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ON THE LARGEST COMPONENT OF A RANDOM GRAPH WITH A SUBPOWER-LAW DEGREE SEQUENCE IN A SUBCRITICAL PHASE

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A uniformly random graph on n vertices with a fixed degree sequence, obeying a γ subpower law, is studied. It is shown that, for $\gamma > 3$, in a subcritical phase with high probability the largest component size does not exceed $n^{1/\gamma+\varepsilon_n}$, $\varepsilon_n = O(\ln\ln n/\ln n)$, $1/\gamma$ being the best power for this random graph. This is similar to the best possible $n^{1/(\gamma-1)}$ bound for a different model of the random graph, one with independent vertex degrees, conjectured by Durrett, and proved recently by Janson.

1. Introduction. In a recently published book ([5], Section 1.2), Durrett formulated the following conjecture.

Let $\mathbf{p} = \{p_j\}_{j\ge 1}$ be a probability distribution. Let $D_1, \dots, D_n$ be i.i.d. random variables, each having the distribution $\mathbf{p}$. Consider a graph on the vertex set $[n]$, chosen uniformly at random among all graphs with the degree sequence $(D_1, \dots, D_n)$. For such a set of graphs to be nonempty, it is necessary that $\max D_i < n$ and $\sum_i D_i$ is even. (The first condition holds with probability approaching 1 if $\mathrm{E}[D] < \infty$, and the second condition holds with probability approaching $1/2$ if $\mathrm{E}[D^2] < \infty$, and g.c.d.$\{j : p_j > 0\}$ is odd.) Durrett states that at a vicinity of a generic vertex v the random graph looks like a tree rooted at v, and the number of direct descendants of every descendant of v has a distribution $\mathbf{q} = \{q_j\}_{j\ge 0}$,

$$q_j = \frac{(j+1)p_{j+1}}{\sum_{k\ge 1} kp_k}, \qquad j \ge 0. \tag{1.1}$$

If so, under the condition

$$\nu := \sum_j jq_j < 1,$$

one should expect that the component containing v, and even the largest component, are likely to be small compared to n. Specifically, Durrett conjectured that

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for the power-law distribution,

$$p_j = Cj^{-\gamma}, \qquad \gamma > 3,$$

the likely size of the largest component should be of order $n^{1/(\gamma-1)}$, exactly. In other words, the largest component has size of order of the maximum vertex degree. Janson [6] has recently proved Durrett's conjecture.

In this paper we consider a different model of the random graph, in which a degree sequence is fixed. There is given a tuple $\mathbf{d} = \mathbf{d}(n) = (d_1, \dots, d_n)$ of positive integers $d_1, \dots, d_n < n$ such that $d_1 + \cdots + d_n$ is even. We consider a sample space $\mathcal{G}_{n,\mathbf{d}}$ of all graphs on $[n]$ with the degree sequence $\mathbf{d}$. Introduce the *empirical* degree distribution

$$\mathbf{p} = \{p_1, \dots, p_{n-1}\}, \qquad p_j := \frac{|\{i \in [n] : d_i = j\}}{n}.$$

Let $\mathbf{q} = \mathbf{q}(\mathbf{p})$ be defined by (1.1). Assuming that $\mathbf{p}$ obeys a subpower law, that is,

$$p_j \le cj^{-\gamma}, \qquad 1 \le j \le n-1, \tag{1.2}$$

with $\gamma > 3$, we show that $\mathcal{G}_{n,\mathbf{d}}$ is nonempty. We prove that, under the condition

$$\sum_{j\ge 1} jq_j \le 1 - \varepsilon, \qquad \varepsilon > 0, \tag{1.3}$$

the largest component in the graph $G_{n,\mathbf{d}}$, chosen uniformly at random from $\mathcal{G}_{n,\mathbf{d}}$, has size $C_n = O_p(n^{1/\gamma} \ln n)$, that is, $C_n/(n^{1/\gamma} \ln n)$ is bounded in probability. Similarly to Janson's result for the independent degrees model, the power $1/\gamma$ is the best possible for the fixed-degree-sequence model, since among the degree sequences $\mathbf{d}$ in question there are those with $\max_{v\in[n]} d_v$ of the exact order $n^{1/\gamma}$.

That, under the condition equivalent to (1.3), $C_n/n \to 0$ in probability, had already been proved by Molloy and Reed [7, 8]. They also proved that, under their form of the condition

$$\sum_{j\ge 1} jq_j \ge 1 + \varepsilon,$$

with high probability the random graph $G_{n,\mathbf{d}}$ has a giant component of size $\Theta(n)$, even being able to establish, under additional conditions, the limit of that size scaled by n.

Following the footsteps of Molloy and Reed, our proof is based on analysis of an algorithm that determines the component containing a given vertex. We construct a collection of exponential supermartingales in order to prove, via the optional sampling theorem, that the random growth of that component follows closely a certain deterministic path. See [1] and [9], where a similar approach was used for analysis of the site (bootstrap) and the bond percolation on a random *regular* graph.

