

## OPTIMAL STRATEGIES FOR A CLASS OF SEQUENTIAL CONTROL PROBLEMS WITH PRECEDENCE RELATIONS

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Consider the following multi-phase project management problem. Each project is divided into several phases. All projects enter the next phase at the same point chosen by the decision maker based on observations up to that point. Within each phase, one can pursue the projects in any order. When pursuing the project with one unit of resource, the project state changes according to a Markov chain. The probability distribution of the Markov chain is known up to an unknown parameter. When pursued, the project generates a random reward depending on the phase and the state of the project and the unknown parameter. The decision maker faces two problems: (a) how to allocate resources to projects within each phase, and (b) when to enter the next phase, so that the total expected reward is as large as possible. In this paper we formulate the preceding problem as a stochastic scheduling problem and propose asymptotic optimal strategies, which minimize the shortfall from perfect information payoff. Concrete examples are given to illustrate our method.

**1. Introduction.** We first formulate the multi-phase project management problem as that of optimally scheduling a number of jobs. Suppose that a single machine is available to process  $\mathcal{U}$  jobs. Each job belongs to one job group and there are  $I$  job groups all together. Within each group, the job can be processed in any order. However, there exists a predetermined order among job groups. That is, after leaving the current job group, there is no return to it in the future. The state of a job under processing evolves as a Markov chain and earns rewards as it is processed, not otherwise. The time-varying reward distribution depends on an unknown parameter  $\theta$ . The objective is to minimize the shortfall from perfect information payoff, which is the difference between the optimal reward when the parameter is known and that when it is unknown. We establish an asymptotic lower bound on this difference and construct policies which attain the lower bound. Clearly the preceding stochastic scheduling problem is the same as the

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multi-phase project management problem when we identify jobs in the same group with projects in the same phase.

To solve the proposed stochastic scheduling problem, we need to resolve two issues. First, our solution must prescribe how to process jobs within the same group. Secondly, the solution needs to stipulate the timing of leaving the current job group and entering the next one. All existing methods address only one of the two issues. As one shall see, to address these two issues simultaneously requires new ideas as well as nontrivial combination of existing methods.

The advantages of efficient strategies constructed in Section 4 are three-fold.

- They address the two crucial issues described in the previous paragraph simultaneously.
- They are still optimal, if we consider constant switching cost from one project to another.
- When the bad set (see Section 2.4 for definition) is empty the strategy is super-efficient in the sense of attaining  $o(\log N)$  regret (see Section 2.2 for definition).

If the parameter  $\theta$  were known, the best policy would be to process only the job with greatest one-step expected reward. In ignorance of  $\theta$ , an optimal policy needs to trade off a reduced reward in exchange for information on  $\theta$ . The key to the optimal trade-off is the construction of a strategy that achieves the asymptotic lower bound for the shortfall from the complete information payoff, which we shall refer to as regret hereafter. Although dynamic programming and the Gittins index rule (cf. [8]) have been developed to solve a general class of adaptive control problems, to which the proposed problem belongs, computational difficulty makes them less applicable. One reason for adopting the approach described here is to obtain an explicit solution which is easy to implement.

This approach was first introduced by Lai and Robbins [16] and generalized by Anantharam, Varaiya and Walrand [3] and Lai [15]. When there is only one job group and the rewards from each job are independent and identically distributed (i.i.d.), the preceding control problem is the classical multi-armed bandit problem; see [4, 8, 21]. When there is only one job in each group and rewards are i.i.d., it is the irreversible multi-arm bandit problem studied by Hu and Wei [13], whereas Hu and Lee [12] considered the same problem under a Bayesian setting. Fuh and Hu [7] investigated the irreversible multi-armed bandit problem with Markovian rewarding. Agrawal, Teneketzis and Anantharam [1, 2] studied controlled i.i.d. processes and Markov chains in finite parameter and state spaces. They introduced the concept of bad sets and showed that it plays an important role in the solution of the adaptive control problem. Other related work can be found in [8–11, 14, 17, 20] and references therein.

The rest of the paper is organized as follows. In Section 2 we describe the components of a statistical model for the proposed problem. The asymptotic lower bound for the regret is derived in Section 3. In Section 4 we propose a class of strategies making use of an adjusted MLE  $\hat{\theta}_a$ . This adjustment is necessary for

consistent estimation of the bad sets of  $\theta$  when the parameter space is continuous. The efficiency of our procedure relies on an initial experimentation stage based on the adjusted MLE estimate to maximize the information content and also on a subsequent testing stage via sequential likelihood ratio tests to reject suboptimal jobs or a whole group of jobs. Unequal allocation of processing time on jobs may occur in the testing stage so that there is more frequent processing of superior jobs. In Section 5 we discuss how our method can be applied to multi-phase project management examples. Most of the technical proofs are deferred to the [Appendix](#).

**2. Preliminaries.**

2.1. *The scheduling problem.* Let  $\mathcal{U} = J_1 + \dots + J_I$  indicate that there are  $I$  groups and  $J_i$  jobs in the  $i$ th group for  $i = 1, \dots, I$ . One is free to process any job within the same group, while jobs must be processed following the order of  $1, \dots, I$  between groups. As processing a job a unit time is equivalent to taking an observation from a statistical population, we have  $\mathcal{U}$  statistical populations  $\Pi_{11}, \dots, \Pi_{IJ_I}$ . For each  $ij$ , the observations from  $\Pi_{ij}$  follow a Markov chain on a state space  $D$  with  $\sigma$ -algebra  $\mathcal{D}$ . It is assumed that the transition probability  $P_{ij}^\theta$  for the Markov chain has a probability density function  $p_{ij}(x, y; \theta)$  with respect to some nondegenerate measure  $Q$ , where  $p_{ij}(x, y; \cdot)$  is known and  $\theta$  is an unknown parameter belonging to a parameter space  $\Theta$ . We assume that the stationary probability distribution for the Markov chain exists and has probability density function  $\pi_{ij}(\cdot; \theta)$  with respect to  $Q$ . At each step, we are required to process one job respecting the partial order  $ij \preceq i'j' \Leftrightarrow i \leq i'$ .

An adaptive policy is a rule that dictates, at each step, which job should be processed based on information from previous observations. We can represent a policy as a sequence of random variables  $\phi = \{\phi_t\}$  taking values in  $\{ij : i = 1, \dots, I; j = 1, \dots, J_i\}$ , such that the event  $\{\phi_t = ij\}$  (process job  $ij$  at step  $t$ ) belongs to the  $\sigma$ -field  $\mathcal{F}_{t-1}$  generated by  $\phi_1, X_1, \dots, \phi_{t-1}, X_{t-1}$ , where  $X_n$  denotes the state of the job being processed at the  $n$ th step. The constraint

$$(2.1) \quad \phi_t \preceq \phi_{t+1} \quad \text{for } 1 \leq t \leq N - 1$$

indicates that once a sample has been taken from  $\Pi_{ij}$ , one can switch to other jobs within group  $i$  or to the jobs in groups  $i + 1$  to  $I$ , but no further sampling is allowed from  $\Pi_{11}, \dots, \Pi_{(i-1)J_{i-1}}$ .

2.2. *The objective function.* Let the initial state of the job  $ij$  under processing be distributed according to  $v_{ij}(\cdot; \theta)$ . Throughout this paper we shall use the notation  $E_\theta (P_\theta)$  to denote expectation (probability) with respect to the initial distribution  $v_{ij}(\cdot; \theta)$ ; similarly,  $E_{\pi(\theta)}$  will denote expectation with respect to  $P_\theta$  and the stationary distribution  $\pi_{ij}(\cdot; \theta)$ . We shall assume that  $\mathcal{V}_{ij} = \{x \in D : v_{ij}(x; \theta) > 0\}$  does not depend on  $\theta$  and

$$(2.2) \quad v_{ij} := \inf_{x \in \mathcal{V}_{ij}} \inf_{\theta, \theta' \in \Theta} [v_{ij}(x; \theta) / v_{ij}(x; \theta')] > 0 \quad \text{for all } i, j.$$

Suppose that  $\int_{x \in D} |g(x)| \pi_{ij}(x; \theta) Q(dx) < \infty$  for some real-valued function (reward)  $g$ . Let

$$\mu_{ij}(\theta) = \int_{x \in D} g(x) \pi_{ij}(x; \theta) Q(dx)$$

be the mean reward under stationary distribution  $\pi_{ij}$  if job  $ij$  is processed once. Let  $N$  be the total processing time for all jobs, and

$$(2.3) \quad T_N(ij) = \sum_{t=1}^N \mathbf{1}_{\{\phi_t=ij\}}$$

be the amount of time that job  $ij$  is processed, where  $\mathbf{1}$  denotes the indicator function. An optimal strategy would be one which maximizes

$$(2.4) \quad W_N(\theta) := \sum_{t=1}^N \sum_{i=1}^I \sum_{j=1}^{J_i} E_\theta \{ E_\theta [g(X_t) \mathbf{1}_{\{\phi_t=ij\}} | \mathcal{F}_{t-1}] \}.$$

In the case of independent rewards, that is, when  $p_{ij}(x, y, ; \theta) = p_{ij}(y; \theta)$  for all  $i, j, x, y$  and  $\theta$ ,  $W_N(\theta) = \sum_{i=1}^I \sum_{j=1}^{J_i} \mu_{ij}(\theta) E_\theta T_N(ij)$ . We shall show in the [Appendix](#) that for Markovian rewards, under regularity conditions A3 and A4 (see Section 2.3), there exists a constant  $C_0 < \infty$  independent of  $\theta \in \Theta$ ,  $N > 0$  and the strategy  $\phi$  such that

$$(2.5) \quad \left| W_N(\theta) - \sum_{i=1}^I \sum_{j=1}^{J_i} \mu_{ij}(\theta) E_\theta T_N(ij) \right| \leq C_0.$$

When the parameter space  $\Theta$  and state space  $D$  are both finite, (2.5) also follows from [3], Lemma 2.1. In light of (2.5), maximizing  $W_N(\theta)$  is asymptotically equivalent [up to an  $O(1)$  term] to minimizing the regret

$$(2.6) \quad \begin{aligned} R_N(\theta) &:= N\mu^*(\theta) - \sum_{i=1}^I \sum_{j=1}^{J_i} \mu_{ij}(\theta) E_\theta T_N(ij) \\ &= \sum_{ij: \mu_{ij}(\theta) < \mu^*(\theta)} [\mu^*(\theta) - \mu_{ij}(\theta)] E_\theta T_N(ij), \end{aligned}$$

where  $\mu^*(\theta) := \max_{1 \leq i \leq I} \max_{1 \leq j \leq J_i} \mu_{ij}(\theta)$ .

Because adaptive strategies  $\phi$  which are optimal for all  $\theta \in \Theta$  and large  $N$  in general do not exist, we consider the class of all (asymptotically) *uniformly good* adaptive strategies under the partial order constraint  $\preceq$ , with regret satisfying

$$(2.7) \quad R_N(\theta) = o(N^\alpha) \quad \text{for all } \alpha > 0 \text{ and } \theta \in \Theta.$$

Such strategies have regret that does not increase too rapidly for any  $\theta \in \Theta$ . We would like to find a strategy that minimizes the increasing rate of the regret within the class of uniformly good adaptive strategies under the partial order constraint  $\preceq$ .

Due to the irreversibility constraint (2.1), a strategy satisfying (2.7) would in general be dependent on  $N$  when there is more than one group of arms. Consider for example the case in which the optimal arm is unique and lies in the first group. Let  $p > 0$  be the probability that a strategy  $\phi$  bypasses the first group of arms before a fixed time  $N_0$ . If the strategy  $\phi$  is independent of  $N$ , then

$$R_N(\theta) \geq p(N - N_0) \left[ \mu^*(\theta) - \max_{2 \leq i \leq I} \max_{1 \leq j \leq J_i} \mu_{ij}(\theta) \right]$$

and (2.7) does not hold. This is unlike the case of a one-group multi-armed bandit considered by Lai and Robbins [16], Anantharam, Varaiya and Walrand [3] and Agrawal, Teneketzis and Anantharam [1, 2] whereby optimal strategies  $\phi$  satisfying (2.7) and not dependent on  $N$  have been constructed.

