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Some skew symmetric inverse reflected distributions

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Abstract. Skew-symmetric distributions are defined based on the reflected gamma, reflected Weibull and the reflected Pareto distributions. Expressions are derived for the probability density function, cumulative distribution functions, moments and the shape. Estimation procedures by the methods of moments and maximum likelihood and Fisher information matrices are provided. Evidence of flexibility of the distributions is shown. An application is illustrated using the Old Faithful Geyser data. Some of the attractive properties of the distributions include multimodality and polynomial tails.

1 Introduction

Let X_1 and X_2 be two independent continuous random variables having the common probability density function (PDF) $g(x)$ which is symmetric about zero. Then for any real number c, both X_1 and cX_2 are independent and have symmetric density about 0. By the property of symmetry

$$\begin{aligned}\frac{1}{2} &= P(X_1 - cX_2 \le 0) = \int_{-\infty}^{\infty} P(X_1 - cX_2 \le 0 | X_2 = x) g(x)\, dx \\ &= \int_{-\infty}^{\infty} G(cx) g(x)\, dx,\end{aligned}$$

where $G(x)$ is the cumulative distribution function (CDF) corresponding to the PDF $g(x)$.

Lemma 1 (Azzalini (1985)). *If R^1 is the support of g and $g(x) = g(-x)$ for all real numbers x then for any real number c,*

$$f(z; c) = 2g(z)G(cz) \tag{1}$$

is a skewed PDF of a continuous random variable Z, which will be denote by $SD(c)$.

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Various skewed distributions can be obtained from (1) by taking $g(\cdot)$ and $G(\cdot)$ to belong to standard parametric families. Of special interest is the skewed normal distribution given by the PDF

$$f(z; c) = 2\phi(z)\Phi(cz), \tag{2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal PDF and the standard normal CDF, respectively. This distribution was introduced in the seminal paper by Azzalini (1985). It has been studied extensively by several authors. Henze (1986), Liseo and Loperfido (2003) and Gupta, Nguyen and Sanqui (2004) provided various characterizations and representations of this distribution. Gupta and Chen (2001) and Monti (2003) considered goodness-of-fit and estimation issues. Arellano-Valle, Gomez and Quintana (2004) and Gupta and Gupta (2004) developed certain generalizations of the skewed normal distribution. Pewsey (2000) developed the wrapped skewed normal distribution for circular data. Azzalini and Chiogna (2004) considered stress–strength modeling using the skewed normal distribution.

Among other skewed distributions arising from (1), see Arnold and Beaver (2000) for skewed Cauchy, Kozubowski and Panorska (2004), Aryal and Nadarajah (2005) for skewed Laplace and Wahed and Ali (2001) for skewed logistic. See also Gupta, Chang and Huang (2002).

All of the skewed distributions arising from (1) have been based on symmetric families. We are aware of no distributions based on other families. In this note, we study properties of (1) when $g(\cdot)$ and $G(\cdot)$ correspond to the reflected versions of the three most popular models for skewed data: gamma, Weibull and the Pareto distributions. The reflected versions of these three distributions have widespread applications. For example, the reflected Weibull distribution has been a popular model for HIV infection (Commenges et al. (1992); Downs, Salamina and Ancellepark (1995); Marston et al. (2005)), complex drug intake process (Bressolle et al. (1994)), tensile impact (Wang and Xia (2000)) and reliability (Nadarajah (2004)). The reflected Pareto distribution has been a popular model for human settlements (Reed (2002)), income (Reed (2003); Chebotarev (2007)), magnitude and frequency of landslides (Guthrie and Evans (2004)), airport network analysis (Li and Cai (2004); Guida and Maria (2007)), size (Reed and Jorgensen (2004); Babbitt, Kiltie and Bolker (2006)) and adjustment of ion source (Han and Hu (2005)). The main feature in (1) is that a new parameter c is introduced to control skewness and kurtosis. So, we can expect the skewed reflected distributions to be useful in many more practical situations.

The contents of this note are organized as follows. Section 2 discusses some general properties of the distribution defined by (1). Sections 3, 4 and 5 derive mathematical properties of the skewed reflected gamma distribution, skewed reflected Weibull distribution and the skewed reflected Pareto distribution. We derive explicit expressions for the PDF, CDF, moments and the shape of each distribution. We provide graphical illustrations to show the flexibility of each distribution. We also provide estimation procedures by the methods of moments and maximum likelihood and the Fisher information matrix. Finally, Section 6 illustrates an application of the proposed skewed distributions.

2 Mathematical preliminaries

2.1 Azzalini's results

Let $F(z;c) = \int_{-\infty}^{z} 2g(t)G(ct)\,dt$ denote the CDF corresponding to (1). The CDF $F(z;c)$ can be expressed as

$$F(z;c) \equiv \int_{-\infty}^{z} f(t;c)\,dt = 2\int_{-\infty}^{z} g(t)G(ct)\,dt = 2\int_{-\infty}^{z}\int_{-\infty}^{ct} g(t)g(s)\,ds\,dt$$
$$= G(z) - 2\int_{z}^{\infty}\int_{0}^{ct} g(t)g(s)\,ds\,dt.$$

Define $I(z;c) = \int_{z}^{\infty}\int_{0}^{ct} g(t)g(s)\,ds\,dt$ for $z > 0$ and $c > 0$.

The following results are due to Azzalini (1985):

Lemma 2 (Azzalini (1985)). (a) *$Z \sim SD(c) \iff -Z \sim SD(-c)$ for any real number c. Especially, $SD(0) \sim g(x)$.*

(b) $F(z;-c) = 1 - F(-z;c)$.

(c) *$F(z;1) = G^2(z)$, where $G(x)$ is the original CDF of X.*

(d) *If $g(x)$ is a.e. twice differentiable function and $d^2g(x)/dx^2 \le 0$ and $(d/dx)\log\{g(x)/G(x)\} \le 0$, then $\log\{2g(z)G(cz)\}$ is a concave function of z, that is, the skewed distribution has always a unique mode.*

(e) *For positive z, $\lim_{c\to\infty} f(z;c)$ becomes a half-distribution of $g(x)$.*

We note from Lemma 2(b) and (c) that if $c = -1$ then $F(z;-1) = 1 - G^2(-z)$.

Lemma 3 (Azzalini (1985)). (a) *$I(z;c)$ is a decreasing function of z.*

(b) $I(z;c) = -I(z;-c)$.

(c) $I(-z;c) = I(z;c)$.

(d) $2I(z;1) = G(z)G(-z)$.

2.2 Definition of skewed inverse reflected distributions

We need the following fact to generate the skewed inverse reflected distributions discussed in Sections 3 to 5.

Fact 1. Let X be a random variable with density $f_X(x) = dF_X(x)/dx$ symmetric about zero and with the real line as the support of f. Then:

(a) $Y = 1/X$ has the density $y^{-2} f_X(1/y)$ which is symmetric about zero.

(b) The CDF of $Y = 1/X$ is

$$F_Y(y) = \frac{1}{2} + \operatorname{sgn}(y)\left\{1 - F_X\left(\frac{1}{|y|}\right)\right\},$$

where $\operatorname{sgn}(y) = 1$ if $y \geq 0$ and $\operatorname{sgn}(y) = -1$ if $y < 0$.

(c) For $y > 0$, $f_Y(y) = 2y^{-2} f_X(1/y)$ is a half-density of the inverse random variable.

(d) By Lemma 1, we define a skewed inverse reflected distribution as one having its PDF specified by

$$f(z; c) = \frac{1}{z^2} f_X\left(\frac{1}{z}\right)\left\{1 + 2\operatorname{sgn}(cz)\left[1 - F_X\left(\frac{1}{|cz|}\right)\right]\right\}$$

for $x \in R^1$ and $c \in R^1$.

