce that the variables  $\langle \bar{\mu}_\infty^{(i)}, h^{(i)} \rangle$  are almost surely equal to zero. Hence

$$\mathbb{E}[|\langle \mu_\infty, h^{(i)} \rangle|^p] = 0 \quad \text{for } i = 0, 1.$$

Iterating the argument we deduce the following property: for any  $j \geq 0$  and  $k \leq 2^j - 1$ , if we define a function on  $[0, 1]$  by  $h^{(j,k)}(x) = h(2^{-j}(x + k))$  we have

$$\mathbb{E}[|\langle \mu_\infty, h^{(j,k)} \rangle|^p] = 0.$$

This is clearly impossible if we choose  $j, k$  such that  $h$  remains positive (or negative) on  $[k2^{-j}, (k + 1)2^{-j}]$ . By the assumptions on  $h$  one can find such an interval, yielding to a contradiction.  $\square$

LEMMA 5. *We have*

$$\mathbb{E}[|\mathcal{S}_{\bar{\mu},g} - \mathbb{E}[\mathcal{S}_{\bar{\mu},g}]|^{1+\epsilon}] \leq C2^{j\chi} \mathbb{E}[|\mathcal{S}_{\mu,g}(j, p)|^{1+\epsilon}],$$

where  $C$  is a constant that depends only on  $\epsilon$ .

PROOF. The proof is the same as for Lemma 1.  $\square$

LEMMA 6. *For any  $\epsilon > 0$  small enough we have*

$$\mathbb{E}[|\mathcal{S}_{\mu,g}(j, p)|^{1+\epsilon}] \leq K2^{-j(1+\epsilon)\tau(p)} \sum_{k=0}^j 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)},$$

where  $K$  is a constant that depends only on  $p$  and  $\epsilon$ .

PROOF. The proof basically follows the same lines as the proof of Lemma 2. The only difficulty, compared to this latter proof, comes from the fact that the quantity  $\langle \mu_\infty, g_{j,k} \rangle$  a priori involves several nodes of level  $j$  of the  $\mathcal{M}$ -cascade. We have to reorganize the sum (40).

Since we are interested in the limit  $j \rightarrow +\infty$ , we can suppose, with no loss of generality that  $j > J$ . In the following, we note  $1^{(n)}$  the  $n$ -uplet

$$1^{(n)} = 11, \dots, 1 \quad \text{where the 1 is repeated } n \text{ times.}$$

The partition function (40) can be written  $S_{\mu,g}(j, p) = \sum_k |\langle \mu_\infty, g_{j,\bar{k}} \rangle|^p$ , where the sum is over  $k \in \{0, 1\}^j$  such that  $k_l = 0$  for some  $l \leq j - J$ . The sum can be regrouped in the following way, where  $a + 1$  denotes the position of the last 0 in the  $j - J$  first components of  $k$ :

$$S_{\mu,g}(j, p) = \sum_{a=0}^{j-J-1} \sum_{\substack{r \in \{0,1\}^a \\ q=r0}} \sum_{s \in \{0,1\}^J} |\langle \mu_\infty, g_{j,q1^{(j-J-1-a)}s} \rangle|^p.$$

We set

$$(54) \quad X_{a,s} = \sum_{\substack{r \in \{0,1\}^a \\ q=r0}} |\langle \mu_\infty, g_{j,q1^{(j-J-1-a)}s} \rangle|^p$$

and consequently

$$S_{\mu,g}(j, p) = \sum_{a=0}^{j-J-1} \sum_{s \in \{0,1\}^J} X_{a,s}.$$

Actually,  $a$  exactly corresponds to the level of the “highest” node that is common for dyadic intervals in the support of  $g_{j,q1^{(j-J-1-a)}s}$ . Indeed, let us prove that

$$(55) \quad a \geq 0, \forall s \in \{0, 1\}^J \quad \text{Supp } g_{j,q1^{(j-J-1-a)}s} \subset I_{a,\bar{r}},$$

where  $q = r0$ . Indeed, according to (39), the support of  $g_{j,q1^{(j-J-1-a)}s}$  is included in  $[2^{-j}q1^{(j-J-1-a)}s, 2^{-j}q1^{(j-J-1-a)}s + 2^{-(j-J)}]$ . Then,

$$\begin{aligned} 2^{-j}q1^{(j-J-1-a)}s &= 2^{-a}\bar{r} + \sum_{i=a+2}^{j-J} 2^{-i} + 2^{-j}\bar{s} \\ &= 2^{-a}\bar{r} + 2^{-a-1} - 2^{-(j-J)} + 2^{-j}\bar{s}. \end{aligned}$$

Since  $\bar{s}$  varies in  $[0, 2^J - 1]$ , and  $a \leq j - J - 1$ , it is easy to show that

$$0 \leq 2^{-a-1} - 2^{-(j-J)} + 2^{-j}\bar{s}$$

and

$$2^{-a-1} + 2^{-j}\bar{s} \leq 2^{-a},$$

which proves (55).

