A GEOMETRICAL STRUCTURE FOR AN INFINITE ORIENTED CLUSTER AND ITS UNIQUENESS

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We consider the supercritical oriented percolation model. Let $\mathcal{K}$ be all the percolation points. For each $u \in \mathcal{K}$, we write γ_u as its rightmost path. Let $G = \bigcup_u \gamma_u$. In this paper, we show that G is a single tree with only one topological end. We also present a relationship between $\mathcal{K}$ and G and construct a bijection between $\mathcal{K}$ and $\mathbb{Z}$ using the preorder traversal algorithm. Through applications of this fundamental graph property, we show the uniqueness of an infinite oriented cluster by ignoring finite vertices.

1. Introduction and statement of the results. We consider the graph with vertices $\mathcal{L} = \{(m, n) \in \mathbb{Z}^2 : m + n \text{ is even}\}$ and oriented edges from (m, n) to $(m + 1, n + 1)$ and to $(m - 1, n + 1)$. The oriented edge from u to v is denoted by $[u, v\rangle$. As usual, each edge is independently open or closed with a probability of p or $1 - p$. We denote by $\mathbb{P}_p$ the corresponding product measure and by $\mathbb{E}_p$ the expectation with respect to $\mathbb{P}_p$. For two vertices $u, v \in \mathcal{L}$, we say v can be reached from u, denoted by $u \to v$, if there is a sequence of vertices and edges $v_0 = u, e_1, v_1, \ldots, v_{m-1}, e_m, v_m = v$ such that $e_i = [v_{i-1}, v_i\rangle$ is open for $1 \le i \le m$. If there is no such sequence, we say v cannot be reached from u and denote it by $u \nrightarrow v$. We define the oriented percolation cluster at $(x, y) \in \mathcal{L}$ by

$$C_{(x,y)} = \{(z, w) \in \mathcal{L} : (x, y) \to (z, w)\}.$$

Let

$$\Omega_{(x,y)} = \{|C_{(x,y)}| = \infty\}.$$

The percolation probability and the critical point are defined by

$$\theta(p) = \mathbb{P}_p(\Omega_{(0,0)}) \quad \text{and} \quad \vec{p}_c = \sup\{p : \theta(p) = 0\}.$$

It is well known that

$$0 < \vec{p}_c < 1.$$

By definition, we know that

$$\theta(p) = 0 \quad \text{if } p < \vec{p}_c \quad \text{and} \quad \theta(p) > 0 \quad \text{if } p > \vec{p}_c.$$

Furthermore, Bezuidenhout and Grimmett [2] showed that

$$(1.1) \qquad\qquad \theta(p) = 0 \Leftrightarrow p \le \vec{p}_c.$$

We define $(x, y) \in \mathcal{L}$ to be a *percolation point* if $|C_{(x,y)}| = \infty$. By (1.1), if $p \le \vec{p}_c$, there is no percolation point, but if $p > \vec{p}_c$, there are infinitely many percolation points. When $p > \vec{p}_c$, we collect all percolation points and denote them by

$$(1.2) \qquad\qquad \mathcal{K} = \{(x, y) \in \mathcal{L} : |C_{(x,y)}| = \infty\}.$$

To understand the oriented clusters, we need to establish their boundaries. For $(x, y) \in \mathcal{K}$, let $\gamma_{(x,y)}$ be the *rightmost* infinite open path starting at (x, y). More precisely, let $\gamma_{(x,y)}$ be the infinite sequence of vertices and open oriented edges $v_0 = (x, y), e_1, v_1, \dots, e_n, v_n, \dots$, with $v_n = (x_n, y_n)$ and $e_n = [v_{n-1}, v_n\rangle$ satisfying

$$\{k \ge 1 : (x, y) \to (x_n + k, y_n) \in \mathcal{K}\} = \varnothing \qquad \text{for all } n \ge 1.$$

Similarly, we may define the *leftmost* infinite open path as $\ell_{(x,y)}$, starting at (x, y) by changing $k \ge 1$ in the above equation to $k \le -1$. With these definitions, note that any infinite oriented path of $C_{(x,y)}$ will stay in the cone between $\gamma_{(x,y)}$ and $\ell_{(x,y)}$.

For any infinite oriented open path Γ and vertex $v \in \Gamma$, let

$$b_r(v, \Gamma) := \{u \in \mathcal{L} \setminus \Gamma : u \text{ lies to the right of } \Gamma \text{ and}$$

$$v \to u \text{ uses no edges of } \Gamma\},$$

(1.3)

$$b_l(v, \Gamma) := \{u \in \mathcal{L} \setminus \Gamma : u \text{ lies to the left of } \Gamma \text{ and}$$

$$v \to u \text{ uses no edges of } \Gamma\}.$$

These will be the right and left *buds* of Γ planted in v (see Figure 2). For two vertices u and v of Γ such that $u \to v$ in Γ, we write $\Gamma(u, v)$ for the finite piece of Γ from u to v, and let

$$C_r(\Gamma(u, v)) := \bigcup_{v' \in \Gamma(u,v) \setminus \{v\}} b_r(v', \Gamma),$$

(1.4)

$$C_l(\Gamma(u, v)) := \bigcup_{v' \in \Gamma(u,v) \setminus \{v\}} b_l(v', \Gamma).$$

Clearly, if Γ is a rightmost (resp., leftmost) path, then all right (resp., left) buds of Γ are finite, which implies that $C_r(\Gamma(u, v))$ [resp., $C_l(\Gamma(u, v))$] is finite.

Let G be the random oriented graph consisting of $\gamma_{(x,y)}$ for $(x, y) \in \mathcal{K}$. In other words, the vertex set of G is in $\mathcal{K}$ and the edges of G are open edges in $\gamma_{(x,y)}$ for $(x, y) \in \mathcal{K}$. Clearly, there are no loops in G, and G is defined as a *forest*.

