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Hitting time of a corner for a reflected diffusion in the square

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Abstract. We discuss the long time behavior of a two-dimensional reflected diffusion in the unit square and investigate more specifically the hitting time of a neighborhood of the origin.

We distinguish three different regimes depending on the sign of the correlation coefficient of the diffusion matrix at the point 0. For a positive correlation coefficient, the expectation of the hitting time is uniformly bounded as the neighborhood shrinks. For a negative one, the expectation explodes in a polynomial way as the diameter of the neighborhood vanishes. In the null case, the expectation explodes at a logarithmic rate. As a by-product, we establish in the different cases the attainability or nonattainability of the origin for the reflected process.

From a practical point of view, the considered hitting time appears as a deadlock time in various resource sharing problems.

Résumé. Nous étudions le comportement en temps long d'une diffusion réfléchie à valeurs dans le carré unité et nous focalisons plus précisément sur le temps d'atteinte d'un voisinage de l'origine.

Nous distinguons trois régimes différents, selon le signe du coefficient de corrélation de la matrice de diffusion prise au point 0. Pour un coefficient de corrélation strictement positif, l'espérance du temps d'atteinte reste bornée lorsque le voisinage se rétrécit. Pour un coefficient strictement négatif, l'espérance explose à vitesse polynomiale lorsque le diamètre du voisinage tend vers zéro. Dans le cas d'un coefficient nul, l'espérance diverge à vitesse logarithmique. Au passage, nous établissons selon les cas la possibilité ou l'impossibilité pour la diffusion réfléchie d'atteindre l'origine.

D'un point de vue pratique, le temps d'atteinte considéré apparaît comme un instant de blocage dans différents problèmes de partage de ressource.

MSC: Primary 60H10; 60G40; secondary 68W15

Keywords: Reflected diffusions; Hitting times; Lyapunov functions; Distributed algorithms

1. Introduction

Background

In the papers [12,24] and [22], Tanaka, Lions and Sznitman and Saisho investigate the strong solvability of stochastic differential equations with reflection on the boundary of a domain. In the normal reflection setting, pathwise existence and uniqueness hold for systems driven by Lipschitz continuous coefficients and for domains that are convex (see [24]) or that satisfy both a uniform exterior sphere condition and a uniform interior cone condition ([12] and [22]). Compared to the usual Itô case, the basic dynamics for a reflected stochastic differential equation (RSDE in short) involve an additional pushing process of bounded variation that prevents the diffusion to escape from the underlying domain. A typical example of this kind is given by the Brownian local time that keeps the reflected Brownian motion in the nonnegative half-plane. The formulation of the reflection phenomenon in terms of pushing processes or of local times is inherited from the earlier definition of the Skorokhod problem and turns out to be very handy (think, for example, of the proof of the Feynman–Kac formula for the Neumann problem). In this paper, we provide another example of application of this technology to investigate the long time behavior of a two-dimensional diffusion process with normal reflection on the boundary of the unit square.

Two-dimensional reflected processes have been widely studied for twenty years. In [26], Varadhan and Williams investigate a submartingale problem for the two-dimensional Brownian motion with an oblique reflection on the boundary of a wedge of angle ξ. They discuss the unique solvability of the problem and the attainability of the corner by the underlying process in terms of the ratio $\alpha \equiv (\theta_1 + \theta_2)/\xi$, θ_1 and θ_2 denoting the angles of reflection (with respect to the underlying normal vectors). For $\alpha \leqslant 0$, the process doesn't reach the corner; for $0 < \alpha < 2$, the process reaches the corner, but the amount of time that the process is at the corner is zero; and, for $\alpha \geqslant 2$, the process reaches the corner and remains there. In [27], Williams shows that the process is transient for $\alpha < 0$ and is recurrent in the sense of Harris for $0 \leqslant \alpha < 2$. The escape rate in the transient framework is investigated in [6]. In [28] and [5], Williams and DeBlassie prove that the Brownian motion in a wedge admits a semimartingale representation for $\alpha < 1$ and $\alpha \geqslant 2$. Finally, Menshikov and Williams [18] and Balaji and Ramasubramanian [1] discuss the passage time moments: denoting by τ_ρ, $\rho > 0$, the hitting time of the ball of radius ρ centered at the corner of the wedge, they prove that $\mathbb{E}[\tau_\rho^p] < +\infty$ for $\alpha > 0$ and $p < \alpha/2$, $\mathbb{E}[\tau_\rho^p] = +\infty$ for $\alpha > 0$, $p > \alpha/2$ and ρ small enough, and $\mathbb{E}[\tau_\rho^p] = +\infty$ for $\alpha = 0$, $p > 0$ and ρ small enough. The monograph by Fayolle, Malyshev and Menshikov [7] focuses on the time-discrete counterpart (see Section 3.3 therein).

Objective and main results

Our own situation is slightly different: the underlying process, denoted in the sequel by $(X_t)_{t\geqslant 0}$, may possess a nonconstant drift and a nonconstant covariance matrix. Moreover, it evolves in the unit square, which is compact. The reflection on the boundary occurs along the inward normal vector, so that the pathwise existence and uniqueness of the process follow from [24]. In light of the previous paragraph, we then ask the following questions: assuming that the underlying diffusion matrix is uniformly elliptic, so that $(X_t)_{t\geqslant 0}$ is recurrent in the sense of Harris, is the hitting time of the origin (or more generally of every corner of the square) finite, and if so, is it of finite expectation? More generally, how does the hitting time of a small neighborhood of the origin behave as its diameter tends to zero?

As an answer, we exhibit in this paper a local criterion. More precisely, we show that the expectation of the hitting time of a neighborhood of the origin highly depends on the correlation coefficient s of $a(0)$, the diffusion matrix at the origin. When positive, the expectation remains bounded as the neighborhood shrinks. In the opposite case, it is finite for every neighborhood but tends to the infinity as the diameter tends to zero: at a logarithmic rate if s is zero and at a polynomial rate if s is negative. As a by-product, we derive the following rule for the initial problem: the hitting time of the origin is of finite expectation for a positive correlation coefficient and is almost surely infinite in the other cases.

The proof is inspired by the recurrence and transience analysis for a classical diffusion process and relies on the construction of suitable Lyapunov functionals for the reflected process. To this end, we benefit from the work of Varadhan and Williams [26]: we successfully transport, by a linear mapping, the original problem from the square to a wedge $\mathcal{W}$ and then bring back from the wedge to the square the Lyapunov functions used in [26] to investigate the reflected Brownian motion. The form of the wedge $\mathcal{W}$ as well as the angles of reflection are given by the correlation coefficient s of the covariance matrix $a(0)$: the underlying angle ξ is equal to $\arccos(-s)$ and the parameter α to $2 - \pi/\arccos(-s)$. In particular, α is always less than 1 and

$$\alpha \in]0, 1[\Leftrightarrow s > 0, \qquad \alpha = 0 \Leftrightarrow s = 0, \qquad \alpha < 0 \Leftrightarrow s < 0.$$

The classification we give for the behavior of the reflected diffusion in the square is coherent with the classification of the reflected Brownian motion in $\mathcal{W}$. On the one hand, the reflected Brownian motion in $\mathcal{W}$ may be seen as the image, by a linear mapping, of the solution of a RSDE in the orthant. For this reason, we expect the reflected Brownian motion to be a semimartingale. By [28] and [5], it actually is. On the other hand, the different regimes for the expectation of the hitting time, by $(X_t)_{t\geqslant 0}$, of a neighborhood of the origin correspond to the recurrence and transience properties of the reflected Brownian motion in $\mathcal{W}$. For $s > 0$, the expectation of the hitting time remains bounded as the neighborhood shrinks and the corresponding Brownian motion reaches the corner of $\mathcal{W}$. In the case $s = 0$, the expectation explodes in a logarithmic way; the corresponding reflected Brownian motion does not reach the corner but is recurrent in the sense of Harris. In the case $s < 0$, the expectation explodes in a polynomial way: the Brownian motion in $\mathcal{W}$ does not reach the corner and is transient.

The extension to the upper dimensional framework remains open. In [11], Kwon and Williams investigate the reflected Brownian motion in a cone, in dimension greater than or equal to three, and extend to this setting the results of Varadhan and Williams [26]. For the moment, we do not know if this work might be adapted to the analysis of a reflected diffusion in a hypercube. There are two main difficulties: first, the basis of the cone, in [11], is assumed to be of class $\mathcal{C}^3$, whereas the boundary of the hypercube is just Lipschitz continuous; second, the Lyapunov functions exhibited by Kwon and Williams are not explicit, as in the two-dimensional framework. There are other works on reflected processes in high dimension, but we are not able at this time, to connect them to our purpose: the weak existence and uniqueness (i.e., à la Varadhan and Williams) of reflected processes in polyhedra are discussed, among others, in [3,4,20,21,25]; for the pathwise analysis, we refer to the recent work [19] and to the references therein.

Application

The paper is motivated by concrete applications to resource sharing problems arising in data processing or in mathematical finance. Consider, for example, the "banker algorithm": two customers C_1 and C_2 share a finite amount ρN, $1 < \rho < 2$, of money lent by a banker, the maximum need for each of them being worth N. At a discrete time n, the sum granted to C_1 and C_2 is represented by the position X_n of a Markov process living inside the square of size N without the right upper triangle $[(N, (\rho - 1)N), ((\rho - 1)N, N), (N, N)]$. The process is reflected on the sides of the square and absorbed on the hypotenuse of the triangle, that is, on the line segment between $(0, \rho N)$ and $(\rho N, 0)$: when X hits the line segment, the allocation system stops since the available resource is exhausted. In other words, the hitting time of the line segment appears as a deadlock time for the allocation system and its evaluation constitutes a relevant challenge.