2.3. *The assumptions.* Denote the Kullback–Leibler information number by

$$(2.8) \quad I_{ij}(\theta, \theta') = \int_{x \in D} \int_{y \in D} \log \left[ \frac{p_{ij}(x, y; \theta)}{p_{ij}(x, y; \theta')} \right] \times p_{ij}(x, y; \theta) \pi_{ij}(x; \theta) Q(dy) Q(dx).$$

Then  $0 \leq I_{ij}(\theta, \theta') \leq \infty$ . We shall assume that  $I_{ij}(\theta, \theta') < \infty$  for all  $i, j$  and  $\theta, \theta' \in \Theta$ . Let  $\mu_i(\theta) = \max_{1 \leq j \leq J_i} \mu_{ij}(\theta)$  be the largest reward in the  $i$ th group of jobs, and

$$(2.9) \quad \Theta_i = \{ \theta \in \Theta : \mu_i(\theta) > \mu_{i'}(\theta) \text{ for all } i' < i \\ \text{and } \mu_i(\theta) \geq \mu_{i'}(\theta) \text{ for all } i' \geq i \}$$

be the set of parameter values such that the first optimal job is in group  $i$ . Let

$$(2.10) \quad \Theta_{ij} = \{ \theta \in \Theta_i : \mu_{ij}(\theta) = \mu_i(\theta) \}$$

be the parameter set such that job  $ij$  is one of the first optimal jobs. Each  $\theta \in \Theta$  belongs to exactly one  $\Theta_i$  but may belong to more than one  $\Theta_{ij}$ . Let

$$(2.11) \quad \Theta_i^* = \{ \theta \in \Theta : \mu_i(\theta) > \mu_{i'}(\theta) \text{ for all } i' \neq i \}$$

be the parameter set in which all the optimal arms lie in group  $i$ . Clearly,  $\Theta_i^* \subset \Theta_i$ , but the reverse relation is not necessarily true.

We now state a set of assumptions that will be used to prove the optimality results in Sections 3 and 4. Let  $\Theta$  be a compact subset of  $\mathbf{R}^d$  for some  $d \geq 1$  and let  $X_{ijt}$  denote the  $t$ th observation taken from arm  $ij$ .

A1.  $\mu_{ij}(\cdot)$  are finite and continuous on  $\Theta$  for all  $i, j$ . Moreover, no job group is redundant in the sense that  $\Theta_i^* \neq \emptyset$  for all  $i = 1, \dots, I$ .

A2.  $\sum_{j=1}^{J_1} I_{1j}(\theta, \theta') > 0$  for all  $\theta' \neq \theta$  and  $\inf_{\theta' \in \Theta_{ij}} I_{ij}(\theta, \theta') > 0$  for all  $1 \leq i < I, 1 \leq j \leq J_i$  and  $\theta \in \bigcup_{\ell > i} \Theta_\ell$ .

A3. For each  $j = 1, \dots, J_i, i = 1, \dots, I$  and  $\theta \in \Theta$ ,  $\{X_{ijt}, t \geq 0\}$  is a Markov chain on a state space  $D$  with  $\sigma$ -algebra  $\mathcal{D}$ , irreducible with respect to a maximal irreducible measure on  $(D, \mathcal{D})$  and aperiodic. Furthermore,  $X_{ijt}$  is Harris recurrent in the sense that there exist a set  $G_{ij} \in \mathcal{D}$ ,  $\alpha_{ij} > 0$  and probability measure  $\varphi_{ij}$  such that  $P_{ij}^\theta\{X_{ijt} \in G_{ij} \text{ i.o.} | X_{ij0} = x\} = 1$  for all  $x \in D$  and

$$(2.12) \quad P_{ij}^\theta\{X_{ij1} \in A | X_{ij0} = x\} \geq \alpha_{ij}\varphi_{ij}(A) \quad \text{for all } x \in G_{ij} \text{ and } A \in \mathcal{D}.$$

A4. There exist constants  $0 < \bar{b} < 1, b > 0$  and drift functions  $V_{ij} : D \rightarrow [1, \infty)$  such that for all  $i = 1, \dots, I$  and  $j = 1, \dots, J_i$ ,

$$(2.13) \quad \sup_{x \in D} |g(x)|/V_{ij}(x) < \infty,$$

and for all  $x \in D$  and  $\theta \in \Theta$ ,

$$(2.14) \quad P_{ij}^\theta V_{ij}(x) \leq (1 - \bar{b})V_{ij}(x) + b\mathbf{1}_{\{x \in G_{ij}\}},$$

where  $G_{ij}$  satisfies (2.12) and  $P_{ij}^\theta V_{ij}(x) = \int_D V_{ij}(y)P_{ij}^\theta(x, dy)$ . Moreover, we require that

$$(2.15) \quad \int_D V_{ij}(x)v_{ij}(x; \theta)Q(dx) < \infty \quad \text{and} \quad V_{ij}^* := \sup_{x \in G_{ij}} V_{ij}(x) < \infty.$$

Let  $\ell_{ij}(x, y; \theta, \theta') = \log[p_{ij}(x, y; \theta)/p_{ij}(x, y; \theta')]$  be the log likelihood ratio between  $P_{ij}^\theta$  and  $P_{ij}^{\theta'}$  and  $N_\delta(\theta) = \{\theta' : \|\theta - \theta'\| < \delta\}$  a ball of radius  $\delta$  around  $\theta$ , where  $\|\cdot\|$  denotes the Euclidean norm.

A5. There exists  $\delta > 0$  such that for all  $\theta, \theta' \in \Theta$ ,

$$(2.16) \quad K_{\theta, \theta'} := \sup_{x \in D} \frac{E_\theta[\sup_{\tilde{\theta} \in N_\delta(\theta')} \ell_{ij}^2(X_{ij0}, X_{ij1}; \theta, \tilde{\theta}) | X_{ij0} = x]}{V_{ij}(x)} < \infty$$

for all  $j = 1, \dots, J_i, i = 1, \dots, I$ . Moreover,

$$(2.17) \quad \sup_{\tilde{\theta} \in N_{\delta'}(\theta')} |\ell_{ij}(x, y; \theta', \tilde{\theta})| \rightarrow 0 \quad \text{as } \delta' \rightarrow 0$$

for all  $x, y \in D$  and  $\theta' \in \Theta$ .

Assumption A1 is for excluding some unrealistic models in which efficient but impractical strategies may exist. A2 is a positive information criterion: the first inequality makes sure that information is available in the first job group to estimate  $\theta$ ; the second inequality allows us to gather information in the  $i$ th job group for moving to the next group when  $\theta \in \Theta_\ell$  for some  $\ell > i$ . Assumption A3 is a recurrence condition and A4 is a drift condition. These two conditions are used to guarantee the stability of the Markov chain so that the strong law of large numbers and Wald's equation hold. A5 is a finite second moment condition that allows us to bound the probability that the MLE of  $\theta$  lies outside a small neighborhood of  $\theta$ .

This bound is important for us to determine the level of unequal allocation of observations that can be permitted in the testing stage of our procedure. The proof of the asymptotic lower bound in Theorem 1 requires only A1–A3; the additional assumptions A4 and A5 are required for the construction of efficient strategies attaining the lower bound.

We now demonstrate an immediate consequence of A3–A5, that for any  $\theta \in \Theta$  and  $\varepsilon > 0$ , there exists  $0 < \delta' < \delta$  such that

$$(2.18) \quad E_{\pi_{ij}(\theta)} \left[ \sup_{\tilde{\theta} \in N_{\delta'}(\theta')} |\ell_{ij}(X_{ij0}, X_{ij1}; \theta', \tilde{\theta})| \right] < \varepsilon$$

for all  $ij$  and  $\theta' \in \Theta$ . Note that the continuity of  $I_{ij}(\theta, \cdot)$  follows from (2.18).

Since  $\pi_{ij} = C' \sum_{k=0}^{\infty} (P_{ij} - \alpha_{ij} \varphi_{ij} \mathbf{1}_{G_{ij}})^k \varphi_{ij}$ , where  $C'$  is a normalizing constant, it follows from (2.14) and (2.15) that  $\int_D V_{ij}(x) \pi_{ij}(x; \theta) Q(dx) < \infty$ . Hence by (2.16) and the relation  $\ell_{ij}(X_{ij0}, X_{ij1}; \theta', \tilde{\theta}) = \ell_{ij}(X_{ij0}, X_{ij1}; \theta, \tilde{\theta}) - \ell_{ij}(X_{ij0}, X_{ij1}; \theta, \theta')$ , we have

$$\begin{aligned} & E_{\pi_{ij}(\theta)} \left[ \sup_{\tilde{\theta} \in N_{\delta}(\theta')} |\ell_{ij}(X_{ij0}, X_{ij1}; \theta', \tilde{\theta})| \right] \\ & \leq E_{\pi_{ij}(\theta)} |\ell_{ij}(X_{ij0}, X_{ij1}; \theta, \theta')| + E_{\pi_{ij}(\theta)} \left[ \sup_{\tilde{\theta} \in N_{\delta}(\theta')} |\ell_{ij}(X_{ij0}, X_{ij1}; \theta, \tilde{\theta})| \right] \\ & < \infty. \end{aligned}$$

As the convergence in (2.17) is monotone decreasing, it follows from the dominated convergence theorem that (2.18) holds.

**2.4. Bad sets.** Bad set is a useful concept for understanding the learning required within the group containing optimal jobs. It is associated with the asymptotic lower bound described in Section 3 and is used explicitly in Section 4 to construct the asymptotically efficient strategy. For  $\theta \in \Theta_\ell$ , define  $J(\theta) = \{j : \mu^*(\theta) = \mu_{\ell j}(\theta)\}$  as the set of optimal jobs in group  $\ell$ . Hence  $\theta \in \Theta_\ell$  if and only if  $j \in J(\theta)$ . We also define the bad set, the set of “bad” parameter values associated with  $\theta$ , as all  $\theta' \in \Theta_\ell$  which cannot be distinguished from  $\theta$  by processing any of the optimal jobs  $\ell j$ . More specifically, the bad set

$$(2.19) \quad B_\ell(\theta) = \left\{ \theta' \in \Theta_\ell \setminus \left( \bigcup_{j \in J(\theta)} \Theta_{\ell j} \right) : I_{\ell j}(\theta, \theta') = 0 \text{ for all } j \in J(\theta) \right\}.$$

We note that if  $I_{\ell j}(\theta, \theta') = 0$ , then the transition probabilities of  $X_{\ell j t}$  are identical under both  $\theta$  and  $\theta'$ . If  $\theta' \in B_\ell(\theta)$ , then by definition,  $\theta' \notin \bigcup_{j \in J(\theta)} \Theta_{\ell j}$  and hence  $J(\theta') \cap J(\theta) = \emptyset$ . Let  $j \in J(\theta)$  and  $j' \in J(\theta')$ . Then  $\mu_{\ell j'}(\theta') > \mu_{\ell j}(\theta) = \mu_{\ell j}(\theta) > \mu_{\ell j'}(\theta)$ . Thus

$$(2.20) \quad I_{\ell j'}(\theta, \theta') > 0 \quad \text{for all } \theta' \in B_\ell(\theta) \text{ and } j' \in J(\theta').$$

The interpretation of (2.20) is as follows. Although we cannot distinguish  $\theta$  from  $\theta' \in B_\ell(\theta)$  when processing the optimal job for  $\theta$ , we can distinguish them by processing the optimal job for  $\theta'$ . This fact explains the necessity of processing nonoptimal jobs to collect information.