Each vertex $u \in \mathcal{L}$ is adjacent to two edges above u, denoted by *upper edges*, and two edges below u, denoted by *lower edges*. Note that G consists of oriented paths without loops, so each vertex u of G is adjacent to only one upper edge in G. We define the other endpoint of the upper edge as the *mother vertex* of u. On the other hand, u is also adjacent to one or two lower edges in G. We define the other vertex or vertices of the lower edge or edges of u as the *daughter vertex* or the *daughter vertices*. By the definition of G, every vertex of G has a mother vertex and at most two daughter vertices. If u has two daughter vertices, they are *sisters*, and the vertex at the left lower edge is the *older sister* and the vertex at the right lower edge is the *younger sister*.

Let $M(u)$ denote the mother vertex of u, and iteratively for $n \geq 1$, let $M^n(u) = M(M^{n-1}(u))$ denote the nth ancestor of u, where $M^0(u) = u$. Define

$$D^n(u, G) := \{v \in \mathcal{K} : M^n(v) = u\} \qquad \text{for } n \geq 0$$

and

$$D(u, G) := \bigcup_{n \geq 0} D^n(u, G),$$

the nth generation and the set of all descendants of u. We call $D(u, G)$ the *branch* of u. If $D(u, G)$ is finite for all $u \in \mathcal{K}$, we say that G has finite branches. We say that two vertices $u, v \in \mathcal{K}$ are connected if they have a common ancestor, that is, there exist nonnegative integers n and m such that $M^n(u) = M^m(v)$. This defines an equivalence relation in $\mathcal{K}$, and the equivalence classes are called *connected components*. Moreover, a connected component of G is obviously a *single tree*.

In graph G, a one-way infinite path is called a *ray*. If we remove finitely many vertices from a ray, the rest of the connected infinite part is still a ray that is called the *tail* of the ray. Two rays R and R' are *equivalent* if their tails differ in at most finitely many sites. This is an equivalence relation on the set of rays in G, and the equivalence classes are called *topological ends* (or, equivalently, graph-theoretical ends) of G. Now, with these definitions, we state a fundamental property for graph G as follows.

THEOREM 1.1.   *For any $p \in (\vec{p}_c, 1)$, and the oriented graph G defined above, we have*:

   (i) *G has a unique connected component*,
  (ii) *G has finite branches*, *and*
 (iii) *Each vertex of G has an ancestor with a younger sister almost surely.*

Obviously, items (i) and (ii) of Theorem 1.1 tell us that any two rays R and R' in G have the same tail. This gives the following corollary.

COROLLARY 1.2. *G has one topological end.*

For any vertex u of G, let $\sigma(u) = 1$ if u is older than her sister, and let $\sigma(u) = 2$ otherwise. We associate to each vertex u the sequence of relative sister-order of its ancestors: let $\sigma_i(u) := \sigma(M^i(u))$, $i \geq 0$. For any vertices u and v of G, because G is a single tree, u and v have common ancestors. Let $z = M^i(u) = M^j(v)$ be their closest common ancestor. We say *u precedes v* if $\sigma_{i-1}(u) < \sigma_{j-1}(v)$, where $\sigma_{-1}(u) = 0$.

On the other hand, we define the *successor* of $u \in \mathcal{K}$ as $u' \in \mathcal{K}$ if u precedes u' and no vertex precedes u and u'. Conversely, u is the *predecessor* of u' if and only if u' is the successor of u.

The successor of a vertex can be found by using the following algorithm. If the vertex has a daughter, we choose the older one. Otherwise, we move up the tree until we hit the first vertex that has a younger sister; this younger sister will be the successor. Note that the existence of such a vertex is guaranteed by Theorem 1.1(iii).

The predecessor vertex can be also found by using this algorithm. If the vertex is the older sister, her mother will be the predecessor vertex. If the vertex has an older sister, we move from her older sister down the tree and choose the younger daughter at each step until we come to a vertex with no daughter. This will be the predecessor. Because G has finite branches, the predecessor vertex can always be found.

We say that there is a *succession line* from u to v if there exists a finite sequence of vertices $u = v_0, \ldots, v_k = v$ such that v_{i-1} is the successor of v_i for $i = 1, 2, \ldots, k$. We say G has a *unique infinite succession line* if, for every couple of vertices u and v, there is a unique succession line either from u to v or from v to u.

With these definitions, by Theorem 1.1, we have the following corollary.

COROLLARY 1.3. *G has a unique infinite succession line. The maps $\Pi(u, G) = u'$ (the successor of u) and $\Pi^{-1}(u', G) = u$ (the predecessor of u') are well defined, and one is the inverse of the other. Furthermore, $\mathcal{K} = \{\Pi^n(u, G) : n \in \mathbb{Z}\}$ for all $u \in \mathcal{K}$.*

The succession line was first studied in Ferrari, Landim and Thorisson [5] for *Poisson trees* defined from the two- or three-dimensional Poisson point processes. By constructing succession lines of Poisson trees, Ferrari, Landim and Thorisson [5] proved the *point-stationary* property of the Palm version of the Poisson point processes. The concept *point-stationary* is defined in Thorisson [10] and is shown to be the characterizing property of the Palm version of any stationary point process in $\mathbb{R}^d$.

In view of point processes, $\mathcal{K}$ is a (discrete version) translation-invariant point process of $\mathbb{R}^2$. On event $\Omega_{(0,0)}$, if we let $\mathcal{K}_0 := \{(x, y) \in \mathcal{L} : |C_{(x,y)}| = |C_{(0,0)}| =$

∞}, then by Theorem 5.1 of [5] and Corollary 1.3, we know that $\mathcal{K}_0$ is point-stationary and $\mathcal{K}_0$ is the Palm version of $\mathcal{K}$.