We are now ready to compute the upper bound for  $\|\mathcal{S}_{\mu,g}(j, p)\|_{L^{1+\epsilon}(\mathbb{P})} \leq \sum_{a=0}^{j-J-1} \sum_{s \in \{0,1\}^J} \|X_{s,a}\|_{L^{1+\epsilon}(\mathbb{P})}$ . Using (54), (55) and Lemma 3, we get

$$X_{a,s} = 2^{-ap} \sum_{r \in \{0,1\}^a} \left( \prod_{i=1}^a W_{r|i}^p \right) |\langle \bar{\mu}_\infty^{(r)}, g_{j,01^{(j-J-1-a)}s} \rangle|^p,$$

where the  $\bar{\mu}_\infty^{(r)}$  are independent cascade measures on  $[0, T]$ .

Let us identify  $X_{a,s}$  with  $X$  as defined in Lemma 2 by (45) in which  $j$  plays the role of  $a$  and  $Z^{(r)}$  of  $\langle \bar{\mu}_\infty^{(r)}, g_{j-a,01^{(j-J-1-a)}s} \rangle^p$ . As in (46), we can decompose  $X_{a,s}^2$  as the sum of the nondiagonal terms  $Y_{a,s}$  and the diagonal terms  $D_{a,s}$

$$(56) \quad X_{a,s}^2 = Y_{a,s} + D_{a,s}.$$

Using the exact same development as the one we used for  $\mathbb{E}[X^{1+\epsilon}]$  starting at equation (44), we get the bound of the nondiagonal terms corresponding to (50) in which the term  $\mathbb{E}[Z^{(r_1)} Z^{(r_2)}]$  has to be replaced by

$$\mathbb{E}[|\langle \mu_\infty^{(r_1,s)}, g_{j-a,01^{(j-J-1-a)}s} \rangle|^p |\langle \mu_\infty^{(r_2,s)}, g_{j-a,01^{(j-J-1-a)}s} \rangle|^p],$$

which can be bounded [using (41)] by  $K 2^{-2(j-a)(\tau(p)+1)}$ . Going on with the same arguments as in Lemma 2, we finally get the bound for the nondiagonal terms corresponding to (51)

$$\begin{aligned} \mathbb{E}[|Y_{a,s}|^{(1+\epsilon)/2}] &\leq K 2^{-j(1+\epsilon)\tau(p)} 2^{(a-j)(1+\epsilon)} \sum_{k=0}^{a-1} 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)} \\ &\leq K 2^{-j(1+\epsilon)\tau(p)} 2^{(a-j)(1+\epsilon)} \sum_{k=0}^{j-1} 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)}. \end{aligned}$$

Following the arguments in Lemma 2 for the diagonal terms, we get

$$(57) \quad \mathbb{E}[D_{a,s}^{(1+\epsilon)/2}] \leq 2^{-j\tau(p(1+\epsilon))} 2^{a-j}.$$

By (56) and (57), we finally get

$$\mathbb{E}[|X_{a,s}|^{1+\epsilon}] \leq K 2^{(a-j)} 2^{-j(1+\epsilon)\tau(p)} \sum_{k=0}^j 2^{-k\tau(p(1+\epsilon))} 2^{k(1+\epsilon)\tau(p)}.$$

Then we write

$$\mathbb{E}[|\mathcal{S}_{\mu,g}(j, p)|^{1+\epsilon}] \leq \|\mathcal{S}_{\mu,g}(j, p)\|_{L^{1+\epsilon}(\mathbb{P})}^{1+\epsilon} \leq \left( \sum_{a=0}^{j-J-1} \sum_{s \in \{0,1\}^J} \|X_{a,s}\|_{L^{1+\epsilon}(\mathbb{P})} \right)^{1+\epsilon},$$

and the lemma follows.  $\square$

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