3 Skewed inverse reflected gamma distribution

From the reflected gamma density in Johnson, Kotz and Balakrishnan (1994, 1995) and Fact 1(a), we can obtain the density of the inverse reflected gamma random variable as

$$g(y) = \frac{1}{2\Gamma(\alpha)} \frac{1}{|y|^{\alpha+1}} e^{-1/|y|} \tag{3}$$

for $y \in R^1$ and $\alpha > 0$. From formula 3.381(3) in Gradshteyn and Ryzhik (1965) and Fact 1(b), the corresponding CDF is

$$G(y) = \frac{1}{2}\left[1 + \operatorname{sgn}(y) \frac{\Gamma(\alpha, 1/|y|)}{\Gamma(\alpha)}\right] \tag{4}$$

for $y \in R^1$, where $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt$. The density (3) is symmetric about zero and for $y > 0$, $2g(y)$ becomes an inverted gamma density with a shape parameter α. Using formula 8.334(3) in Gradshteyn and Ryzhik (1965), the corresponding kth moment of Y is

$$E(Y^k) = \frac{1 + (-1)^k}{2} \frac{\Gamma(\alpha - k)}{\Gamma(\alpha)}$$

for $\alpha > k$. In particular, the variance of Y is

$$\operatorname{Var}(Y) = \frac{1}{(\alpha - 1)(\alpha - 2)}$$

for $\alpha > 2$.

3.1 PDF and CDF

By Fact 1(d), we can define the skewed inverse reflected gamma distribution by the PDF

$$f(z;c) = \frac{1}{2\Gamma(\alpha)|z|^{\alpha+1}} e^{-1/|z|} \left[1 + \operatorname{sgn}(cz) \frac{\Gamma(\alpha, 1/|cz|)}{\Gamma(\alpha)} \right] \tag{5}$$

for $z \in R^1$, $\alpha > 0$ and $c \in R^1$. For $z > 0$ and $c > 0$, the corresponding CDF is

$$F(z;c) = G(z) - 2I(z;c),$$

where

$$I(z;c) = \int_z^\infty \int_0^{ct} g(t) g(s)\, ds\, dt$$

and $g(\cdot)$ and $G(\cdot)$ are given by (3) and (4), respectively. Note that the CDF for negative values of z or c can be obtained using the facts $I(z;c) = -I(z;-c)$ and $I(-z;c) = I(z;c)$.

3.2 Shape

Figure 1 illustrates possible shapes of (5). Note that the shapes are multimodal. The modes are the solutions of the equation

$$\frac{\operatorname{sgn}(cz) e^{-1/|cz|}}{\Gamma(\alpha) + \operatorname{sgn}(cz)\Gamma(\alpha, 1/|cz|)} = |cz|^{\alpha-2} \left(\alpha + 1 - \frac{1}{|z|} \right).$$

Note that if $c > 0$ then $f(z) \sim z^{-\alpha-1}/\Gamma(\alpha)$ as $z \to \infty$ and $f(z) \sim |z|^{-2\alpha-1}/\{2\alpha\Gamma^2(\alpha) \mid c \mid^{\alpha}\}$ as $z \to -\infty$. If $c < 0$ then $f(z) \sim |z|^{-2\alpha-1}/\{2\alpha\Gamma^2(\alpha) \mid c \mid^{\alpha}\}$ as $z \to \infty$ and $f(z) \sim |z|^{-\alpha-1}/\Gamma(\alpha)$ as $z \to -\infty$. Also $f(z) \sim |z|^{-\alpha-1} e^{-1/|z|}/\{2\Gamma(\alpha)\}$ as $z \to 0$.

3.3 Moments

Using formula 8.354(2) in Gradshteyn and Ryzhik (1965), we can express $I(z;c)$ as

$$I(z;c) = \frac{1}{4\Gamma(\alpha)} \gamma\left(\alpha, \frac{1}{z}\right) - \frac{1}{4\Gamma^2(\alpha)} \sum_0^\infty \frac{(-1)^i}{i!(\alpha+i)c^{\alpha+i}} \gamma\left(2\alpha + i, \frac{1}{z}\right)$$

for $c > 0$ and $z > 0$, where $\gamma(\alpha, u) = \int_0^u y^{\alpha-1} e^{-y}\, dy$. Using formulas 3.1513.41 in Oberhettinger (1974) and 8.354(2) in Gradshteyn and Ryzhik (1965), the kth moment corresponding to (5) can be expressed as

$$E(Z^k;c) = \frac{\{1 + (-1)^k\}\Gamma(\alpha - k)}{2\Gamma(\alpha)} + \frac{\{1 - (-1)^k\} c^{\alpha-k} \Gamma(2\alpha - k)}{2\Gamma^2(\alpha)(\alpha - k)(1+c)^{2\alpha-k}} {}_2F_1\left(1, 2\alpha - k; \alpha - k + 1; \frac{c}{c+1}\right)$$

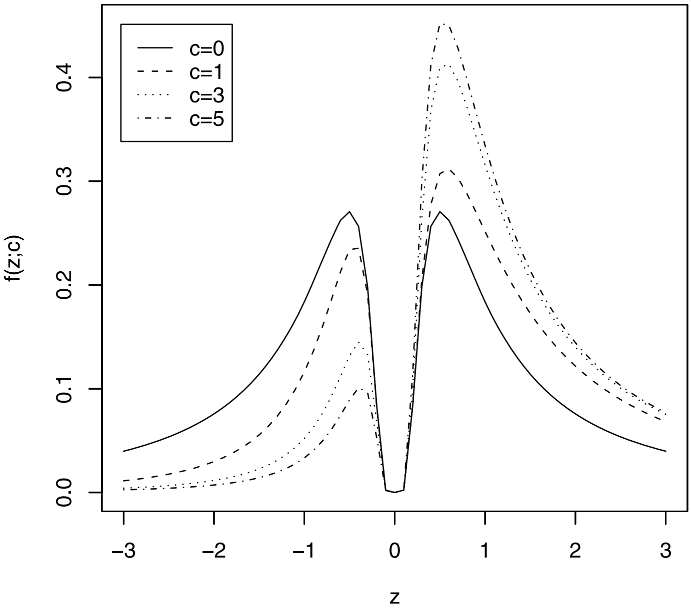

Figure 1 *The pdf of the skewed inverse reflected gamma distribution (given by equation (5)) for* $c = 0, 1, 3, 5$ *and* $\alpha = 1$.

for $c > 0$ and $\alpha > k$, where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. An alternative representation in terms of the gamma functions is

$$E(Z^k; c) = \frac{\Gamma(\alpha - k)}{\Gamma(\alpha)} - \frac{(1 - (-1)^k)}{2\Gamma^2(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(\alpha + i)c^{\alpha+i}} \Gamma(2\alpha + i - k).$$