With Theorem 1.1 in hand, we may try to ask the question: when properly rescaled, does G converge to the Brownian Web? (See the definition of the Brownian Web and the related theorems by Fontes, Isopi, Newman and Ravishankar [6, 7].) Note that by using the convergence criteria given in [6, 7], Ferrari, Fontes and Wu [4] proved that the two-dimensional Poisson trees converge to the Brownian Web. In fact, one of the original motivations for this paper was to investigate the convergence of the percolation system to the Brownian Web. At this point, we are unable to show this argument.

By applying this fundamental graph property of G, we will try to characterize infinite oriented clusters. In fact, one of the most important questions in percolation models is to investigate the uniqueness of infinite clusters. For two different percolation points (x_1, y_1) and (x_2, y_2), Grimmett and Hiemer [8] worked on the first step of uniqueness to show that, for some $(x_3, y_3) \in \mathcal{K}$,

$$C_{(x_1, y_1)} \cap C_{(x_2, y_2)} \supset C_{(x_3, y_3)}.$$

Recall that for a general percolation model, Aizeman, Kesten and Newman [1] showed the uniqueness of infinite clusters. We may ask the uniqueness of infinite oriented clusters. Clearly, $C_{(x_1, y_1)} \neq C_{(x_2, y_2)}$ for two percolation points (x_1, y_1) and (x_2, y_2), since we are investigating oriented paths. However, even though two infinite oriented clusters are always different, they might be different only in finitely many vertices. In other words, the main infinite parts of two oriented clusters are the same. With this observation, we may modify the definition of uniqueness to investigate infinite parts of $C_{(x_1, y_1)}$ and $C_{(x_2, y_2)}$. More precisely, for two percolation points (x_1, y_1) and (x_2, y_2), we say

$$C_{(x_1, y_1)} \asymp C_{(x_2, y_2)} \qquad \text{if } C_{(x_1, y_1)} = C_{(x_2, y_2)} \text{ except finitely many vertices.}$$

With this new "$\asymp$," we may ask what the uniqueness of infinite oriented clusters is. As expected, we show the following result.

THEOREM 1.4. *Under the definition of "$\asymp$," there is only one infinite oriented cluster.*

2. Kuczek's construction, coalescing random walks and CLT. We use the notation in Kuczek [9] in Section 2. For $A \subset (-\infty, \infty)$, we denote a random subset by

$$\xi_n^A = \{x : \exists x' \in A \text{ such that } (x', 0) \to (x, n)\}, \qquad n > 0.$$

The right edge of $\xi_n^{(-\infty, 0]}$ is defined by

$$r_n = \sup \xi_n^{(-\infty, 0]} \qquad (\text{where } \sup \varnothing = -\infty).$$

We know (see page 1004 in [3]) by using a subadditive argument that there exists a nonrandom constant $\alpha(p)$ such that

$$\lim_{n \to \infty} \frac{r_n}{n} = \inf_n \left\{ \frac{\mathbb{E}_p(r_n)}{n} \right\} = \alpha(p) \qquad \text{a.s. and in } L_1.$$

It has been proved in Benzuidenhout and Grimmett [2] and Durrett [3] that

$$\alpha(p) = -\infty \quad \text{if } p < \vec{p}_c \quad \text{and} \quad \alpha(\vec{p}_c) = 0 \quad \text{and} \quad 1 \geq \alpha(p) > 0 \quad \text{if } p > \vec{p}_c.$$

Moreover, $\alpha(p)$ is infinitely differentiable for all $p \in (\vec{p}_c, 1)$ (see Zhang [11]).

Let us now denote

$$\xi_0' = \xi_0^{\{0\}},$$

and for all $n \geq 0$,

$$\xi_{n+1}' = \begin{cases} \{x : (y, n) \to (x, n+1) \text{ for some } y \in \xi_n'\}, & \text{if this set is nonempty;} \\ \{n+1\}, & \text{if otherwise.} \end{cases}$$

Let $r_n' = \sup \xi_n'$. On the event $\Omega_{(0,0)}$, we know that

$$r_n' = r_n \geq \gamma_{(0,0)}(n),$$

where $\gamma_{(0,0)}(n) \in \mathbb{Z}$ satisfies $(\gamma_{(0,0)}(n), n) \in \gamma_{(0,0)}$.

Let $T_0 = 0$ and $T_m = \inf\{n \geq T_{m-1} + 1 : (r_n', n) \in \mathcal{K}\}$ for $m \geq 1$. Define $\tau_0 = 0$ and

$$\tau_1 = T_1, \qquad \tau_2 = T_2 - T_1, \ldots, \tau_m = T_m - T_{m-1}, \ldots,$$

where $\tau_i = 0$ if T_i and T_{i-1} are infinity. Also define $X_0 = 0$ and

$$X_1 = r_{T_1}', \qquad X_2 = r_{T_2}' - r_{T_1}', \ldots, X_m = r_{T_m}' - r_{T_{m-1}}', \ldots,$$

where $X_i = 0$ if T_i and T_{i-1} are infinity. The collection of points (r_{T_m}', T_m), $m \geq 0$, are called *break points* for point $(0, 0)$.

In the case of $p \in (\vec{p}_c, 1)$, with these definitions, Kuczek [9] proved the following Central Limit Theorem (CLT).

PROPOSITION 2.1. *For $p \in (\vec{p}_c, 1)$, on $\Omega_{(0,0)}$, $\{(X_m, \tau_m) : m \geq 1\}$ are independently identically distributed with all moments, and $\frac{\gamma_{(0,0)}(n) - \alpha(p)n}{\sqrt{n\sigma^2}}$ converges to $N(0, 1)$ in distribution as $n \to \infty$, where $\sigma^2 = \mathbb{E}(X_1 \mathbb{E}\tau_1 - \tau_1 \mathbb{E}X_1)^2 > 0$ and $\mathbb{E}$ is the expectation with respect to the conditional measure $\mathbb{P}_p(\cdot \mid \Omega_{(0,0)})$.*

Actually, Kuczek [9] proved the above CLT for r_n; we point out here that the same argument is also valid for $\gamma_{(0,0)}(n)$.