In a series of papers due (among others) to Louchard et al. [13–15] and to Maier et al. [16] and [17] and more recently to Guillotin and Schott [9] and Comets, Delarue and Schott [2], it is shown in various contexts how to reduce, through a normalization procedure of factor N, the analysis for large values of N to an absorption problem for a reflected diffusion process living in the unit square. The absorption occurs on the right upper triangle $[(1, \rho - 1), (\rho - 1, 1), (1, 1)]$, that is on the right upper triangle delimited by the line segment $[(0, \rho), (\rho, 0)]$: it is a neighborhood of $(1, 1)$ with radius $2 - \rho$. Therefore, the situation in the limit framework corresponds, up to a rotation of center $(1/2, 1/2)$ and of angle π, to the one investigated in the paper. When the diffusion matrix of the limit RSDE is diagonal and constant and the drift is null, several explicit computations in terms of Bessel functions are conceivable for the mean of the hitting time (see [15]). In the general case, this strategy fails and our current work provides relevant estimates, especially for ρ close to two.

Organization

The proof is organized as follows. In Section 2, we expose the basic background for our analysis as well as the main result of the paper. The proof is given in Section 3.

2. Notation and main results

For $d \geqslant 1$, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean scalar product and the Euclidean norm on $\mathbb{R}^d$ and, for $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the (open) Euclidean ball of center x and radius r.

2.1. *Reflected SDE*

Let $\mathcal{M}_2(\mathbb{R})$ denote the set of 2×2 real matrices. We consider, for a given triple $(\kappa, \lambda, \Lambda) \in (\mathbb{R}_+^*)^3$, a couple of κ-Lipschitz continuous coefficients $(b, \sigma) : [0, 1]^2 \to \mathbb{R}^2 \times \mathcal{M}_2(\mathbb{R})$ such that for all $(\xi, x) \in (\mathbb{R}^2)^2$:

$$\lambda |\xi|^2 \leqslant \langle \xi, a(x)\xi \rangle \leqslant \Lambda |\xi|^2,$$

where $a(x)$ denotes the symmetric positive matrix $\sigma\sigma^*(x)$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a two-dimensional Brownian motion $(B_t)_{t \geqslant 0}$, whose natural filtration, augmented with $\mathbb{P}$-null sets, is denoted by $(\mathcal{F}_t)_{t \geqslant 0}$. We consider the RSDE with normal reflection on the boundary of the unit square $[0,1]^2$ associated to the pair (b,σ) and to an initial condition $x_0 \in [0,1]^2 \setminus \{0\}$. In other words, we are seeking a triple (X,H,K) of continuous and $(\mathcal{F}_t)_{t\geqslant 0}$-adapted processes with values in $[0,1]^2 \times (\mathbb{R}_+)^2 \times (\mathbb{R}_+)^2$ such that:

RSDE(1) The coordinates of H and K are nondecreasing processes.
RSDE(2) For $i \in \{1,2\}$, the ith coordinate process H^i is nonincreasing on the set $\{t \geqslant 0, X_t^i > 0\}$ and the ith coordinate process K^i is nonincreasing on the set $\{t \geqslant 0, X_t^i < 1\}$, so that:

$$\forall t \geqslant 0, \quad \int_0^t \mathbf{1}_{\{X_r^i > 0\}} \,\mathrm{d}H_r^i = 0, \qquad \int_0^t \mathbf{1}_{\{X_r^i < 1\}} \,\mathrm{d}K_r^i = 0.$$

RSDE(3) X is an Itô process satisfying for $t \geqslant 0$:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}H_t - \mathrm{d}K_t, \quad \text{with } X_0 = x_0.$$

Thanks to Theorem 4.1 in [24], the equation **RSDE(1-2-3)** admits a unique solution (set, for $i = 1,2$, $\mathrm{d}H_t^i = \mathbf{1}_{\{X_t^i=0\}}\,\mathrm{d}\phi_t^i$ and $\mathrm{d}K_t^i = -\mathbf{1}_{\{X_t^i=1\}}\,\mathrm{d}\phi_t^i$ under the notations of [24]).

We remind the reader of the Itô rule for $(X_t)_{t\geqslant 0}$. For a function f of class $\mathcal{C}^2$ on $[0,1]^2$, we have for all $t \geqslant 0$,

$$\begin{aligned}\mathrm{d}f(X_t) &= \frac{1}{2}\sum_{i,j=1}^{2} a_{i,j}(X_t)\frac{\partial^2 f}{\partial x_i\,\partial x_j}(X_t)\,\mathrm{d}t + \sum_{i=1}^{2} b_i(X_t)\frac{\partial f}{\partial x_i}(X_t)\,\mathrm{d}t + \langle \nabla f(X_t), \sigma(X_t)\,\mathrm{d}B_t\rangle \\ &\quad + \frac{\partial f}{\partial x_1}\big(0,X_t^2\big)\,\mathrm{d}H_t^1 + \frac{\partial f}{\partial x_2}\big(X_t^1,0\big)\,\mathrm{d}H_t^2 - \frac{\partial f}{\partial x_1}\big(1,X_t^2\big)\,\mathrm{d}K_t^1 - \frac{\partial f}{\partial x_2}\big(X_t^1,1\big)\,\mathrm{d}K_t^2.\end{aligned}$$

We are now interested in the attainability of the origin or more generally of a small neighborhood of the origin for the process X.

2.2. *Harris recurrence*

In the previous framework, our first question is: is the process X Harris recurrent?

Due to the uniform ellipticity of the matrix a, it actually is: the process X hits with probability one every Borel subset of $[0,1]^2$ of nonzero Lebesgue measure, so that the hitting time of every neighborhood of the origin is almost-surely finite.

2.3. *Absorption at or around the origin*

Now, our second question is: does the process X hit the origin? Or more generally, how does the hitting time of a small neighborhood of the origin behave as its diameter decreases?

To answer the second question, we focus on the expectation of the hitting time of a small neighborhood of the origin and investigate its behavior as the neighborhood shrinks. We then derive as by-product the attainability or nonattainability of $(0,0)$.

In practical applications (up to a rotation of center $(1/2,1/2)$, see the references mentioned in Introduction), the neighborhood is delimited by the line of equation $x_1 + x_2 = \rho$, for $0 < \rho < 1$. We thus focus on the hitting time $T_\rho \equiv \inf\{t \geqslant 0, X_t^1 + X_t^2 \leqslant \rho\}$ and more specifically on $\mathbb{E}(T_\rho)$. It is then plain to bound the expectation of the hitting time of any other small neighborhood of the origin by the expectations of $T_{\rho'}$ and $T_{\rho''}$, for two suitable parameters ρ' and ρ''. Of course, if $\rho > (x_0)_1 + (x_0)_2$, $\mathbb{E}(T_\rho) = 0$.

In this paper, we manage to distinguish three different asymptotic regimes for the expectation $\mathbb{E}(T_\rho)$ as the parameter ρ tends to zero, each of these regimes depending on the covariance matrix $a(0)$, and more precisely, on the sign of its off-diagonal components. Indeed, the matrix $a(0)$ can be written:

$$a(0) = \begin{pmatrix} \rho_1^2 & s\rho_1\rho_2 \\ s\rho_1\rho_2 & \rho_2^2 \end{pmatrix}, \tag{2.1}$$

with $\rho_1, \rho_2 > 0$ and $s \in]-1, 1[$ (since a is uniformly elliptic). It admits two eigenvalues:

$$\begin{cases} \lambda_1 = [\rho_1^2 + \rho_2^2 + \delta]/2, \\ \lambda_2 = [\rho_1^2 + \rho_2^2 - \delta]/2, \end{cases} \quad \text{with } \delta \equiv \big(\rho_1^4 + \rho_2^4 - 2(1-2s^2)\rho_1^2\rho_2^2\big)^{1/2}. \tag{2.2}$$

We denote by E_1 and E_2 the associated eigenvectors (up to a multiplicative constant). For $s \neq 0$, $E_1 = (1, (2s\rho_1\rho_2)^{-1}(\delta + \rho_2^2 - \rho_1^2))^t$ and $E_2 = (-(2s\rho_1\rho_2)^{-1}(\delta + \rho_2^2 - \rho_1^2), 1)^t$. Since $\delta + \rho_2^2 - \rho_1^2 \geqslant 0$, the signs of the nontrivial coordinates of E_1 and E_2 are given by the sign of s. The main eigenvector (i.e., E_1) has two positive components for $s > 0$, and a positive one and a negative one for $s < 0$. Of course, if s vanishes, E_1 and E_2 are equal to the vectors of the canonical basis.

The three different regimes for $\mathbb{E}(T_\rho)$ can be distinguished as follows:

Positive case. *If $s > 0$, the main eigenvector of $a(0)$ (i.e., E_1) is globally oriented from $(0,0)$ to the neighborhood of the corner $(1,1)$, or, up to a change of sign, from $(1,1)$ to the origin, and thus tends to push the reflected diffusion towards the absorption area. The reflection on the boundary cancels most of the effects of the second eigenvalue and keeps on bringing back the diffusion towards the main axis. As a consequence, the hitting time of the border line is rather small and the following asymptotic holds for the diffusion:*

$$\sup_{0<\rho<1} \mathbb{E}(T_\rho) < +\infty.$$

We illustrate this phenomenon when b vanishes and a is the constant matrix given by $\rho_1 = \rho_2 = 1$ and $s = 0.9$ by plotting below (see Fig. 1, first case) a simulated trajectory of the reflected diffusion process, starting from $(1,1)$ at time 0, and running from time 0 to time 5 in the box $[0,1]^2$. The algorithm used to simulate the reflected process is given in Słomiński [23]. The eigenvector E_1 is equal to $(1,1)^t$.