For $c < 0$, the kth moment can be evaluated by the fact that $E(Z^k; c) = (-1)^k E(Z^k; -c)$. For noninteger $x > 0$, the formula $\Gamma(1 - x) = \frac{\pi}{\Gamma(x) \sin \pi x}$ can be used for the evaluation of the kth moment. The first four moments of Z are

$$E(Z) = \begin{cases} \frac{1}{\alpha - 1} + \frac{1}{\Gamma^2(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(\alpha + i)c^{\alpha+i}} \Gamma(2\alpha + i - 1), & \text{if } c > 0, \\ -\frac{1}{\alpha - 1} - \frac{1}{\Gamma^2(\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(\alpha + i)(-c)^{\alpha+i}} \Gamma(2\alpha + i - 1), & \text{if } c < 0, \end{cases}$$

$$E(Z^2) = \frac{1}{(\alpha - 1)(\alpha - 2)},$$

$$E(Z^3) = \begin{cases} \dfrac{1}{(\alpha-1)(\alpha-2)(\alpha-3)} \\ \qquad + \dfrac{1}{\Gamma^2(\alpha)} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{i!(\alpha+i)c^{\alpha+i}} \Gamma(2\alpha+i-3), & \text{if } c > 0, \\ -\dfrac{1}{(\alpha-1)(\alpha-2)(\alpha-3)} \\ \qquad - \dfrac{1}{\Gamma^2(\alpha)} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{i!(\alpha+i)(-c)^{\alpha+i}} \Gamma(2\alpha+i-3), & \text{if } c < 0, \end{cases}$$

and

$$E(Z^4) = \frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}.$$

The skewness and the kurtosis of Z can be calculated by using the relationships that

$$\mathit{Skewness}(Z) = \frac{E(Z^3) - 3E(Z)E(Z^2) + 2E^3(Z)}{\{E(Z^2) - E^2(Z)\}^{3/2}} \tag{6}$$

and

$$\mathit{Kurtosis}(Z) = \frac{E(Z^4) - 4E(Z)E(Z^3) + 6E(Z^2)E^2(Z) - 3E^4(Z)}{\{E(Z^2) - E^2(Z)\}^2}. \tag{7}$$

Figure 2 shows the flexibility of (5) over (3) in terms of skewness and kurtosis.

3.4 Maximum likelihood estimation

Suppose $\{z_1, z_2, \ldots, z_n\}$ is a random sample from (5). Then the maximum likelihood estimators of c and α are the simultaneous solutions of the equations

$$\sum_{i=1}^{n} \operatorname{sgn}(cz_i) \frac{\Gamma(\alpha)\frac{\partial}{\partial\alpha}\Gamma(\alpha, 1/|cz_i|) - \Gamma'(\alpha)\Gamma(\alpha, 1/|cz_i|)}{\Gamma(\alpha) + \operatorname{sgn}(cz_i)\Gamma(\alpha, 1/|cz_i|)}$$
$$= \Gamma(\alpha)\left\{ n\psi(\alpha) + \sum_{i=1}^{n} \log|z_i| \right\}$$

and

$$\sum_{i=1}^{n} \frac{z_i |cz_i|^{1-\alpha} e^{-1/|cz_i|}}{\Gamma(\alpha) + \operatorname{sgn}(cz_i)\Gamma(\alpha, 1/|cz_i|)} = 0,$$

where $\psi(x) = d \log \Gamma(x)/dx$ is the digamma function. The Fisher information matrix is given by the elements

$$E\left(-\frac{\partial^2 \log L}{\partial\alpha^2}\right) = n\psi'(\alpha) - nI_1 + nI_2,$$

$$E\left(-\frac{\partial^2 \log L}{\partial\alpha\partial c}\right) = nI_3 + nI_4$$

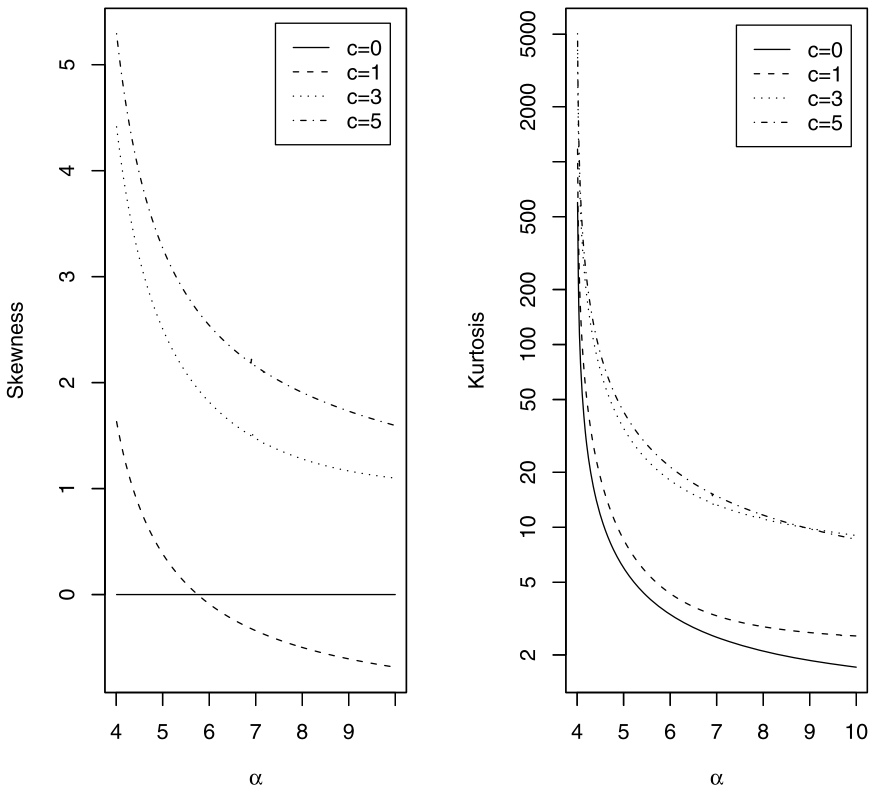

Figure 2 *Skewness and kurtosis of the skewed inverse reflected gamma distribution* (*given by equation* (5)) *versus* α *for* $c = 0, 1, 3, 5$.

and

$$E\left(-\frac{\partial^2 \log L}{\partial c^2}\right) = -nI_5 + nI_6,$$

where

$$I_1 = E\left[\operatorname{sgn}(cZ)\left(\frac{\partial^2}{\partial \alpha^2}\Gamma\left(\alpha, \frac{1}{|cZ|}\right) - \Gamma''(\alpha)\Gamma\left(\alpha, \frac{1}{|cZ|}\right) - \Gamma'(\alpha)\frac{\partial}{\partial \alpha}\Gamma\left(\alpha, \frac{1}{|cZ|}\right)\right)\left(\Gamma(\alpha) + \operatorname{sgn}(cZ)\Gamma\left(\alpha, \frac{1}{|cZ|}\right)\right)^{-1}\right],$$

$$I_2 = E\left[\operatorname{sgn}(cZ)\left\{\Gamma'(\alpha) + \operatorname{sgn}(cZ)\frac{\partial}{\partial \alpha}\Gamma\left(\alpha, \frac{1}{|cZ|}\right)\right\} \times \left\{\Gamma(\alpha)\frac{\partial}{\partial \alpha}\Gamma\left(\alpha, \frac{1}{|cZ|}\right) - \Gamma'(\alpha)\Gamma\left(\alpha, \frac{1}{|cZ|}\right)\right\}\right.$$

$$\times \left\{ \Gamma(\alpha) + \operatorname{sgn}(cZ)\Gamma\left(\alpha, \frac{1}{|cZ|}\right) \right\}^{-2} \Bigg],$$

$$I_3 = E\left[\frac{Z \log|cZ||cZ|^{1-\alpha} e^{-1/|cZ|}}{\Gamma(\alpha) + \operatorname{sgn}(cZ)\Gamma(\alpha, 1/|cZ|)} \right],$$

$$I_4 = E\left[\frac{Z|cZ|^{1-\alpha} e^{-1/|cZ|} \{\Gamma'(\alpha) + \operatorname{sgn}(cZ)\frac{\partial}{\partial\alpha}\Gamma(\alpha, 1/|cZ|)\}}{\{\Gamma(\alpha) + \operatorname{sgn}(cZ)\Gamma(\alpha, 1/|cZ|)\}^2} \right],$$

$$I_5 = E\left[\frac{Z^2 \operatorname{sgn}(cZ)|cZ|^{-1-\alpha} e^{-1/|cZ|}\{(1-\alpha)|cZ| + 1\}}{\Gamma(\alpha) + \operatorname{sgn}(cZ)\Gamma(\alpha, 1/|cZ|)} \right]$$

and

$$I_6 = E\left[\frac{Z^2|cZ|^{2-2\alpha} e^{-2/|cZ|}}{\{\Gamma(\alpha) + \operatorname{sgn}(cZ)\Gamma(\alpha, 1/|cZ|)\}^2} \right].$$