Now we turn to our rightmost infinite paths. On $\Omega_{(0,0)}$, the rightmost infinite path $\gamma_{(0,0)}$ is well defined and all break points (r_{T_m}, T_m), $m \geq 1$, are well embedded

in $\gamma_{(0,0)}$. By Proposition 2.1, on event $\Omega_{(0,0)}$, we define an integer-valued random walk $\zeta_{(0,0)} = \{\zeta_{(0,0)}(t) : t \geq 0\}$ as follows:

$$\zeta_{(0,0)}(0) := 0 \quad \text{and} \quad \zeta_{(0,0)}(t) := \sum_{i=0}^{N(t)} X_i \qquad \text{for } t > 0,$$

where $N(t)$ is the largest integer m such that $T_m \leq t$. Note that a break point defined above is also a jump point of $\zeta_{(0,0)}$. In the same way, we define the random walk $\zeta_{(x,y)} := \{\zeta_{(x,y)}(t) : t \geq y\}$ on event $\Omega_{(x,y)}$, $(x, y) \in \mathcal{L}$. For any vertices $u_1 = (x_1, y_1), u_2 = (x_2, y_2), \ldots, u_k = (x_k, y_k)$ in $\mathcal{L}$, on event $\Omega_{u_1} \cap \cdots \cap \Omega_{u_k}$, we say random walks $\zeta_{u_{i_1}}, \zeta_{u_{i_2}}$ *meet* if two walks *jump synchronously* to the same position at some time $t_0 (\in \mathbb{Z}) \geq y_{i_1} \vee y_{i_2}$. By this definition of meeting, once two walks meet, they will coalesce into one thenceforth. This defines a finite system of coalescing random walks. For random walks and our rightmost infinite paths, we have the following proposition to describe the relationship between the jump points in two random walks and the meeting points in two rightmost open paths.

PROPOSITION 2.2. *For any $p \in (\vec{p}_c, 1)$ and any pair $u_1, u_2 \in \mathcal{L}$, conditioned on $\Omega_{u_1} \cap \Omega_{u_2}$, the following two statements are equivalent*:

(i) ζ_{u_1} *meets* ζ_{u_2},
(ii) *the rightmost infinite paths γ_{u_1} and γ_{u_2} meet*.

PROOF.   It is clear that (i) implies (ii), so it suffices to prove that (ii) implies (i).
Let Γ_1 and Γ_2 be two realizations of γ_{u_1} and γ_{u_2}, respectively. Without loss of generality, we may assume that Γ_1 and Γ_2 meet at $u_{1,2} \in \mathcal{K}$ and u_1 *precedes* u_2. Recall that the concept of *precedes* is defined in Section 1.
Note that $u_{1,2}$ is a break point for u_1, so it is also a jump point for ζ_{u_1}. To prove Proposition 2.2, it suffices to find a point $v_{1,2} \in \Gamma_1 \cap \Gamma_2$ preceding $u_{1,2}$ such that $v_{1,2}$ is a common jump point of ζ_{u_1} and ζ_{u_2}.
Now let $\{v_m = (x_m, y_m) : m \geq 0\}$ be jump (or break) points of ζ_{u_2} such that $v_0 = u_2$, v_1 precedes $v_0, \ldots$, and v_m precedes v_{m-1}. If $u_{1,2}$ is also one of the jump points for ζ_{u_2}, then $u_{1,2}$ should be the meeting point of ζ_{u_1} and ζ_{u_2}. Proposition 2.2 follows by taking $v_{1,2} = u_{1,2}$. If $u_{1,2}$ is not a jump point for ζ_{u_2}, then there exists some $k \geq 1$ such that $u_{1,2} \in \Gamma_2(v_{k-1}, v_k)$. By the definition of break point, y_k, which is the second coordinate of v_k, is greater than y for any $(x, y) \in C_r(\Gamma_2(v_{k-1}, v_k))$, where $C_r(\Gamma_2(v_{k-1}, v_k))$ is defined in (1.4). Note that $C_r(\Gamma_1(u_{1,2}, v_k)) = C_r(\Gamma_2(u_{1,2}, v_k)) \subset C_r(\Gamma_2(v_{k-1}, v_k))$, so $y_k > y$ for any $(x, y) \in C_r(\Gamma_1(u_{1,2}, v_k))$. By the fact that $u_{1,2}$ is a jump (or break) point for ζ_{u_1} and by the definition of break point, we know that v_k is a jump (or break) point of ζ_{u_1}. Proposition 2.2 follows by taking $v_{1,2} = v_k$.   □

Write $R_{\alpha(p)}$ for the line in $\mathbb{R}^2$ with the equation $y = x/\alpha(p)$. Conditioned on $\Omega_{(0,0)}$, let us consider the behavior of the rightmost infinite path $\gamma_{(0,0)}$. By Proposition 2.1, we have the following proposition.