Negative case. *If $s < 0$, the main eigenvector of $a(0)$ is globally oriented from $(1,0)$ to the neighborhood of the corner $(0,1)$ and attracts the diffusion away from the border line. Again, the reflection on the boundary cancels most of the effects of the second eigenvalue, and thus, acts now as a trap: the diffusion stays for a long time along the main axis and hardly touches the boundary. The hitting time satisfies the following asymptotic behavior:*

$$\exists c_1 \geqslant 1,\ \exists c_2 > 0,\ \forall \rho \in]0,1[, \quad c_1^{-1}\rho^{-c_2} - c_1 \leqslant \mathbb{E}(T_\rho) \leqslant c_1\rho^{-c_2} + c_1.$$

This point is illustrated by the second case in Fig. 1 when b vanishes and a is equal to the constant matrix given by $\rho_1 = \rho_2 = 1$ and $s = -0.9$ (again, $x_0 = (1,1)$). The eigenvector E_1 is given, in this case, by $(1,-1)^t$.

Null case. *The case $s = 0$ is intermediate. Eigenvectors are parallel to the axes and the behavior of the diffusion is close to the behavior of the two-dimensional Brownian motion. We have*

$$\exists c_1 \geqslant 1,\ \forall \rho \in]0,1[, \quad -c_1^{-1}\ln(\rho) - c_1 \leqslant \mathbb{E}(T_\rho) \leqslant -c_1\ln(\rho) + c_1.$$

This is illustrated by the third point in Fig. 1 when b vanishes and a is equal to the identity matrix ($x_0 = (1,1)$).

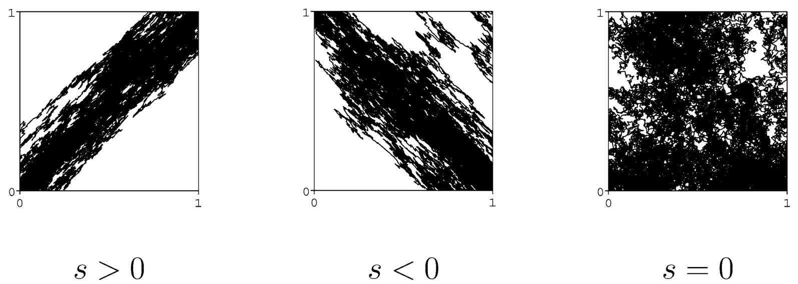

Fig. 1. Trajectories of the reflected process in function of s.

2.4. *Main results*

The following theorem summarizes the different cases detailed in the previous subsection:

Theorem 2.1. *There exists a constant $D \geqslant 1$, depending only on the known parameters λ, Λ, K, ρ_1, ρ_2 and s, such that, for all $\rho \in]0,1[$,*

$$\begin{aligned}
&\text{if } s>0, \quad \mathbb{E}(T_\rho) \leqslant D,\\
&\text{if } s<0, \quad D^{-1}\rho^\alpha - D|x_0|^\alpha \leqslant \mathbb{E}(T_\rho) \leqslant D\rho^\alpha + D, \quad \text{with } \alpha \equiv 2 - \frac{\pi}{\arccos(-s)} < 0,\\
&\text{if } s=0, \quad -D^{-1}\ln(\rho) + D\big[\ln\big(|x_0|\big) - 1\big] \leqslant \mathbb{E}(T_\rho) \leqslant -D\ln(\rho) + D.
\end{aligned}$$

We emphasize that D does not depend on x_0.

As a by-product of the proof of Theorem 2.1, we can establish the following criterion for the attainability of the origin:

Proposition 2.2. *Denote by T_0 the first hitting time of the origin by the process X. Then, $\mathbb{E}(T_0) < +\infty$ if $s > 0$ and $\mathbb{P}\{T_0 = +\infty\} = 1$ if $s \leqslant 0$.*

3. Proofs

In the whole proof of Theorem 2.1 and Proposition 2.2, the constants appearing in various estimates only depend on the known parameters λ, Λ, K, ρ_1, ρ_2 and s. Even if denoted by the same letter, their values may vary from line to line.

The common approach to investigate the recurrence and transience properties of a diffusion process is exposed in [8], Ch. IX. The strategy consists in exhibiting superharmonic functions (or equivalently subharmonic functions) for the generator of the diffusion process: these supersolutions (or subsolutions) play the role of Lyapunov functions (see [10], Ch. I, for the definition of a Lyapunov function for an ordinary differential equation). In our framework, the generator is given by:

$$\mathcal{L} \equiv \frac{1}{2}\sum_{i,j=1}^{2} a_{i,j}(\cdot)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{2} b_i(\cdot)\frac{\partial}{\partial x_i}.$$

We are then seeking a supersolution (resp. subsolution) $h \in \mathcal{C}^2([0,1]^2 \setminus \{0\})$ satisfying

$$\mathcal{L}h(x) \leqslant (\text{resp.} \geqslant)\ C, \quad x \in [0,1]^2 \setminus \{0\}, \tag{3.1}$$

for a constant $C > 0$, verifying a zero Neumann boundary condition on the edges $](0,0),(0,1)]$ and $](0,0),(1,0)]$, that is

$$\begin{aligned}
&\frac{\partial h}{\partial x_1}(0,x_2) = 0 \quad \text{for } 0 < x_2 \leqslant 1,\\
&\frac{\partial h}{\partial x_2}(x_1,0) = 0 \quad \text{for } 0 < x_1 \leqslant 1,
\end{aligned} \tag{3.2}$$

and having positive (resp. negative) derivatives along the outward normal vectors on the other sides, that is

$$\begin{aligned}
&\frac{\partial h}{\partial x_1}(1,x_2) > (\text{resp.} <)\, 0 \quad \text{for } 0 < x_2 \leqslant 1,\\
&\frac{\partial h}{\partial x_2}(x_1,1) > (\text{resp.} <)\, 0 \quad \text{for } 0 < x_1 \leqslant 1.
\end{aligned} \tag{3.3}$$

If we can find such a function h, we deduce by the Itô formula and (3.1), (3.2) and (3.3)

$$\mathbb{E}\big(h(X_{T_\rho})\big) - \mathbb{E}\big(h(X_0)\big) \leqslant \text{(resp. } \geqslant\text{)}\, C\mathbb{E}(T_\rho).$$

The asymptotic trend of $\mathbb{E}(T_\rho)$ is then given by the growth of h at the origin. The growth at the origin of the supersolution (resp. subsolution) that we construct in the sequel is governed by the sign of s: it is bounded if $s > 0$, it is of logarithmic growth if $s = 0$ (i.e. $|h(x)| \asymp |\ln(|x|)|$ for $|x|$ small: $\exists c \geqslant 1$, $c^{-1}|\ln(|x|)| \leqslant |h(x)| \leqslant c|\ln(|x|)|$ for $|x|$ small) and it is of polynomial growth, with exponent α, if $s < 0$ (i.e. $|h(x)| \asymp |x|^\alpha$ for $|x|$ small). This classification coincides with the one given in the statement of Theorem 2.1.

We proceed in two steps to construct h. Using the results of Varadhan and Williams [26], we manage to construct a superharmonic (resp. subharmonic) function f on $[0,1]^2 \setminus \{0\}$ satisfying a zero Neumann boundary condition on the edges $](0,0),(0,1)]$ and $](0,0),(1,0)]$ and having negative derivatives along the outward normal vectors on the other sides. The growth of f is governed at the origin by the sign of s, as explained for h.

Moreover, we exhibit a bounded Lyapunov function g on $[0,1]^2$ satisfying

$$\mathcal{L}g(x) \leqslant \text{(resp. } \geqslant\text{)}\, C, \quad x \in [0,1]^2 \setminus \{0\},$$

for a constant $C > 0$, verifying a zero Neumann boundary condition on the edges $[(0,0),(0,1)]$ and $[(0,0),(1,0)]$ and having positive derivatives along the outward normal vectors on the other sides.

We then construct h as a linear combination of f and g.

3.1. *Harmonic function for the reflected Brownian motion in a wedge*

We remind the reader of the results established in [26]. For an angle $\xi \in]0, 2\pi[$, let $\mathcal{W}$ denote the subset of $\mathbb{R}^2$ given, in polar coordinates, by $\mathcal{W} \equiv \{(r,\theta), 0 \leqslant \theta \leqslant \xi, r \geqslant 0\}$, and let $\partial\mathcal{W}_1 \equiv \{(r,\theta), \theta = 0, r \geqslant 0\}$ and $\partial\mathcal{W}_2 \equiv \{(r,\theta), \theta = \xi, r \geqslant 0\}$ denote the two sides of the boundary of $\mathcal{W}$. For $j = 1,2$, let n_j denote the unit inward normal vector to $\partial\mathcal{W}_j$.

For $j = 1,2$, we are given a unit vector v_j such that $\langle v_j, n_j\rangle > 0$: the angle between n_j and v_j, measured such that it is positive if and only if v_j points towards the origin, is denoted by θ_j (see Fig. 2).

