

ON GENERALIZED WIDE DIAMETER OF GRAPHS

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Abstract. The wide diameter of a graph is a natural generalization of diameter in a graph when we take account of the connectivity of the graph. In this paper, we define the generalized wide diameter of a graph and show that every k -regular k -connected graph on n vertices has generalized k -diameter at most $n/2$ and this upper bound cannot be improved when $n = 4k - 6 + i(2k - 4)$.

1. INTRODUCTION

The wide diameter of a graph is a natural generalization of diameter in a graph when we take account of the connectivity of the graph. The concept of wide diameter has been discussed and used in practical applications, especially in the distributed and parallel computer networks (see [3] for the survey of this special subject). The problem of wide diameter of graph G considers the wide distance between any two vertices of G . The wide distance between any pair of subsets of $V(G)$, is considered by the generalized wide diameter of G .

Let G be a graph without self-loops or multiple edges unless defined otherwise. The terminology and notations of graph theory follow Bondy and Murty [2]. For any pair of subsets of vertices $S \subset V(G)$ and $T \subseteq V(G) \setminus S$ with $|S| \cdot |T| \leq k(G)$. Let $\bar{P}_{|T|}(S, T)$ be a family of $|T|$ vertex disjoint paths between S and T , i.e.

$$\bar{P}_{|T|}(S, T) = \{P_1, P_2, \dots, P_{|T|}\}, \quad |P_1| \cdot |P_2| \cdot \dots \cdot |P_{|T|}|,$$

and for any vertex of S or T , it is at least one end of one path. The generalized $|T|$ -wide distance (or simply generalized $|T|$ -distance) between S and T , written as $d_{|T|}(S, T)$, is the minimum $|P_{|T|}|$ among all $\bar{P}_{|T|}(S, T)$ and the generalized $|T|$ -wide diameter (or simply generalized $|T|$ -diameter) of G , denoted by $\tilde{d}_{|T|}(G)$, is defined

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as the maximum generalized $|T|$ -wide distance $d_{|T|}(S, T)$ over all pairs $\{S, T\}$ of subsets of $V(G)$.

Clearly, when $|S| = |T| = 1$, the generalized wide diameter $\bar{d}_{|T|}(G) = d(G)$, which is the diameter of the graph G . Hence it is easy to see that $\bar{d}_{|T|}(G) \geq d(G)$.

Let G be a k -connected graph. In this paper, we mainly investigate the generalized wide diameter of G when $S = \{s\}$ and $|T| = k$ and denote it by $\bar{d}_k(G)$ (this is also called k -Rabin number, defined by Hsu [3] and discussed widely by Rabin [9], Liaw and Chang [5, 6, 7] and Liaw, Chang, Cao and Hsu [8], etc.). In section 2, we derive some properties of $\bar{d}_k(G)$ and show that every k -regular k -connected graph on n vertices has generalized k -diameter at most $n/2$ and this upper bound is tight when $n = 4k - 6 + i(2k - 4)$.

2. THE BASIC PROPERTIES OF $\bar{d}_k(G)$ AND THE GENERALIZED WIDE DIAMETERS OF k -REGULAR k -CONNECTED GRAPHS

We start with some simple observations concerning generalized k -diameter of k -connected graphs.

Proposition 2.1. *If G is k -connected then*

$$\bar{d}_k(G) \geq \bar{d}_{k-1}(G) \geq \cdots \bar{d}_1(G) = d(G).$$

Moreover, there exist graphs G for which $\bar{d}_k(G) = d(G)$.

Proof. The inequality is trivial. For the second part we take G to be a complete graph on $k + 1$ vertices. Then $\bar{d}_k(G) = d(G) = 1$. ■

Proposition 2.2. *If G is k -connected then $\bar{d}_k(G) \leq n - k$. Furthermore, for every k, n such that $1 \leq k \leq n - 1$, there exist k -connected graphs for which $\bar{d}_k(G) = n - k$.*