The partial derivatives of $\Gamma(a, z)$ can be calculated by using the facts

$$\frac{\partial \Gamma(a,z)}{\partial a} = \Gamma^2(a) z^a {}_2F_2(a, a; a+1; a+1; -z) + \Gamma(a)\psi(a) - \gamma(a,z)\log z$$

and

$$\begin{aligned} \frac{\partial^2 \Gamma(a,x)}{\partial a^2} = (\log x)^2 \Gamma(a,x) + \Gamma(a)\{\psi^2(a) + \psi'(a) - (\log z)^2\} \\ - \frac{2x^a}{a^3}\{{}_3F_3(a, a, a; a+1, a+1, a+1; -x) \\ - a(\log z)\, {}_2F_2(a, a; a+1, a+1; -x)\}, \end{aligned}$$

where ${}_2F_2(a, b; c, d; z)$ and ${}_3F_3(a, b, c; d, e, f; z)$ denote hypergeometric functions. The method of moments estimators can be obtained as the simultaneous solutions of $E(Z; c) = (1/n)\sum_{i=1}^n z_i$ and $E(Z^2; c) = (1/n)\sum_{i=1}^n z_i^2$ if $c > 0$ and those of $-E(Z; -c) = (1/n)\sum_{i=1}^n z_i$ and $E(Z^2; -c) = (1/n)\sum_{i=1}^n z_i^2$ if $c < 0$.

4 Skewed inverse reflected Weibull distribution

From the reflected Weibull density in Johnson, Kotz and Balakrishnan (1994, 1995) and Fact 1(b), the density of the inverse reflected Weibull random variable can be obtained as

$$g(y) = \frac{\alpha}{2} \frac{1}{|y|^{\alpha+1}} e^{-1/|y|^\alpha} \tag{8}$$

for $y \in R^1$ and $\alpha > 0$, The corresponding CDF is

$$G(y) = \frac{1}{2}\left[1 + \operatorname{sgn}(y) e^{-1/|y|^\alpha}\right]$$

for $y \in R^1$. From formulas 3.381(4) and 8.334(3) in Gradshteyn and Ryzhik (1965), we can obtain the corresponding kth moment of Y as

$$E(Y^k) = \frac{1+(-1)^k}{2}\Gamma\left(1-\frac{k}{\alpha}\right)$$

for $\alpha > k$. In particular, the variance of Y is

$$\mathrm{Var}(Y) = \Gamma\left(1-\frac{2}{\alpha}\right)$$

for $\alpha > 2$.

4.1 PDF and CDF

By Fact 1(d), we can define the skewed inverse reflected Weibull distribution by the PDF

$$f(z;c) = \frac{\alpha}{2}\frac{1}{|z|^{\alpha+1}}e^{-1/|z|^\alpha}\left[1+\mathrm{sgn}(cz)e^{-1/|cz|^\alpha}\right] \tag{9}$$

for $z \in R^1$, $a > 0$ and $c \in R^1$. The corresponding CDF can be expressed as

$$F(z;c) = G(z) - 2I(z;c),$$

where

$$I(z;c) = \frac{1}{4}\frac{c^\alpha}{1+c^\alpha}\left[1-e^{-(1+c^\alpha)/(c^\alpha z^\alpha)}\right]$$

for $z > 0$ and $c > 0$.

4.2 Shape

Figure 3 illustrates possible shapes of (9). Note that the shapes are multimodal. The modes are the solutions of the equation

$$\frac{\mathrm{sgn}(cz)\log|cz|e^{-1/|cz|^\alpha}}{1+\mathrm{sgn}(cz)e^{-1/|cz|^\alpha}} = \frac{\alpha+1}{|z|} - \frac{\alpha}{|z|^{\alpha+1}}.$$

Note that if $c > 0$ then $f(z) \sim \alpha z^{-\alpha-1}$ as $z \to \infty$ and $f(z) \sim \alpha|z|^{-2\alpha-1}/\{2 \mid c \mid^\alpha\}$ as $z \to -\infty$. If $c < 0$ then $f(z) \sim \alpha|z|^{-2\alpha-1}/\{2 \mid c \mid^\alpha\}$ as $z \to \infty$ and $f(z) \sim \alpha|z|^{-\alpha-1}$ as $z \to -\infty$. Also $f(z) \sim (\alpha/2)|z|^{-\alpha-1}e^{-1/|z|^\alpha}$ as $z \to 0$.

4.3 Moments

Using formulas 3.381(4) and 8.334(3) in Gradshteyn and Ryzhik (1965), the corresponding kth moment of Z can be obtained as

$$E(Z^k;c) = \frac{1}{2}\Gamma\left(1-\frac{k}{\alpha}\right)\left[1+(-1)^k+(1-(-1)^k)\left(\frac{c^\alpha}{1+c^\alpha}\right)^{1-k/\alpha}\right]$$

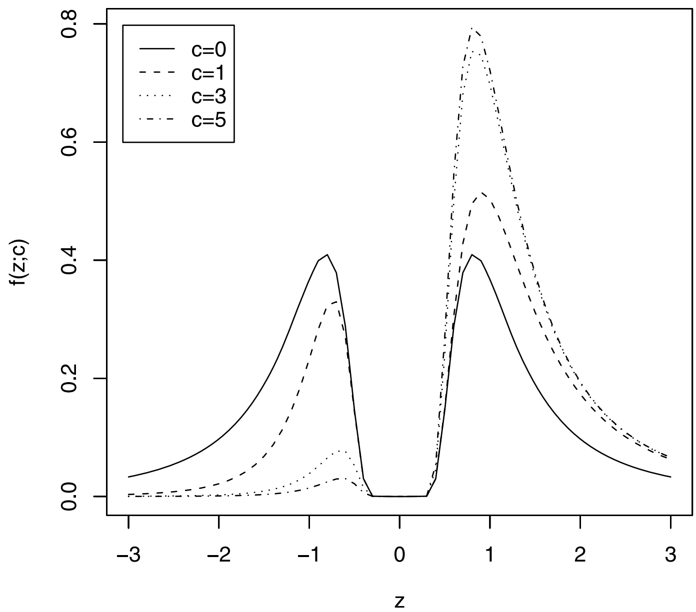

Figure 3 *The pdf of the skewed inverse reflected Weibull distribution* (*given by equation* (9)) *for* $c = 0, 1, 3, 5$ *and* $\alpha = 2$.

for $c > 0$ and $\alpha > k$. If $c < 0$, then we can obtain the kth moment using the fact $E(Z^k; c) = (-1)^k E(Z^k; -c)$. The first four moments of Z are

$$E(Z) = \begin{cases} \Gamma\left(1 - \frac{1}{\alpha}\right)\left(\frac{c^\alpha}{1 + c^\alpha}\right)^{1-1/\alpha}, & \text{if } c > 0, \\ -\Gamma\left(1 - \frac{1}{\alpha}\right)\left(\frac{(-c)^\alpha}{1 + (-c)^\alpha}\right)^{1-1/\alpha}, & \text{if } c < 0, \end{cases}$$

$$E(Z^2) = \Gamma\left(1 - \frac{2}{\alpha}\right),$$

$$E(Z^3) = \begin{cases} \Gamma\left(1 - \frac{3}{\alpha}\right)\left(\frac{c^\alpha}{1 + c^\alpha}\right)^{1-3/\alpha}, & \text{if } c > 0, \\ -\Gamma\left(1 - \frac{3}{\alpha}\right)\left(\frac{(-c)^\alpha}{1 + (-c)^\alpha}\right)^{1-3/\alpha}, & \text{if } c < 0, \end{cases}$$

and

$$E(Z^4) = \Gamma\left(1 - \frac{4}{\alpha}\right).$$

The skewness and the kurtosis of Z can be calculated by using (6) and (7), respectively. Figure 4 shows the flexibility of (9) over (8) in terms of skewness and kurtosis.

