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Uniform weak convergence of the time-dependent poverty measures for continuous longitudinal data

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Abstract. The poverty analysis may require the observation of the same set of households over the time in order to explain the evolution of the poverty situation and to try to explain their behavior. In this case, the poverty measures have to be determined continuously in some interval $[0, T]$ and the sample poverty index becomes time-dependent. In this paper, we settle the global problem of the weak convergence of the time-dependent poverty measures in the functional space of continuous functions defined on $[0, T]$. We entirely describe the uniform asymptotic normality of the class of nonweighted poverty indices including the Foster–Greer–Thorbecke and Chakravarty ones, which both have the special property of satisfying all the needed axioms for a poverty index.

1 Introduction

In this paper, we are concerned with the statistical analysis of poverty indices. These are defined as follows. We consider a population of individuals, each of which having a random income or expenditure Y with distribution function $G(y) = \mathbb{P}(Y \leq y)$. An individual is classified as poor whenever his income or expenditure Y fulfills $Y < Z$, where Z is a specified treshold level (the poverty line).

Consider now a random sample $Y_1, Y_2, \ldots, Y_n$ of size n of incomes, with empirical distribution function $G_n(y) = n^{-1}\#\{Y_i \leq y : 1 \leq i \leq 1\}$. The number of poor individuals within the sample is then equal to $Q_n = nG_n(Z)$. And, from now on, all the random elements used in the paper are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Given these preliminaries, we introduce measurable functions $A(p, q, z)$, $w(t)$, and $d(t)$ of $p, q \in \mathbb{N}$, and $z, t \in \mathbb{R}$. The meaning of these functions will be discussed later on. Set $B(q) = \sum_{i=1}^{q} w(i)$.

Let now $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$ be the order statistics of the sample $Y_1, Y_2, \ldots, Y_n$ of Y. We consider general poverty indices of the form

$$GPI_n = \frac{A(Q_n, n, Z)}{nB(Q_n)} \sum_{j=1}^{Q_n} w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4) d\left(\frac{Z - Y_{j,n}}{Z}\right), \qquad (1.1)$$

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where $\mu_1, \mu_2, \mu_3, \mu_4$ are constants. *In the sequel,* (1.1) *will be called a poverty index (indices in the plural) or simply a poverty measure according to the economists terminology.*

A number of poverty indices have been introduced in the literature since the pioneering work of Nobel Prize winner Amartya Sen (1976), who first derived poverty measures (see [9]) from an axiomatic point of view. A survey of these indices is to be found in Zheng [12], who also discussed their introduction from an axiomatic point of view. The determination of the poverty line is also of major interest. In the domain of welfare research, several models of this type have been introduced. At times, such models may be incompatible (see [6]). In [1,3,5,7, 8], a general methodology of investigation of these models have been developed, through the appropriate modern technology. In the general frame proposed in [7], the statistic (1.1) converges in probability to

$$GPI = \int_0^Z L_1(u, G) d\left(\frac{Z-u}{Z}\right) dG(u), \tag{1.2}$$

where L_1 is some weight function depending on the distribution function, under some very mild conditions. These results have natural applications to derive asymptotic confidence intervals for indices based on data collected within developing countries. These methods turn out to be successful in the statistical monitoring of poverty factors.

This model is, however, unpractical, in the sense that it is not time-dependent and is then not appropriate to handle continuous panel data. In practice, the income of individuals varies with time. We may be faced with continuous data in the form of $Y(t), 0 < t < T$, and some modification is needed in the definition of indices to take this into account. We are led to consider the time-dependent general poverty indices defined by

$$\begin{aligned} GPI_n(t) = {} & \frac{A(Q_n(t), n, Z)}{nB(Q_n)} \\ & \times \sum_{j=1}^{Q_n(t)} w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) d\left(\frac{Z - Y_{j,n}(t)}{Z}\right), \end{aligned} \tag{1.3}$$

with $0 \leq t \leq T$ and $T \in \mathbb{R}$.