Proof. The equality is trivial. For second part, we construct graphs G with $\bar{d}_k(G) = n - k$. Since for a path on n vertices and a cycle of length n , we have $\bar{d}_1(P_n) = d(P_n) = n - 1$ and $\bar{d}_2(C_n) = n - 2$ and the result holds for $k = 1$ and $k = 2$. Thus suppose that $k \geq 3$ and define G as $C_{n-k+2} \cdot H$, where H is a graph on $k - 2$ vertices, i.e. G is a graph with vertex set $\{v_1, v_2, \dots, v_{n-k+2}, w_1, w_2, \dots, w_{k-2}\}$ such that subset $\{v_1, v_2, \dots, v_{n-k+2}\}$ spans C_{n-k+2} and subgraph induced by $\{w_1, w_2, \dots, w_{k-2}\}$ is isomorphic to H with v_i adjacent to w_j for all $i = 1, 2, \dots, n - k + 2$ and $j = 1, 2, \dots, k - 2$. One can easily see that if H is $k - (n - k + 2)$ connected then G is k -connected and the generalized k -distance between vertex v_1 and subset $T = \{v_2, v_3\} \cup V(H)$ is equal to $n - k$. Thus, in order to get G with $\bar{d}_k(G) = n - k$, it suffices to take H a graph with no edges when either $k = 3$ or $n = 2k - 2$, and

any l -connected graph with $k - 2$ vertices, where $l = \max\{2, 2k - n - 2\}$ in all other cases. ■

Let

$$\bar{f}(n, k) = \max\{\bar{d}_k(G) : G \text{ is } k\text{-regular } k\text{-connected graph with } n \text{ vertices}\}.$$

Clearly, $\bar{f}(n, 2) = n - 2$, and $\bar{f}(n, k) \leq n - k$. Moreover, Proposition 2.3 provides the value of $\bar{f}(n, k)$ for large k .

Proposition 2.3. *If either kn is even and $5 \leq n/2 + 2 \leq k \leq n - 1$ or $n = 2k - 2$, then $\bar{f}(n, k) = n - k$.*

Proof. We prove our result following Proposition 2.2. Since $5 \leq n/2 + 2 \leq k \leq n - 1$, so $2k - n - 2 \geq 2$. Take $l = 2k - n - 2$. Since kn is even, so is $l(k - 2)$ and we can take H a l -connected and l -regular graph on $k - 2$ vertices. Thus, it is easy to check that G is a k -regular, k -connected graph with $\bar{d}_k(G) = n - k$.

The result follows from $\bar{f}(n, k) \leq n - k$. ■

The following theorem states that even for small k , $\bar{f}(n, k)$ is bounded by $n/2$.

Theorem 2.4. *If kn is even and $k \geq 3$, then $\bar{f}(n, k) \leq n/2$.*

Proof. Let G be a k -regular k -connected graph on n vertices, vertex $s \in V(G)$ and subset $T \subseteq V(G) \setminus \{s\}$ with $|T| = k$, such that $\bar{d}_k(s, T) = \bar{d}_k(G)$ and

$$\bar{P}_k(s, T) = \{P_1, P_2, \dots, P_k\}, |P_1| + |P_2| + \dots + |P_k| = \bar{d}_k(G)$$

be a family of k vertex disjoint paths between s and T that for every other family

$$\bar{P}'_k(s, T) = \{P'_1, P'_2, \dots, P'_k\}, |P'_1| + |P'_2| + \dots + |P'_k| = \bar{d}_k(G),$$

we have $\sum_{i=1}^k |P'_i| \geq \sum_{i=1}^k |P_i|$. Moreover, let A denotes the subset of all vertices of G which belong to none of the paths P_1, P_2, \dots, P_k . Since G has n vertices, then

$$(*) \quad 1 + \sum_{i=1}^k |P_i| + |A| = n.$$

We estimate from below the number of edges in G . The number of edges which belong to paths from $\bar{P}_k(s, T)$ is equal to $\sum_{i=1}^k |P_i|$. Furthermore, no two vertices which belong to path P_k are joined by an edge which does not belong to path P_k (otherwise P_k would be replaced by a shorter path contradicting the choice of

$\bar{P}_k(s, T)$), so there exist precisely $(k-2)|P_k| + 1$ edges incident to vertices from path P_k which are not contained in it. We shall show that there exist at least $|A|$ edges which are neither contained in one of the paths from $P_k(s, T)$ nor incident to vertices of P_k .

Let H be a component of a subgraph induced in G by set A and let $|H|$ be the number of vertices of H . We shall prove that at least $|H|$ edges of G are incident to vertices from H and not incident to vertices from P_k . If H contains a cycle then it contains at least $|H|$ edges. It suffices to consider the case that H is a tree.

Case 1: $k = 3$.