PROPOSITION 2.3. *Suppose that $p \in (\vec{p}_c, 1)$; then, conditioned on $\Omega_{(0,0)}$, almost surely, the rightmost infinite path $\gamma_{(0,0)}$ crosses the line $R_{\alpha(p)}$ infinitely many times.*

3. Proof of Theorem 1.1. Before our proofs, we need to introduce a few notations (see Figure 1). For any $u = (x, y) \in \mathcal{L}$, we define

$$\bigvee_u := \{v \in \mathcal{L} : \text{there is an oriented path from } u \text{ to } v\}.$$

Note that the oriented path in the definition of $\bigvee_u$ does not need to be open, so $C_u \subset \bigvee_u$. Similarly, we define

$$\bigwedge_u := \{v \in \mathcal{L} : \text{there is an oriented path from } v \text{ to } u\}.$$

For any $n \geq 0$ and $u = (x, y)$, let

$$\bigwedge_u(n) := \left\{ v = (x', y') \in \bigwedge_u : y - y' \leq n \right\}.$$

For a finite set $A \subset \mathcal{L}$ contained in a horizontal line, we define

$$\bigwedge_A := \{v \in \mathcal{L} : \text{there is an oriented path from } v \text{ to some point } u \text{ of } A\}$$

and

$$\bigwedge_A(n) := \left\{ v = (x', y') \in \bigwedge_A : y - y' \leq n \right\} \qquad \text{for } n \geq 0,$$
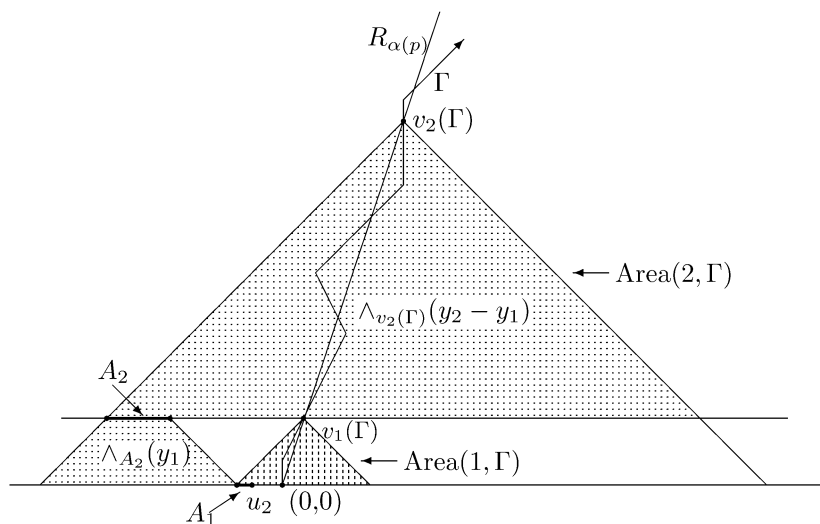

FIG. 1. *This figure reveals that* Area$(1, \Gamma)$ *and* Area$(2, \Gamma)$ *are edge-disjoint areas. The situation for large k is similar.*

where y is the second coordinate of some vertex in A.

In our proofs, we need to use the following *anti-oriented open path*. Given u and $v \in \bigwedge_u$, we say there is an anti-oriented open path from u to v, if $v \to u$. For any $u \in \mathcal{L}$, let

$$C_u^{\text{anti}} := \{v \in \mathcal{L} : v \to u\};$$

clearly, C_u^{anti} is a random subset of $\bigwedge_u$. On event $\{|C_u^{\text{anti}}| = \infty\}$, we write ℓ_u^{anti} for the *leftmost* anti-oriented infinite open path from u.

When $p > \vec{p}_c$, we have

$$(3.1) \qquad \mathbb{P}_p(|C_u^{\text{anti}}| = |C_u| = \infty) = \theta(p)^2 > 0 \qquad \forall u \in \mathcal{L}.$$

Vertex u, satisfying $|C_u^{\text{anti}}| = |C_u| = \infty$, is called a *bidirectional percolation point*. Let $\tilde{\mathcal{K}}$ denote the set of all bidirectional percolation points.

PROOF OF THEOREM 1.1(i). It suffices to prove that, for any vertices $u_1 = (x_1, y_1), u_2 = (x_2, y_2) \in \mathcal{L}$ with $y_1 = y_2$, conditioned on $\Omega_{u_1} \cap \Omega_{u_2}$, γ_{u_1} and γ_{u_2} will meet almost surely. In fact, in the case that $y_1 \neq y_2$, on $\Omega_{u_1} \cap \Omega_{u_2}$, there exists some $u_i' = (x_i', y_i')$, $i = 1, 2$, almost surely in $\mathcal{K}$ with $y_1' = y_2' = b$ such that

$$x_1' < \min\left\{x : (x, b) \in \bigvee_{u_1} \cup \bigvee_{u_2}\right\} \leq \max\left\{x : (x, b) \in \bigvee_{u_1} \cup \bigvee_{u_2}\right\} < x_2'.$$

If $\gamma_{u_1'}$ and $\gamma_{u_2'}$ meet then γ_{u_1} and γ_{u_2} meet.

By translation invariance, we choose $u_2 = (0, 0)$ and $u_1 = (-n_0, 0)$ for some $n_0 \geq 1$. Let us focus on $\gamma_{(0,0)}$. For any realization of Γ of $\gamma_{(0,0)}$, by Proposition 2.3, we may assume that Γ crosses the line $R_{\alpha(p)}$ infinitely many times. For some vertex $v \in \Gamma$, let $e = [u, v\rangle$ be the lower (oriented) edge of v in Γ. We call v a *crossing point* if $[u, v\rangle \cap R_{\alpha(p)} \neq \varnothing$.

Given such a realization of Γ, we define a series of independent events $E(k, \Gamma)$, $k \geq 1$, as follows.