This is the case where the poverty situation is analysed over the continuous period of time $[0, T]$. We then have to move from a fixed-time poverty analysis to a continuous poverty analysis, where the same households are continuously observed over the time and give the longitidunal observations of $Y \in C([0, T])$,

$$\{Y_1(t), \dots, Y_n(t), 0 < t < T\}.$$

This paper is aimed to settle the uniform weak convergence of such statistics, that is, the asymptotic theory of the time-dependent poverty measures (1.3), in the space $C([0,T])$ of real continuous functions defined on $[0,T]$. As a first step, the class of nonweighted poverty measures

$$J_n(t) = \frac{1}{n}\sum_{j=1}^{Q_n(t)} d\left(\frac{Z - Y_{j,n}(t)}{Z}\right), \qquad 0 < t < T, \tag{1.4}$$

is studied. This class is very important for the poverty analysis since it includes the Foster–Greer–Thorbecke and the Chakravarty indices which both are very powerful ones since they fulfill all the desirable axioms (see [12]). Our best result is the complete description of the uniform asymptotic weak laws of (1.4). Moreover, we get as side effects tools that will be useful to handle the weighted and general cases.

The paper is organized as follows. In Section 2, we review some important cases of poverty measures as illustrations for the reader. In Section 3, we give our results on the time-dependent poverty index (1.4) and discuss the hypotheses from a parametric view. Section 4 is devoted to the proofs, while the conclusion is given in Section 5.

2 Examples

One may devide the poverty indices into two classes. The first includes the nonweighted ones. The most popular of them is the Foster–Greer–Thorbecke (1984, [4]) class which is defined for $\alpha \geq 0$, by

$$FGT(\alpha) = \frac{1}{n}\sum_{j=1}^{Q_n}\left(\frac{Z - Y_{j,n}}{Z}\right)^{\alpha}. \tag{2.1}$$

For $\alpha = 0$, (2.1) reduces to Q_n/n, the headcount of poor individuals. For $\alpha = 1$ and $\alpha = 2$, it is respectively interpreted as the severity of poverty and the depth in poverty. (2.1) is obtained from (1.1) by taking

$$w(u) \equiv 1 : A(Q_n, n, Z) = Q_n : B(Q_n) = Q_n : d(u) = u^{\alpha}.$$

Next, we have for $\alpha \geq 0$, by

$$C(\alpha) = \frac{1}{n}\sum_{j=1}^{Q_n}\left(1 - \left(\frac{Y_{j,n}}{Z}\right)^{\alpha}\right),$$

the Chakravarty family class of poverty measures (see [2]).

The second class consists of the weighted indices. We mention here two of its famous members. The Sen (1976) index (see [9])

$$P_{SE,n} = \frac{2}{n(Q_n+1)}\sum_{j=1}^{Q_n}(Q_n - j + 1)\left(\frac{Z - Y_{j,n}}{Z}\right). \tag{2.2}$$

(2.2) is obtained from (1.1), by taking

$$d(u) = u : w(u) = u : A(Q_n, n, Z) = Q_n,$$
$$B(Q_n) = Q_n(Q_n + 1)/2 : \mu_1 = 0; \qquad \mu_3 = \mu_2 = \mu_4 = 1.$$

The Shorrocks (1995) index (see [10])

$$P_{SH,n} = \frac{1}{n^2} \sum_{j=1}^{Q_n} (2n - 2j + 1)\left(\frac{Z - Y_{j,n}}{Z}\right). \tag{2.3}$$

(2.3) is obtained from (1.1) by taking

$$d(u) = u, \qquad w(u) = u, \qquad A(Q_n, n, Z) = Q_n(Q_n + 1)/(n),$$

and

$$B(Q_n) = Q_n(Q_n + 1), \qquad \mu_1 = \mu_3 = 2, \qquad \mu_2 = 0, \qquad \mu_4 = 1.$$

(2.2) and (2.3) evaluate the poverty intensity by giving a more important weight on the poorest individuals. This means that a small decrease of the intensity indicates most improvement in the population.