Then, the function ψ given, in polar coordinates, by $\psi(r,\theta) \equiv r^\alpha \cos(\alpha\theta - \theta_1)$ for $(r,\theta) \in \mathcal{W}$, with $\alpha \equiv (\theta_1 + \theta_2)/\xi$, satisfies, if $\alpha \neq 0$, the Neumann problem $\Delta\psi = 0$ in $\mathcal{W} \setminus \{0\}$, with the boundary conditions $\langle v_1, \nabla\psi\rangle = 0$ on $\partial\mathcal{W}_1 \setminus \{0\}$ and $\langle v_2, \nabla\psi\rangle = 0$ on $\partial\mathcal{W}_2 \setminus \{0\}$ (∇ and Δ denote the gradient and Laplace operators).

3.2. *Transposition of the original problem to a wedge*

We set $a_0 \equiv a(0)$ and we consider, for the operator $\sum_{i,j=1}^2 (a_0)_{i,j}[\partial^2/\partial x_i\,\partial x_j]$, the associated Neumann problem in the orthant $\mathcal{W}_0 \equiv \{x \in \mathbb{R}^2, x_1, x_2 \geqslant 0\}$ with a zero boundary condition on the sides. With a suitable change of variable, we now reduce this Neumann problem to a Neumann problem in a wedge for the Laplace operator.

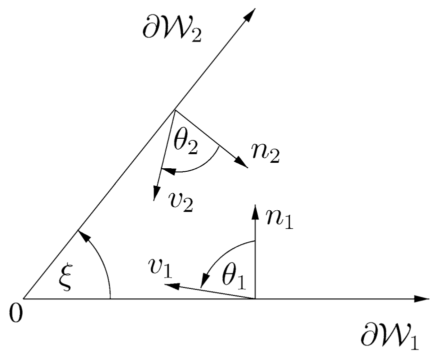

Fig. 2. Wedge $\mathcal{W}$.

The matrix a_0^{-1} has the form

$$a_0^{-1} = (1-s^2)^{-1}\begin{pmatrix} \rho_1^{-2} & -s\rho_1^{-1}\rho_2^{-1} \\ -s\rho_1^{-1}\rho_2^{-1} & \rho_2^{-2} \end{pmatrix}.$$

We denote by $\varsigma \equiv a_0^{-1/2}$ the symmetric square root of a_0^{-1}. We can associate to ς the norm:

$$\forall x \in \mathbb{R}^2, \quad r(x) \equiv |\varsigma x| = \left[\langle x, a_0^{-1}x\rangle\right]^{1/2} = (1-s^2)^{-1/2}\left[\rho_1^{-2}x_1^2 + \rho_2^{-2}x_2^2 - 2s\rho_1^{-1}\rho_2^{-1}x_1x_2\right]^{1/2}.$$

By ellipticity of a, we deduce that

$$\forall x \in \mathbb{R}^2, \quad \Lambda^{-1/2}|x| \leqslant r(x) \leqslant \lambda^{-1/2}|x|. \tag{3.4}$$

Setting $u_1 \equiv \varsigma e_1/|\varsigma e_1|$ and $u_2 \equiv \varsigma e_2/|\varsigma e_2|$, the image of the wedge $\mathcal{W}_0$ by the matrix ς may be expressed as $\varsigma\mathcal{W}_0 = \{\lambda u_1 + \mu u_2,\ \lambda, \mu \geqslant 0\}$. Since ς is a positive symmetric matrix, we can prove that the angle (u_1, u_2) belongs to $]0, \pi[$. Indeed, we can find an orthogonal matrix M such that

$$\varsigma = MDM^*, \quad D \equiv \begin{pmatrix} \lambda_1^{-1/2} & 0 \\ 0 & \lambda_2^{-1/2} \end{pmatrix},$$

λ_1 and λ_2 being the eigenvalues of a_0 (see (2.2)). The angle (M^*e_1, M^*e_2) is equal to $\pi/2$. Since the diagonal coefficients of D are positive, we easily deduce that the angle (DM^*e_1, DM^*e_2) belongs to $]0, \pi[$. Since M is orthogonal, the angle (u_1, u_2), which is equal to the angle (MDM^*e_1, MDM^*e_2), belongs to $]0, \pi[$. In particular, the angle (u_1, u_2) is equal to $\xi \equiv \arccos(\langle u_1, u_2\rangle) = \arccos(-s)$.

Moreover, $\langle u_1, e_1\rangle = \langle \varsigma e_1, e_1\rangle > 0$, $\langle u_2, e_2\rangle = \langle \varsigma e_2, e_2\rangle > 0$ and $\langle u_1, e_2\rangle = (|\varsigma e_2|/|\varsigma e_1|)\langle u_2, e_1\rangle$ so that $\langle u_1, e_2\rangle$ and $\langle u_2, e_1\rangle$ have the same sign (see Fig. 3).

We denote by $\mathcal{W}$ the wedge formed by the vectors u_1 and u_2. The inward normal vectors to the two sides are denoted by n_1 and n_2 (n_1 is the unitary orthogonal vector to u_1 generating a positive angle with it and n_2 is the unitary orthogonal vector to u_2 generating a negative angle with it, see Fig. 3). Along the side generated by u_1, the vector u_2 plays the role of the vector v_1 in the description of Section 3.1. Similarly, the vector u_1 plays, along the side generated by u_2, the role of the vector v_2. Moreover, the angle θ_1 between the vectors n_1 and u_2 is equal to the angle θ_2 between the vectors n_2 and u_1 and their common value is $\theta_1 = \theta_2 = \xi - \pi/2$.

The parameter α introduced in the previous subsection is equal, in this setting, to $\alpha = 2 - \pi/\xi = 2 - \pi/\arccos(-s)$. It is plain to prove that

$$\alpha < 0 \Leftrightarrow s < 0, \qquad \alpha = 0 \Leftrightarrow s = 0, \qquad \alpha > 0 \Leftrightarrow s > 0.$$

We denote by (y_1, y_2) the Cartesian coordinates and by (r, θ) the polar coordinates in the basis (u_1, n_1), and we set $\psi(r, \theta) \equiv r^\alpha \cos(\alpha\theta - \theta_1)$. By Section 3.1, we deduce that ψ satisfies, if $s \neq 0$, the Neumann problem $\Delta\psi = 0$

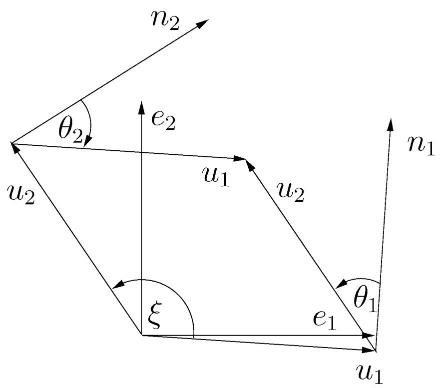

Fig. 3. Image of the wedge $\mathcal{W}_0$ by ς.

in $\mathcal{W} \setminus \{0\}$, with the boundary conditions $\langle u_2, \nabla\psi\rangle = 0$ on $\partial\mathcal{W}_1 \setminus \{0\}$ and $\langle u_1, \nabla\psi\rangle = 0$ on $\partial\mathcal{W}_2 \setminus \{0\}$ (recall that the Laplace operator Δ is independent of the choice of the Cartesian coordinates).

We assume for a while that $s \neq 0$. For $x \in \mathcal{W}_0$, we set $\varphi(x) \equiv \psi(\varsigma x)$. We let the reader check that, for $x \in \mathcal{W}_0 \setminus \{0\}$, $\nabla\varphi(x) = \varsigma\nabla\psi(\varsigma x)$ and $\nabla^2\varphi(x) = \varsigma\nabla^2\psi(\varsigma x)\varsigma$ ($\nabla^2\varphi$ stands for the Hessian matrix of φ). The function φ may be expressed as

$$\varphi(x) = \big[r(x)\big]^\alpha \cos\big(\alpha\theta(x) - \theta_1\big),$$

$\theta(x)$ denoting the angle of ςx in the basis (u_1, n_1).

The function φ satisfies the Neumann problem:

$$\sum_{i,j=1}^{2} (a_0)_{i,j} \frac{\partial^2\varphi}{\partial x_i\,\partial x_j}(x) = 0 \quad \text{for } x \in \mathcal{W}_0 \setminus \{0\}, \tag{3.5}$$

with the boundary conditions $\langle e_2, \nabla\varphi(x)\rangle = 0$ if $x_1 > 0$ and $x_2 = 0$ and $\langle e_1, \nabla\varphi(x)\rangle = 0$ if $x_2 > 0$ and $x_1 = 0$.

Proposition 3.1. *Assume that $s \neq 0$. Then, there exists a constant $\eta \geqslant 1$ such that*

$$\forall x \in \mathcal{W}_0 \setminus \{0\}, \quad \begin{cases} \eta^{-1}\big[r(x)\big]^\alpha \leqslant \varphi(x) \leqslant \big[r(x)\big]^\alpha, \\ \eta^{-1}\big[r(x)\big]^{\alpha-1} \leqslant \big|\nabla\varphi(x)\big| \leqslant \eta\big[r(x)\big]^{\alpha-1}, \\ \big|\nabla^2\varphi(x)\big| \leqslant \eta\big[r(x)\big]^{\alpha-2}. \end{cases}$$

Moreover,

$$\alpha \inf_{x_2\in[0,1]} \frac{\partial\varphi}{\partial x_1}(1, x_2) > 0, \qquad \alpha \inf_{x_1\in[0,1]} \frac{\partial\varphi}{\partial x_2}(x_1, 1) > 0.$$

Proof. We first establish the bounds for the function φ and its derivatives. The upper bound for φ is obvious. We prove the lower bound. By construction, we know that, for all $x \in \mathcal{W}_0 \setminus \{0\}$, $0 \leqslant \theta(x) \leqslant \xi$. If $\alpha > 0$, $-\theta_1 \leqslant \alpha\theta(x) - \theta_1 \leqslant \alpha\xi - \theta_1$, that is $\pi/2 - \xi \leqslant \alpha\theta(x) - \theta_1 \leqslant \xi - \pi/2$ ($\theta_1 = \xi - \pi/2$ and $\alpha = 2 - \pi/\xi$). We deduce the lower bound for $\alpha > 0$. The same argument holds for $\alpha < 0$.