We fix $\epsilon_0 > 0$ such that

$$\int_{-\infty}^{-\epsilon_0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx > \frac{1}{3}.$$

By Proposition 2.2, we choose N_0 large enough such that

$$(3.2) \quad \mathbb{P}_p\big(\gamma_{(0,0)}(n^2) - \alpha(p)n^2 < -n\sigma\epsilon_0 \mid \Omega_{(0,0)}\big) \geq \tfrac{1}{3} \qquad \text{for all } n \geq N_0.$$

Let $v_0(\Gamma) = (x_0, y_0) = (0, 0)$. We go along Γ from $(0, 0)$ to meet $v_1(\Gamma) = (x_1, y_1)$ (see Figure 1), one of crossing points, with $y_1 > \max\{n_0/(\epsilon_0\sigma), N_0\}^2$. Iteratively, we go along Γ from $v_{k-1}(\Gamma)$ to meet $v_k(\Gamma) = (x_k, y_k)$, one of crossing points, with

$$(3.3) \qquad\qquad y_k - y_{k-1} > \max\{2y_{k-1}/(\epsilon_0\sigma), N_0\}^2.$$

Now we define $E(1, \Gamma)$ to be the event that there is an anti-oriented open path from $v_1(\Gamma)$ to the half-line $(-\infty, -n_0] \times \{0\}$. Iteratively, for $k \geq 2$, we define

$E(k, \Gamma)$ to be the event that there is an anti-oriented open path from $v_k(\Gamma)$ to the half-line $(-\infty, x_k - 2y_{k-1}] \times \{y_{k-1}\}$ and then to the line $(-\infty, +\infty) \times \{0\}$. By this definition, $E(k, \Gamma)$, $k \geq 1$, depends on the edges (see Figure 1) in Area$(k, \Gamma) = \bigwedge_{v_k(\Gamma)}(y_k - y_{k-1}) \cup \bigwedge_{A_k}(y_{k-1})$, where

$$A_1 = \left\{ u = (x, y) \in \bigwedge\nolimits_{v_1(\Gamma)}(y_1) : x \leq -n_0, \, y = 0 \right\} \quad \left(= \bigwedge\nolimits_{A_1}(y_0) = \bigwedge\nolimits_{A_1}(0) \right),$$

$$A_k = \left\{ u = (x, y) \in \bigwedge\nolimits_{v_k(\Gamma)}(y_k - y_{k-1}) : x \leq x_{k-1} - 2y_{k-1}, \, y = y_{k-1} \right\}, \qquad k \geq 2.$$

Note that Area(k, Γ), $k \geq 1$, are edge-disjoint areas, so for a fixed oriented path Γ, $E(k, \Gamma)$, $k \geq 1$, are independent (see Figure 1). Furthermore, by (3.2), (3.3) and by using Proposition 2.2 for $\ell_{v_k(\Gamma)}^{\text{anti}}$, the leftmost anti-oriented infinite open path from $v_k(\Gamma)$, we have

$$\mathbb{P}_p(E(k, \Gamma)) \geq \mathbb{P}_p\big(|C_{v_k(\Gamma)}^{\text{anti}}| = \infty \text{ and } \ell_{v_k(\Gamma)}^{\text{anti}} \cap A_k \neq \varnothing\big)$$

(3.4)
$$= \mathbb{P}_p\big(\ell_{v_k(\Gamma)}^{\text{anti}} \cap A_k \neq \varnothing \,\big|\, |C_{v_k(\Gamma)}^{\text{anti}}| = \infty\big) \cdot \mathbb{P}_p\big(|C_{v_k(\Gamma)}^{\text{anti}}| = \infty\big)$$

$$\geq \tfrac{1}{3}\theta(p) > 0.$$

We point out here that Γ is only used to determine the vertex set $\{v_k(\Gamma) : k \geq 0\}$. Furthermore, we need to work on the event family $\{E(k, \Gamma) : k \geq 1\}$ on event $\{\gamma_{(0,0)} = \Gamma\}$. Now, on event $\{\gamma_{(0,0)} = \Gamma\}$, let $E^*(1, \Gamma)$ be the event that there is an anti-oriented open path from $\Gamma(0, v_1(\Gamma))$ to A_1, and let $E^*(k, \Gamma)$, $k \geq 2$, be the event that there is an anti-oriented open path from $\Gamma(v_{k-1}(\Gamma), v_k(\Gamma))$ to A_k and then to $(-\infty, +\infty) \times \{0\}$. On $\{\gamma_{(0,0)} = \Gamma\}$, note that Γ is open, so event $E^*(k, \Gamma)$ only depends on the configurations of the edges of Area(k, Γ) lying on the left side of Γ (see Figure 1). Note also that event $\{\gamma_{(0,0)} = \Gamma\}$ can be decomposed into the intersection of the following two events:

1. $\mathcal{A}(\Gamma) = \{\Gamma \text{ is open}\}$,
2. $\mathcal{B}(\Gamma) = \{v \nrightarrow \infty \text{ in } R(\Gamma) \text{ for each vertex } v \in \Gamma\}$,

where $R(\Gamma)$ is the edge set to the right of Γ. It follows from the definition that $\mathcal{B}(\Gamma)$ only depends on the configurations of the edges in $R(\Gamma)$; this implies that $E^*(k, \Gamma)$, $k \geq 1$, are independent of $\mathcal{B}(\Gamma)$ conditioned on $\mathcal{A}(\Gamma)$. By this decomposition and the fact that $E^*(k, \Gamma) \cap \mathcal{A}(\Gamma) = E(k, \Gamma) \cap \mathcal{A}(\Gamma)$, we have

$$\mathbb{P}_p\big(E^*(k, \Gamma) \mid \gamma_{(0,0)} = \Gamma\big) = \mathbb{P}_p\big(E^*(k, \Gamma) \mid \mathcal{A}(\Gamma)\big)$$

(3.5)
$$= \mathbb{P}_p\big(E(k, \Gamma) \mid \mathcal{A}(\Gamma)\big).$$

For any $k \geq 1$, let $\mathcal{A}_k(\Gamma)$ be the event that all edges in $\Gamma(v_{k-1}(\Gamma), v_k(\Gamma))$ are open. Note that $\mathcal{A}_k(\Gamma)$, $k \geq 1$, are increasing events, so by the FKG inequality,