2. Main result and proofs. Let $d_1, \dots, d_n$ be positive integers, such that $d_1 + \dots + d_n$ is even. Let $\mathcal{G}_{n,\mathbf{d}}$ denote the sample space of all graphs on the vertex set $[n]$ that have the degree sequence $\mathbf{d} = (d_1, \dots, d_n)$. Denote by $G_{n,\mathbf{d}}$ the random graph which is distributed uniformly on $\mathcal{G}_{n,\mathbf{d}}$.

In parallel, let $M\mathcal{G}_{n,\mathbf{d}}$ denote the sample space of all multigraphs with the degree sequence $\mathbf{d}$. Let us describe the random multigraph $MG_{n,\mathbf{d}}$ suggested first by Bollobás [3]. Consider the disjoint sets $S_1, \dots, S_n$ of cardinalities $d_1, \dots, d_n$; set S_i representing vertex $i \in [n]$. (Some people prefer assigning d_i "half-edges" to a vertex $i \in [n]$, instead of sets S_i, but the difference is purely linguistic.) We know that $S := \bigcup_i S_i$ has an even cardinality $2m := d_1 + \dots + d_n$. Introduce the sample space $\mathcal{P}_{n,\mathbf{d}}$ of all $(2m-1)!! = 1 \cdot 3 \cdots (2m-1)$ *pairings* on S. Let $P_{n,\mathbf{d}}$ be the random pairing distributed uniformly on $\mathcal{P}_{n,\mathbf{d}}$. Define $MG_{n,\mathbf{d}}$ as follows: two vertices $i, j \in [n]$ are joined by an edge iff there are $s' \in S_i$, $s'' \in S_j$ such that $\{s', s''\}$ is one of the pairs in $P_{n,\mathbf{d}}$. Obviously $MG_{n,\mathbf{d}}$ may well have loops and multiple edges. And it is *not* uniformly distributed on $M\mathcal{G}_{n,\mathbf{d}}$. However, conditioned on the event $A_n := \{\text{no loops, no multiple edges}\}$, $MG_{n,\mathbf{d}}$ is a simple graph distributed uniformly on $\mathcal{G}_{n,\mathbf{d}}$, hence can be viewed as the random graph $G_{n,\mathbf{d}}$. (This connection is due to the observation that every $G \in \mathcal{G}_{n,\mathbf{d}}$ induces the same number, $d_1! \cdots d_n!$, of pairings in $\mathcal{P}_{n,\mathbf{d}}$.)

Suppose that $\mathbf{d} = \mathbf{d}(n)$ is such that

$$\liminf \mathrm{P}(A_n) > 0. \tag{2.1}$$

Under (2.1), any asymptotically rare (sure) event for $MG_{n,\mathbf{d}}$ is an asymptotically rare (sure) event for $G_{n,\mathbf{d}}$. And we will see that the probability estimates for the events in $M\mathcal{G}_{n,\mathbf{d}}$ become quite manageable once translated into the language of the space $\mathcal{P}_{n,\mathbf{d}}$.

Introduce

$$\nu = \nu(n) = \frac{\sum_{i \in [n]} d_i(d_i - 1)}{\sum_{i \in [n]} d_i}. \tag{2.2}$$

ν can be interpreted as the expected *outdegree* of a nonroot vertex in a tree rooted at a given vertex v, which, heuristically, is how $G_{n,\mathbf{d}}$ looks like in a vicinity of v. Let

$$p_j = p_j(n) := \frac{1}{n} |\{i \in [n] : d_i = j\}|, \qquad j \in [n-1].$$

Then (2.2) becomes

$$\nu = \frac{\sum_{j \in [n-1]} j(j-1) p_j}{\sum_{j \in [n-1]} j p_j},$$

which is the ratio of the first two factorial moments of the distribution $\{p_j\}$. We denote the first moment, the average vertex degree, by $d = d(n)$.

We assume that $\mathbf{d} = \mathbf{d}(n)$ is such that

$$\limsup_{n\to\infty} \sum_{j\in[n-1]} j^2 p_j < \infty, \qquad \lim_{n\to\infty} n^{-1} \sum_{j\in[n-1]} j^4 p_j = 0. \tag{2.3}$$

In fact, we assume a stronger condition, namely that $\{p_j\}$ is a subpower-law distribution, that is,

$$p_j \le c\, j^{-\gamma}, \qquad \gamma > 3. \tag{2.4}$$

In this case, since

$$|\{i \in [n] : d_i = j\}| = np_j \le c\frac{n}{j^\gamma} < 1 \qquad \forall j > j_n := \lfloor A(\gamma, c) n^{1/\gamma} \rfloor,$$

we see that

$$|\{i \in [n] : d_i = j\}| = 0, \qquad p_j = 0\ \forall j > j_n.$$

In other words, $\max_{i\in[n]} d_i$, the largest vertex degree, is j_n, at most. That the first condition in (2.3) is met under (2.4) is obvious; the second condition holds true, since (2.4) implies

$$n^{-1} \sum_{j\in[n-1]} j^4 p_j = o(n^{-1/3});$$

see (2.19).