Assumption A2 says when sampling from the optimal arm one *can* distinguish any  $\theta$  value whose optimal arm is in a *future* group. But having a nonempty bad set says that when sampling from the optimal arm one *cannot* distinguish some  $\theta$  value whose optimal arm is in the *current* group. These two statements are compatible. We now provide two examples from the celebrated multi-armed bandit problem to illustrate the idea of bad sets.

EXAMPLE 1. *Independent armed-bandit problem.* Let  $\Pi_{11}, \dots, \Pi_{1J}$  denote  $J$  statistical populations specified, respectively, by density functions  $p(x; \theta_j)$  with respect to some measure  $Q$ . For simplicity, assume that  $x = 0, 1$  and  $p(0; \theta_j) = 1 - \theta_j, p(1; \theta_j) = \theta_j$ , where  $\theta_j$  are unknown parameters taking values in  $[0, 1]$ . A multi-armed bandit problem searches for strategies to sample  $X_1, X_2, \dots$ , sequentially from these  $J$  populations in order to maximize the expected value of the sum  $S_N = \sum_{t=1}^N X_t$  as  $N \rightarrow \infty$ .

Let  $\theta = (\theta_1, \dots, \theta_J)$ . If  $\theta = (0.2, 0.1)$ , then the set of optimal arms  $J(\theta) = \{1\}$  and the bad set  $B_1(\theta) = \{(0.2, \theta'_2) : 0.2 < \theta'_2 \leq 1\}$ . Even though arm 1 is optimal, experimentation from arm 2 is required to make sure that the true parameter value does not lie in  $B_1(\theta)$ .

The two-armed bandit problem studied by Feldman [6] has  $\Theta = \{(\theta_1, \theta_2), (\theta_2, \theta_1)\}$  with  $\theta_1 \neq \theta_2$ . It follows that  $B_1(\theta) = \emptyset$  for all  $\theta \in \Theta$ . This leads to remarkably low regret,  $R_N(\theta) = O(1)$ .

EXAMPLE 2. *Correlated armed-bandit problem.* Consider bivariate normal populations  $\Pi_{11}, \Pi_{12}, \Pi_{13}$  with respective mean vectors  $(\mu_1, \lambda), (\mu_2, \mu_3)$  and  $(\mu_3, \mu_2 + \lambda)$ , where  $\mu_1, \mu_2, \mu_3, \lambda$  are unknown parameters. The problem is to sample the random vectors sequentially to maximize the expected value of the first component of the observed sum,  $\sum_{t=1}^N X_t$ , as  $N \rightarrow \infty$ . Let  $\theta = (\mu_1, \mu_2, \mu_3, \lambda)$ . If  $J(\theta) = \{1\}$ , then

$$B_1(\theta) = \{\theta' \in \Theta : \mu_1 = \mu'_1, \lambda = \lambda', \max(\mu'_2, \mu'_3) > \mu'_1\}.$$

**3. A lower bound for the regret.** The following theorem gives an asymptotic lower bound for the regret (2.6) of uniformly good adaptive strategies under the partial order constraint  $\preceq$ .

THEOREM 1. *Assume A1–A3 and let  $\theta \in \Theta_\ell$ . For any uniformly good adaptive strategy  $\phi$  under the partial order constraint  $\preceq$ ,*

$$(3.1) \quad \liminf_{N \rightarrow \infty} R_N(\theta) / \log N \geq z(\theta, \ell),$$

where  $z(\theta, \ell)$  is a solution of the following minimization problem.

$$(3.2) \quad \text{Minimize } \sum_{i < \ell} \sum_{j=1}^{J_i} [\mu^*(\theta) - \mu_{ij}(\theta)] z_{ij}(\theta) + \sum_{j \notin J(\theta)} [\mu^*(\theta) - \mu_{\ell j}(\theta)] z_{\ell j}(\theta),$$

subject to  $z_{ij}(\theta) \geq 0, j = 1, \dots, J_i$  if  $i < \ell; j \notin J(\theta)$  if  $i = \ell$ ; and

$$(3.3) \quad \left\{ \begin{array}{l} \inf_{\theta' \in \Theta_1} \left\{ \sum_{j=1}^{J_1} I_{1j}(\theta, \theta') z_{1j}(\theta) \right\} \geq 1, \\ \inf_{\theta' \in \Theta_2} \left\{ \sum_{j=1}^{J_1} I_{1j}(\theta, \theta') z_{1j}(\theta) + \sum_{j=1}^{J_2} I_{2j}(\theta, \theta') z_{2j}(\theta) \right\} \geq 1, \\ \vdots \\ \inf_{\theta' \in \Theta_{\ell-1}} \left\{ \sum_{j=1}^{J_1} I_{1j}(\theta, \theta') z_{1j}(\theta) + \dots + \sum_{j=1}^{J_{\ell-1}} I_{(\ell-1)j}(\theta, \theta') z_{(\ell-1)j}(\theta) \right\} \geq 1, \\ \inf_{\theta' \in B_\ell(\theta)} \left\{ \sum_{i < \ell} \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') z_{ij}(\theta) + \sum_{j \notin J(\theta)} I_{\ell j}(\theta, \theta') z_{\ell j}(\theta) \right\} \geq 1. \end{array} \right.$$

The first  $\ell - 1$  inequalities in (3.3) are due to the partial order constraints. When there is no partial order constraint and the jobs are independent, the solution of Problem A reduces to the lower bound given in Theorem 1 of [16].

Under the assumptions of Theorem 1, the strategies that satisfy, for  $\theta \in \Theta_\ell$ ,

$$(3.4) \quad \lim_{N \rightarrow \infty} R_N(\theta) / \log N = z(\theta, \ell),$$

are said to be *asymptotically efficient*. If  $B_\ell(\theta) = \emptyset$ , then the last inequality of (3.3) is removed. In particular, when  $\theta \in \Theta_1$ , (3.4) implies that

$$(3.5) \quad R_N(\theta) = \begin{cases} O(\log N), & \text{if } B_1(\theta) \neq \emptyset, \\ o(\log N), & \text{if } B_1(\theta) = \emptyset. \end{cases}$$

We shall assume that  $B_\ell(\theta)$  is nonempty for the underlying  $\theta \in \Theta_\ell$ , which is true for most applications. The case of  $B_\ell(\theta) = \emptyset$  is treated in [5].

The following lemma will be used to prove Theorem 1. The proofs of both Lemma 1 and Theorem 1 will be given in the Appendix.

LEMMA 1. Assume A2–A3. Let  $\phi$  be a uniformly good adaptive strategy under the partial order constraint  $\preceq$ . If  $\theta \in \Theta_\ell$ , then for every  $\theta' \in \Theta_k^*, k < \ell$ ,

$$(3.6) \quad \liminf_{N \rightarrow \infty} \left\{ \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') E_\theta T_N(ij) \right\} / \log N \geq 1,$$

and for every  $\theta' \in B_\ell(\theta)$ ,

$$(3.7) \quad \liminf_{N \rightarrow \infty} \left\{ \sum_{i < \ell} \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') E_\theta T_N(ij) + \sum_{j \notin J(\theta)} I_{\ell j}(\theta, \theta') E_\theta T_N(\ell j) \right\} / \log N \geq 1.$$

**4. Construction of asymptotically efficient strategies.**

4.1. *Outline of the construction.* The goal of any reasonable strategy is to determine whether the job currently under processing is optimal or not based on sequential observations. The job under processing, say job  $ij$ , is optimal if  $\theta \in \Theta_{ij}$ . Thus, the problem of constructing an efficient adaptive strategy reduces to that of finding a procedure to determine whether  $\theta \in \Theta_{ij}$  is true or not based on a sequential sample. The asymptotic lower bound discussed in Section 3 gives us valuable information about the size of the sequential sample. In particular, it suggests that for  $\theta \in \Theta_\ell$ , the amount of processing time for job  $ij$ ,  $j = 1, \dots, J_i$ ,  $i < \ell$ , and  $j \notin J(\theta)$  if  $i = \ell$  should be  $[z_{ij}(\theta) + o(1)] \log N$ , where  $z_{ij}(\theta)$  solves the minimization problem (3.2).

In view of Theorem 1, the sample size  $[z_{ij}(\theta) + o(1)] \log N$  represents the minimum amount of learning about job  $ij$  in order for the strategy to be uniformly good. Because of the partial order constraint  $\leq$ , we also need a sequential test to ensure that the optimal job is passed over with probability not exceeding  $N^{-1}$ . These two facts are important guidelines for the construction of asymptotically efficient strategies so that the two crucial issues mentioned in the abstract and Section 1 can be addressed.

Let  $n_0, n_1$  be positive integers that increase to infinity with respect to  $N$  such that  $n_0 = o(\log N)$  and  $n_1 = o(n_0)$ . We shall now describe the asymptotically efficient strategy  $\phi^*$  by dividing it into three distinct stages; estimation, experimentation and testing.

In the estimation stage,  $n_0 = o(\log N)$  observations are taken from each job in group 1 for estimating the parameter  $\theta \in \Theta_\ell$ . If  $\ell > 1$  or  $\ell = 1$  and  $B_1(\theta) \neq \emptyset$ , then an order of  $\log N$  observations are taken in the experimental stage which contribute  $[z(\theta, \ell) + o(1)] \log N$  to the regret; see (3.1). Finally, in the testing phase,  $o(\log N)$  observations are taken from each of the suboptimal jobs. We first consider the optimal strategy for the case of finite  $\Theta$ , which captures the essential ingredients without too much technical detail. We then extend the strategy to infinite  $\Theta$  followed by a formal statement of optimality in Theorem 2.

4.2. *Optimal strategy for finite  $\Theta$ .*

1. *Estimation.* Take an initial sample of  $n_0$  observations from each job in group 1. Let  $\hat{\theta}$  be the maximum likelihood estimate (MLE) of  $\theta$  defined by

$$(4.1) \quad L(\theta) = \sum_{j=1}^{J_1} \sum_{t=1}^{n_0} \log p_{1j}(X_{1j(t-1)}, X_{1jt}; \theta), \quad \hat{\theta} = \arg \max_{\theta \in \Theta} L(\theta).$$

Let  $k = 1$ .

2. *Experimentation.* Let  $\lfloor \cdot \rfloor$  denote the greatest integer function.

(a) If  $\hat{\theta} \in \bigcup_{i>k} \Theta_i$ : Take  $\lfloor z_{kj}(\hat{\theta}) \log N \rfloor$  observations from job  $kj$  for  $j = 1, \dots, J_k$ .

(b) If  $\hat{\theta} \in \Theta_k$ : Take  $\lfloor z_{kj}(\hat{\theta}) \log N \rfloor$  observations from job  $kj$  for  $j \notin J(\hat{\theta})$ .

(c) If  $\hat{\theta} \in \bigcup_{i<k} \Theta_i$ : Skip the experimentation phase.

3. *Testing.* Start with a full set  $\{k1, \dots, kJ_k\}$  of unrejected jobs. Let  $\mathbf{n} = (n_{11}, \dots, n_{kJ_k})$ , where  $n_{ij}$  denotes the number of observations taken from arm  $ij$  so far. The rejection of a job is based on the following test statistic. Let  $F_k$ ,  $1 \leq k \leq I$ , be a probability distribution with positive probability on all open subsets of  $\bigcup_{i=k}^I \Theta_i$ . Define

$$\begin{aligned}
 &U_k(\mathbf{n}; \lambda) \\
 (4.2) \quad &= \frac{\int_{\bigcup_{i=k}^I \Theta_i} \prod_{j=1}^k \prod_{t=1}^{J_i} v_{ij}(X_{ij0}; \theta) \prod_{t=1}^{n_{ij}} p_{ij}(X_{ij(t-1)}, X_{ijt}; \theta) dF_k(\theta)}{\prod_{j=1}^k \prod_{t=1}^{J_i} v_{ij}(X_{ij0}; \lambda) \prod_{t=1}^{n_{ij}} p_{ij}(X_{ij(t-1)}, X_{ijt}; \lambda)}
 \end{aligned}$$

for all  $\lambda \in \Theta_k$ .