We now prove the bounds for the gradient of φ. By ellipticity of the matrix ς, it is sufficient to investigate $\nabla\psi$. We denote by $\nabla_y\psi = (\partial\psi/\partial y_1, \partial\psi/\partial y_2)$, with $\partial\psi/\partial y_1 = \langle\nabla\psi, u_1\rangle$ and $\partial\psi/\partial y_2 = \langle\nabla\psi, n_1\rangle$, the coordinates of $\nabla\psi$ in the basis (u_1, n_1). For a point $y \in \mathcal{W} \setminus \{0\}$, we can express $\nabla_y\psi(y)$ in terms of the polar coordinates (r, θ) of y, $r > 0$ and $0 \leqslant \theta \leqslant \xi$. We obtain:

$$\nabla_y\psi(r,\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \partial_r\psi(r,\theta) \\ r^{-1}\partial_\theta\psi(r,\theta) \end{pmatrix}.$$

Since $\partial_r\psi(r,\theta) = \alpha r^{\alpha-1}\cos(\alpha\theta - \theta_1)$ and $\partial_\theta\psi(r,\theta) = -\alpha r^\alpha \sin(\alpha\theta - \theta_1)$, we obtain the following expression (in polar coordinates) for $\nabla_y\psi$:

$$\begin{aligned} \nabla_y\psi(r,\theta) &= \alpha r^{\alpha-1} \begin{pmatrix} \cos(\theta)\cos(\alpha\theta - \theta_1) + \sin(\theta)\sin(\alpha\theta - \theta_1) \\ \sin(\theta)\cos(\alpha\theta - \theta_1) - \cos(\theta)\sin(\alpha\theta - \theta_1) \end{pmatrix} \\ &= \alpha r^{\alpha-1} \begin{pmatrix} \cos[(\alpha-1)\theta - \theta_1] \\ -\sin[(\alpha-1)\theta - \theta_1] \end{pmatrix}. \end{aligned} \tag{3.6}$$

The bounds for $|\nabla\varphi|$ easily follows ($r = |y| = |\varsigma x| = r(x)$). By a similar argument, we establish the bound for the second-order derivatives of φ.

We now investigate the sign of $\alpha[\partial\varphi/\partial x_1](x) = \alpha\langle\nabla\varphi(x), e_1\rangle$ for $x_1 = 1$ and $0 \leqslant x_2 \leqslant 1$. It is sufficient to investigate the sign of $\alpha\langle\nabla\psi(y), \varsigma e_1\rangle$ for $y = \varsigma x$, that is the sign of $\alpha\langle\nabla\psi(y), u_1\rangle$. We denote to this end by (r, θ) the polar coordinates of y. We deduce from (3.6) that

$$\alpha\big\langle\nabla_y\psi(r,\theta), u_1\big\rangle = \alpha^2 r^{\alpha-1}\cos\big[(\alpha-1)\theta - \theta_1\big].$$

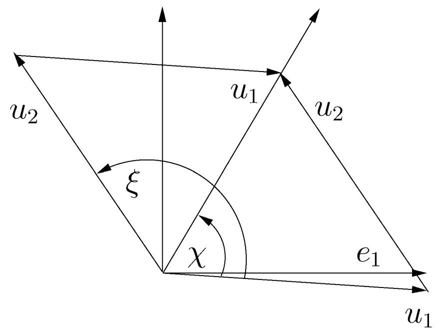

Fig. 4. Angle χ.

Since $x_1 = 1$ and $0 \leqslant x_2 \leqslant 1$, there exists $0 < \chi < \xi$ such that $0 \leqslant \theta \leqslant \chi$ (see Fig. 4). Moreover, $\alpha - 1 = 1 - \pi/\xi < 0$ because $\xi < \pi$. We deduce that

$$-\frac{\pi}{2} = (\alpha - 1)\xi - \theta_1 < (\alpha - 1)\chi - \theta_1 \leqslant (\alpha - 1)\theta - \theta_1 \leqslant -\theta_1 = \frac{\pi}{2} - \xi < \frac{\pi}{2}.$$

We deduce that

$$\alpha \inf_{x_2 \in [0,1]} \frac{\partial \varphi}{\partial x_1}(1, x_2) > 0.$$

We investigate in the same way the sign of $\alpha[\partial\varphi/\partial x_2](x)$ for $x_2 = 1$ and $0 \leqslant x_1 \leqslant 1$. Again, it is sufficient to investigate the sign of $\alpha\langle\nabla\psi(y), u_2\rangle$ for $y = \varsigma x$. The polar coordinates of y are denoted by (r, θ), with $r > 0$ and $\chi \leqslant \theta \leqslant \xi$. We deduce from (3.6):

$$\alpha\bigl\langle\nabla\psi(r, \theta), u_2\bigr\rangle = \alpha^2 r^{\alpha-1} \cos\bigl[(\alpha - 1)\theta + \xi - \theta_1\bigr].$$

As above,

$$-\frac{\pi}{2} < \xi - \frac{\pi}{2} = \alpha\xi - \theta_1 \leqslant (\alpha - 1)\theta + \xi - \theta_1 \leqslant (\alpha - 1)\chi + \xi - \theta_1 < \xi - \theta_1 = \frac{\pi}{2}.$$

We deduce that

$$\alpha \inf_{x_1 \in [0,1]} \frac{\partial \varphi}{\partial x_2}(x_1, 1) > 0. \qquad \square$$

We now investigate the case $s = 0$. We let the reader check.

Proposition 3.2. *Assume that $s = 0$. Then, the function $\varphi(x) \equiv -\ln(r(x)) + \ln(r(1, 1))$, with $r(x) \equiv (\rho_1^{-2}x_1^2 + \rho_2^{-2}x_2^2)^{1/2}$, is nonnegative on the square $[0, 1]^2$ and satisfies*

$$\sum_{i,j=1}^{2} (a_0)_{i,j} \frac{\partial^2 \varphi}{\partial x_i \, \partial x_j}(x) = 0 \quad \textit{for } x \in \mathcal{W}_0 \setminus \{0\},$$

with the boundary conditions $\langle e_1, \nabla\varphi(x)\rangle = 0$ if $x_1 = 0$ and $x_2 > 0$ and $\langle e_2, \nabla\varphi(x)\rangle = 0$ if $x_2 = 0$ and $x_1 > 0$. There exists a constant $\eta \geqslant 1$ such that

$$\forall x \in \mathcal{W}_0 \setminus \{0\}, \quad \begin{cases} \eta^{-1}\bigl[r(x)\bigr]^{-1} \leqslant \bigl|\nabla\varphi(x)\bigr| \leqslant \eta\bigl[r(x)\bigr]^{-1}, \\ \bigl|\nabla^2\varphi(x)\bigr| \leqslant \eta\bigl[r(x)\bigr]^{-2}. \end{cases}$$

Moreover,

$$\inf_{x_2\in[0,1]} \frac{\partial\varphi}{\partial x_1}(1,x_2) < 0, \qquad \inf_{x_1\in[0,1]} \frac{\partial\varphi}{\partial x_2}(x_1,1) < 0.$$

3.3. *Lyapunov functions*

For all $N \geqslant 1$, we set $\zeta_N \equiv \inf\{t \geqslant 0,\ |X_t| \leqslant N^{-1}\}$.

Proposition 3.3. *If $\alpha > 0$, there exist a local martingale $(M_t)_{t\geqslant 0}$, a smooth function F from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $c > 0$ and $\gamma > 0$ such that, for all $r > 0$, $r\exp(-\gamma r^{1/\alpha}) \leqslant F(r) \leqslant r$, for all $N \geqslant 1$, $(M_t)_{0\leqslant t\leqslant \zeta_N}$ is square integrable, and for all $t \in [0,\zeta_N]$,*

$$\mathrm{d}\big[-F\big(\varphi(X_t)\big)\big] \geqslant \mathrm{d}M_t + c\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