LEMMA. *Under the condition* (2.3),

$$\liminf \mathrm{P}(A_n) \ge \exp(-\hat{\nu}/2 - \hat{\nu}^2/4) > 0, \qquad \hat{\nu} := \limsup \nu.$$

NOTE. Applied to the degree sequence $\mathbf{d} = (d, \dots, d)$, this lemma yields a well-known asymptotic formula for the number of all d-regular graphs, due to Bender and Canfield [2].

We prove Lemma in the Appendix.

THEOREM. *Let C_n denote the size of the largest component* (*cluster*) *of the random graph $G_{n,\mathbf{d}}$. Under the condition* (2.4), *for $\lambda = \lambda(n) \to \infty$ however slowly,*

$$\lim_{n\to\infty} \mathrm{P}\{C_n \le \lambda n^{1/\gamma} \ln n\} = 1,$$

provided that

$$\limsup \nu < 1; \tag{2.5}$$

in short, $C_n = O_P(n^{1/\gamma} \ln n)$.

PROOF. We will prove the bound by upper-bounding the likely size of a component that contains a generic vertex $v \in [n]$. In view of Lemma, it suffices to bound the size of the component of the random multigraph $MG_{n,\mathbf{d}}$ that contains vertex v.

Notice that a subset V of $[n]$ is the vertex set of this component iff for every $u \in V$ there exist $w_1, \dots, w_k \in [n]$ such that, for some $s_0 \in S_v, s_1 \in S_{w_1}, \dots, s_k \in S_{w_k}, s_{k+1} \in S_u$, all the pairs $\{s_0, s_1\}, \dots, \{s_k, s_{k+1}\}$ are in $P_{n,\mathbf{d}}$. So we may, and will, deal with the corresponding "component" in $P_{n,\mathbf{d}}$ itself. We determine this component algorithmically, by adding to a current cluster of pairs exactly one new pair $\{s', s''\} \in P_{n,\mathbf{d}}$, where a *point* s' is not in the current cluster of pairs, but has the same vertex "host" as one of the points in those pairs. [We will call them the (currently) active points.] If the point s'', the partner of the point s', is hosted by a fresh vertex, u, then u joins the current vertex cluster, and the $d_u - 1$ still unexplored points hosted by u become active. As a result, the number of active points changes by $(-1) + (d_u - 1) = d_u - 2$. If s'' happens to be hosted by a vertex from the current vertex cluster, then the vertex cluster remains the same, but the number of active points decreases by 2.

Importantly, instead of generating the uniformly random paring $P_{n,\mathbf{d}}$ in advance, we can generate it one pair at a time, as called for by the algorithmic process. Namely, given a total ordering of the points in S, as s' we pick the first, say, active point and pair it with a point s'', chosen uniformly at random among all points not in the pairs.

Let $A(t)$, $I_j(t)$ denote the total number of the currently active points and the number of the currently inactive (not in the current cluster, i.e.) vertices after t steps of the algorithm. In particular,

$$A(0) = d_v, \qquad I_j(0) = np_j - \delta_{j,d_v}, \qquad j \le j_n. \tag{2.6}$$

Introduce

$$I(t) = \sum_{j \le j_n} j I_j(t),$$

the total number of inactive *points*. From the discussion above,

$$A(t+1) + I(t+1) = A(t) + I(t) - 2,$$

so that, by (2.6),

$$A(t) + I(t) = A(0) + I(0) - 2t = nd - 2t, \tag{2.7}$$

where

$$d = n^{-1} \sum_{i \le n} d_i = \sum_{j \le j_n} j p_j$$

is the average vertex degree. From (2.7), the process will terminate no later than by time $t \le nd/2$.