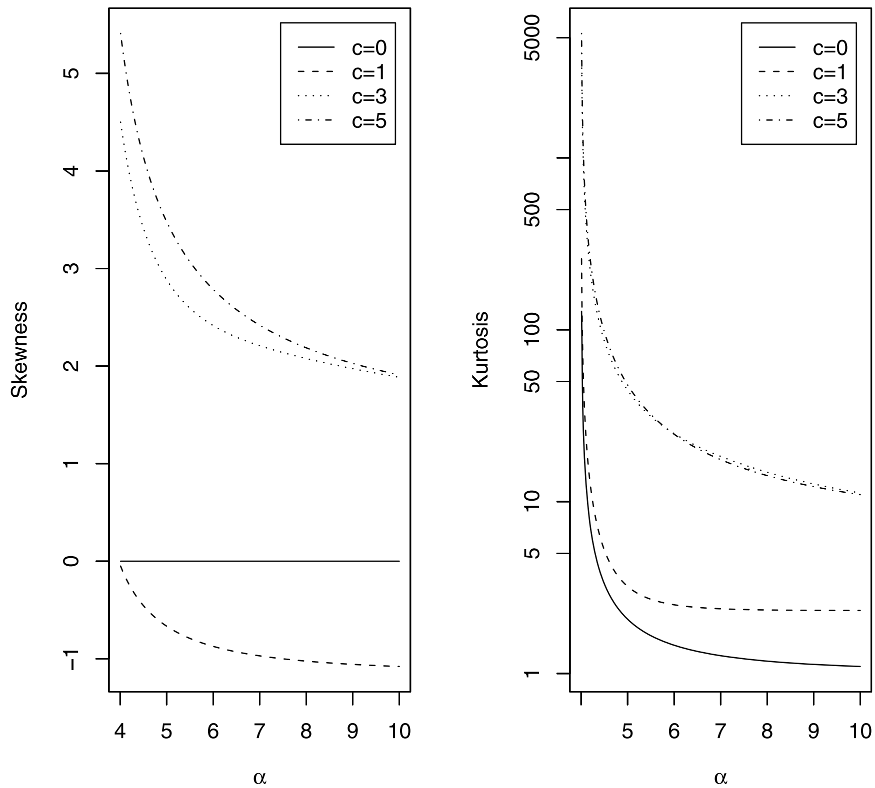

Figure 4 *Skewness and kurtosis of the skewed inverse reflected Weibull distribution (given by equation (9)) versus α for $c = 0, 1, 3, 5$.*

4.4 Maximum likelihood estimation

Suppose $\{z_1, z_2, \ldots, z_n\}$ is a random sample from (9). Then the maximum likelihood estimators of c and α are the simultaneous solutions of the equations

$$\sum_{i=1}^{n} \frac{\operatorname{sgn}(cz_i) \log |cz_i| e^{-1/|cz_i|^\alpha}}{|cz_i|^\alpha [1 + \operatorname{sgn}(cz_i) e^{-1/|cz_i|^\alpha}]} = \sum_{i=1}^{n} \log |z_i| (1 - |z_i|^{-\alpha}) - \frac{n}{\alpha}$$

and

$$\sum_{i=1}^{n} \frac{z_i |cz_i|^{-\alpha-1} e^{-1/|cz_i|^\alpha}}{[1 + \operatorname{sgn}(cz_i) e^{-1/|cz_i|^\alpha}]} = 0.$$

The Fisher information matrix is given by the elements

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\alpha^{-2} + nE[|Z|^{-\alpha} (\log |Z|)^2] - nI_1 + nI_2,$$

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha \, \partial c}\right) = -nI_3 + nI_4$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial c^2}\right) = -nI_5 + nI_6,$$

where

$$I_1 = E\left[\operatorname{sgn}(cZ)\frac{(\log|cZ|)^2|cZ|^{-2\alpha}e^{-1/|cZ|^\alpha}\{1-|cZ|^\alpha\}}{1+\operatorname{sgn}(cZ)e^{-1/|cZ|^\alpha}}\right],$$

$$I_2 = E\left[\frac{(\log|cZ|)^2|cZ|^{-2\alpha}e^{-2/|cZ|^\alpha}}{\{1+\operatorname{sgn}(cZ)e^{-1/|cZ|^\alpha}\}^2}\right],$$

$$I_3 = E\left[\frac{Z|cZ|^{-\alpha-1}e^{-1/|cZ|^\alpha}\{1-\alpha+\alpha|cZ|^{-\alpha}\log|cZ|\}}{1+\operatorname{sgn}(cZ)e^{-1/|cZ|^\alpha}}\right],$$

$$I_4 = \alpha E\left[\operatorname{sgn}(cZ)\frac{Z\log|cZ||cZ|^{-2\alpha-1}e^{-2/|cZ|^\alpha}}{\{1+\operatorname{sgn}(cZ)e^{-1/|cZ|^\alpha}\}^2}\right],$$

$$I_5 = \alpha E\left[\operatorname{sgn}\frac{Z^2(cZ)|cZ|^{-\alpha-2}e^{-1/|cZ|^\alpha}\{-1-\alpha+\alpha|cZ|^{-\alpha}\}}{1+\operatorname{sgn}(cZ)e^{-1/|cZ|^\alpha}}\right]$$

and

$$I_6 = \alpha^2 E\left[\frac{Z^2|cZ|^{-2\alpha-2}e^{-2/|cZ|^\alpha}}{\{1+\operatorname{sgn}(cZ)e^{-1/|cZ|^\alpha}\}^2}\right].$$

The method of moments estimators can be obtained as the simultaneous solutions of $E(Z;c) = (1/n)\sum_{i=1}^n z_i$ and $E(Z^2;c) = (1/n)\sum_{i=1}^n z_i^2$ if $c > 0$ and those of $-E(Z;-c) = (1/n)\sum_{i=1}^n z_i$ and $E(Z^2;-c) = (1/n)\sum_{i=1}^n z_i^2$ if $c < 0$.

5 Skewed inverse reflected Pareto distribution

If a random variable X follows a beta distribution with parameter (α,α) then $Y = (1-X)/X$ follows the inverse reflected Pareto distribution. The PDF of an inverse reflected Pareto random variable is

$$g(y) = \frac{1}{2B(\alpha,\alpha)}|y|^{\alpha-1}(1+|y|)^{-2\alpha} \tag{10}$$

for $y \in R^1$ and $\alpha > 0$. Using formulas 8.391 and 8.392 in Gradshteyn and Ryzhik (1965), the corresponding CDF can be obtained as

$$G(y) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(y)[1 - I_{1/(1+|y|)}(\alpha,\alpha)],$$

where $I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}$ and $B_x(a,b)$ is an incomplete beta function. Using formula 2.19 in Oberhettinger (1974), the corresponding kth moment of Y

is

$$E(Y^k) = \frac{1 + (-1)^k}{2} \frac{\Gamma(\alpha + k)\Gamma(\alpha - k)}{\Gamma^2(\alpha)}$$

for $\alpha > k$.