These are only some of the most important indices used by economists for monitoring the poverty situation. But our form (1.1) concerns an unlimited number of possible indices, covering almost all the available ones. Our statistical results, in this paper, are given for the general form and will work, at once, for almost the used measures.

3 Our results

We suppose that the d.f. $G_t(x) = \mathbb{P}(Y_j(t) \le x)$ admits a derivative $g_t(x)$. Put $G_{s,t}(u) = \mathbb{P}(X(t) \le u, X(s) \le u)$. Our results will rely on the following hypotheses.

(H1) There exits a positive real number $r > 0$, and there exists a positive function g such that for $0 \le s, t \le T$,

$$\sup_{u \ge 0} |g_t(u) - g_s(u)| \le g(u)|t - s|^{1+r}$$

and

$$\int_0^Z g(u)\, du = K_1 < \infty.$$

(H2) For $0 \le s, t \le T$, for some constant K_2,

$$\sup_{u \ge 0} |G_{t,s}(u) - G_s(u)| \le K_2|t - s|^{1+r}.$$

(H3) For $0 \leq s, t \leq T$, for some constant K_3,

$$\mathbb{E}|Y(t) - Y(s)|^2 \leq K_3 |t - s|^{1+r}.$$

(H4) d is bounded by one and is differentiable with derivative function d' bounded by M:

$$0 \leq d \leq 1, \qquad |d'| \leq M.$$

We now state our main result.

Theorem 1. *Let* (H1), (H2), (H3), (H4) *hold. Then the stochastic process* $\{\sqrt{n}(J_n(t) - \mu(t)), 0 \leq t \leq T\}$ *converges in* $C([0, T])$ *to a centred Gaussian process with covariance function*

$$\begin{aligned}\Gamma(s,t) &= \int_0^\infty \int_0^\infty \left(d\left(\frac{Z-u}{Z}\right)1_{(u\leq Z)} - \mu(t)\right) \\ &\qquad \times \left(d\left(\frac{Z-v}{Z}\right)1_{(v\leq Z)} - \mu(s)\right) g_t(u) g_s(v)\, du\, dv \\ &= \int_0^Z \int_0^Z d\left(\frac{Z-u}{Z}\right) d\left(\frac{Z-v}{Z}\right) g_t(u) g_s(v)\, du\, dv - \mu(t)\mu(s),\end{aligned}$$

where

$$\mu(t) = \int_0^Z d\left(\frac{Z-u}{Z}\right) g_t(u)\, du.$$

Let us make some remarks on the hypotheses before the proofs.

Remark 1. In handling the Senegalese data, we have seen that the variable $X = 1/(y_0 - Y)$, where Y is the income and y_0 the lowest income, is well fitted to the lognormal law. For three periods, the variance remains almost constant. In the case of longitidunal data, we may then suppose that the variable $X(t)$ follows a lognormal law of mean $m(t)$ and variance $\sigma^2(t)$. If we suppose, as mentioned above that the variance is constant and equal to σ^2 and that the lowest income is also fixed to y_0, we have

$$G_t(u) = \phi\big((-\log(u - y_0) - m(t))/\sigma\big) 1_{(u \geq y_0)}$$

and

$$g_t(u) = \frac{-\log(u - y_0)}{\sigma} \phi'\big((-\log(u - y_0) - m(t))/\sigma\big) 1_{(u \geq y_0)},$$

where ϕ is the distribution function of the standard normal random variable. Since ϕ'' is bounded (say by K_0), we arrive at

$$\sup_{u \geq 0} |g_t(u) - g_s(u)| \leq K_0 \frac{-\log(u - y_0)}{\sigma^2} 1_{(u \geq y_0)} |m(t) - m(s)|.$$

The condition (H4) is then plausible since $\int_{y_0}^{Z} -\log(u - y_0)\,du$ (naturally with $y_0 \leq Z$) is finite and the condition $|m(t) - m(s)| \leq |t - s|^{1+r}$ is not so restrictive.