Note that H is adjacent to at most $|H| + 2$ vertices of path $P_k = v_0v_1 \cdots v_k$, say $v_{l+1}, v_{l+2}, \dots, v_{l+|H|+2}$, where $v_k \in T$. Indeed, otherwise one could find vertices v_i and v_j with $j-i \geq |H| + 2$, both adjacent to H , and replace P_k by a shorter path using vertices of H instead of $v_{i+1}v_{i+2} \cdots v_{j-1}$. Furthermore, at least one of the vertices $v_{l+2}, v_{l+3}, \dots, v_{l+|H|+1}$ must have a neighbor outside H since otherwise graph G could be disconnected by deleting vertices v_{l+1} and $v_{l+|H|+2}$. Thus, P_k sends to H at most $|H| + 2 - 1 = |H| + 1$ edges, so at least

$$3|H| - (|H| - 1) - (|H| + 1) = |H|,$$

edges incident to H are not incident to vertices from P_k .

Case 2: $k = 4$ and H is a path.

Similarly as in the previous case, H must be adjacent to at most $|H| + 2$ vertices of path $P_k = v_0v_1 \cdots v_k$, say $v_{l+1}v_{l+2} \cdots v_{l+|H|+2}$, where at least two of the vertices $v_{l+2}, v_{l+3}, \dots, v_{l+|H|+1}$ have neighbors outside H . Furthermore, it is not hard to see that both vertices v_{l+1} and $v_{l+|H|+2}$ can be adjacent to only one vertex of the path H , namely to one of its ends. Hence, the number of edges between P_k and H is bounded above by $2 + 2|H| - 2$, so at least

$$4|H| - 2|H| - (|H| - 1) = |H| + 1,$$

edges incident to H are not incident to vertices from P_k .

Case 3: $k = 4$ and H is not a path.

Since the diameter of H is less than $|H| - 1$, it is adjacent only to at most $|H| + 1$ vertices of path P_k , from which at least two have neighbors outside H . Thus, as in the previous two cases, the number of edges incident to H but not to P_k is bounded below by

$$4|H| - 2(|H| + 1) + 2 - (|H| - 1) = |H| + 1.$$

Case 4: $k \geq 5$.

Note that no vertex from H is adjacent to more than three vertices from P_k since otherwise path P_k could be replaced by a shorter one. Hence, G contains at least

$$k|H| - 3|H| - (|H| - 1) \geq |H| + 1.$$

edges incident to vertices from H not incident to vertices from P_k .

Thus we have shown that there are at least $|A|$ edges in G which are neither contained in some k paths nor incident to vertices from P_k , so

$$(**) \quad \sum_{i=1}^k |P_i| + (k-2)|P_k| + 1 + |A| \cdot nk/2.$$

Now subtracting (*) from (**) and dividing by $k-2$ gives $n/2$ as the upper bound for $|P_k|$. ■

Remark: Note that from the proof it follows that, when $k > 5$, $\bar{d}_k(s, T) = n/2$ only if all vertices of G lies on some path from $\bar{P}_k(s, T)$ and all edges of G either belong to a path from $\bar{P}_k(s, T)$ or are incident to some vertices from P_k .

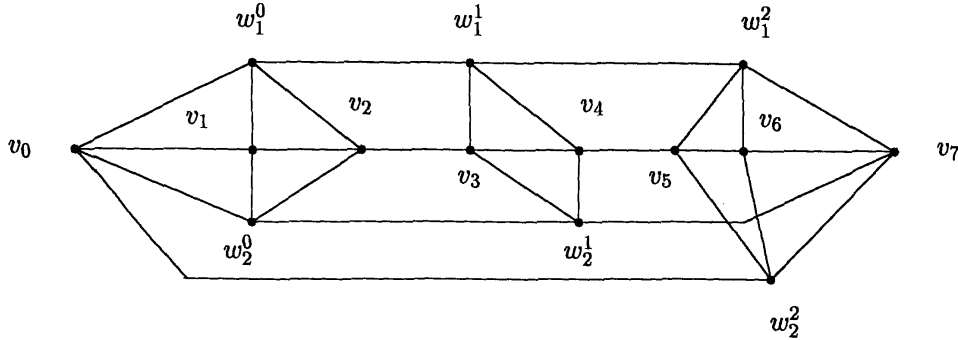
The above bound for $\bar{f}(n, k)$ cannot be improved in general case. In fact, the equality $\bar{f}(n, k) = \lfloor n/2 \rfloor$ holds for infinitely many pairs k and n .