5.1 PDF and CDF

By Fact 1(d), we can define the skewed inverse reflected Pareto distribution by the PDF

$$f(z; c) = \frac{1}{2B(\alpha, \alpha)} |z|^{\alpha-1}(1 + |z|)^{-2\alpha}\left[1 + \mathrm{sgn}(cz)\left(1 - I_{1/(1+|cz|)}(\alpha, \alpha)\right)\right] \quad (11)$$

for $z \in R^1$ and $c \in R^1$. The corresponding CDF can be expressed as

$$F(z; c) = G(z) - 2I(z; c), \quad (12)$$

where

$$I(z; c) = \frac{1}{4\alpha B(\alpha, \alpha)} \frac{z^\alpha}{(1 + z)^{2\alpha}} {}_2F_1\left(1, 2\alpha; \alpha + 1; \frac{z}{1 + z}\right) - \frac{1}{4\alpha B^2(\alpha, \alpha)} \sum_{i=0}^{\infty} \frac{(\alpha)_i (1 - \alpha)_i}{i!(\alpha + 1)_i} J_i(z; a, c)$$

for $z > 0$ and $c > 0$, where $(a)_i \equiv a(a + 1)(a + 2) \cdots (a + i - 1)$, $(a)_0 \equiv 1$ and

$$J_i(z; a, c) = \int_z^\infty \frac{t^{\alpha-1}}{(1 + t)^{2\alpha}(1 + ct)^{\alpha+i}} \, dt.$$

From formula 2.29 in Oberhettinger (1974), since sum of the powers in denominator is greater than that in the numerator, the integral $J_i(z; a, c)$ converges.

5.2 Shape

Figure 5 illustrates possible shapes of (11). Note that the shapes are multimodal. The modes are the solutions of the equation

$$\frac{|c| \, \mathrm{sgn}(cz)|cz|^{\alpha-1}}{1 + \mathrm{sgn}(cz) I_{|cz|/(1+|cz|)}(\alpha, \alpha)} = B(\alpha, \alpha)(1 + |cz|)^{2\alpha}\left(\frac{2\alpha}{1 + |z|} - \frac{\alpha - 1}{|z|}\right).$$

Note that if $c > 0$ then $f(z) \sim |z|^{-\alpha-1}/B(\alpha, \alpha)$ as $z \to \infty$ and $f(z) \sim |z|^{-2\alpha-1}/\{2\alpha B^2(\alpha, \alpha)|c|^\alpha\}$ as $z \to -\infty$. If $c < 0$ then $f(z) \sim |z|^{-2\alpha-1}/\{2\alpha B^2(\alpha, \alpha)|c|^\alpha\}$ as $z \to \infty$ and $f(z) \sim |z|^{-\alpha-1}/B(\alpha, \alpha)$ as $z \to -\infty$. Also $f(z) \sim |z|^{\alpha-1}/\{2B(\alpha, \alpha)\}$ as $z \to 0$.

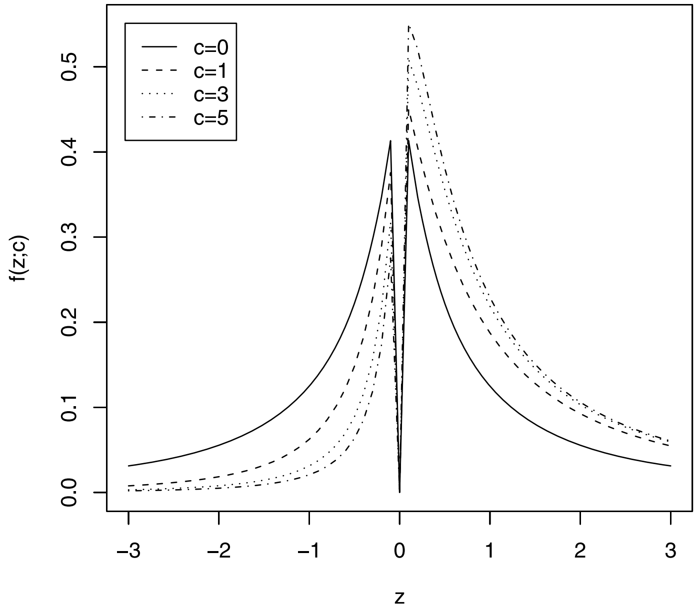

Figure 5 *The pdf of the skewed inverse reflected Pareto distribution* (*given by equation* (11)) *for* $c = 0, 1, 3, 5$ *and* $\alpha = 1$.

5.3 Moments

Using formulas 2.19 and 2.29 in Oberhettinger (1974), formula 8.391 in Gradshteyn and Ryzhik (1965) and the representation

$$ {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, $$

the kth moment of Z corresponding to (11) can be obtained as

$$ E(Z^k; c) = \frac{3 - (-1)^k}{2} \frac{\Gamma(\alpha + k)\Gamma(\alpha - k)}{\Gamma^2(\alpha)} + \frac{(-1)^k - 1}{\alpha B^2(\alpha, \alpha)} I $$

for $\alpha > k$ and $c > 0$, where I is given by

$$ I = \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{(\alpha)_i (1-\alpha)_i}{i!(\alpha+1)_i} B(k+\alpha, 2\alpha+i-k) \\ \qquad \times {}_2F_1(-\alpha - i, \alpha + k; 3\alpha + i; 1 - c), & \text{if } 0 < c < 1, \\ \displaystyle\sum_{i=0}^{\infty} \frac{(\alpha)_i (1-\alpha)_i}{i!(\alpha+1)_i} c^{-k-1} B(\alpha+k, 2\alpha+i-k) \\ \qquad \times {}_2F_1\left(-2\alpha, k+\alpha; 3\alpha+i; 1 - \frac{1}{c}\right), & \text{if } c > 1. \end{cases} $$

If c is negative, then the kth moment can be evaluated using the fact $E(Z^k; c) = (-1)^k E(Z^k; -c)$. The first four moments of Z are

$$E(Z) = \begin{cases} \dfrac{2\alpha}{\alpha - 1} - \dfrac{2}{\alpha B^2(\alpha, \alpha)} I(c), & \text{if } c > 0, \\ -\dfrac{2\alpha}{\alpha - 1} + \dfrac{2}{\alpha B^2(\alpha, \alpha)} I(-c), & \text{if } c < 0, \end{cases}$$

$$E(Z^2) = \frac{\alpha(\alpha + 1)}{(\alpha - 1)(\alpha - 2)},$$

$$E(Z^3) = \begin{cases} \dfrac{2\alpha(\alpha + 1)(\alpha + 2)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} - \dfrac{2}{\alpha B^2(\alpha, \alpha)} I(c), & \text{if } c > 0, \\ -\dfrac{2\alpha(\alpha + 1)(\alpha + 2)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} + \dfrac{2}{\alpha B^2(\alpha, \alpha)} I(-c), & \text{if } c < 0, \end{cases}$$

and

$$E(Z^4) = \frac{\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)}.$$

The skewness and the kurtosis of Z can be calculated by using (6) and (7), respectively. Figure 6 shows the flexibility of (11) over (10) in terms of skewness and kurtosis.