Clearly $\{A(t), \{I_j(t)\}_{j \le j_n}\}_{t \ge 0}$ is a Markov chain, and if

$$t < T = T_v := \min\{\tau > 0 : \min\{A(\tau), I(\tau)\} = 0\},$$

then

$$\mathrm{P}[I_j(t+1) = I_j(t) - 1 \mid \mathcal{F}_t] = -\frac{jI_j(t)}{A(t) + I(t) - 1}, \tag{2.8}$$

$$\mathrm{E}[I_j(t+1) \mid \mathcal{F}_t] = I_j(t) + (-1)\frac{jI_j(t)}{A(t)+I(t)-1},$$

$$\begin{aligned} \mathrm{E}[A(t+1) \mid \mathcal{F}_t] &= A(t) + (-2)\frac{A(t)-1}{A(t)+I(t)-1} \\ &\quad + \sum_{j \ge 1} \frac{jI_j(t)}{A(t)+I(t)-1}(j-2); \end{aligned} \tag{2.9}$$

here $\mathrm{P}[\cdot \mid \mathcal{F}_t]$, $\mathrm{E}[\cdot \mid \mathcal{F}_t]$ denote the probability and the expectation conditioned on $\{A(t), \{I_j(t)\}_{j \le j_n}\}$. Since T is a stopping time, it follows from (2.8) and (2.7) that, for each $j \le j_n$,

$$X_j(t) := \begin{cases} \dfrac{I_j(t)}{\prod_{\tau=0}^{t-1}(1 - j/(nd - 2\tau - 1))}, & t \le T, \\ X_j(T), & t > T, \end{cases}$$

is a martingale, with

$$\mathrm{E}[X_j(t)] \equiv X_j(0) = I_j(0) = np_j - \delta_{j,d_v}. \tag{2.10}$$

We want to show that, for $t = O(n^\alpha)$, $\alpha \in (\gamma^{-1}, 1 - \gamma^{-1})$, $X_j(t)$ is relatively close to $X_j(0)$, with probability very close to 1. (Of course, we focus on α close to γ^{-1}, since we expect the process to terminate around a time close to $n^{\gamma^{-1}}$.) To this end, first let us prove that the sequence

$$Q_j(t) := \exp[n^{\beta_j} X_j(t)/n], \qquad t \le n^\alpha,$$

is "almost" a (super)martingale, provided that

$$\begin{aligned} &\gamma^{-1} + \alpha < 1, \\ &2(\beta_j - 1) = \min\left\{0, -\alpha + (\gamma - 1)\frac{\ln j}{\ln n}\right\}. \end{aligned} \tag{2.11}$$

Let $t < T$. Observe that, for $j \le j_n$,

$$\begin{aligned} \prod_{\tau=0}^{t}\left(1 - \frac{j}{nd - 2\tau - 1}\right) &= \exp\left(-\frac{j}{2}\int_{nd-2t}^{nd} \frac{dx}{x} + O(j/n)\right) \\ &= \left(1 - \frac{2t}{nd}\right)^{j/2}(1 + O(j/n)) \end{aligned} \tag{2.12}$$

$$= 1 + O(jt/n)$$
$$= 1 + O(n^{\gamma^{-1}+\alpha-1}) \to 1.$$

Consequently, using

$$0 \le I_j(t) - I_j(t+1) \le 1, \qquad jI_j(t) \le jI_j(0) = jp_jn \le dn,$$

we obtain

$$\begin{aligned} & n^{-1}X_j(t+1) - n^{-1}X_j(t) \\ & \quad = n^{-1}\frac{I_j(t+1) - I_j(t)}{\prod_{\tau=0}^{t}(1 - j/(nd - 2\tau - 1))} \\ & \qquad - n^{-1}\frac{I_j(t)j/(nd - 2t - 1)}{\prod_{\tau=0}^{t}(1 - j/(nd - 2\tau - 1))} \\ & \quad = O\big(n^{-1}|I_j(t+1) - I_j(t)|\big) + O(jI_j(t)n^{-2}) \\ & \quad = O(n^{-1}). \end{aligned} \tag{2.13}$$

Therefore, as $\beta_j \le 1$,

$$\begin{aligned} \frac{Q_j(t+1)}{Q_j(t)} &= \exp\big[n^{\beta_j}\big(n^{-1}X_j(t+1) - n^{-1}X_j(t)\big)\big] \\ &= 1 + n^{\beta_j-1}\big(X_j(t+1) - X_j(t)\big) \\ &\quad + O\big(n^{2(\beta_j-1)}\big(X_j(t+1) - X_j(t)\big)^2\big). \end{aligned}$$

Since $X_j(t)$ is a martingale, we have

$$\mathrm{E}[X_j(t+1) - X_j(t) \mid \mathcal{F}_t] = 0.$$

Further, by (2.13) and (2.8),

$$\begin{aligned} & \mathrm{E}\big[\big(X_j(t+1) - X_j(t)\big)^2 \mid \mathcal{F}_t\big] \\ & \quad = O\big(\mathrm{E}\big[\big(I_j(t+1) - I_j(t)\big)^2 \mid \mathcal{F}_t\big]\big) + O((jp_j)^2) \\ & \quad = O\big(jp_j + (jp_j)^2\big) = O(jp_j). \end{aligned}$$