(a) If  $\hat{\theta} \in \bigcup_{i>k} \Theta_i$ : Add one observation from each unrejected job. Reject the parameter  $\lambda$  if  $U_k(\mathbf{n}; \lambda) \geq N$ . Reject a job  $kj$  if all  $\lambda \in \Theta_{kj}$  have been rejected at some point in the testing stage. If there is a job in group  $k$  left unrejected and the total number of observations is less than  $N$ , repeat 3(a). Otherwise go to step 4.

(b) If  $\hat{\theta} \in \Theta_k$ : Add  $n_1$  observations from each unrejected job  $kj$ ,  $j \in J(\hat{\theta})$ , and one observation from each unrejected job  $kj$ ,  $j \notin J(\hat{\theta})$ . Reject a job  $kj$  if all  $\lambda \in \Theta_{kj}$  have been rejected at some point in the testing phase. If there is a job in group  $k$  left unrejected and the total number of observations is less than  $N$ , repeat 3(b). Otherwise, go to step 4.

(c) If  $\hat{\theta} \in \bigcup_{i<k} \Theta_i$ : Adopt the procedure of 3(a).

4. *Moving to the next group and termination.* The strategy terminates once  $N$  observations have been collected. Otherwise, if  $k < I$ , increment  $k$  by 1 and go to step 2; if  $k = I$ , select all remaining observations from a job  $Ij$  satisfying  $\mu_{Ij}(\hat{\theta}) = \max_{1 \leq h \leq J_I} \mu_{Ih}(\hat{\theta})$ .

We shall now describe how each feature of the proposed strategy leads to asymptotic optimality in Theorem 2. The positive information assumption in the first half of A2 allows us to estimate  $\theta$  consistently and hence enables us to determine the optimal sample size  $z_{1j}(\theta)$  in the experimental stage of group 1. The assumption is important because once we move to the next group of jobs, irreversibility would prevent us from making up any shortfall in the optimal sample size required from group 1. By selecting  $n_0 \rightarrow \infty$ , we ensure the consistency of  $\hat{\theta}$ , while by choosing  $n_0 = o(\log N)$ , the estimation of  $\theta$  incurs negligible contribution to the regret.

Let  $k$  be the current group of jobs under sampling. Consider first  $\hat{\theta} \in \Theta_\ell$  for some  $\ell \geq k$ . We are instructed to select  $\lfloor z_{kj}(\hat{\theta}) \log N \rfloor$  observations from each job

in the experimental stage. By Theorem 1 and the consistency of  $\widehat{\theta}$ , this is optimal for learning. If  $\widehat{\theta} \in \Theta_\ell$  for some  $\ell < k$ , then the estimate  $\widehat{\theta}$  says that we have overshot the optimal group, and the estimate  $\widehat{\theta}$  cannot be trusted. In both cases, our strategy then is to rely on the testing stage to decide if we should stay within the current job group.

The testing stage is important in stopping us from moving beyond the first group of optimal jobs. The rationale is that by irreversibility, the penalty for moving beyond the first group of optimal jobs can be of order  $N$ , which is large compared to the desired regret of  $O(\log N)$ . The usefulness of the testing stage in this aspect can be seen from (4.6) below, which guarantees that the regret due to overshooting the optimal job group is  $O(1)$ . The positive information assumption in the second half of A2 is necessary for the testing stage to be successful.

Let us now consider the strategy in 3(b). If  $\widehat{\theta} = \theta$ , then  $\lfloor z_{kj}(\theta) \log N \rfloor$  observations from arm  $kj$  are taken in the experimental stage and hence by the last inequality of (3.3),  $o(\log N)$  observations from jobs with positive information are needed to reject  $\lambda \in B_\ell(\theta)$  in the testing stage but we may still need an order of  $\log N$  observations to reject  $\lambda \in \Theta_\ell \setminus B_\ell(\theta)$ . Since we would like  $o(\log N)$  observations from suboptimal jobs in the testing phase, sampling equally from all jobs would be undesirable here. We consider instead the selection of  $n_1$  observations from job  $\ell j$ ,  $j \in J(\widehat{\theta})$ , for each observation from the other jobs, where  $n_1$  goes to infinity with  $N$ , so that  $O(n_1^{-1} \log N) = o(\log N)$  observations are taken from suboptimal jobs when  $\widehat{\theta} = \theta$ . When  $\widehat{\theta} \neq \theta$ , it might be possible that each job  $kj$ ,  $j \in J(\widehat{\theta})$ , would provide no information to reject some  $\lambda \in \Theta_k \setminus \bigcup_{j \in J(\theta)} \Theta_{kj}$ . Our procedure would then allocate  $O(n_1 \log N)$  observations from suboptimal jobs in the testing phase conditional on this happening. By A5 and Chebyshev's inequality, the probability of providing an incorrect estimate of  $\theta$  is  $O(n_0^{-1})$  and hence by specifying  $n_1 = o(n_0)$ , we ensure that the average contribution from suboptimal jobs is  $O(n_0^{-1} n_1 \log N) = o(\log N)$ .

The final case  $\widehat{\theta} \in \bigcup_{i < k} \Theta_i$  occurs with probability  $o(1)$  which together with the  $O(\log N)$  observations taken in the nonoptimal jobs in the testing stage when this happens results in an overall  $o(\log N)$  contribution to the regret.

The last step is to proceed to the next group of jobs when all parameters in  $\Theta_k$  have been rejected. The exception is when  $k = I$ . To be at stage 4 when  $k = I$ , all  $\theta \in \Theta$  have been rejected at some point in time. Clearly, the true parameter has been rejected as well but this occurs with very small probability and the contribution to the regret in this case is asymptotically negligible.

4.3. *Extension to infinite  $\Theta$ .* Let  $\theta \in \Theta_\ell$  be the true underlying parameter. When  $\Theta$  is finite, consistency of  $\widehat{\theta}$  would imply that  $\widehat{\theta} = \theta$  with probability close to 1 when  $N$  is large. Hence  $B_\ell(\widehat{\theta})$  and  $J(\widehat{\theta})$  would be good substitutes for the unknown  $B_\ell(\theta)$  and  $J(\theta)$ , respectively. Complications arise when  $\Theta$  is infinite. First, it is possible that  $B_\ell(\theta)$  is nonempty while  $B_\ell(\theta')$  is empty for all  $\theta'$  arbitrarily

close to  $\theta$ . Secondly, by continuity of  $\mu_{ij}(\cdot)$ , it follows that there exists  $\delta > 0$  such that

$$J(\theta') \subset J(\theta) \quad \text{for all } \theta' \in N_\delta(\theta) \cap \Theta_\ell,$$

but the preceding statement with  $\subset$  replaced by  $=$  is not necessarily true. Hence  $B_\ell(\hat{\theta})$  and  $J(\hat{\theta})$  are in general poor substitutes for  $B_\ell(\theta)$  and  $J(\theta)$  when  $\Theta$  is infinite. Moreover, if  $\theta$  lies on the boundary of  $\Theta_\ell$ , then  $(\bigcup_{i>\ell} \Theta_i) \cap N_\delta(\theta)$  can be nonempty for all small  $\delta > 0$ . This implies that  $z_{kj}(\hat{\theta})$  may be inconsistent for  $z_{kj}(\theta)$ . This would not happen when  $\Theta$  is finite.

Our strategy in extending the optimal procedure from finite  $\Theta$  to infinite  $\Theta$  is not to select  $\hat{\theta}$  during the estimation phase but rather to select some appropriate adjusted estimate  $\hat{\theta}_a \in N_{\delta/2}(\hat{\theta})$  where  $\delta \rightarrow 0$  as  $N \rightarrow \infty$  at a rate that is specified in Theorem 2 below. We require first that

$$(4.3) \quad \hat{\theta}_a \in N_{\delta/2}(\hat{\theta}) \cap \Theta_\ell \quad \text{where } \ell = \min\{i : \Theta_i \cap N_{\delta/2}(\hat{\theta}) \neq \emptyset\}.$$

This condition ensures that if  $\theta$  lies in the boundary of  $\Theta_\ell$ , then the probability that  $\hat{\theta}_a \in \Theta_\ell$  tends to 1 as  $N \rightarrow \infty$ . Our next condition would ensure that the probability that  $J(\hat{\theta}_a) = J(\theta)$  tends to 1 as  $N \rightarrow \infty$ . Let  $|\cdot|$  denote the number of elements in a finite set and

$$\mathbf{J} = \max\{|J(\theta')| : \theta' \in N_{\delta/2}(\hat{\theta}) \cap \Theta_\ell\}.$$

We require in addition to (4.3), that

$$(4.4) \quad \hat{\theta}_a \in H := \{\theta \in N_{\delta/2}(\hat{\theta}) \cap \Theta_\ell : |J(\theta)| = \mathbf{J}\},$$

where  $\ell$  is defined in (4.3).

If  $\Theta$  is finite, then for  $\delta > 0$  small enough,  $N_{\delta/2}(\hat{\theta}) = \{\hat{\theta}\}$  and hence by (4.3) and (4.4),  $\hat{\theta}_a = \hat{\theta}$ . Therefore the selection of  $\hat{\theta}_a$  for infinite  $\Theta$  is consistent with the selection procedure for finite  $\Theta$  when  $N$  is large. The final thing left to do is the estimation of  $B_\ell(\theta)$ . This can be done by taking a union of  $B_\ell(\theta')$  over  $\theta' \in H$ . We thus have the following modification of the optimal strategy for infinite  $\Theta$ , which reduces to the optimal strategy for finite  $\Theta$  for  $\delta > 0$  small enough.

*Optimal strategy for infinite  $\Theta$ .*

1'. *Estimation.* Let  $k = 1$  and  $\hat{\theta}_a$  be an adjusted MLE satisfying (4.3) and (4.4).

2'. *Experimentation.* Let  $\hat{z}_{kj}$  be the solution to Problem A with parameter  $\hat{\theta}_a$  and with the bad set  $B_\ell(\theta)$  replaced by  $\bigcup_{\theta' \in H} B_\ell(\theta')$ .

(a') If  $\hat{\theta}_a \in \bigcup_{i>k} \Theta_i$ : Take  $\lceil \hat{z}_{kj} \log N \rceil$  observations from job  $kj$ ,  $j = 1, \dots, J_k$ .

(b') If  $\hat{\theta}_a \in \Theta_k$ : Take  $\lceil \hat{z}_{kj} \log N \rceil$  observations from job  $kj$  for  $j \notin J(\hat{\theta}_a)$ .

(c') If  $\hat{\theta}_a \in \bigcup_{i<k} \Theta_i$ : Skip the experimentation phase.

3'. and 4'. Identical to the strategy for finite  $\Theta$ , with  $\hat{\theta}_a$  replacing  $\hat{\theta}$ .

In view of (4.3) and (4.4), the modified strategy  $\phi^*$  described above will lead to asymptotic efficiency for infinite  $\Theta$  as stated in Theorem 2 below. It is also

convenient, when  $\Theta$  is infinite, to decide on the rejection of a job in step 3 based on the current sample rather than to keep track of which  $\lambda$  have been rejected previously. Hence for practical use, we can also make the following modification to the rejection of jobs in step 3':

Let  $U_{kj}(\mathbf{n}) = \inf_{\lambda \in \Theta_{kj}} U_k(\mathbf{n}; \lambda)$ . Reject job  $kj$  if  $U_{kj}(\mathbf{n}) \geq N$ .