Remark 2. The condition (H1) is reasonable since $G_{s,t} \uparrow G_s$ for $s \uparrow t$. In summary, in future papers, the hypotheses have to be largely expressed in terms of parametric models of the income, in the sense painted here, such as lognormal and Singh–Maddala families which both seem to be very adequate for poverty data.

4 Proofs

Our proofs will rely of the modern theory of the empirical processes weak convergence as stated in [11]. We then begin by noticing that $\{\sqrt{n}(J_n(t) - \mu(t)), 0 \leq t \leq T\}$ is a functional empirical process

$$\alpha_n(f) = \sqrt{n}\left\{\frac{1}{n}\sum_{j=1}^{n} f(Y_j) - \mathbb{E}f(Y_j)\right\},$$

for $f \in \mathcal{F}_0 = \{f_t : x \mapsto d(\frac{Z-x(t)}{Z})1_{(x(t)\leq Z)}, 0 \leq t \leq T\}$. We have, in view of (H4),

$$\sup_{0\leq t\leq T} |f_t(x) - \mathbb{E}f_t(Y)| = \sup_{0\leq t\leq T}\left|d\left(\frac{Z - x(t)}{Z}\right)1_{(x(t)\leq Z)} - \mu(t)\right| \leq 2. \tag{4.1}$$

By following [11] (page 81), the functional weak law of $\{\alpha_n(f), f \in \mathcal{F}\}$, for a given family $\mathcal{F}$ of measurable functions defined on $[0, T]$, may be handled in the general frame of empirical processes theory, whenever

$$\sup_{f\in\mathcal{F}} |f(x) - \mathbb{E}f(Y)| < \infty, \tag{4.2}$$

for any x$\in C([0, T])$. Indeed this is the case for our family because of (4.1). This allows to study $\{\alpha_n(f), f \in \mathcal{F}_0\}$ as a stochastic process indexed by functions $f \in \mathcal{F}_0$ with states in $l^{\infty}(C([0, T]))$, the space of bounded functions defined on $C([0, T])$. Next, since the functions $f \in \mathcal{F}$ are square integrable with respect to $\mathbb{P}_Y$, that is,

$$\forall t \in [0, T] \qquad \mathbb{E}f_t(Y)^2 = \mathbb{P}_Y(f_t^2) < 1,$$

we get, by the results in [11] (page 81), that the sequence of stochastic processes $\{\sqrt{n}(J_n(t) - \mu(t)), 0 \leq t \leq T\}$ converges in finite distributions to a Gaussian process $\{X(t), 0 \leq t \leq T\}$ with covariance function $\Gamma(s, t)$. This, in fact, easily follows from the multivariate central limit theorem. The second step of the proof is to show that $\{\sqrt{n}(J_n(t) - \mu(t)), 0 \leq t \leq T\}$ weakly converges to $\{X(t), 0 \leq t \leq T\}$. At this step, we shall use an application of Prohorov's theorem through Theorem 1.5.4 in [11], which states that the sequence of stochastic processes

$$X_n(t) = \alpha_n(f_t) = \frac{1}{\sqrt{n}}\sum_{j}\left\{d\left(\frac{Z - Y_j(t)}{Z}\right)1_{(Y_j(t)\leq Z)} - \mu(t)\right\}$$

weakly converges X, whenever the finite distributions of X_n weakly converge to those of X and $\{X_n(t), 0 \leq t \leq T\}$ is asymptotically tight. Since finite-distribution convergence holds, we have to establish the asymptotic tightness of X_n. This, in turn, requires, in view of Theorem 1.5.7 of [11], that $X_n(t)$ is asymptotically tight for every t, and that there exists a semi-metric ρ on $[0, T]$, such that $([0, T], \rho)$ is totally bounded and

$$\forall \varepsilon > 0, \eta > 0, \exists \delta > 0, \qquad \limsup_{n \to +\infty} \mathbb{P}^*\Big(\sup_{\rho(s,t)<\delta} |X_n(s) - X_n(t)| > \varepsilon \Big) < \eta, \quad (4.3)$$