5.4 Maximum likelihood estimation

Suppose $\{z_1, z_2, \ldots, z_n\}$ is a random sample from (11). Then the maximum likelihood estimators of c and α are the simultaneous solutions of the equations

$$\sum_{i=1}^{n} \frac{\operatorname{sgn}(cz_i) \frac{\partial}{\partial \alpha} I_{|cz_i|/(1+|cz_i|)}(\alpha, \alpha)}{1 + \operatorname{sgn}(cz_i) I_{|cz_i|/(1+|cz_i|)}(\alpha, \alpha)} = 2n\psi(\alpha) - 2n\psi(2\alpha) - \sum_{i=1}^{n} \log |z_i| + 2 \sum_{i=1}^{n} \log(1 + |z_i|)$$

and

$$\sum_{i=1}^{n} \frac{z_i |cz_i|^{\alpha - 1}}{B(\alpha, \alpha)(1 + |cz_i|)^{2\alpha} [1 + \operatorname{sgn}(cz_i) I_{|cz_i|/(1+|cz_i|)}(\alpha, \alpha)]} = 0.$$

The Fisher information matrix is given by the elements

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) = 2n\psi'(\alpha) - 4n\psi'(2\alpha) - nI_1 + nI_2,$$

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha \, \partial c}\right) = -nI_3 + nI_4$$

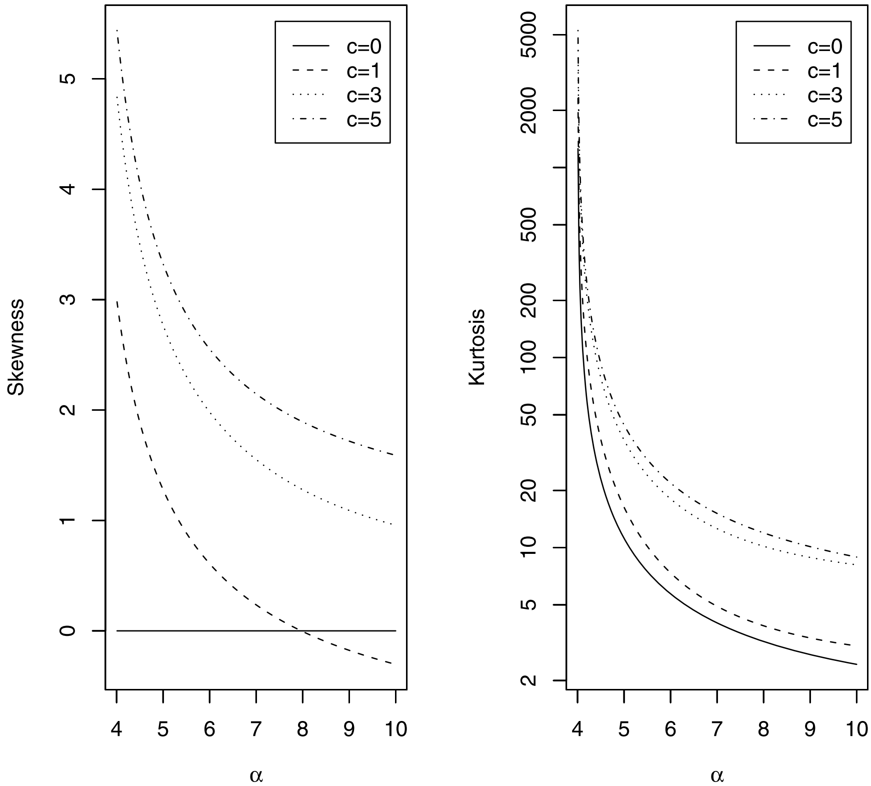

Figure 6 *Skewness and kurtosis of the skewed inverse reflected Pareto distribution* (*given by equation* (11)) *versus* α *for* $c = 0, 1, 3, 5$.

and

$$E\left(-\frac{\partial^2 \log L}{\partial c^2}\right) = -nI_5 + nI_6,$$

where

$$I_1 = E\left[\operatorname{sgn}(cZ)\frac{\frac{\partial^2}{\partial\alpha^2} I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)}{1+\operatorname{sgn}(cZ) I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)}\right],$$

$$I_2 = E\left[\frac{\{\frac{\partial}{\partial\alpha} I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)\}^2}{\{1+\operatorname{sgn}(cZ) I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)\}^2}\right],$$

$$I_3 = E\left[\operatorname{sgn}(cZ)\frac{(\partial/\partial\alpha)\{B^{-1}(\alpha,\alpha) Z|cZ|^{\alpha-1}(1+|cZ|)^{-2\alpha}\}}{1+\operatorname{sgn}(cZ) I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)}\right],$$

$$I_4 = E\left[\operatorname{sgn}(cZ)\frac{Z|cZ|^{\alpha-1}(1+|cZ|)^{-2\alpha}\frac{\partial}{\partial\alpha} I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)}{B(\alpha,\alpha)\{1+\operatorname{sgn}(cZ) I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)\}^2}\right],$$

$$I_5 = E\left[\operatorname{sgn}(cZ) \frac{Z^2|cZ|^{\alpha-2}(1+|cZ|)^{-2\alpha-1}\{(\alpha-1)(1+|cZ|)-2\alpha|cZ|\}}{B(\alpha,\alpha)\{1+\operatorname{sgn}(cZ) I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)\}}\right]$$

and

$$I_6 = E\left[\frac{Z^2|cZ|^{2\alpha-2}(1+|cZ|)^{-4\alpha}}{B^2(\alpha,\alpha)\{1+\operatorname{sgn}(cZ) I_{|cZ|/(1+|cZ|)}(\alpha,\alpha)\}^2}\right].$$

The partial derivatives of $I_z(a,b)$ can be calculated by using the facts

$$\begin{aligned}
\frac{\partial I_z(a,b)}{\partial a} &= \{\log z - \psi(a) + \psi(a+b)\} I_z(a,b) \\
&\quad - \frac{\Gamma(a)\Gamma(a+b)}{\Gamma(b)} z^a {}_3F_2(a,a,1-b;a+1;a+1;z), \\
\frac{\partial I_z(a,b)}{\partial b} &= \frac{\Gamma(b)\Gamma(a+b)}{\Gamma(a)} (1-z)^b {}_3F_2(b,b,1-a;b+1;b+1;1-z) \\
&\quad + \{\psi(b) - \psi(a+b) - \log(1-z)\} I_{1-z}(b,a), \\
\frac{\partial^2 I_z(a,b)}{\partial a^2} &= \frac{2\Gamma(a)\Gamma(a+b)}{\Gamma(b)} \\
&\quad \times z^a [\Gamma(a) {}_4F_3(a,a,a,1-b;a+1,a+1,a+1;z) \\
&\qquad - \{\log z - \psi(a) + \psi(a+b)\} \\
&\qquad\quad \times {}_3F_2(a,a,1-b;a+1,a+1;z)] \\
&\quad + I_z(a,b)[\log^2 z + 2\psi(a+b)\log z + \psi^2(a) + \psi^2(a+b) \\
&\qquad\qquad - 2\psi(a)\{\log z + \psi(a+b)\} - \psi'(a) + \psi'(a+b)]
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 I_z(a,b)}{\partial b^2} &= \frac{2\Gamma(b)\Gamma(a+b)}{\Gamma(a)(1-z)^{-b}} \\
&\quad \times [\{\log(1-z) - \psi(b) + \psi(a+b)\} \\
&\qquad \times {}_3F_2(b,b,1-a;b+1,b+1;1-z) \\
&\qquad - \Gamma(b) {}_4F_3(b,b,b,1-a;b+1,b+1,b+1;1-z)] \\
&\quad - I_{1-z}(b,a)[\log^2(1-z) + 2\psi(a+b)\log(1-z) \\
&\qquad\qquad + \psi^2(b) + \psi^2(a+b) \\
&\qquad\qquad - 2\psi(b)\{\log(1-z) + \psi(a+b)\} \\
&\qquad\qquad - \psi'(b) + \psi'(a+b)],
\end{aligned}$$

where ${}_3F_2(a,b,c;d,e;z)$ and ${}_4F_3(a,b,c,d;e,f,g;z)$ denote hypergeometric functions. The method of moments estimators can be obtained as the simultaneous solutions of $E(Z;c) = (1/n)\sum_{i=1}^{n} z_i$ and $E(Z^2;c) = (1/n)\sum_{i=1}^{n} z_i^2$ if $c > 0$ and those of $-E(Z;-c) = (1/n)\sum_{i=1}^{n} z_i$ and $E(Z^2;-c) = (1/n)\sum_{i=1}^{n} z_i^2$ if $c < 0$.

6 Application

As mentioned in Section 1, the most well known of the skewed symmetric distributions is the skewed normal distribution in (2) due to Azzalini (1985). The aim here is to illustrate the usefulness of the skewed symmetric distributions proposed in Sections 3 to 5 over the skewed normal distribution. We use the following real data: the eruption times in minutes for the Old Faithful Geyser in Yellowstone National Park, Wyoming, USA. This data has been studied by many authors; see Azzalini and Bowman (1990), for instance. It is known to be bimodal.