**THEOREM 2.** *Assume A1–A5. The strategy  $\phi^*$  has error probabilities from the estimation stage that satisfy the following property. Let  $n_0 \rightarrow \infty$  with  $n_0 = o(\log N)$  and  $n_1 \rightarrow \infty$  such that  $n_1 = o(n_0)$ . Then there exists  $\delta (= \delta_N) \downarrow 0$  such that*

$$(4.5) \quad P_\theta\{\hat{\theta}_a \in \Theta \setminus N_\delta(\theta)\} = o(n_1^{-1}) \quad \text{as } N \rightarrow \infty.$$

Let  $\theta \in \Theta_\ell$ . Then the regret of  $\phi^*$  due to overshoot in the testing stage is  $O(1)$  because

$$(4.6) \quad \sum_{i>\ell} \sum_{j=1}^{J_i} E_\theta T_N(ij) \leq 1.$$

Therefore, the total regret

$$(4.7) \quad \lim_{N \rightarrow \infty} R_N(\theta) / \log N = z(\theta, \ell).$$

**REMARK 1.** Theorem 2 extends Fuh and Hu [7] to situations where more than one job in each group are available for processing. Theorem 2 generalizes the results of Lai [15] and Agrawal, Teneketzis and Anantharam [1, 2] to the case of infinite state and parameter spaces and more than one job group.

**REMARK 2.** If there is a constant switching cost each time we switch from one job to another, it can be shown that the strategy  $\phi^*$  has switching cost of order  $o(\log N)$ . Hence  $\phi^*$  is still efficient considering switching cost. The details are given in [5].

**REMARK 3.** We consider a nonempty bad set in this paper. It can be shown that the proposed strategy  $\phi^*$  can achieve  $o(\log N)$  regret when the bad set is empty and  $I = 1$ . In general, within the optimal group, the contribution to the regret from jobs optimal for parameter values outside the bad set is  $o(\log N)$ . The essence of the proof for this fact is contained in Section 6. We provide detailed justification in [5]. The upshot is that the strategy  $\phi^*$  can achieve superefficient results outside of bad sets.

**5. Examples.**

EXAMPLE 3. *Multi-phase project management.* To illustrate how our method can be applied, we discuss a few examples. Our purpose here is not to provide an accurate statistical model for a particular situation, but rather to supply concrete examples of parameter spaces and probability distributions such that the assumptions in Section 2.3 are satisfied.

Consider the management of  $N$  research and development (R&D) projects. When a project is pursued with one unit of resource, the reward is a normal random variable  $X$  with mean  $\mu_t(\theta)$  and variance  $\sigma_t^2(\theta)$ . Given the parameter value  $\theta$ , the mean  $\mu_t(\theta)$  reflects, at time  $t$ , the level of existing technology and knowledge relevant to the concerned projects as well as the competition in the market. Let  $\theta = (\alpha, \beta)$  and

$$(5.1) \quad \mu_t(\theta) = \frac{f(t, \alpha)}{h(t, \beta)}, \quad \sigma_t(\theta) = \frac{1}{h(t, \beta)},$$

where both  $f(t, \alpha)$  (reflecting technology and knowledge) and  $h(t, \beta)$  (reflecting competition) are increasing functions of time  $t$ . Observe that under (5.1) the coefficient of variation,  $\sigma_t/\mu_t = 1/f(t, \alpha)$ , is a decreasing function of  $t$ , which can be interpreted as follows. Because the products from the project will be gradually superseded by more advanced ones through competition in the market, therefore not only does the mean reward become smaller but we are also more certain of it due to time moving on. If we take  $f$  and  $h$  to be

$$(5.2) \quad f(t, \alpha) = \alpha t^2 \quad \text{and} \quad h(t, \beta) = e^{t\beta} - 1,$$

then the maximal value of  $\mu_t(\theta)$ , for a fixed value of  $\theta$ , is attained uniquely at  $t$  such that  $t\beta = \text{constant} \approx 1.5936$ .

Designate  $I$  phases indexed by time points  $0 < t_1 < \dots < t_I$  during which pursuit of a project can take place. And there are  $J$  different types of projects that can be pursued at any phase  $i = 1, \dots, I$ . To accommodate  $I$  phases and  $J$  types of projects, we expand the parameter vector to  $\theta = (\alpha_1, \dots, \alpha_J, \beta)$ . Given (5.1) and (5.2), let the rewards  $X_{ijk}$  from the pursuit with the  $k$ th unit of resource of the type  $j$  project in phase  $i$  be i.i.d. normal with means and standard deviations

$$(5.3) \quad \mu_{ij}(\theta) = \frac{\alpha_j t_i^2}{e^{t_i \beta} - 1}, \quad \sigma_i(\theta) = \frac{1}{e^{t_i \beta} - 1},$$

respectively.

By selecting  $\Theta = [\underline{\alpha}, \bar{\alpha}]^J \times [\underline{\beta}, \bar{\beta}]$  where  $0 < \underline{\alpha} < \bar{\alpha} < \infty$  and  $0 < \underline{\beta} < \bar{\beta} < \infty$ , condition A1 is easily seen to hold. Let  $\theta' = (\alpha'_1, \dots, \alpha'_J, \beta')$ . Then

$$I_{ij}(\theta, \theta') = \log \left[ \frac{\sigma_i(\theta')}{\sigma_i(\theta)} \right] + \frac{\sigma_i^2(\theta) - \sigma_i^2(\theta') + [\mu_{ij}(\theta) - \mu_{ij}(\theta')]^2}{2\sigma_i^2(\theta')}$$

equals zero if and only if  $\mu_{ij}(\theta) = \mu_{ij}(\theta')$  and  $\sigma_i^2(\theta) = \sigma_i^2(\theta')$ , or equivalently,  $\alpha_j = \alpha'_j$  and  $\beta = \beta'$ , the information assumption A2 is also satisfied. It can be shown that there exist  $(\underline{\beta} =)\beta_I < \beta_{I-1} < \dots < \beta_1 < \beta_0(=\overline{\beta})$  such that

$$(5.4) \quad \Theta_i = \begin{cases} \{\theta \in \Theta : \beta \in [\beta_1, \beta_0]\}, & \text{for } i = 1, \\ \{\theta \in \Theta : \beta \in [\beta_i, \beta_{i-1}]\}, & \text{for } 2 \leq i \leq I \end{cases}$$

and

$$(5.5) \quad \Theta_{ij} = \left\{ \theta \in \Theta_i : \alpha_j = \max_{1 \leq k \leq J} \alpha_k \right\}.$$

Since the observations  $X_{ijk}$  are independent, the assumptions A3–A5 are satisfied by selecting  $G_{ij} = \mathbf{R}$ ,  $V_{ij}(x) = |x| + 1$  and  $b > \sup_{i,j,\theta} E_\theta |X_{ij1}| + 1$ . Consequently, the strategies described in Section 4 are efficient in the sense of attaining the regret lower bound given by Theorem 1.

EXAMPLE 4. *Multi-phase project management with Markovian reward.* Continuing from Example 3, instead of i.i.d. rewards, we assume that  $k$ th pursuit of a project of type  $j$  at time  $t_i$  follows an AR(1) process

$$X_{ijk} = a_i X_{ij(k-1)} + \epsilon_{ijk},$$

where  $|a_i| < 1$  and  $\epsilon_{ijk} \sim N(\mu_{ij}(\theta), \sigma_i^2(\theta))$  with  $\mu_{ij}$  and  $\sigma_i$  given by (5.3). Let  $G_{ij} = [-c, c]$  for some  $c > 0$ . Since  $\epsilon_{ij1}$  has a positive density on the real line, A3 is satisfied. Let  $V_{ij}(x) = |x| + 1$ . From [18], page 380,  $\{X_{ijk}\}_{k \geq 0}$  is geometric ergodic and A4 holds with  $0 < \bar{b} < 1 - \max_i |a_i|$  and  $b, c$  large enough.

The stationary distribution is normally distributed with mean and variance given by  $(1 - a_i)^{-1} \mu_{ij}(\theta)$  and  $(1 - a_i^2)^{-1} \sigma_i^2(\theta)$ . It can be checked that (5.4) and (5.5), which reveal the structure of the parameter space, still hold for AR(1) rewards. Consequently, A1 is true for AR(1) rewards. To simplify the presentation of the Kullback–Leibler information number, we drop the indices  $i, j$  and use  $\mu', \sigma'$  to denote  $\mu(\theta'), \sigma(\theta')$ , respectively,

$$\begin{aligned} I(\theta, \theta') &= \log\left(\frac{\sigma'}{\sigma}\right) + \frac{\sigma^2 - \sigma'^2 + (\mu - \mu')^2}{2\sigma'^2} + \\ &+ \frac{(a - a')^2 \{\mu^2(1 - a)^{-2} + \sigma^2(1 - a^2)^{-1}\}}{2\sigma'^2} \\ &+ \frac{2(a - a')(\mu - \mu')\mu(1 - a)^{-1}}{2\sigma'^2}. \end{aligned}$$

It is clear that the Kullback–Leibler number is greater than zero if  $\theta \neq \theta'$ . From the preceding equation, we can verify that A2 and A5 hold.

**6. Proof of asymptotic efficiency.** We shall demonstrate the asymptotic efficiency of  $\phi^*$  by proving (4.5)–(4.7). A change-of-measure argument is first used to prove (4.6). As the proofs of (4.5) and (4.7) are too involved for one reading, we prove them in Section 6.1 for the restricted case of finite  $\Theta$  and extend the proofs to infinite  $\Theta$  in Section 6.2.

**PROOF OF (4.6).** Let  $\tilde{P}$  be the measure which generates  $\mathbf{X}_n := \{X_{ijt}\}$  for  $j = 1, \dots, J_i$  and  $i = 1, \dots, \ell$ ,  $t = 1, \dots, n_{ij}$  in the following manner. First generate  $\theta'$  randomly from  $F_\ell$ . Using the strategy  $\phi^*$  to select the jobs to be processed, generate  $X_{ij0}$  from  $v_{ij}(\cdot; \theta')$  and  $X_{ijt}$ ,  $t \geq 1$ , according to the transition density  $p_{ij}(X_{ij(t-1)}, \cdot; \theta')$  when at job  $ij$ . Let  $\theta \in \Theta_{\ell j}$ . Then

$$\begin{aligned} \frac{d\tilde{P}}{dP_\theta}(\mathbf{X}_n) &= \frac{\int_{\cup_{i=\ell}^1 \Theta_i} \prod_{i=1}^\ell \prod_{j=1}^{J_i} v_{ij}(X_{ij0}; \theta') \prod_{t=1}^{n_{ij}} p_{ij}(X_{ij(t-1)}, X_{ijt}; \theta') F_\ell(d\theta')}{\prod_{i=1}^\ell \prod_{j=1}^{J_i} v_{ij}(X_{ij0}; \theta) \prod_{t=1}^{n_{ij}} p_{ij}(X_{ij(t-1)}, X_{ijt}; \theta)} \\ &= U_\ell(\mathbf{n}; \theta). \end{aligned}$$

Let  $\mathbf{T} = (T_N(11), \dots, T_N(\ell J_\ell))$  and  $A = \{U_\ell(\mathbf{T}; \theta) \geq N\}$ . Then  $P_\theta\{\sum_{i>\ell} T_N(i) > 0\}$  is bounded by

$$(6.1) \quad P_\theta(A) = E_{\tilde{P}} \left[ \frac{dP_\theta}{d\tilde{P}}(\mathbf{X}_\mathbf{T}) \mathbf{1}_A \right] \leq N^{-1}.$$

Hence (4.6) follows from (6.1) and the bound  $\sum_{i>\ell} T_N(i) \leq N$ .  $\square$

**6.1. Finite parameter space.** Let  $\Theta = \{\theta_0, \dots, \theta_h\}$ . Let  $\theta_0 \in \Theta_{\ell j_0}$  be the true parameter value. For  $1 \leq q \leq h$ , define

$$(6.2) \quad \xi_{ijt}(q) = \log[p_{ij}(X_{ij(t-1)}, X_{ijt}; \theta_0) / p_{ij}(X_{ij(t-1)}, X_{ijt}; \theta_q)].$$

Then  $E_{\pi(\theta_0)} \xi_{ijt}(q) = I_{ij}(\theta_0, \theta_q)$ . To get the essence of the strategy without being overly involved in cumbersome notation, let us consider the specific case  $\ell = 2$ ,  $J_1 = J_2 = 2$ ,  $\theta_0 \in \Theta_{21}$  and  $J(\theta_0) = \{1\}$ .