Consequently, for $t < T$, and trivially for $t \ge T$,

$$\begin{aligned} \frac{1}{Q_j(t)}\mathrm{E}[Q_j(t+1) \mid \mathcal{F}_t] &= 1 + O\big(n^{2(\beta_j-1)}jp_j\big) \\ &= 1 + O\big(n^{2(\beta_j-1)}j^{-\gamma+1}\big) \\ &= 1 + O(n^{-\alpha}), \end{aligned}$$

the third equality and the fourth equality following from (2.4) and the definition of β_j in (2.11), respectively. Thus, there exists $\varepsilon_n > 0$, $\varepsilon_n = O(n^{-\alpha})$, such that

$$\mathrm{E}[(1+\varepsilon_n)^{-t-1} Q_j(t+1) \mid \mathcal{F}_t] \le (1+\varepsilon_n)^{-t} Q_j(t), \qquad t \le n^{\alpha}.$$

It makes

$$\hat{Q}_j(t) := (1+\varepsilon_n)^{-t} Q_j(t), \qquad t \le n^{\alpha},$$

a supermartingale, that differs from $Q_j(t)$ by a factor bounded away from both zero and infinity.

Given $j \le j_n$, and $z > 0$, introduce a stopping time $\mathcal{T}_j(z)$, the first $t \le n^{\alpha} \wedge T$ such that

$$\left| \frac{I_j(t)}{n} \prod_{\tau=0}^{t-1} \left(1 - \frac{j}{nd - 2\tau - 1}\right)^{-1} - \frac{I_j(0)}{n} \right| > \frac{z}{n^{\beta_j}},$$

and set $\mathcal{T}_j(z) = \lceil n^{\alpha} + 1 \rceil$ if no such t exists. By (2.10), for $t \le n^{\alpha} \wedge T$ and $t < \min_j \mathcal{T}_j(z)$, we have

$$I_j(t) = (np_j - \delta_{j,d_v}) \prod_{\tau=0}^{t-1} \left(1 - \frac{j}{nd - 2\tau - 1}\right) + O(n^{1-\beta_j} z). \tag{2.14}$$

Applying the Optional Sampling Theorem to the supermartingale

$$\frac{\hat{Q}_j(t)}{\hat{Q}_j(0)} = (1+\varepsilon_n)^{-t} \exp\bigl[n^{\beta_j}\bigl(X_j(t)/n - X_j(0)/n\bigr)\bigr], \qquad t \le n^{\alpha},$$

the stopping time $\mathcal{T}_j(z)$ (Durrett [4], Section 4.7), and using Markov inequality, we have: *uniformly* for $z > 0$, and $j \le j_n$,

$$\mathrm{P}\{\mathcal{T}_j(z) = \lceil n^{\alpha} + 1 \rceil\} = O(e^{-z}).$$

Choosing $z = \chi \ln n$, $(\chi > \gamma^{-1})$, and introducing

$$B_n = \left\{ \min_{j \le j_n} \mathcal{T}_j(z) = \lceil n^{\alpha} + 1 \rceil \right\},$$

we obtain then

$$\mathrm{P}(B_n) \ge 1 - O(j_n e^{-z}) = 1 - O(n^{\gamma^{-1}} e^{-\chi \ln n}) = 1 - O(n^{-\chi + \gamma^{-1}}). \tag{2.15}$$

Notice that, on the likely event B_n, (2.14) holds for all $t \le n^{\alpha} \wedge T$.

Armed with (2.14), we turn our attention to (2.9) for

$$\mathrm{E}[A(t+1)\mid \mathcal{F}_t], \qquad t\le n^{\alpha}\wedge T \text{ and } t<\min_j T_j(z).$$

By (2.14), for the sum in (2.10) we can write

$$\begin{aligned}\sum_{j\le j_n} j(j-2)I_j(t) &= n\sum_{j\le j_n} j(j-2)p_j\prod_{\tau=0}^{t-1}\left(1-\frac{j}{nd-2\tau-1}\right)\\ &\quad + O(d_v^2)+O\left(\ln n\sum_{j\le j_n} j^2 n^{1-\beta_j}\right).\end{aligned}\tag{2.16}$$

Here $d_v^2=O(n^{2\gamma^{-1}})$, and by the definition of β_j,

$$\begin{aligned}\sum_{j\le j_n} j^2 n^{1-\beta_j} &\le \sum_{j\le n^{\alpha/(\gamma-1)}} j^2 n^{1-\beta_j}+\sum_{j\le j_n} j^2\\ &= n^{\alpha/2}\sum_{j\le n^{\alpha/(\gamma-1)}}\frac{1}{j^{(\gamma-5)/2}}+O(j_n^3)\\ &= \begin{cases} O\big(n^{3\alpha/(\gamma-1)}\ln n\big)+O(n^{3\gamma^{-1}}), & \gamma\le 7,\\ O(n^{\alpha/2})+O(n^{3\gamma^{-1}}), & \gamma>7.\end{cases}\end{aligned}$$