where $\mathbb{P}^*$ denotes the outer probability defined by $\mathbb{P}^*(A) = inf\{\mathbb{P}(B), B \supseteq A, B \in \mathcal{A}\}$. To prove that X_n is asymptotically tight, notice first that each $X_n(t)$ weakly converges to Gaussian random variable $\mathcal{N}(0, \Gamma(t,t))$. It follows, in view of Lemma 1.3.8 in [11], that $X_n(t)$ is asymptotically tight, since any probability on $\mathbb{R}$ is tight and then asympotically tight. Since X_n has almost sure continuous paths, $\sup_{\rho(s,t)<\delta} |X_n(s) - X_n(t)|$ is measurable and (4.3) becomes

$$\forall \varepsilon > 0, \eta > 0, \exists \delta > 0, \qquad \limsup_{n \to +\infty} \mathbb{P}^*\Big(\sup_{\rho(s,t)<\delta} |X_n(s) - X_n(t)| > \varepsilon \Big) < \eta. \quad (4.4)$$

Now, one may use techniques very similar to those of Example 2.2.12 in [11] to conclude. We prove in Lemma 1 below that (4.4) is implied by

$$\mathbb{E}|X_n(t) - X_n(s)|^2 \leq K_0 |t-s|^{1+r} \quad (4.5)$$

with $\rho(s,t) = |s-t|$. Then, the rest of the proof consists in establishing (4.5), given Lemma 1 to be proved later. In order to do that, let

$$\overline{M}_j(t) = d\left(\frac{Z - Y_j(t)}{Z}\right) 1_{(Y_j(t) \leq Z)} - \mu(t)$$

and

$$M_j(t) = d\left(\frac{Z - Y_j(t)}{Z}\right) 1_{(Y_j(t) \leq Z)}.$$

Since the random elements of $C([0, T])$, $\overline{M}_j(s) - \overline{M}_j(t)$ are independent and centred, for any $(s,t) \in [0, T]^2$, we get

$$\mathbb{E}|X_n(t) - X_n(s)|^2 = \frac{1}{n} \sum \mathbb{E}\big(\overline{M}_j(t) - \overline{M}_j(s)\big)^2.$$

By the c$_2$-inequality ($|a+b|^r \leq 2^{r-1}(|a|^r + |b|^r)$ for $r \geq 1$),

$$\mathbb{E}\big(\overline{M}_j(t) - \overline{M}_j(s)\big)^2 \leq 2\big(\mathbb{E}\big(M_j(t) - M_j(s)\big)^2 + 2\big(\mu(s) - \mu(t)\big)\big)^2.$$

First by (H1) and (H4),

$$\begin{aligned} |\mu(t) - \mu(t)| &\leq \int_0^Z d\left(\frac{Z-u}{Z}\right) |g_t(u) - g_s(u)|\, du \\ &\leq \left(\int_0^Z g(u)\, d\left(\frac{Z-u}{Z}\right)\right) |t-s|^{1+r} = K_1 |t-s|^{1+r}. \end{aligned}$$

This in turn implies

$$|\mu(t)-\mu(t)|^2 \leq K_1^2 T^{1+r}|t-s|^{1+r}.$$

Next

$$M_j(t)-M_j(s)=d\left(\frac{Z-Y_j(t)}{Z}\right)1_{(Y_j(t)\leq Z)}-d\left(\frac{Z-Y_j(s)}{Z}\right)1_{(Y_j(s)\leq Z)}.$$

Add and substract

$$d\left(\frac{Z-Y_j(t)}{Z}\right)1_{(Y_j(s)\leq Z)}$$

in the formula above and use the c$_2$-inequality again and the one bound of the function $d(\cdot)$, to get

$$\begin{aligned}\big(M_j(t)-M_j(s)\big)^2 &\leq 2\left|d\left(\frac{Z-Y_j(t)}{Z}\right)-d\left(\frac{Z-Y_j(s)}{Z}\right)\right|^2\\ &\quad+2\left|1_{(Y_j(t)\leq Z)}-1_{(Y_j(s)\leq Z)}\right|^2\\ &\leq 2\frac{M^2}{Z^2}\big(Y_j(t)-Y_j(s)\big)^2+2\left|1_{(Y_j(t)\leq Z)}-1_{(Y_j(s)\leq Z)}\right|^2.\end{aligned}$$