If $\alpha < 0$, there exist a local martingale $(\hat{M}_t)_{t\geqslant 0}$, a smooth function $\hat{F}$ from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $\hat{c} > 0$ and $\hat{\gamma} > 0$ such that, for all $r > 0$, $\hat{\gamma}(r-1)^+ \leqslant \hat{F}(r) \leqslant r$, for all $N \geqslant 1$, $(\hat{M}_t)_{0\leqslant t\leqslant \zeta_N}$ is square integrable, and for all $t \in [0,\zeta_N]$,

$$\mathrm{d}\hat{F}\big(\varphi(X_t)\big) \geqslant \mathrm{d}\hat{M}_t + \hat{c}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Moreover, setting $m = \inf_{[0,1]^2\setminus\{0\}} \varphi > 0$, there exist a square integrable martingale $(\check{M}_t)_{t\geqslant 0}$, a smooth function $\check{F}$ from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $\check{c} > 0$ and $\check{\gamma} > 0$ such that, for all $r \geqslant m$, $r - m \leqslant \check{F}(r) \leqslant \check{\gamma} r$, for all $N \geqslant 1$, $(\check{M}_t)_{0\leqslant t\leqslant \zeta_N}$ is square integrable, and for all $t \in [0,\zeta_N]$,

$$\mathrm{d}\check{F}\big(\varphi(X_t)\big) \leqslant \mathrm{d}\check{M}_t + \check{c}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Proof. In a first time, we do not take into account the sign of α. Whatever the sign of α is, the function φ is smooth on $\mathcal{W}_0 \setminus \{0\}$ and we can apply the Itô formula to $(\varphi(X_t))_{0\leqslant t\leqslant \zeta_N}$. For a fixed $N \geqslant 1$ and for all $t \in [0,\zeta_N]$, we deduce from (3.5) and from the Neumann boundary condition satisfied by φ:

$$\begin{aligned}
\mathrm{d}\varphi(X_t) &= \big\langle \nabla\varphi(X_t), b(X_t)\big\rangle \mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^{2} a_{i,j}(X_t)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}(X_t)\,\mathrm{d}t \\
&\quad + \big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle + \big\langle \nabla\varphi(X_t), \mathrm{d}H_t\big\rangle - \big\langle \nabla\varphi(X_t), \mathrm{d}K_t\big\rangle \\
&= \big\langle \nabla\varphi(X_t), b(X_t)\big\rangle \mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^{2} \big[a_{i,j}(X_t) - a_{i,j}(0)\big]\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}(X_t)\,\mathrm{d}t \\
&\quad + \big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle \nabla\varphi(X_t), \mathrm{d}K_t\big\rangle.
\end{aligned} \tag{3.7}$$

By the Lipschitz continuity of a and the boundedness of b, and by Proposition 3.1, we can find a bounded function Γ from $\mathbb{R}^2$ to $\mathbb{R}$, independent of N, such that

$$\mathrm{d}\varphi(X_t) = \big(\varphi(X_t)\big)^{(\alpha-1)/\alpha}\Gamma(X_t)\,\mathrm{d}t + \big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle \nabla\varphi(X_t), \mathrm{d}K_t\big\rangle.$$

Let F be a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}$. Again, the Itô formula yields for all $t \in [0,\zeta_N]$:

$$\begin{aligned}
\mathrm{d}F\big(\varphi(X_t)\big) &= F'\big(\varphi(X_t)\big)\big(\varphi(X_t)\big)^{(\alpha-1)/\alpha}\Gamma(X_t)\,\mathrm{d}t \\
&\quad + \frac{1}{2}F''\big(\varphi(X_t)\big)\big\langle \nabla\varphi(X_t), a(X_t)\nabla\varphi(X_t)\big\rangle \mathrm{d}t \\
&\quad + F'\big(\varphi(X_t)\big)\big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - F'\big(\varphi(X_t)\big)\big\langle \nabla\varphi(X_t), \mathrm{d}K_t\big\rangle.
\end{aligned}$$

For a real C, we choose $F'(r) = \exp(Cr^{1/\alpha})$ for $r > 0$, so that $F''(r) = (C/\alpha)r^{1/\alpha - 1}\exp(Cr^{1/\alpha})$. We obtain for all $t \in [0, \zeta_N]$:

$$\begin{aligned} \mathrm{d}F\big(\varphi(X_t)\big) = {} & \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big[\big(\varphi(X_t)\big)^{(\alpha-1)/\alpha}\Gamma(X_t) \\ & + \big(C/(2\alpha)\big)\big(\varphi(X_t)\big)^{(1-\alpha)/\alpha}\big\langle\nabla\varphi(X_t), a(X_t)\nabla\varphi(X_t)\big\rangle\big]\,\mathrm{d}t \\ & + \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big[\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle\nabla\varphi(X_t), \mathrm{d}K_t\big\rangle\big]. \end{aligned} \tag{3.8}$$

We now assume that $\alpha > 0$. By Proposition 3.1 and (3.8), we can find a constant $C < 0$, independent of N, such that, for all $t \in [0, \zeta_N]$,

$$\begin{aligned} \mathrm{d}F\big(\varphi(X_t)\big) & \leqslant \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big[\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle\nabla\varphi(X_t), \mathrm{d}K_t\big\rangle\big] \\ & = \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\bigg[\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \frac{\partial\varphi}{\partial x_1}\big(1, X_t^2\big)\,\mathrm{d}K_t^1 - \frac{\partial\varphi}{\partial x_2}\big(X_t^1, 1\big)\,\mathrm{d}K_t^2\bigg]. \end{aligned}$$

By Proposition 3.1, we can find a constant $c > 0$, independent of N, such that, for all $t \in [0, \zeta_N]$,

$$\mathrm{d}\big[-F\big(\varphi(X_t)\big)\big] \geqslant -\exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle + c\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Moreover, we can choose $F(r) = \int_0^r \exp(Cs^{1/\alpha})\,\mathrm{d}s$. Then, for all $r > 0$, we have

$$r\exp\big(Cr^{1/\alpha}\big) \leqslant F(r) \leqslant r.$$

This completes the proof in the case $\alpha > 0$.

We turn to the case $\alpha < 0$. By Proposition 3.1 and (3.8), we can find a constant $C < 0$, independent of N, such that, for all $t \in [0, \zeta_N]$,

$$\mathrm{d}F\big(\varphi(X_t)\big) \geqslant \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big[\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle\nabla\varphi(X_t), \mathrm{d}K_t\big\rangle\big].$$

By Proposition 3.1, we can find a constant $c > 0$, independent of N, such that, for all $t \in [0, \zeta_N]$,

$$\mathrm{d}F\big(\varphi(X_t)\big) \geqslant \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle + c\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Choosing $F(r) = \int_0^r \exp(Cs^{1/\alpha})\,\mathrm{d}s$, we have for all $r > 0$

$$(r-1)^+\exp(C) \leqslant F(r) \leqslant r.$$

This completes the proof of the lower bound in the case $\alpha < 0$.

We finally prove the upper bound. By Proposition 3.1 and (3.8), we can find a constant $C > 0$, independent of N, such that, for all $t \in [0, \zeta_N]$,

$$\mathrm{d}F\big(\varphi(X_t)\big) \leqslant \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big[\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle\nabla\varphi(X_t), \mathrm{d}K_t\big\rangle\big].$$

By Proposition 3.1, we can find a constant $d > 0$, independent of N, such that, for all $t \in [0, \zeta_N]$,

$$\mathrm{d}F\big(\varphi(X_t)\big) \leqslant \exp\big(C\big(\varphi(X_t)\big)^{1/\alpha}\big)\big\langle\nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle + d\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Choosing $F(r) = \int_m^r \exp(Cs^{1/\alpha})\,\mathrm{d}s$, we have for all $r \geqslant m$

$$r - m \leqslant F(r) \leqslant r\exp\big(Cm^{1/\alpha}\big).$$

This completes the proof. □

We now investigate the case $\alpha = 0$.

Proposition 3.4. *If $\alpha = 0$, there exist a local martingale $(\hat{M}_t)_{t\geqslant 0}$, a smooth function $\hat{F}$ from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $\hat{c} > 0$ and $\hat{\gamma} > 0$ such that, for all $r > 0$, $\hat{\gamma} r \leqslant \hat{F}(r) \leqslant r$, for all $N \geqslant 1$, $(\hat{M}_t)_{0\leqslant t\leqslant \zeta_N}$ is square integrable, and for all $t \in [0, \zeta_N]$,*

$$\mathrm{d}\hat{F}\big(\varphi(X_t)\big) \geqslant \mathrm{d}\hat{M}_t + \hat{c}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Moreover, there exist a square integrable martingale $(\check{M}_t)_{t\geqslant 0}$, a smooth function $\check{F}$ from $\mathbb{R}_+$ to $\mathbb{R}$ and two constants $\check{c} > 0$ and $\check{\gamma} > 0$ such that, for all $r > 0$, $r \leqslant \check{F}(r) \leqslant \check{\gamma} r$, for all $N \geqslant 1$, $(\check{M}_t)_{0\leqslant t\leqslant \zeta_N}$ is square integrable, and for all $t \in [0, \zeta_N]$,

$$\mathrm{d}\check{F}\big(\varphi(X_t)\big) \leqslant \mathrm{d}\check{M}_t + \check{c}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Proof. We follow the proof of Proposition 3.3. Applying the Itô formula to $(\varphi(X_t))_{0\leqslant t\leqslant \zeta_N}$, $N \geqslant 1$, we obtain (3.7). By Proposition 3.2, we can find a bounded function Γ from $\mathbb{R}^2$ to $\mathbb{R}$, independent of N, such that for all $t \in [0, \zeta_N]$,

$$\mathrm{d}\varphi(X_t) = \exp\big(\varphi(X_t)\big)\Gamma(X_t)\,\mathrm{d}t + \big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle \nabla\varphi(X_t), \mathrm{d}K_t\big\rangle.$$

For a real C, we consider a smooth function F from $\mathbb{R}_+^*$ to $\mathbb{R}$ such that $F'(r) = \exp(C\exp(-r))$. The Itô formula yields for all $t \in [0, \zeta_N]$:

$$\begin{aligned}\mathrm{d}F\big(\varphi(X_t)\big) &= \exp\big(C\exp\big(-\varphi(X_t)\big)\big)\big[\exp\big(\varphi(X_t)\big)\Gamma(X_t) - (C/2)\exp\big(-\varphi(X_t)\big)\big\langle \nabla\varphi(X_t), a(X_t)\nabla\varphi(X_t)\big\rangle\big]\,\mathrm{d}t \\ &\quad + \exp\big(C\exp\big(-\varphi(X_t)\big)\big)\big[\big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle - \big\langle \nabla\varphi(X_t), \mathrm{d}K_t\big\rangle\big].\end{aligned} \tag{3.9}$$

By Proposition 3.2, we can find two constants $C < 0$ and $c > 0$, independent of N, such that

$$\mathrm{d}F\big(\varphi(X_t)\big) \geqslant \exp\big(C\exp\big(-\varphi(X_t)\big)\big)\big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle + c\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Choosing $F(r) = \int_0^r \exp(C\exp(-s))\,\mathrm{d}s$, we have for all $r > 0$,

$$\exp(C)r \leqslant F(r) \leqslant r.$$

This completes the proof of the lower bound.