Or

$$\sum_{j\le j_n} j^2 n^{1-\beta_j}=O(n^{3\gamma^{-1}}),\tag{2.17}$$

for α close to γ^{-1}. Furthermore, using (2.12),

$$\begin{aligned}&n\sum_{j\le j_n} j(j-2)p_j\prod_{\tau=0}^{t-1}\left(1-\frac{j}{nd-2\tau-1}\right)\\ &\qquad = n\sum_{j\le j_n} j(j-2)p_j\left(1-\frac{2t}{nd}\right)^{j/2}+O\left(\sum_{j\le j_n} j^3p_j\right).\end{aligned}\tag{2.18}$$

Here, by

$$1-mx\le (1-x)^m\le 1-mx+\binom{m}{2}x^2, \qquad x\ge 0,$$

we have

$$\begin{aligned}1-\frac{2t}{nd}\frac{j}{2} &\le \left(1-\frac{2t}{nd}\right)^{j/2}\\ &\le 1-\frac{2t}{nd}\frac{j}{2}+O(n^{-2}j^2t^2);\end{aligned}$$

for $j=1$ we need the lower bound as $j(j-2)|_{j=1}<0$. Therefore

$$
\begin{aligned}
&n\sum_{j\le j_n} j(j-2)p_j\left(1-\frac{2t}{nd}\right)^{j/2}\\
&\qquad\le n\sum_{j\le j_n} j(j-2)p_j+O\left(t\sum_{j\le j_n} j^3p_j+n^{-1}t^2\sum_{j\le j_n} j^4p_j\right)\\
&\qquad= n\sum_{j\le j_n} j(j-2)p_j+O\bigl(n^{\alpha+\gamma^{-1}}+n^{-1+2(\alpha+\gamma^{-1})}\bigr)\\
&\qquad= n\sum_{j\le j_n} j(j-2)p_j+O(n^{\alpha+\gamma^{-1}}),
\end{aligned}\tag{2.19}
$$

as $\alpha+\gamma^{-1}<1$; see (2.11). [We have used the bounds

$$
\begin{aligned}
\sum_{j\le j_n} j^3p_j&=O\bigl(n^{\max\{0,(4-\gamma)\gamma^{-1}\}}\ln n\bigr),\\
\sum_{j\le j_n} j^4p_j&=O\bigl(n^{\max\{0,(5-\gamma)\gamma^{-1}\}}\ln n\bigr),
\end{aligned}\tag{2.20}
$$

which easily follow from (2.4).]

Combining (2.16)–(2.19), we obtain: for $t\le n^\alpha\wedge T$, $t<\min_j \mathcal{T}_j(z)$,

$$
\sum_{j\le j_n} j(j-2)I_j(t)\le n\sum_{j\le j_n} j(j-2)p_j+O(n^{3\gamma^{-1}}\ln n),\tag{2.21}
$$

if α is close to γ^{-1}. Notice that

$$
\sum_j j(j-2)p_j=\left(\sum_j jp_j\right)\left(\frac{\sum_j j(j-1)p_j}{\sum_j jp_j}-1\right)=d(\nu-1),
$$

so that

$$
\limsup\sum_j j(j-2)p_j<0,
$$

which is the Molloy–Reed condition for the subcritical phase. So, by (2.21) and the condition $\gamma>3$, (2.9) implies that

$$
\begin{aligned}
&\mathrm{E}[A(t+1)\mid \mathcal{F}_t]\le A(t)-a\qquad \Bigl[t\le T\wedge n^\alpha \text{ and } t<\min_j \mathcal{T}_j(z)\Bigr];\\
&\qquad a:=\tfrac{1}{2}\limsup(1-\nu)>0,
\end{aligned}\tag{2.22}
$$

for all n large enough.

The rest is short. Set

$$A(t+1) = A(t) - a \qquad \Big[t > T \wedge n^\alpha \text{ or } t \ge \min_j \mathcal{T}_j(z)\Big].$$

Clearly the extended sequence $\{A(t)\}$ satisfies (2.22) for all t. Besides, since $T = T_v$ is the first time τ when

$$\min\{A(\tau), I(\tau)\} = 0,$$

we have

$$\begin{aligned}\{n^\alpha < T\} \cap \Big\{\min_j \mathcal{T}_j(z) = \lceil n^\alpha + 1\rceil\Big\} &= \{n^\alpha < T\} \cap B_n \\ &\subseteq \{A([n^\alpha]) > 0\}.\end{aligned} \tag{2.23}$$

Furthermore, since the maximum vertex degree is j_n at most,

$$A(0) = d_v \le j_n, \qquad |A(t+1) - A(t)| \le j_n; \qquad j_n = O(n^{\gamma^{-1}}).$$