Recall that

$$|1_A-1_B|^2=1_{AB^c}+1_{A^cB}.$$

Then

$$\left|1_{(Y_j(t)\leq Z)}-1_{(Y_j(s)\leq Z)}\right|^2=1_{(Y_j(t)\leq Z<Y_j(s))}+1_{(Y_j(s)\leq Z<Y_j(t))}$$

and

$$\begin{aligned}\mathbb{E}\big(1_{(Y_j(t)\leq Z<Y_j(s))}\big)&=\mathbb{P}\big(Y_j(t)\leq Z\big)-\mathbb{P}\big(Y_j(t)\leq Z,Y_j(s)\leq Z\big)\\ &=G_t(Z)-G_{s,t}(Z).\end{aligned}\tag{4.6}$$

By using (H2), we arrive at

$$\mathbb{E}|X_n(t)-X_n(s)|^2\leq\left(\frac{4M^2}{Z^2}K_3+8K_2+K_1^2T^{1+r}\right)|t-s|^{1+r}.\tag{4.7}$$

This proves, in view of Lemma 1 that $\{\sqrt{n}(J_n(t)-\mu(t)), 0\leq t\leq T\}$ is asymptotically tight. The uniform weak convergence follows, that is, $\mathcal{F}_0=\{f_t: x\mapsto d(\frac{Z-x(t)}{Z})1_{(x(t)\leq Z)}, 0\leq t\leq T\}$ is a Donsker Class.
As announced, the proof will be complete after of the following lemma.

Lemma 1. *Let $T>0$ and $\{X_n(t), 0\leq t\leq T\}$ be a sequence of separable stochastic processes such that there exists $r>0$ so that*

$$\forall n\geq 1\qquad \mathbb{E}|X_n(t)-X_n(s)|^2\leq K_0|t-s|^{1+r}.\tag{4.8}$$

Then,

$$\forall n > 0, \forall \varepsilon > 0, \forall \nu > 0, \exists \rho > 0, \qquad \mathbb{P}\Big(\sup_{|s-t| \leq \delta^{1/\alpha}} |X_n(s) - X_n(t)| \geq \varepsilon \Big) \leq \nu.$$

Proof. Assume that (4.8) holds. Let $\alpha = \min(1, (1+r)/2)$ and consider the semi-metric $d(s,t) = |s-t|^\alpha$ on $[0,T]$. We have

$$\|X_n(s) - X_n(t)\|_2 \leq \max\big(K^{1/2}, T^{(r-1)/2}\big) d(s,t). \tag{4.9}$$

This inequality is obvious for $\alpha = (1+r)/2$. As for $\alpha = 1 \leq (1+r)/2$, it is based on

$$|s-t|^{(1+r)/2} \leq T^{(1+r)/2} \left| \frac{s-t}{T} \right|^{(1+r)/2} \leq T^{(1+r)/2} \left| \frac{s-t}{T} \right| \leq T^{(r-1)/2} |s-t|,$$

for $(s,t) \in [0,T]^2$, since the function $x \mapsto a^x$ is decreasing for $0 < a < 1$. Next $[t - \varepsilon^{1/\alpha}, t + \varepsilon^{1/\alpha}]$ is a d-ball of radius ε centred on $t \in [0,T]$. It follows that one can cover $[0,T]$ with $(T/2)\varepsilon^{-1/\varepsilon}$ d-balls of radius ε so that the minimum number of d-balls of radius needed to cover $[0,T]$, denoted $N(d,\varepsilon)$ is less or equal to $(T/2)\varepsilon^{-1/\alpha}$. Considering the L^p-*norm* as an Orlicz one with $\|\cdot\|_p = \|\cdot\|_\psi$ and $\psi(x) = x^2$, which complies with the condition of Theorem 2.2.4 in [11]. Applying this theorem, we get, for all $\eta > 0$ and for $\delta > 0$, for some $K_4 > 0$ only depending on $C_0 = \min(K^{1/2}, T^{(r-1)/2)})$,