We first prove (4.5). Let us consider the inequality

$$P_{\theta_0}\{\hat{\theta} \neq \theta_0\} = \sum_{q=1}^h P_{\theta_0}\{\hat{\theta} = \theta_q\} \leq \sum_{q=1}^h P_{\theta_0} \left\{ \sum_{t=1}^{n_0} \xi_{11t}(q) + \xi_{12t}(q) < 0 \right\}.$$

By A5 and Chebyshev’s inequality,

$$\begin{aligned} &P_{\theta_0} \left\{ \sum_{t=1}^{n_0} \xi_{11t}(q) + \xi_{12t}(q) < 0 \right\} \\ &\leq \frac{\text{Var}_{\theta_0}(\sum_{t=1}^{n_0} \xi_{11t}(q) + \xi_{12t}(q))}{[E_{\theta_0}(\sum_{t=1}^{n_0} \xi_{1jt}(q) + \xi_{11t}(q))]^2} \\ &\leq (1 + o(1)) \frac{E_{\pi(\theta_0)}[\xi_{11t}^2(q) + \xi_{12t}^2(q)] + 2I_{11}(\theta_0, \theta_q)I_{12}(\theta_0, \theta_q)}{n_0[I_{11}(\theta_0, \theta_q) + I_{12}(\theta_0, \theta_q)]} = O(n_0^{-1}). \end{aligned}$$

This completes the proof of (4.5) for the finite parameter case.

We now undertake the proof of (4.7). For  $q \geq 1$ , let  $\theta_q \in \Theta_{kj'}$  where either (i)  $k < \ell$  or (ii)  $k = \ell$  and  $j' \notin J(\theta_0)$ . Let  $\tau_{kj}(q)$  be the number of observations selected from job  $kj$  in the testing phase of group  $k$  before parameter  $\theta_q$  is rejected. To show (4.7), it suffices to prove that

$$(6.3) \quad \begin{aligned} E_{\theta_0} \left[ \sum_{j=1}^{J_k} \tau_{kj}(q) \right] &= o(\log N) \quad \text{if } k < \ell \quad \text{and} \\ E_{\theta_0} \left[ \sum_{j \notin J(\theta_0)} \tau_{\ell j}(q) \right] &= o(\log N), \end{aligned}$$

because (6.3) implies that the regret in the testing phase before leaving the optimal group is  $o(\log N)$ , and the regret due to overshooting the optimal group is also  $o(\log N)$  by the established (4.6), thus completing the justification.

Select  $C > 0$  large enough such that  $\xi'_{ijt}(q) = \xi_{ijt}(q) \wedge C$  has positive expectation under  $\pi(\theta_0)$  for all  $i, j, q$  satisfying  $I_{ij}(\theta_0, \theta_q) > 0$ . Let  $\mathbf{n} = (n_{11}, n_{12})$ . We will first show that the first half of (6.3) is satisfied when  $\theta_q \in \Theta_1$ . By (4.2),

$$(6.4) \quad \begin{aligned} \log U_1(\mathbf{n}; \theta_q) &\geq \sum_{t=1}^{n_{11}} \xi_{11t}(q) + \sum_{t=1}^{n_{12}} \xi_{12t}(q) + \log v_{11} \\ &\quad + \log v_{12} + \log F_1(\theta_0), \end{aligned}$$

where  $v_{ij} = \inf_{x, \theta, \lambda} [v_{ij}(x; \theta) / v_{ij}(x; \lambda)] > 0$  as assumed in (2.2). Hence by (4.2), rejection of  $\theta_q$  has occurred when

$$(6.5) \quad \begin{aligned} \sum_{t=1}^{m_{11}} \xi_{11t}(q) + \sum_{t=1}^{m_{12}} \xi_{12t}(q) + \sum_{t=m_{11}+1}^{n_{11}} \xi'_{11t}(q) + \sum_{t=m_{12}+1}^{n_{12}} \xi'_{12t}(q) \\ > c := \log N - \log v_{11} - \log v_{12} - \log F_1(\theta_0), \end{aligned}$$

where  $\mathbf{m} = (m_{11}, m_{12}) = (n_0 + \lfloor z_{11}(\hat{\theta}) \log N \rfloor, n_0 + \lfloor z_{12}(\hat{\theta}) \log N \rfloor)$  is the sample size at the beginning of the testing phase. Since  $\xi'_{ijt}(q)$  is bounded above by  $C$ , it follows that at  $\mathbf{n} = (n'_{11}, n'_{12})$  for which the boundary is first crossed by  $\xi'_{ijt}$ 's

$$(6.6) \quad \begin{aligned} E_{\theta_0} \left[ \sum_{t=1}^{m_{11}} \xi_{11t}(q) + \sum_{t=1}^{m_{12}} \xi_{12t}(q) \right] + E_{\theta_0} \left[ \sum_{t=m_{11}+1}^{n'_{11}} \xi'_{11t}(q) + \sum_{t=m_{12}+1}^{n'_{12}} \xi'_{12t}(q) \right] \\ \leq c(1 + o(1)). \end{aligned}$$

By (4.5), the condition  $n_0 = o(\log N)$ , (6.5) and the constraint  $I_{11}(\theta_0, \theta_q) \times z_{11}(\theta_0) + I_{12}(\theta_0, \theta_q) z_{12}(\theta_0) \geq 1$  from (3.3), it follows that

$$(6.7) \quad E_{\theta_0} \left[ \sum_{t=1}^{m_{11}} \xi_{11t}(q) + \sum_{t=1}^{m_{12}} \xi_{12t}(q) \right] \geq (1 + o(1))c.$$

Subtracting (6.7) from (6.6), we have

$$(6.8) \quad E_{\theta_0} \left[ \sum_{t=m_{11}+1}^{n_{11}} \xi'_{11t}(q) + \sum_{t=m_{12}+1}^{n_{12}} \xi'_{12t}(q) \right] = o(c).$$

By Wald’s equation for Markov processes, the left-hand side of (6.8) equals

$$(6.9) \quad (1 + o(1)) [E_{\pi(\theta_0)} \xi'_{11t}(q) E_{\theta_0}(n_{11} - m_{11}) + E_{\pi(\theta_0)} \xi'_{12t}(q) E_{\theta_0}(n_{12} - m_{12})].$$

The proof of Wald’s equation for Markov process is given in the [Appendix](#). By A2 and the choice of  $C$  for  $\xi'_{ijt}(q)$ ,  $E_{\pi(\theta_0)} \xi'_{1jt}(q) > 0$ . In view of the sample size in the testing stage  $\tau_{1j}(q) \leq n_{1j} - m_{1j}$ , it follows from (6.8) and (6.9) that  $E_{\theta_0}[\tau_{11}(q) + \tau_{12}(q)] = o(c)$  for all  $\theta_q \in \Theta_1$ . Hence the rejection of both  $\Theta_{11}$  and  $\Theta_{12}$  involves only  $o(\log N)$  observations and the first half of (6.3) holds.

Next we show that the second half of (6.3) holds when  $\theta_q \in \Theta_{22}$ . We divide into two cases,  $\theta_q \in B_2(\theta_0)$  and  $\theta_q \notin B_2(\theta_0)$ . Consider the first case. By (2.20),  $I_{22}(\theta_0, \theta_q) > 0$ . We then follow the arguments above using (4.5) and the last inequality of (3.3) to show that  $E_{\theta_0} \tau_{22}(q) = o(c)$ .

The second scenario involves  $\theta_q \notin B_2(\theta_0)$ . The key observation is  $I_{21}(\theta_0, \theta_q) > 0$  by (2.19). In other words, information is always collected and no additional regret is incurred when we sample from job 21. Under unequal sampling,

$$(6.10) \quad \begin{aligned} E_{\theta_0} \tau_{22}(q) &= E_{\theta_0} [\tau_{22}(q) \mathbf{1}_{\{J(\hat{\theta})=\{1\}\}}] + E_{\theta_0} [\tau_{22}(q) \mathbf{1}_{\{J(\hat{\theta})=\{1,2\}\}}] \\ &\quad + E_{\theta_0} [\tau_{22}(q) \mathbf{1}_{\{J(\hat{\theta})=\{2\}\}}] \\ &= n_1^{-1} E_{\theta_0} [\tau_{21}(q) \mathbf{1}_{\{J(\hat{\theta})=\{1\}\}}] + E_{\theta_0} [\tau_{21}(q) \mathbf{1}_{\{J(\hat{\theta})=\{1,2\}\}}] \\ &\quad + n_1 E_{\theta_0} [\tau_{21}(q) \mathbf{1}_{\{J(\hat{\theta})=\{2\}\}}]. \end{aligned}$$

Since  $E_{\theta_0} [\tau_{21}(q) \mathbf{1}_{\{J(\hat{\theta})=A\}}] \leq (1 + o(1))c P_{\theta_0}\{J(\hat{\theta}) = A\} / I_{21}(\theta_0, \theta_q)$  for  $A = \{1\}, \{2\}$  and  $\{1, 2\}$  and as  $n_1 \rightarrow \infty$ , the first term on the right-hand side of (6.10) is  $o(c)$  while (4.5) ensures the second term is  $o(c)$ . By (4.5),  $n_1 P_{\theta_0}\{J(\hat{\theta}) \neq \{1\}\} \leq n_1 P_{\theta_0}\{\hat{\theta} \neq \theta_0\} = o(1)$  and thus the third term on the right-hand side of (6.10) is  $o(c)$ . We can conclude that  $E_{\theta_0} \tau_{22}(q) = o(c)$  or the second half of (6.3) holds.

6.2. *Extension to infinite parameter space.* We preface the extension with the following lemma. The proof of this lemma is given in the [Appendix](#) in Section A.4. We shall let  $\bar{A}$  denote the closure of a set  $A$ .

LEMMA 2. *Let  $\theta_0 \in \Theta_\ell$ . Assume A1–A5 and let  $n_0 \rightarrow \infty, n_1 = o(n_0)$ .*

(a) *Let  $\theta' \neq \theta_0$  and let  $\hat{\theta}$  be the MLE estimate (4.1). Then there exists  $\delta' > 0$  small enough such that*

$$(6.11) \quad P_{\theta_0} \{\hat{\theta} \in N_{\delta'}(\theta')\} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

(b) Let  $\theta' \in \bar{\Theta}_k$  for some  $k < \ell$  or  $\theta' \in \bigcup_{j \notin J(\theta_0)} \bar{\Theta}_{\ell j}$ . Let  $\delta' > 0$  and let  $\tau_{kj}$  ( $\tau_{\ell j}$ ) be the number of observations selected from job  $kj$  ( $\ell j$ ) in the testing phase of group  $k$  ( $\ell$ ) before all parameters in the set  $N_{\delta'}(\theta')$  are rejected. Then for  $\delta' > 0$  small enough,

$$(6.12) \quad \begin{aligned} E_{\theta_0} \left( \sum_{j=1}^{J_k} \tau_{kj} \right) &= o(\log N) \quad \text{if } k < \ell \quad \text{and} \\ E_{\theta_0} \left( \sum_{j \notin J(\theta_0)} \tau_{\ell j} \right) &= o(\log N). \end{aligned}$$

We now apply Lemma 2 to extend the proof of Theorem 2. By the compactness of  $\Theta \setminus N_{\delta/2}(\theta_0)$ ,  $\delta > 0$ , there exist a finite set  $\{\theta_1, \dots, \theta_h\}$  and constants  $\delta_q > 0$  such that (6.11) holds for  $\theta' = \theta_q$  and  $\delta' = \delta_q$  for all  $1 \leq q \leq h$  and  $\Theta \setminus \{\theta_0\} \supset \bigcup_{q=1}^h N_{\delta_q}(\theta_q) \supset \Theta \setminus N_{\delta/2}(\theta_0)$ . Then by (6.11),  $P_{\theta_0} \{\hat{\theta} \in \Theta \setminus N_{\delta/2}(\theta_0)\} \rightarrow 0$  as  $N \rightarrow \infty$  and the result (4.5) follows from (4.3) because  $\|\hat{\theta}_a - \hat{\theta}\| < \delta/2$ .