Also, reading out the conditional distribution $\mathrm{P}\{A(t+1) - A(t) = i \,|\, \mathcal{F}_t\}$ from (2.10), and using (2.20),

$$\mathrm{E}\big[\big(A(t+1) - A(t)\big)^2 \mid \mathcal{F}_t\big] \le 4 + \frac{2}{d}\sum_{j \le j_n} j^3 p_j = O\big(n^{\max\{0,(4-\gamma)\gamma^{-1}\}} \ln n\big),$$

if $t \le T \wedge n^\alpha$ and $t < \min_j \mathcal{T}_j(z)$. And the bound holds trivially for the larger values of t. Then

$$\begin{aligned}&\mathrm{E}\big[\exp\big(n^{-\gamma^{-1}}\big(A(t+1) - A(t)\big)\big) \mid \mathcal{F}_t\big] \\ &\quad = 1 + n^{-\gamma^{-1}} \mathrm{E}[A(t+1) - A(t) \mid \mathcal{F}_t] \\ &\qquad + O\big(n^{-2\gamma^{-1}} \mathrm{E}\big[\big(A(t+1) - A(t)\big)^2 \mid \mathcal{F}_t\big]\big) \\ &\quad \le 1 - an^{-\gamma^{-1}} + O\big(n^{\max\{-2\gamma^{-1},(2-\gamma)\gamma^{-1}\}} \ln n\big) \\ &\quad \le 1 - bn^{-\gamma^{-1}}, \qquad b < a,\end{aligned}$$

since $\gamma > 3$. Therefore

$$\mathrm{E}\big[\exp\big(n^{-\gamma^{-1}}\big(A(t+1) - A(t)\big)\big) \mid \mathcal{F}_t\big] \le \exp(-bn^{-\gamma^{-1}}),$$

and then

$$\mathrm{E}[\exp(n^{-\gamma^{-1}} A(t))] = O(\exp(-tbn^{-\gamma^{-1}})).$$

Hence

$$\mathrm{P}\{A(t) > 0\} = O(\exp(-tbn^{-\gamma^{-1}})).$$

In particular, choosing

$$\alpha = \gamma^{-1} + \frac{\ln\ln n}{\ln n} + \frac{\eta}{\ln n}, \qquad \eta > 0, \tag{2.24}$$

which certainly satisfies the inequality $\gamma^{-1} + \alpha < 1$ in (2.11) for $n \ge n(\eta)$, we obtain

$$\mathrm{P}\{A([n^{\alpha}]) > 0\} = O(n^{-b\eta}).$$

By (2.23), we have then

$$\mathrm{P}\{(n^{\alpha} < T_v) \cap B_n\} = O(n^{-b\eta}),$$

and combining this estimate with (2.15) we conclude: for any fixed $\chi > 0$ and $\eta > 0$,

$$\mathrm{P}\{n^{\alpha} < T_v\} = \mathrm{P}\{e^{\eta} n^{\gamma^{-1}} \ln n < T_v\} = O(n^{-\chi+\gamma^{-1}} + n^{-b\eta}).$$

Of course, a bounded constant factor implicit in the big-Oh notation depends on χ and η. Thus, given $K > 0$, there exists $L = L(K)$ such that

$$\mathrm{P}\{L n^{\gamma^{-1}} \ln n < T_v\} \le n^{-K-1}, \qquad v \in [n],$$

whence

$$\mathrm{P}\left\{\max_{v \in [n]} T_v > L n^{\gamma^{-1}} \ln n\right\} \le n^{-K}.$$

It remains to notice that the component containing the vertex v has size T_v at most, so that C_n, the size of the largest component, is $\max_v T_v$, at most.

NOTE. If, instead of (2.24), we had set

$$\alpha = \gamma^{-1} + \frac{\omega(n)}{\ln n}, \qquad \omega(n) \to \infty, \qquad \omega(n) = o(\ln n),$$

we would have proved that

$$\mathrm{P}\{\omega(n) n^{\gamma^{-1}} < T_v\} = O\big(n^{-\chi+\gamma^{-1}} + e^{-b\omega(n)}\big), \tag{2.25}$$

so that $T_v = O_p(n^{\gamma^{-1}})$. However, the $e^{-b\omega(n)}$ term in (2.25) would not have allowed us to deduce that $\max_{v \in [n]} T_v = O_p(n^{\gamma^{-1}})$ as well. $\square$

APPENDIX

PROOF OF LEMMA. Our argument is patterned after Bollobás's proof [3] of a similar, but more general, result for $\max_i d_i = O(1)$.