We prove the upper bound in the same way. By Proposition 3.2 and (3.9), we can find two constants $C > 0$ and $c > 0$, independent of N, such that

$$\mathrm{d}F\big(\varphi(X_t)\big) \leqslant \exp\big(C\exp\big(-\varphi(X_t)\big)\big)\big\langle \nabla\varphi(X_t), \sigma(X_t)\,\mathrm{d}B_t\big\rangle + c\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Choosing $F(r) = \int_0^r \exp(C\exp(-s))\,\mathrm{d}s$, we have for all $r > 0$,

$$r \leqslant F(r) \leqslant \exp(C)r.$$

This completes the proof. □

Proposition 3.5. *We set, for all $t \geqslant 0$, $S_t \equiv |X_t|^2$ (so that $0 \leqslant S_t \leqslant 2$). Then, there exist a square integrable martingale $(\hat{N}_t)_{t\geqslant 0}$, a smooth function $\hat{G} : \mathbb{R}_+ \to \mathbb{R}$ and two constants $\hat{C} \geqslant 1$ and $\hat{\delta} \geqslant 1$, such that, for all $0 \leqslant r \leqslant 2$, $r \leqslant \hat{G}(r) \leqslant \hat{\delta} r$, and for all $t \geqslant 0$,*

$$\mathrm{d}\hat{G}(S_t) \geqslant \hat{C}^{-1}\,\mathrm{d}t + \mathrm{d}\hat{N}_t - \hat{C}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Moreover, there exist a square integrable martingale $(\check{N}_t)_{t\geqslant 0}$, a smooth function $\check{G} : \mathbb{R}_+ \to \mathbb{R}$ and two constants $\check{C} \geqslant 1$ and $\check{\delta} \geqslant 1$, such that, for all $0 \leqslant r \leqslant 2$, $\check{\delta}^{-1} r \leqslant \check{G}(r) \leqslant r$, and for all $t \geqslant 0$,

$$\mathrm{d}\check{G}(S_t) \leqslant \check{C}\,\mathrm{d}t + \mathrm{d}\check{N}_t - \check{C}^{-1}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big).$$

Proof. The function $q : x \in \mathbb{R}^2 \mapsto |x|^2$ satisfies $[\partial q/\partial x_1](0, x_2) = 0$ and $[\partial q/\partial x_1](1, x_2) = 2$ for $x_2 \in \mathbb{R}$ and $[\partial q/\partial x_2](x_1, 0) = 0$ and $[\partial q/\partial x_2](x_1, 1) = 2$ for $x_1 \in \mathbb{R}$. By the Lipschitz property of a and by the boundedness of b, we can find a bounded function Δ from $\mathbb{R}^2$ to $\mathbb{R}$, such that, for $t \geqslant 0$,

$$\begin{aligned} \mathrm{d}S_t &= \operatorname{trace}\bigl(a(X_t)\bigr)\,\mathrm{d}t + 2\bigl\langle X_t, b(X_t)\bigr\rangle \mathrm{d}t - 2\bigl[\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr] + 2\bigl\langle X_t, \sigma(X_t)\,\mathrm{d}B_t\bigr\rangle \\ &= \operatorname{trace}\bigl(a(0)\bigr)\,\mathrm{d}t + \Delta(X_t)S_t^{1/2}\,\mathrm{d}t - 2\bigl[\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr] + 2\bigl\langle X_t, \sigma(X_t)\,\mathrm{d}B_t\bigr\rangle. \end{aligned}$$

For a given $D \in \mathbb{R}$, we set, for all $r \geqslant 0$, $G(r) = \int_0^r \exp(Dv^{1/2})\,\mathrm{d}v$. For all $r \geqslant 0$, $G'(r) = \exp(Dr^{1/2})$ and, for all $r > 0$, $G''(r) = (D/2)r^{-1/2}G'(r)$.

For $r \in [0, 2]$, $r \leqslant G(r) \leqslant r\exp(2^{1/2}D)$ if $D \geqslant 0$ and $r\exp(2^{1/2}D) \leqslant G(r) \leqslant r$ if $D \leqslant 0$. Moreover, for all $t \geqslant 0$,

$$\begin{aligned} \mathrm{d}G(S_t) &= G'(S_t)\operatorname{trace}\bigl(a(0)\bigr)\,\mathrm{d}t + G'(S_t)\bigl[DS_t^{-1/2}\bigl|\sigma^*(X_t)X_t\bigr|^2 + \Delta(X_t)S_t^{1/2}\bigr]\,\mathrm{d}t \\ &\quad - 2G'(S_t)\bigl[\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr] + 2G'(S_t)\bigl\langle X_t, \sigma(X_t)\,\mathrm{d}B_t\bigr\rangle. \end{aligned}$$

For $D = -\lambda^{-1}\|\Delta\|_\infty$,

$$\mathrm{d}G(S_t) \leqslant \bigl(\rho_1^2 + \rho_2^2\bigr)\,\mathrm{d}t - 2\exp\bigl(-2^{1/2}\lambda^{-1}\|\Delta\|_\infty\bigr)\bigl[\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr] + 2\exp\bigl(DS_t^{1/2}\bigr)\bigl\langle X_t, \sigma(X_t)\,\mathrm{d}B_t\bigr\rangle.$$

For $D = \lambda^{-1}\|\Delta\|_\infty$,

$$\mathrm{d}G(S_t) \geqslant \bigl(\rho_1^2 + \rho_2^2\bigr)\,\mathrm{d}t - 2\exp\bigl(2^{1/2}\lambda^{-1}\|\Delta\|_\infty\bigr)\bigl[\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr] + 2\exp\bigl(DS_t^{1/2}\bigr)\bigl\langle X_t, \sigma(X_t)\,\mathrm{d}B_t\bigr\rangle.$$

This completes the proof. □

3.4. *Proof of Theorem* 2.1

To establish Theorem 2.1, it is sufficient to investigate the asymptotic behavior of $\mathbb{E}(\zeta_N)$ as $N \to +\infty$. Indeed, we can easily check that there exists a parameter $\beta > 1$ such that, for $\rho > 0$, $\zeta_{\lfloor\beta\rho^{-1}\rfloor} \leqslant T_\rho \leqslant \zeta_{\lfloor\beta^{-1}\rho^{-1}\rfloor}$.

We start with the case $\alpha > 0$. By Proposition 3.3, there exist a local martingale $(M_t)_{t\geqslant 0}$, a smooth function F from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $c > 0$ and $\gamma > 0$ such that, for all $r > 0$, $r\exp(-\gamma r^{1/\alpha}) \leqslant F(r) \leqslant r$, for all $N \geqslant 1$, $(M_t)_{0\leqslant t\leqslant\zeta_N}$ is square integrable, and for all $t \in [0, \zeta_N]$,

$$\mathrm{d}\bigl[-F\bigl(\varphi(X_t)\bigr)\bigr] \geqslant \mathrm{d}M_t + c\bigl(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr). \tag{3.10}$$

Moreover, by Proposition 3.5, we can find a square integrable martingale $(\hat{N}_t)_{t\geqslant 0}$, a smooth function $\hat{G} : \mathbb{R}_+ \to \mathbb{R}$, two constants $\hat{C} \geqslant 1$ and $\hat{\delta} \geqslant 1$ such that, for all $0 \leqslant r \leqslant 2$, $r \leqslant \hat{G}(r) \leqslant \hat{\delta}r$, and for all $t \geqslant 0$,

$$\mathrm{d}\hat{G}(S_t) \geqslant \hat{C}^{-1}\,\mathrm{d}t + \mathrm{d}\hat{N}_t - \hat{C}\bigl(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr). \tag{3.11}$$

Gathering (3.10) and (3.11), we obtain, for all $N \geqslant 1$ and $t \geqslant 0$,

$$\mathbb{E}\bigl[-\hat{C}F\bigl(\varphi(X_{t\wedge\zeta_N})\bigr) + c\hat{G}(S_{t\wedge\zeta_N})\bigr] - \mathbb{E}\bigl[-\hat{C}F\bigl(\varphi(X_0)\bigr) + c\hat{G}(S_0)\bigr] \geqslant c\hat{C}^{-1}\mathbb{E}[t \wedge \zeta_N].$$

Using (3.4), Proposition 3.1 (growth of φ) and the bounds for F and $\hat{G}$ and letting $t \to +\infty$, we deduce that for all $N \geqslant 1$,

$$c\hat{C}^{-1}\mathbb{E}[\zeta_N] \leqslant 2c\hat{\delta} + \hat{C}\bigl(\varphi(X_0)\bigr)^\alpha \leqslant 2c\hat{\delta} + 2^{\alpha/2}\hat{C}\lambda^{-\alpha/2}.$$

This completes the proof for $\alpha > 0$.