It remains to show that the number of observations taken from each nonoptimal job in the testing phase is  $o(\log N)$ . Consider  $k < \ell$ ,  $j = 1, \dots, J_k$  or  $k = \ell$  with  $j \notin J(\theta_0)$ . Since  $\bar{\Theta}_{kj}$  is compact, there exist a finite set  $\{\theta_1, \dots, \theta_h\}$  and constants  $\delta_q > 0$  such that (6.12) is satisfied for  $\theta' = \theta_q$ ,  $\delta' = \delta_q$  for all  $1 \leq q \leq h$  and  $\bigcup_{q=1}^h N_{\delta_q}(\theta_q) \supset \bar{\Theta}_{kj}$ , and hence by (6.12), the number of times job  $kj$  is processed in the testing phase is  $o(\log N)$  as required.

APPENDIX

**A.1. Proof of (2.5).** Let  $X_{ijt}$  denotes the  $t$ th observation taken from arm  $ij$ . Then

$$(A.1) \quad \left| W_N(\theta) - \sum_{i=1}^I \sum_{j=1}^{J_i} \mu_{ij}(\theta) E_{\theta} T_N(ij) \right| \leq \sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{t=1}^{\infty} |E_{\theta} g(X_{ijt}) - \mu_{ij}(\theta)|.$$

For any signed measure  $\lambda$  on  $(D, \mathcal{D})$ , let

$$(A.2) \quad \|\lambda\|_{V_{ij}} = \sup_{h: |h| \leq V_{ij}} \left| \int h(x) \lambda(dx) \right|.$$

It follows from [18], page 367 and Theorem 16.0.1, that under A3 and the geometric drift condition (2.14),

$$(A.3) \quad \omega_{ij} := \sup_{\theta \in \Theta, x \in D} \sum_{t=1}^{\infty} \|P_{ijt}^{\theta}(x, \cdot) - \pi_{ij}(\cdot; \theta)\|_{V_{ij}} / V_{ij}(x) < \infty,$$

where  $P_{ijt}^{\theta}(x, \cdot)$  denotes the distribution of  $X_{ijt}$  conditioned on  $X_{ij0} = x$  and  $\pi_{ij}(\cdot; \theta)$  denotes the stationary distribution of  $X_{ijt}$  under parameter  $\theta$ . By (2.13),

there exists  $\kappa > 0$  such that  $\kappa|g(x)| \leq V_{ij}(x)$  for all  $x \in D$  and hence it follows from (A.2) and (A.3) that

$$(A.4) \quad \kappa \sum_{t=1}^{\infty} |E_{\theta,x}g(X_{ijt}) - \mu_{ij}(\theta)| \leq \omega_{ij}V_{ij}(x),$$

where  $E_{\theta,x}$  denotes expectation with respect to  $P_{\theta}$  and the initial distribution  $X_{ij0} = x$ .

In general, for any initial distribution  $v_{ij}(\cdot; \theta)$ , it follows from (2.15) and (A.4) that

$$\sum_{t=1}^{\infty} |E_{\theta}g(X_{ijt}) - \mu_{ij}(\theta)| \leq \int \sum_{t=1}^{\infty} |E_{\theta,x}g(X_{ijt}) - \mu_{ij}(\theta)|v_{ij}(x; \theta)Q(dx) < \infty$$

uniformly over  $\theta \in \Theta$  and hence (2.5) follows from (A.1).

**A.2. Proof of Lemma 1.** To prove (3.6), it suffices to show that for every  $\theta' \in \Theta_k^*$ ,  $k < \ell$  and for  $\delta > \alpha > 0$ ,

$$(A.5) \quad \lim_{N \rightarrow \infty} P_{\theta'} \left\{ \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') T_N(ij) < (1 - \delta) \log N \right\} = 0.$$

Because  $\phi$  is uniformly good and  $\theta' \in \Theta_k^*$ , it follows from (2.7) that  $E_{\theta'}[N - \sum_{j \in J(\theta')} T_N(kj)] = o(N^{\alpha})$  for  $\alpha > 0$ . By A2,  $I_{kj}(\theta, \theta') > 0$  for all  $j \in J(\theta')$  and hence  $I_0 := \min_{j \in J(\theta')} I_{kj}(\theta, \theta') > 0$ . It then follows from Chebyshev's inequality that

$$(A.6) \quad \begin{aligned} & P_{\theta'} \left\{ \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') T_N(ij) < (1 - \delta) \log N \right\} \\ & \leq P_{\theta'} \left\{ I_0 \sum_{j \in J(\theta')} T_N(kj) < (1 - \delta) \log N \right\} \\ & = P_{\theta'} \left\{ \left[ N - \sum_{j \in J(\theta')} T_N(kj) \right] > N - (1 - \delta)(\log N)/I_0 \right\} \\ & = O(N^{-1}) E_{\theta'} \left[ N - \sum_{j \in J(\theta')} T_N(kj) \right] = o(N^{\alpha-1}). \end{aligned}$$

Let  $\mathbf{n} = (n_{11}, \dots, n_{kJ_k})$  and  $\mathbf{T}_N = (T_N(11), \dots, T_N(kJ_k))$ . Let

$$L_{\mathbf{n}} = \sum_{i=1}^k \sum_{j=1}^{J_i} \left\{ \log[v_{ij}(X_{ij0}; \theta)/v_{ij}(X_{ij0}; \theta')] + \sum_{t=1}^{n_{ij}} \ell_{ij}(X_{ij(t-1)}, X_{ijt}; \theta, \theta') \right\}$$

be the log likelihood ratio of  $\theta$  with respect to  $\theta'$ , and denote

$$G_N = \left\{ \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') T_N(ij) < (1 - \delta) \log N \text{ and } L_{\mathbf{T}} \leq (1 - \alpha) \log N \right\}.$$

Then by (A.6),  $P_{\theta'}(G_N) = o(N^{\alpha-1})$ . By Wald's likelihood ratio identity for Markov chains,

$$\begin{aligned} P_{\theta'}\{\mathbf{T}_N = \mathbf{n}, L_{\mathbf{n}} \leq (1 - \alpha) \log N\} &= E_{\theta}[\exp(-L_{\mathbf{n}}) \mathbf{1}_{\{\mathbf{T}_N = \mathbf{n}, L_{\mathbf{n}} \leq (1 - \alpha) \log N\}}] \\ &\geq N^{\alpha-1} P_{\theta}\{\mathbf{T}_N = \mathbf{n}, L_{\mathbf{n}} \leq (1 - \alpha) \log N\}. \end{aligned}$$

By summing the preceding inequality over all  $\mathbf{n}$ , we have

$$(A.7) \quad P_{\theta}(G_N) \leq N^{1-\alpha} P_{\theta'}(G_N) = N^{1-\alpha} o(N^{\alpha-1}) = o(1).$$

By A3 and the strong law of large numbers for Markov chains (cf. Theorem 17.0.1 of [18]),

$$\left| L_{\mathbf{n}} - \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') n_{ij} \right| = o\left( \sum_{i=1}^k \sum_{j=1}^{J_i} n_{ij} \right) \quad P_{\theta} \text{ a.s. as } \sum_{i=1}^k \sum_{j=1}^{J_i} n_{ij} \rightarrow \infty.$$

Thus,

$$\lim_{m \rightarrow \infty} \left\{ \max_{\mathbf{n}: \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') n_{ij} \leq m} \left[ L_{\mathbf{n}} - \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') n_{ij} \right] / m \right\} \rightarrow 0$$

a.s. under  $P_{\theta}$ .

Because  $1 - \alpha > 1 - \delta$ , it then follows that as  $N \rightarrow \infty$ ,

$$\begin{aligned} P_{\theta} \left\{ L_{\mathbf{n}} > (1 - \alpha) \log N, \text{ for some } \mathbf{n} \right. \\ \left. \text{such that } \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') n_{ij} < (1 - \delta) \log N \right\} \rightarrow 0. \end{aligned}$$

Therefore, as  $N \rightarrow \infty$ ,

$$P_{\theta} \left\{ \sum_{i=1}^k \sum_{j=1}^{J_i} I_{ij}(\theta, \theta') T_N(ij) < (1 - \delta) \log N \text{ and } L_{\mathbf{T}} > (1 - \alpha) \log N \right\} \rightarrow 0.$$

This combined with (A.7) gives (A.5), from which (3.6) follows by letting  $\delta \downarrow 0$ .

We now consider the case  $\theta' \in B_{\ell}(\theta)$ . By (2.20),  $\min_{j \in J(\theta')} I_{\ell j}(\theta, \theta') > 0$ . The proof proceeds as before with  $k = \ell$ , which leads us to (3.6) with  $k = \ell$ . Since  $I_{\ell j}(\theta, \theta') = 0$  for all  $j \in J(\theta)$  by (2.19), (3.7) follows.

**A.3. Proof of Theorem 1.** As we mentioned after (3.5) that  $B_\ell(\theta) \neq \emptyset$ , by A1,  $\Lambda_\ell = \Theta_1^* \times \dots \times \Theta_{\ell-1}^* \times B_\ell(\theta)$  is nonempty. For each  $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \Lambda_\ell$  and  $\theta \in \Theta_\ell$ , we define  $z(\theta, \ell, \lambda)$  to be the minimal value of (3.2) with (3.3) replaced by

$$(A.8) \quad \begin{cases} \sum_{j=1}^{J_1} I_{1j}(\theta, \lambda_1) z_{1j}(\theta) \geq 1, \\ \vdots \\ \sum_{i=1}^{\ell-1} \sum_{j=1}^{J_i} I_{ij}(\theta, \lambda_{\ell-1}) z_{ij}(\theta) \geq 1, \\ \sum_{i < \ell} \sum_{j=1}^{J_i} I_{ij}(\theta, \lambda_\ell) z_{ij}(\theta) + \sum_{j \notin J(\theta)} I_{\ell j}(\theta, \lambda_\ell) z_{\ell j}(\theta) \geq 1. \end{cases}$$

By Lemma 1, (A.8) is true for all  $\lambda \in \Lambda_\ell$ . Therefore,  $\liminf_{N \rightarrow \infty} R_N(\theta) / \log N \geq \sup_{\lambda \in \Lambda_\ell} z(\theta, \ell, \lambda)$ , for all  $\theta \in \Theta_\ell$ . The proof is completed if we can show that

$$(A.9) \quad z(\theta, \ell) = \sup_{\lambda \in \Lambda_\ell} z(\theta, \ell, \lambda).$$

If  $Z = \{z_{ij}(\theta) : j = 1, \dots, J_i \text{ for } i < \ell, \text{ and } j \notin J(\theta), i = \ell\}$  satisfies (3.3), then  $Z$  also satisfies (A.8). Thus

$$(A.10) \quad z(\theta, \ell) \geq \sup_{\lambda \in \Lambda_\ell} z(\theta, \ell, \lambda).$$

Because  $I_{ij}(\theta, \theta')$  are continuous with respect to  $\theta'$ , the infima in (3.3) are attained for some  $\bar{\lambda} \in \bar{\Lambda}_\ell$ , the closure of  $\Lambda_\ell$ . Choose a sequence of  $\lambda(n) = (\lambda_1(n), \dots, \lambda_\ell(n)) \in \Lambda_\ell$  such that it converges to  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_\ell)$ . Note that  $\bar{\lambda}$  depends on some feasible  $z$  satisfying (3.3).