$$\Big\| \sup_{d(s,t) \leq \delta} |X_n(s) - X_n(t)| \Big\|_2 \leq K_4 \Big[\int_0^\eta \psi^{-1}(D(d,\varepsilon))\, d\varepsilon + \delta \psi^{-1}(D^2(d,\eta)) \Big],$$

where $D(d,\varepsilon)$ is the maximal lenght of ε-separated chains in $([0,T],d)$. Recall that a sequence of points $x_1, x_2, \ldots$ is a ε-separated chain of points of $([0,T],d)$ if $d(x_i,x_j) > \varepsilon$ for $1 \leq i \neq j$. Using the relation $D(d,\varepsilon) \leq N(d,\varepsilon/2)$, (see [11], page 98). we then get

$$\Big\| \sup_{d(s,t) \leq \delta} |X_n(s) - X_n(t)| \Big\|_2 \leq K_4 \Big[\int_0^\eta \psi^{-1}\big(N(d,\varepsilon/2)\big)\, d\varepsilon + \delta \psi^{-1}\big(N^2(d,\eta/2)\big) \Big].$$

Using the expression of $\psi^{-1}(x) = x^{1/2}$, we arrive at

$$\begin{aligned}
&\Big\| \sup_{|s-t| \leq \delta^{1/\alpha}} |X_n(s) - X_n(t)| \Big\|_2 \\
&\quad \leq K_4 \Big[\Big(\frac{T}{2^{1+1/\alpha}} \Big)^{1/2} \int_0^\eta \varepsilon^{-1/2\alpha}\, d\varepsilon + \delta \frac{T}{2^{1+1/\alpha}} \eta^{-1/\alpha} \Big] \\
&\quad = K_4 \Big[\Big(\frac{T}{2^{1+1/\alpha}} \Big)^{1/2} \int_0^\eta \varepsilon^{-1/2\alpha}\, d\varepsilon + \delta \frac{T}{2^{1+1/\alpha}} \varepsilon^{1/\alpha} \Big] \\
&\quad = K_4 \Big[\Big(\frac{T}{2^{1+1/\alpha}} \Big)^{1/2} \frac{\eta^{1-1/(2\alpha)}}{1 - 1/(2\alpha)} + \delta \frac{T}{2^{1+1/\alpha}} \eta^{-1/\alpha} \Big] = A(\eta,\delta).
\end{aligned}$$

Now it is evident that we may choose first η and next δ to make $A(\eta, \delta)$ arbitrary small. To finish, we use Markov's inequality for $x > 0$

$$\mathbb{P}\Big(\sup_{|s-t|\leq\delta^{1/\alpha}} |X_n(s) - X_n(t)| \geq x\Big) \leq \frac{A(\eta, \delta)^2}{x^2}.$$

Fix $x = \varepsilon > 0$ and $\nu > 0$, and choose first η and next δ such that $x^{-2}A(\eta, \delta) \leq \nu$. Putting $\rho = \delta^{1/\alpha}$, we conclude

$$\forall\varepsilon > 0, \forall\nu > 0, \exists\rho > 0, \qquad P\Big(\sup_{|s-t|\leq\rho} |X_n(s) - X_n(t)| \geq \varepsilon\Big) \leq \nu. \qquad \square$$

5 Conclusion

Our results yield tools to handle discrete and continuous longitudinal data. In applied cases, the hypotheses must be specialised in terms of parametric families suitable to the concerned data. In the Senegalese case, the lognormal and Sing–Maddala families seem to be very adapted and the hypotheses proved not to be very restrictive. Data driven research works are under way in this topic. The results do not cover the so important weighted measures such as the Sen and Shorrocks one. They will be handled in future papers as extension of the results given here.

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