The skewed symmetric distributions in this paper are defined over the entire real line. So, to have them as possible models for the data, we define the standardized eruption time = (eruption time $- m)/s$, where m and s are the observed mean and standard deviation, respectively.

We fitted scale variations of (2), (5), (9) and (11) to the standardized eruption time data. The maximum likelihood procedures described in Sections 3.4, 4.4 and 5.4 were used. The results were that each of (5), (9) and (11) provided a significantly better fit than (2). We give some details comparing (2) with (11).

Our first comparison is based on the negative log likelihoods for (2) and (11): 408.8 and 385.5, respectively. These two models, with the former having one less parameter, are not nested. However, the log likelihood values can be compared by using Akaike's information criteria. Since the difference, $2 \times (408.8 - 385.5)$, is so large it follows that (11) provides a significantly better fit.

Our second comparison is based on probability plots. A probability plot consists of plots of the observed probabilities against the probabilities predicted by the fitted model. In case of the skewed inverse reflected Pareto distribution, for example, this amounts to plotting $F(x_{(i)})$ computed using (12) versus $(i - 0.375)/(n + 0.25)$, $i = 1, 2, \dots, n$ [as recommended by Blom (1958) and Chambers et al. (1983)], where $x_{(i)}$ are the sorted values of the standardized eruption times in the ascending order and n is the number of observations. The probability plot comparing (2) with (11) is shown in Figure 7. The closeness of the plotted points to the 45 degree line is a measure of the goodness of fit of the model. A numerical measure of closeness is the sum of the absolute differences between the observed and expected probabilities. The values of this numerical measure for (2) and (11) are 24.5 and 18.7, respectively. This supports the conclusions of our first method, that is, the skewed inverse reflected Pareto distribution gives a better fit than the skewed normal distribution.

Our third and final method for comparison is based on density plots. A density plot simply compares the fitted estimates of (2) and (11) with the histogram of the standardized eruption time data; see Figure 8. The closeness of the fitted densities

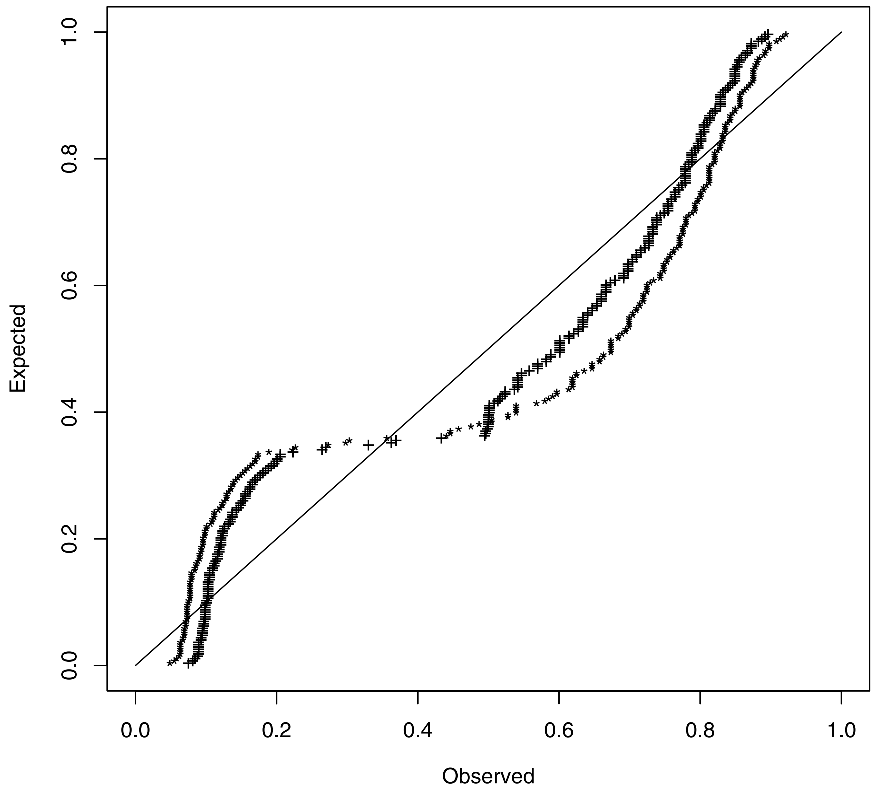

Figure 7 *Probability plots of the skewed normal distribution* ($\star$) *and the skewed inverse reflected Pareto distribution* ($+$) *for the Old Faithful geyser data.*

to the histogram is a measure of the goodness of fit of the model. A visual inspection of the closeness suggests that the conclusions from our first two methods are supported.

7 Conclusions

We have introduced three skew symmetric distributions, referred to as the the skewed reflected gamma distribution, skewed reflected Weibull distribution and the skewed reflected Pareto distribution. We have derived their mathematical properties (PDF, CDF, moments and shape), provided estimation procedures along with the Fisher information matrices, shown evidence of their flexibility and provided an illustration using real data. Some of the attractive properties are the multimodality and polynomials tails (lower and upper). None of the existing skew symmetric distributions have both multimodality and polynomials tails.

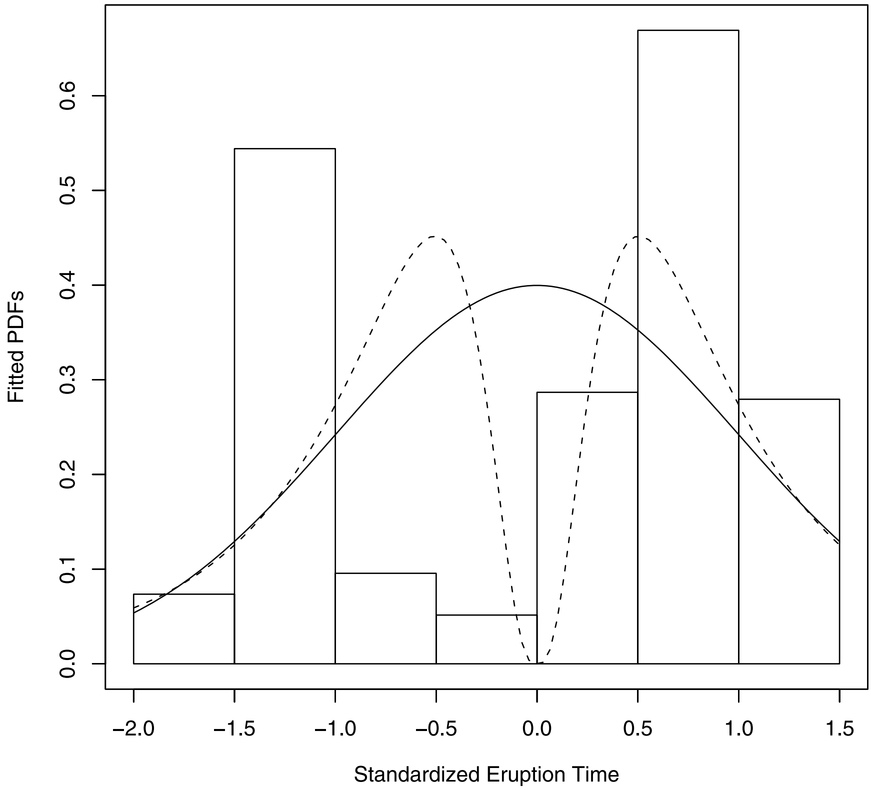

Figure 8 *Fitted densities of the skewed normal distribution* (*solid curve*) *and the skewed inverse reflected Pareto distribution* (*broken curve*) *for the Old Faithful geyser data.*

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