(3.6) $$\mathbb{P}_p\big(E(k, \Gamma) \mid \mathcal{A}(\Gamma)\big) = \mathbb{P}_p\big(E(k, \Gamma) \mid \mathcal{A}_k(\Gamma)\big) \geq \mathbb{P}_p\big(E(k, \Gamma)\big).$$

On the other hand, we know that $E^*(k, \Gamma)$ for $k \geq 1$ depend on the different edge layers to the left of Γ (see Figure 1). Therefore, $E^*(k, \Gamma)$ for $k \geq 1$ are independent and, by (3.5), (3.6) and then (3.4), have probabilities bounded away from $\theta(p)/3$ on $\{\gamma_{(0,0)} = \Gamma\}$. With these observations, by the second Borel–Cantelli lemma, on $\{\gamma_{(0,0)} = \Gamma\}$, $E^*(k, \Gamma)$, $k \geq 1$, occur infinitely often almost surely. Using the definition of rightmost open path, we know that if $E^*(k, \Gamma)$ occurs for some k, then γ_{u_1} will meet $\gamma_{(0,0)} = \Gamma$ in $\Gamma(0, v_k(\Gamma))$. Thus, this shows that γ_{u_1} and γ_{u_2} meet, so Theorem 1.1(i) follows. $\square$

PROOF OF THEOREM 1.1(ii).   For any $u \in \mathcal{K}$, by the definition of $D(u, G)$, we know that $D(u, G) \subset C_u^{\text{anti}}$. Note that if $u \in \mathcal{K} \setminus \tilde{\mathcal{K}}$, that is, u is a percolation point but not a bidirectional percolation point, then $|C_u^{\text{anti}}| < \infty$ and $|D(u, G)| < \infty$, so it suffices to prove that $|D(u, G)| < \infty$ for $u \in \tilde{\mathcal{K}}$.

By translation invariance, it suffices to prove that $|D((0, 0), G)| < \infty$ almost surely when $(0, 0) \in \tilde{\mathcal{K}}$.

Let $\ell_{(0,0)}^{\text{anti}}$ be the leftmost anti-oriented infinite open path from $(0, 0)$ and let L^{anti} be a possible realization of ℓ_0^{anti} crossing $R_{\alpha(p)}$ infinitely many times. Then, it suffices to prove that, on $\ell_{(0,0)}^{\text{anti}} = L^{\text{anti}}$, $|D((0, 0), G)| < \infty$ almost surely.

By the proof of Theorem 1.1(i), we know that, on $\ell_{(0,0)}^{\text{anti}} = L^{\text{anti}}$, with probability 1, for any $n \geq 1$, there exists some point $v_n(L^{\text{anti}})$ in L^{anti} from which there is an oriented open path to $[n, \infty) \times \{0\}$. On the other hand, by (3.1) and the standard ergodic theorem, with probability 1, there exist infinitely many $m > 0$ such that $(m, 0) \in \tilde{\mathcal{K}}$. These observations imply that, on $\ell_{(0,0)}^{\text{anti}} = L^{\text{anti}}$, with probability 1, there exist some $m_0 > 0$ and $v' \in L^{\text{anti}}$ such that $(m_0, 0) \in \tilde{\mathcal{K}}$ and there is an oriented open path from v' to some $v'' \in [m_0, \infty) \times \{0\}$. We denote this oriented open path by $\pi = \pi(v', v'')$.

Let

$$\bigwedge\nolimits_{(0,0)} (L^{\text{anti}}, \pi) := \left\{ u \in \bigwedge\nolimits_{(0,0)} \setminus L^{\text{anti}} : u \text{ lies to the right of } L^{\text{anti}} \text{ and above } \pi \right\}$$

and

$$\bar{C}_l(L^{\text{anti}}(v', (0, 0))) := \left\{ w \notin L^{\text{anti}} : \begin{array}{c} w \text{ lies to the left of } L^{\text{anti}} \text{ and for some} \\ z \in L^{\text{anti}}(v', (0, 0)), \ z \neq v', \ w \to z \\ \text{uses no open edges of } L^{\text{anti}} \end{array} \right\}.$$

We declare that

$$(3.7) \quad D((0, 0), G) \subset \bar{C}_l(L^{\text{anti}}(v', (0, 0))) \cup L^{\text{anti}}(v', (0, 0)) \cup \bigwedge\nolimits_{(0,0)} (L^{\text{anti}}, \pi).$$

In fact, for any vertex u in $C_{(0,0)}^{\text{anti}} (\subset \mathcal{K})$ but outside the set of the right-hand side of (3.7), it is easy to find an open oriented path from u to $(m_0, 0)$. This finding implies $u \notin D((0, 0), G)$.

Now, by the definition of leftmost anti-oriented infinite open path, we have

$$(3.8) \qquad |\bar{C}_l(L^{\text{anti}}(v', (0, 0)))| < \infty.$$

Using (3.7), (3.8) and the following fact:

$$\left| L^{\text{anti}}(v', (0, 0)) \cup \bigwedge\nolimits_{(0,0)} (L^{\text{anti}}, \pi) \right| < \infty,$$

we have $|D((0, 0), G)| < \infty$, so Theorem 1.1(ii) follows. $\square$

PROOF OF THEOREM 1.1(iii). For any $u = (x, y) \in \mathcal{K}$, by the standard ergodic theorem, there exists some $u' = (x', y') \in \mathcal{K}$ such that $x' > x$ and $y' = y$ almost surely. It follows from the proof of Theorem 1.1(i) that γ_u will meet $\gamma_{u'}$ at some point v of $\mathcal{K}$ almost surely. Thus, v has two daughters such that the older one is just the ancestor of u and the other one is her younger sister. $\square$

4. Proof of Theorem 1.4. It suffices to prove that, for any $u_1, u_2 \in \mathcal{K}$, $|C_{u_1} \Delta C_{u_2}| < \infty$, where $C_{u_1} \Delta C_{u_2}$ is the symmetric difference of C_{u_1} and C_{u_2}. By Theorem 1.1, with probability 1,

$$\gamma_{u_1} \cap \gamma_{u_2} \neq \varnothing \quad \text{and} \quad \ell_{u_1} \cap \ell_{u_2} \neq \varnothing.$$

Let

$$v_{1,2}^r = (x_r, y_r) \in \gamma_{u_1} \cap \gamma_{u_2} \quad \text{and} \quad v_{1,2}^l = (x_l, y_l) \in \ell_{u_1} \cap \ell_{u_2}$$

be the vertices with the smallest second coordinates. With these definitions, we will prove that

$$(4.1) \qquad |C_{u_1} \setminus C_{u_2}| < \infty.$$

By (4.1) and symmetry, we also have $|C_{u_2} \setminus C_{u_1}| < \infty$, so Theorem 1.4 follows.