Let  $z_n = (z_{11}(n), \dots, z_{\ell J_\ell}(n))$  be the solution of (3.2) satisfying (A.8) with  $\lambda = \lambda(n)$ . Set

$$c_{ij}(n) = \max\{I_{ij}(\theta, \lambda_1(n)) / I_{ij}(\theta, \bar{\lambda}_1), \dots, I_{ij}(\theta, \lambda_\ell(n)) / I_{ij}(\theta, \bar{\lambda}_\ell)\}.$$

By the continuity of  $I_{ij}$ , we have

$$(A.11) \quad \lim_{n \rightarrow \infty} c_{ij}(n) = 1, \quad \text{for } 1 \leq i \leq \ell.$$

In view of  $\sum_{ij} c_{ij}(n) z_{ij}(n) I_{ij}(\theta, \bar{\lambda}_i) = \sum_{ij} z_{ij}(n) I_{ij}(\theta, \lambda_i(n))$  for  $i, j$  in an appropriate index set, we see that  $\{c_{ij}(n) z_{ij}(n)\}$  satisfies (3.3). Hence,

$$\begin{aligned} \left[ \max_{1 \leq i \leq \ell, 1 \leq j \leq J_i} c_{ij}(n) \right] z(\theta, \ell, \lambda_n) &\geq \sum_{i < \ell} \sum_{j=1}^{J_i} [\mu^*(\theta) - \mu_{ij}(\theta)] c_{ij}(n) z_{ij}(n) \\ &\quad + \sum_{j \notin J(\theta)} [\mu^*(\theta) - \mu_{\ell j}(\theta)] c_{\ell j}(n) z_{\ell j}(n) \\ &\geq z(\theta, \ell). \end{aligned}$$

By (A.11), we have  $\sup_{\lambda \in \Lambda_\ell} z(\theta, \ell, \lambda) \geq z(\theta, \ell)$ , which combined with (A.10) implies (A.9).

**A.4. Proof of Lemma 2.** By (2.18), there exists  $\delta' > 0$  such that

$$(A.12) \quad E_{\pi_{ij}(\theta_0)} \left[ \sup_{\tilde{\theta} \in N_{\delta'}(\theta')} |\ell_{ij}(X_{ij0}, X_{ij1}; \theta', \tilde{\theta})| \right] < \varepsilon$$

for all  $i, j$  and  $\theta' \in \Theta$ ,  $\varepsilon > 0$  to be specified later. Let

$$(A.13) \quad \begin{aligned} \tilde{\xi}_{1jt} &= \inf_{\lambda \in N_{\delta'}(\theta')} \ell_{1j}(X_{1j(t-1)}, X_{1jt}; \theta_0, \lambda) \\ &= \ell_{1j}(X_{1j(t-1)}, X_{1jt}; \theta_0, \theta') \\ &\quad - \sup_{\lambda \in N_{\delta'}(\theta')} \ell_{1j}(X_{1j(t-1)}, X_{1jt}; \theta', \lambda). \end{aligned}$$

Since  $\eta := \sum_{j=1}^{J_1} I_{1j}(\theta_0, \theta') > 0$ , we can select  $\delta' > 0$  to satisfy (A.12) with  $\varepsilon < \eta/J_1$ . Then by (A.12) and (A.13), it follows that

$$n_0^{-1} E_{\pi(\theta_0)} \left[ \sum_{j=1}^{J_1} \sum_{t=1}^{n_0} \tilde{\xi}_{1jt} \right] \geq \sum_{j=1}^{J_1} I_{1j}(\theta_0, \theta') - J_1 \varepsilon \geq \eta - J_1 \varepsilon > 0.$$

By the Harris recurrence condition A3 and the law of large numbers, it follows that

$$P(A) \rightarrow 1 \quad \text{as } N \rightarrow \infty, \text{ where } A = \left\{ \sum_{j=1}^{J_1} \sum_{t=1}^{n_0} \tilde{\xi}_{1jt} > 0 \right\}.$$

In the event  $A$ , the likelihood at  $\theta_0$  is larger than all  $\lambda \in N_{\delta'}(\theta')$  and hence (6.11) holds.

To prove (6.12), we extend (6.2) and define

$$(A.14) \quad \begin{aligned} \check{\xi}_{ijt} &= \inf_{\theta \in N_{\delta'}(\theta_0), \lambda \in N_{\delta'}(\theta')} \log \left[ \frac{p_{ij}(X_{ij(t-1)}, X_{ijt}; \theta)}{p_{ij}(X_{ij(t-1)}, X_{ijt}; \lambda)} \right] \\ &\geq \ell_{ij}(X_{ij(t-1)}, X_{ijt}; \theta_0, \theta') \\ &\quad - \sup_{\theta \in N_{\delta'}(\theta_0)} |\ell_{ij}(X_{ij(t-1)}, X_{ijt}; \theta_0, \theta)| \\ &\quad - \sup_{\lambda \in N_{\delta'}(\theta')} |\ell_{ij}(X_{ij(t-1)}, X_{ijt}; \theta', \lambda)|. \end{aligned}$$

Let  $\theta' \in \bar{\Theta}_{kj_0}$  for some  $k < \ell$ . By A2, we can select  $0 < \varepsilon < I_{kj_0}(\theta_0, \theta')/2J_k$  and hence by (A.12) and (A.14), we have  $E_{\pi(\theta_0)}(\sum_{j=1}^{J_k} \check{\xi}_{kjt}) \geq I_{kj_0}(\theta_0, \theta') - 2J_k \varepsilon > 0$ .

By (4.2) and (A.14), it follows that

$$(A.15) \quad \inf_{\lambda \in N_{\delta'}(\theta')} \log U_k(\mathbf{n}; \lambda) \geq \log F_k(N_{\delta'}(\theta_0)) + \sum_{i=1}^k \sum_{j=1}^{J_i} \log v_{ij} + \sum_{i=1}^k \sum_{j=1}^{J_i} \sum_{t=1}^{n_{ij}} \xi_{ijt},$$

where  $v_{ij} = \inf_{x, \theta, \lambda} [v_{ij}(x; \theta) / v_{ij}(x; \lambda)]$ . By (A.15),  $\tau_{kj} \leq n_{kj} - m_{kj}$ , where  $\mathbf{n} = (n_{ij})$  is the sample size needed for  $\sum_{i=1}^k \sum_{j=1}^{J_i} \sum_{t=1}^{n_{ij}} \xi_{ijt}$  to cross the threshold  $c := \log N - \sum_{i=1}^k \sum_{j=1}^{J_i} \log v_{ij} - \log F_k(N_{\delta'}(\theta_0))$  and  $\mathbf{m} = (m_{ij})$  is the sample size at the start of the testing phase. Now follow arguments analogous to (6.4)–(6.9), and we can prove the first half of (6.12).

Next, let us consider  $k = \ell$ . Let  $f(\theta) = \mu_{\ell j_0}(\theta) - \sup_{j \in J(\theta_0)} \mu_{\ell j}(\theta)$  for some  $j_0 \notin J(\theta_0)$ . Then  $f(\theta_0) < 0$ . Conversely,  $f(\theta') \geq 0$  for any  $\theta' \in \Theta_{\ell j_0}$ . By A1,  $f$  is continuous with respect to  $\theta$  and hence  $\inf_{\theta' \in \Theta_{\ell j_0}} \|\theta_0 - \theta'\| > 0$ . The proof for the second half of (6.12) then follows from arguments similar to those in the last two paragraphs of Section 6.1.

**A.5. Extension of Wald’s equation to Markovian rewards.** As we will be focusing on a single job  $ij$  and fixed parameters  $\theta_0, \theta_q$  such that  $\mu := I_{ij}(\theta_0, \theta_q) > 0$ , we will drop some of the references to  $i, j, \theta_0, \theta_q$  and  $q$  in this subsection. This applies also to the notation in assumptions A3–A5. Moreover, we shall use the notation  $E(\cdot)$  as a short form of  $E_{\theta_0}(\cdot)$  and  $E_x(\cdot)$  as a short form of  $E_{\theta_0}(\cdot | X_0 = x)$ . Let  $S_n = \xi_1 + \dots + \xi_n$ , where  $\xi_k = \log[p_{ij}(X_{k-1}, X_k; \theta_0) / p_{ij}(X_{k-1}, X_k; \theta_q)]$  has stationary mean under  $P_{\theta_0}$  and let  $\tau$  be a stopping time. We shall establish Wald’s equation

$$(A.16) \quad ES_\tau = [\mu + o(1)]E\tau$$

for Markovian rewards.

By (2.12), we can augment the Markov additive process and create a split chain containing an atom, so that increments in  $S_n$  between visits to the atom are independent. More specifically, we construct stopping times  $0 < \kappa(1) < \kappa(2) < \dots$  using an auxiliary randomization procedure such that

$$(A.17) \quad P\{X_{n+1} \in A, \kappa(i) = n + 1 | X_n = x, \kappa(i) > n \geq \kappa(i - 1)\} = \begin{cases} \alpha\varphi(A), & \text{if } x \in G, \\ 0, & \text{otherwise.} \end{cases}$$

Then by Lemma 3.1 of [19],

- (i)  $\{\kappa(i + 1) - \kappa(i) : i = 1, 2, \dots\}$  are i.i.d. random variables;
  - (ii) the random blocks  $\{X_{\kappa(i)}, \dots, X_{\kappa(i+1)-1}\}, i = 1, 2, \dots,$  are independent;
- and

(iii)  $P\{X_{\kappa(i)} \in A | \mathcal{F}_{\kappa(i)-1}\} = \varphi(A)$ , where  $\mathcal{F}_n =$  the  $\sigma$ -field generated by  $\{X_0, \dots, X_n\}$ .

Define  $\kappa = \kappa(1)$ . By (ii) and (iii),  $E_\varphi(S_\kappa - \kappa\mu) = 0$ . We preface the proof of (A.16) with the following preliminary lemmas, whose proofs are given in [5].

LEMMA A.1. *Let  $\gamma(x) = E_x(S_\kappa - \kappa\mu)$ . Then  $Z_n = (S_n - n\mu) + \gamma(X_n)$  is a martingale with respect to  $\mathcal{F}_n$ . Hence*

$$(A.18) \quad ES_\tau = \mu(E\tau) - E[\gamma(X_\tau)] + E[\gamma(X_0)].$$

LEMMA A.2. *Under A3–A5,*

$$|\gamma(x)| \leq \bar{b}^{-1}[V(x) + b + (V^* + b)V^*(\alpha^{-1} + 1)](K + 1 + |\mu|),$$

where  $\alpha$  satisfies (2.12),  $V^*$  is defined in (2.15) and  $K$  is defined in (2.16).

Let  $W_i = |\gamma(X_{\kappa(i)})| + \dots + |\gamma(X_{\kappa(i+1)-1})|$  for  $i \geq 1$ . Then by A3–A5, Lemma A.2 and (i)–(iii),  $W_1, W_2, \dots$  are i.i.d. with finite mean while by (2.15),  $W_0 := |\gamma(X_0)| + \dots + |\gamma(X_{\kappa(1)-1})|$  also has finite mean.

LEMMA A.3. *Let  $M_n = \max_{1 \leq k \leq n} W_k$ . Then for any stopping-time  $\tau$ ,  $E(M_\tau) = o(E\tau)$ .*

PROOF OF (A.16). By Lemma A.3,  $E|\gamma(X_\tau)| + E|\gamma(X_0)| = o(E\tau)$ , and (A.16) follows from (A.18).  $\square$

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