We now investigate the case $\alpha < 0$. By Proposition 3.3, there exist a local martingale $(\hat{M}_t)_{t\geqslant 0}$, a smooth function $\hat{F}$ from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $\hat{c} > 0$ and $\hat{\gamma} > 0$ such that, for all $r > 0$, $\hat{\gamma}(r-1)^+ \leqslant \hat{F}(r) \leqslant r$, for all $N \geqslant 1$, $(\hat{M}_t)_{0\leqslant t\leqslant\zeta_N}$ is square integrable, and for all $t \in [0, \zeta_N]$,

$$\mathrm{d}\hat{F}\bigl(\varphi(X_t)\bigr) \geqslant \mathrm{d}\hat{M}_t + \hat{c}\bigl(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\bigr). \tag{3.12}$$

Gathering (3.12) and (3.11), we deduce that for, all $t \geqslant 0$,

$$\mathbb{E}\big[\hat{C}\hat{F}\big(\varphi(X_{t\wedge\zeta_N})\big) + \hat{c}\hat{G}(S_{t\wedge\zeta_N})\big] - \mathbb{E}\big[\hat{C}\hat{F}\big(\varphi(X_0)\big) + \hat{c}\hat{G}(S_0)\big] \geqslant \hat{c}\hat{C}^{-1}\mathbb{E}[t \wedge \zeta_N].$$

Using (3.4), Proposition 3.1 and the bounds for $\hat{F}$ and $\hat{G}$ and letting $t \to +\infty$, we deduce that

$$\hat{c}\hat{C}^{-1}\mathbb{E}[\zeta_N] \leqslant \hat{C}\lambda^{-\alpha/2}N^{-\alpha} + 2\hat{c}\hat{\delta}.$$

This proves the upper bound for $\mathbb{E}[\zeta_N]$.

We now establish the lower bound. By Proposition 3.3, there exist a local martingale $(\check{M}_t)_{t\geqslant 0}$, a smooth function $\check{F}$ from $\mathbb{R}_+^*$ to $\mathbb{R}$ and two constants $\check{c} > 0$ and $\check{\gamma} > 0$ such that, for all $r \geqslant m$, $r - m \leqslant \check{F}(r) \leqslant \check{\gamma} r$ (with $m = \inf_{[0,1]^2\setminus\{0\}} \varphi$), for all $N \geqslant 1$, $(M_t)_{0\leqslant t\leqslant \zeta_N}$ is square integrable, and for all $t \in [0, \zeta_N]$,

$$\mathrm{d}\check{F}\big(\varphi(X_t)\big) \leqslant \mathrm{d}\check{M}_t + \check{c}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big). \tag{3.13}$$

By Proposition 3.5, we can find a square integrable martingale $(\check{N}_t)_{t\geqslant 0}$, a smooth function $\check{G}$ from $\mathbb{R}_+$ to $\mathbb{R}$ and two constants $\check{C} \geqslant 1$ and $\check{\delta} \geqslant 1$, such that, for all $r \geqslant 0$, $\check{\delta}^{-1} r \leqslant \check{G}(r) \leqslant r$, and for all $t \geqslant 0$,

$$\mathrm{d}\check{G}(S_t) \leqslant \check{C}\,\mathrm{d}t + \mathrm{d}\check{N}_t - \check{C}^{-1}\big(\mathrm{d}K_t^1 + \mathrm{d}K_t^2\big). \tag{3.14}$$

Gathering (3.13) and (3.14), we obtain, for all $t \geqslant 0$,

$$\mathbb{E}\big[\check{C}^{-1}\check{F}\big(\varphi(X_{t\wedge\zeta_N})\big) + \check{c}\check{G}(S_{t\wedge\zeta_N})\big] - \mathbb{E}\big[\check{C}^{-1}\check{F}\big(\varphi(X_0)\big) + \check{c}\check{G}(S_0)\big] \leqslant \check{c}\check{C}\mathbb{E}[t \wedge \zeta_N]. \tag{3.15}$$

Using (3.4), Proposition 3.1 and the bounds for $\check{F}$ and $\check{G}$, we deduce that

$$\check{c}\check{C}\mathbb{E}[\zeta_N] \geqslant \check{C}^{-1}\big(\eta^{-1}\Lambda^{-\alpha/2}N^{-\alpha} - m\big) - \check{C}^{-1}\check{\gamma}\lambda^{-\alpha/2}|X_0|^\alpha - 2\check{c}.$$

This completes the proof for $\alpha < 0$. We can investigate in the same way the case $\alpha = 0$ by means of Proposition 3.4.

3.5. *Proof of Proposition* 2.2

For $s > 0$, the assertion follows from Theorem 2.1 and from the Beppo–Levi theorem. For $s < 0$, (3.15) yields

$$\check{C}^{-1}\big(\eta^{-1}\Lambda^{-\alpha/2}N^{-\alpha} - m\big)\mathbb{P}\{\zeta_N \leqslant t\} \leqslant \check{c}\check{C}t + \check{C}^{-1}\check{\gamma}\lambda^{-\alpha/2}|X_0|^\alpha + 2\check{c}.$$

As $N \to +\infty$, $\zeta_N \to T_0$ almost surely, so that $\mathbb{P}\{T_0 \leqslant t\} = 0$ for all $t \geqslant 0$. A similar argument holds for $s = 0$.

References

[1] S. Balaji and S. Ramasubramanian. Passage time moments for multidimensional diffusions. *J. Appl. Probab.* **37** (2000) 246–251. MR1761674

[2] F. Comets, F. Delarue and R. Schott. Distributed algorithms in an ergodic Markovian environment. *Random Structures Algorithms* **30** (2007) 131–167. MR2283226

[3] J. G. Dai and R. J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedra. *Theory Probab. Appl.* **40** (1996) 1–40. MR1346729

[4] J. G. Dai and R. J. Williams. Letter to the editors: Remarks on our paper "existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedra". *Theory Probab. Appl.* **50** (2006) 346–347. MR2222685

[5] R. D. DeBlassie. Explicit semimartingale representation of Brownian motion in a wedge. *Stochastic Process. Appl.* **34** (1990) 67–97. MR1039563

[6] R. D. DeBlassie, D. Hobson, E. A. Housworth and E. H. Toby. Escape rates for transient reflected Brownian motion in wedges and cones. *Stochastics Stochastics Rep.* **57** (1996) 199–211. MR1425365

[7] G. Fayolle, V. A. Malyshev and M. V. Menshikov. *Topics in the Constructive Theory of Countable Markov Chains*. Cambridge University Press, Cambridge, 1995. MR1331145

[8] A. Friedman. *Stochastic Differential Equations and Applications*. Vol. 1. Probability and Mathematical Statistics, Academic Press, New York–London, 1975. MR0494490

[9] N. Guillotin-Plantard and R. Schott. Distributed algorithms with dynamic random transitions. *Random Structures Algorithms* **21** (2002) 371–396. MR1945376

[10] R. Z. Hasminskii. *Stochastic Stability of Differential Equations*. Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis, Sijthoff & Noordhoff, Alphen aan den Rijn–Germantown, Md, 1980. MR0600653

[11] Y. Kwon and R. J. Williams. Reflected Brownian motion in a cone with radially homogeneous reflection field. *Trans. Amer. Math. Soc.* **327** (1991) 739–780. MR1028760

[12] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** (1984) 511–537. MR0745330

[13] G. Louchard. Some distributed algorithms revisited. *Commun. Statist. Stochastic Models* **11** (1995) 563–586. MR1358937

[14] G. Louchard and R. Schott. Probabilistic analysis of some distributed algorithms. *Random Structures Algorithms* **2** (1991) 151–186. MR1099798

[15] G. Louchard, R. Schott, M. Tolley and P. Zimmermann. Random walks, heat equations and distributed algorithms. *J. Comput. Appl. Math.* **53** (1994) 243–274. MR1306128

[16] R. S. Maier. Colliding stacks: A large deviations analysis. *Random Structures Algorithms* **2** (1991) 379–420. MR1125956

[17] R. S. Maier and R. Schott. Exhaustion of shared memory: stochastic results. In: Proceedings of WADS'93, LNCS No 709, Springer Verlag, 1993, pp. 494–505.

[18] M. Menshikov and R. J. Williams. Passage-time moments for continuous non-negative stochastic processes and applications. *Adv. in Appl. Probab.* **28** (1996) 747–762. MR1404308

[19] K. Ramanan. Reflected diffusions defined *via* the extended Skorokhod map. *Electron. J. Probab.* **11** (2006) 934–992. MR2261058

[20] M. I. Reiman and R. J. Williams. A boundary property of semimartingale reflecting Brownian motions. *Probab. Theory Related Fields* **77** (1988) 87–97. MR0921820

[21] M. I. Reiman and R. J. Williams. Correction to: "A boundary property of semimartingale reflecting Brownian motions" [*Probab. Theory Related Fields* **77** 87–97]. *Probab. Theory Related Fields* **80** (1989) 633. MR980691

[22] Y. Saisho. Stochastic differential equations for multidimensional domain with reflecting boundary. *Probab. Theory Related Fields* **74** (1987) 455–477. MR0873889

[23] L. Słomiński. Euler's approximations of solutions of SDEs with reflecting boundary. *Stochastic Process. Appl.* **94** (2001) 317–337. MR1840835

[24] H. Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math.* **9** (1979) 163–177. MR0529332

[25] L. M. Taylor and R. J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields* **96** (1993) 283–317. MR1231926

[26] S. R. S. Varadhan and R. J. Williams. Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** (1985) 405–443. MR0792398

[27] R. J. Williams. Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *Ann. Probab.* **13** (1985) 758–778. MR0799421

[28] R. J. Williams. Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahrsch. Verw. Gebiete* **69** (1985) 161–176. MR0779455