Let X_n and Y_n denote the total number of loops and the total number of pairs of parallel pairs in the random pairing $P_{n,\mathbf{d}}$. We want to show that, for every fixed k and ℓ,

$$\mathrm{E}[(X_n)_k (Y_n)_\ell] \sim \left(\frac{\nu}{2}\right)^{k+2\ell}, \qquad n \to \infty,$$

where $(a)_b$ stands for the falling factorial $a(a-1)\cdots(a-b+1)$. This would imply that X_n and Y_n are asymptotically independent, and Poisson distributed, with parameter $\nu/2$ and $(\nu/2)^2$, respectively, and the statement would follow, since

$$\mathrm{P}(A_n) = \mathrm{P}\{X_n = 0, Y_n = 0\}.$$

Combinatorially, $(X_n)_k (Y_n)_\ell$ is the total number of samples, with order and without replacement, of k loops and of ℓ pairs of parallel edges from the random pairing $P_{n,\mathbf{d}}$. Given any such sample, let $S_{i_1}, \dots, S_{i_{k+2\ell}}$ be the ordered sequence of sets such that S_{i_j}, $j \le k$, contains the jth loop, and, for $1 \le t \le \ell$, the tth pair of parallel pairs is (s_1, s_2), (s_3, s_4), where $s_1, s_2 \in S_{i_{k+2t-1}}$, $s_3, s_4 \in S_{i_{k+2t}}$. We write

$$\mathrm{E}[(X_n)_k (Y_n)_\ell] = E_1 + E_2.$$

Here E_1 is the expected number of the samples such that $i_1 \ne \cdots \ne i_{k+2\ell}$, and E_2 is the expected number of all other samples, when at least two indices among i_j, $1 \le j \le k + 2\ell$, coincide. Then

$$E_1 = \frac{(nd - 2k - 4\ell - 1)!!}{(nd-1)!!} \sum_{i_1 \ne \cdots \ne i_{k+2\ell}} \prod_{s=1}^{k+2\ell} \binom{d_{i_s}}{2}.$$

EXPLANATION. Let a sequence $i_1 \ne i_2 \ne \cdots \ne i_{k+2\ell}$ be given. From each set S_{i_j} we choose two points, in $\prod_{s=1}^{k+2\ell} \binom{d_{i_s}}{2}$ ways overall. We pair two points from each S_{i_j}, $j \le k$, thus forming k loops. For each $t \in [1, \ell]$, we match two chosen points in S_{k+2t-1} with two chosen points in S_{k+2t}, in 2^ℓ ways overall, and then divide by 2^ℓ to account for irrelevance of the order in which every two sets, S_{k+2t-1} and S_{k+2t}, appear in the sequence $S_{i_{k+1}}, \dots, S_{i_{k+2\ell}}$.

Introduce

$$\Sigma_1 = \sum_{i_1, \dots, i_{k+2\ell}} \prod_{s=1}^{k+2\ell} \binom{d_{i_s}}{2},$$

$$\Sigma_2 = \binom{k+2\ell}{2} \sum_{i_1 = i_2, i_3, \dots, i_{k+2\ell}} \prod_{s=1}^{k+2\ell} \binom{d_{i_s}}{2};$$

so Σ_1 is a counterpart of E_1, with the indices $i_1, \dots, i_{k+2\ell}$ allowed to coincide, and Σ_2 is an upper bound of the total sum of terms in Σ_1, but not in E_1. Clearly then

$$\frac{1}{(nd)^{k+2\ell}} (\Sigma_1 - \Sigma_2) \lesssim E_1 \lesssim \frac{1}{(nd)^{k+2\ell}} \Sigma_1.$$

Further

$$\Sigma_1 = \left(\sum_i \binom{d_i}{2}\right)^{k+2\ell} = (nd)^{k+2\ell}\left(\frac{1}{2d}\sum_j j(j-1)p_j\right)^{k+2\ell},$$

so that

$$\frac{\Sigma_1}{(nd)^{k+2\ell}} = \left(\frac{\nu}{2}\right)^{k+2\ell}.$$

Next

$$\Sigma_2 = \binom{k+2\ell}{2}\left(\sum_i \binom{d_i}{2}^2\right)\left(\sum_{i'} \binom{d_{i'}}{2}\right)^{k+2\ell-2},$$

so that

$$\frac{\Sigma_2}{(nd)^{k+2\ell}} = O\left(n^{-2}\sum_i \binom{d_i}{2}^2 \left(\frac{\nu}{2}\right)^{k+2\ell-2}\right) \to 0,$$

since, by (2.20),

$$n^{-2}\sum_i d_i^4 = n^{-1}\sum_{j \le j_n} j^4 p_j = O(n^{-1/3}).$$

Therefore

$$E_1 \sim \left(\frac{\nu}{2}\right)^{k+2\ell}, \qquad n \to \infty.$$

Finally

$$E_2 \le \frac{(nd-2k-4\ell-1)!!}{(nd-1)!!}\cdot \Sigma_2 = O\left(n^{-2}\sum_i d_i^4\right) \to 0.$$

Therefore

$$\mathrm{E}[(X_n)_k (Y_n)_\ell] = E_1 + O(E_2) \sim \left(\frac{\nu}{2}\right)^{k+2\ell}, \qquad n \to \infty.$$

□

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