Theorem 2.5. *If $n = 2k - 3 + i(k - 2)$, where $i = 0, 1, \dots$ and $3 \leq k \leq n$, then $\bar{f}(2n, k) = n$. In particular, $\bar{f}(2n, 3) = n$ for $n \geq 3$.*

Proof. We shall construct a k -regular k -connected graph $G(2n, k)$ with $2n = 4k - 6 + i(2k - 4)$ vertices for which $\bar{d}_k(G(2n, k))$ contains vertices v_j , $j = 0, 1, \dots, n$ and w_l^m , where $l = 1, 2, \dots, k - 2$ and $m = 0, 1, \dots, i, i + 1$. The set of edges of $G(2n, k)$ consists of the following pairs of vertices:

- (a) $\{v_j, v_{j+1}\}$ for $j = 0, 1, \dots, n - 1$,
- (b) $\{v_0, w_l^0\}$ for $l = 1, 2, \dots, k - 2$, and $\{v_0, w_{k-2}^{i+1}\}$,
- (c) $\{v_n, w_l^{m+1}\}$ for $l = 1, 2, \dots, k - 2$ and $\{v_n, w_{k-2}^i\}$,
- (d) $\{w_l^m, w_l^{m+1}\}$ for $l = 1, 2, \dots, k - 3$, $m = 0, 1, \dots, i$,
- (e) $\{w_{k-2}^m, w_{k-2}^{m+1}\}$ for $m = 0, 1, \dots, i - 1$,
- (f) $\{w_l^m, v_{m(k-2)+s}\}$ for $l = 1, 2, \dots, k - 2$, $m = 0, 1, \dots, i, i + 1$ and $s = 1, 2, \dots, k - 2$. Graph $G(14, 4)$ is given in Figure 1.

Let $T = \{w_l^{i+1} | l = 1, 2, \dots, k - 2\} \cup \{w_{k-2}^i, v_n\}$. One can easily check that $G(2n, k)$ is k -regular k -connected and the only family of k vertex disjoint paths between vertex v_0 and subset T consists of paths $v_0 w_{k-2}^{i+1}$, $v_0 v_1 \dots v_n$, $v_0 w_{k-2}^0$ and $k - 3$ paths $v_0 w_l^0 w_l^1 \dots w_l^{i+1}$, $l = 1, 2, \dots, k - 3$.

FIG. 1. $G(14, 4)$.

One might expect that equality $\bar{f}(n, k) = \lfloor n/2 \rfloor$ holds for every n and k such that nk is even and $3 \leq k \leq \lfloor n/2 \rfloor$. The next result shows that it is not true.

Theorem 2.6. *If $n \geq 8$ and $n/2 + 2 \leq k \leq n - 1$ then $\bar{f}(2n, k) < n$.*

Proof. Due to the observation after the proof of Theorem 2.5, the equality $\bar{f}(2n, k) = n$ can hold only if for some vertex s and subset T , a family of paths $\bar{P}_k(s, T)$ contains all vertices of the graph and each edge of the graph which does not belong to paths from $\bar{P}_k(s, T)$ is incident to P_k . Suppose that for some vertex s and subset T with $|T| = k$ of a k -regular k -connected graph on $2n$ vertices we have $\bar{d}_k(s, T) = n$. Then, G contains $n - 1$ vertices outside path $P_k = sv_1v_2 \dots v_{n-1}v_n$, where $v_n \in T$. So, since $k \leq n - 1$, $P_{k-1} = sw_1w_2 \dots w_l$ for some $l \geq 2$. Vertex w_1 has $k - 2 > (n - 1)/2$ neighbors lying on P_k , and $k - 1 > (n - 1)/2$ neighbors lying on P_k for vertex w_l , so w_1 is adjacent to some vertex v_i with $i > (n - 1)/2$ and w_l is adjacent to vertex v_j with $j < (n - 1)/2$. Thus, paths P_{k-1} and P_k could be replaced by $P' = sv_1v_2 \dots v_jw_l$ and $P'' = sw_1v_iv_{i+1} \dots v_{n-1}v_n$, of lengths $j + 2 < n$ and $n - i + 2 < n$, and $\bar{d}_k(s, T) < n$. This contradicts to $\bar{d}_k(s, T) = n$. ■

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