Now it remains to show (4.1). Without loss of generality, we divide the problem into the following two cases (see Figure 2):

1. u_2 lies within the cone between ℓ_{u_1} and γ_{u_1};
2. u_2 does not lie within the cone between ℓ_{u_1} and γ_{u_1}.

We focus on case 1 [see Figure 2(a)]. Let B_{u_1, u_2} be the finite *butterfly* shape enclosed by $\gamma_{u_2}(u_2, v_{1,2}^r)$, $\gamma_{u_1}(u_1, v_{1,2}^r)$, $\ell_{u_2}(u_2, v_{1,2}^l)$, $\ell_{u_1}(u_1, v_{1,2}^l)$, and the vertices surrounded by them. It is clear that

$$C_{u_1} \setminus C_{u_2} \subset B_{u_1, u_2} \cup C_r(\gamma_{u_1}(u_1, v_{1,2}^r)) \cup C_l(\ell_{u_1}(u_1, v_{1,2}^l)).$$

Note that $C_r(\gamma_{u_1}(u_1, v_{1,2}^r))$ and $C_l(\ell_{u_1}(u_1, v_{1,2}^l))$ are defined in (1.4). By the definition of γ_{u_1} and ℓ_{u_1}, we have

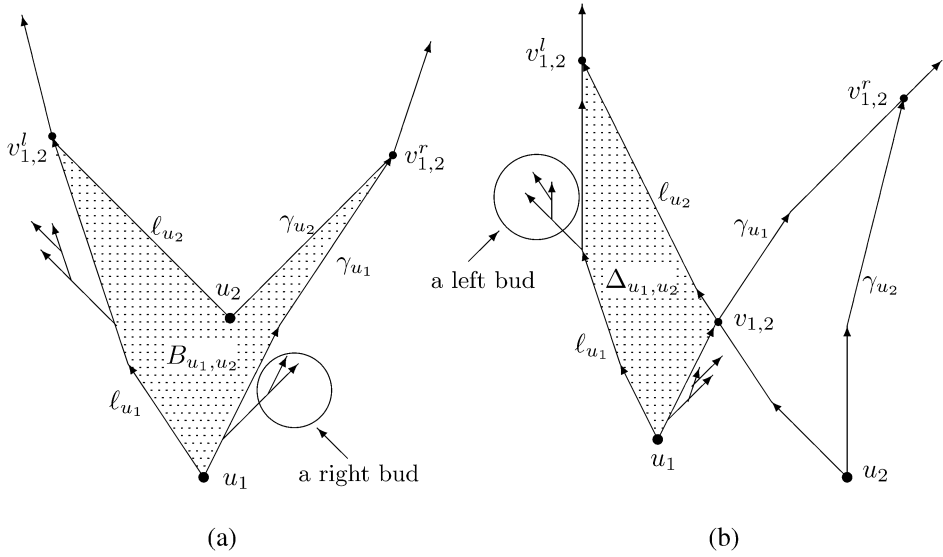$$|C_r(\gamma_{u_1}(u_1, v_{1,2}^r))| < \infty, \qquad |C_l(\ell_{u_1}(u_1, v_{1,2}^l))| < \infty.$$

FIG. 2. B_{u_1,u_2}, Δ_{u_1,u_2}, and the left (resp., right) buds planted in ℓ_{u_1} (resp., γ_{u_1}) are all finite. Note that $C_l(\ell_{u_1}(u_1, v^l_{1,2}))$ consists of all such finite left buds planted in $\ell_{u_1}(u_1, v^l_{1,2}) \setminus \{v^l_{1,2}\}$, so it is finite. The situations for $C_r(\gamma_{u_1}(u_1, v^r_{1,2}))$ and $C_r(\gamma_{u_1}(u_1, v_{1,2}))$ are the same.

This tells us that $|C_{u_1} \setminus C_{u_2}| < \infty$, so (4.1) follows when case 1 holds.

Let us focus on case 2 [see Figure 2(b)]. Without loss of generality, we may further assume that u_1 and u_2 have the relative position such that $\gamma_{u_1} \cap \ell_{u_2} \neq \varnothing$. Let $v_{1,2} \in \gamma_{u_1} \cap \ell_{u_2}$ be the vertex with the smallest second coordinate. Moreover, let Δ_{u_1,u_2} be the finite *triangle* shape enclosed by $\gamma_{u_1}(u_1, v_{1,2})$, $\ell_{u_1}(u_1, v^l_{1,2})$, $\ell_{u_1}(v_{1,2}, v^l_{1,2})$, and the vertices surrounded by them. It is clear that

$$C_{u_1} \setminus C_{u_2} \subset \Delta(u_1, u_2) \cup C_r(\gamma_{u_1}(u_1, v_{1,2})) \cup C_l(\ell_{u_1}(u_1, v^l_{1,2})).$$

The same argument for the first case tells us that $|C_{u_1} \setminus C_{u_2}| < \infty$, so (4.1) also follows when case 2 holds.

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