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STRONG CONVERGENCE THEOREM BY AN EXTRAGRADIENT METHOD FOR FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. The iterative process is based on so-called extragradient method. We obtain a strong convergence theorem for two sequences generated by this process.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let $P_C : H \to C$ be the metric projection of H onto C.

Definition 1.1. Let $A : C \to H$ be a mapping of C into H.

(i) A is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0 \quad \forall u, v \in C;$$

(ii) A is called α-inverse-strongly-monotone (see [1], [3]) if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2 \quad \forall u, v \in C.$$

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It is easy to see that an α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. We consider the following variational inequality problem (VI(A, C)): find a $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \quad \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by Ω. A mapping $S : C \to C$ is called nonexpansive (see [7]) if

$$\|Su - Sv\| \leq \|u - v\| \quad \forall u, v \in C.$$

We denote by $F(S)$ the set of fixed points of S.

For finding an element of $F(S) \cap \Omega$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse-strongly-monotone, Takahashi and Toyoda [8] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \quad \forall n \geq 0, \tag{1.1}$$

where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap \Omega$ is nonempty, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to some $z \in F(S) \cap \Omega$. On the other hand for solving the variational inequality problem in a finite-dimensional Euclidean space R^n under the assumption that a set $C \subset \mathrm{R}^n$ is nonempty, closed and convex, a mapping $A : C \to \mathrm{R}^n$ is monotone and k-Lipschitz continuous and Ω is nonempty, Korpelevich [2] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in \mathrm{R}^n, \\ \bar{x}_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda A\bar{x}_n) \ \forall n \geq 0 \end{cases} \tag{1.2}$$

where $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{\bar{x}_n\}$ generated by (1.2) converge to the same point $z \in \Omega$.

Further motivated by the idea of Korpelevich's extragradient method, Nadezhkina and Takahashi [10] introduced an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They proved the following weak convergence theorem for two sequences generated by this process.

Theorem 1.1 [10, Theorem 3.1]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous* *mapping and* $S : C \to C$ *be a nonexpansive mapping such that* $F(S) \cap \Omega \neq \emptyset$. *Let* $\{x_n\},\ \{y_n\}$ *be the sequences generated by*

$$
\begin{cases}
x_0 = x \in H, \\
y_n = P_C(x_n - \lambda_n A x_n), \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n) \ \forall n \geq 0
\end{cases}
\tag{1.3}
$$

where $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$ *and* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$. *Then the sequences* $\{x_n\},\ \{y_n\}$ *converge weakly to the same point* $z \in F(S) \cap \Omega$ *where*

$$z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n.$$

In this paper inspired by Nadezhkina and Takahashi's iterative process (1.3), we introduce the following iterative process

$$
\begin{cases}
x_0 = x \in H, \\
y_n = P_C(x_n - \lambda_n A x_n), \\
x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n) \ \forall n \geq 0
\end{cases}
\tag{$*$}
$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ for some $\delta \in (0, 1)$;

(b) $\{\alpha_n\} \subset (0, 1),\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0.$

It is shown that the sequences $\{x_n\},\ \{y_n\}$ generated by $(*)$ converge strongly to the same point $P_{F(S)\cap\Omega}(x_0)$ provided $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|\ \forall y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties (see [7] for more details): $P_C x \in C$ and for all $x \in H,\ y \in C$,

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.1}$$

$$\|x-y\|^2 \geq \|x-P_Cx\|^2 + \|y-P_Cx\|^2. \tag{2.2}$$

Let $A : C \to H$ be a mapping. It is easy to see from (2.2) that the following implications hold:

$$u \in \Omega \;\Leftrightarrow\; u = P_C(u-\lambda Au)\ \forall \lambda > 0. \tag{2.3}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H,\ f \in Tx$ and $g \in Ty$, we have $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H,\ \langle x-y, f-g\rangle \geq 0$ for all $(y,g) \in G(T)$, then $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and N_Cv be the normal cone to C at $v \in C$, i.e., $N_Cv = \{w \in H : \langle v-u, w\rangle \geq 0,\ \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_Cv, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see [5].

In order to prove the main result in Section 3, we shall use the following lemmas in the sequel.

Lemma 2.1 [6, Lemma 2.1]. *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions: $s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n,\ \forall\ n \geq 0$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that*

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,*
$\prod_{n=0}^{\infty}(1-\alpha_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1-\alpha_k) = 0;$

(ii) $\limsup_{n\to\infty}\beta_n \leq 0$, *or*

(ii') $\sum_n \alpha_n\beta_n$ *is convergent.*

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2 [4]. *Demiclosedness Principle. Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If $F(S) \neq \emptyset$, then $I-S$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-S)x_n\}$ strongly converges to some y, it follows that $(I-S)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.3. *In a real Hilbert space H, there holds the inequality:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \quad \forall x, y \in H.$$

3. Strong Convergence Theorem

Now we can state and prove the main result in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\},\ \{y_n\}$ be the sequences generated by*

$$(*) \qquad \begin{cases} x_0 = x \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n) \ \forall n \geq 0 \end{cases}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$;

(b) $\{\alpha_n\} \subset (0,1),\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0.$

Then the sequences $\{x_n\},\ \{y_n\}$ converge strongly to the same point $P_{F(S)\cap\Omega}(x_0)$ provided

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded and so is $\{t_n\}$ where $t_n = P_C(x_n - \lambda_n A y_n)\ \forall n \geq 0$. Indeed let $u \in F(S) \cap \Omega$. From (2.2) it follows that

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 \\ &\quad + 2\lambda_n(\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle) \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle. \end{aligned}$$

Further from (2.1) we obtain

$$\begin{aligned} &\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\ &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \end{aligned}$$

$$\leq \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n \rangle$$
$$\leq \lambda_n k \|x_n - y_n\| \|t_n - y_n\|.$$

Hence we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\ &\leq \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\ &\leq \|x_n - u\|^2. \end{aligned} \tag{3.1}$$

Now by induction, we have

$$\|x_n - u\| \leq \|x_0 - u\| \quad \forall n \geq 0. \tag{3.2}$$

Indeed when $n = 0$, it follows from (3.1) that

$$\begin{aligned} \|x_1 - u\| &= \|\alpha_0 x_0 + (1 - \alpha_0) S t_0 - u\| \\ &= \|\alpha_0 (x_0 - u) + (1 - \alpha_0)(S t_0 - u)\| \\ &\leq \alpha_0 \|x_0 - u\| + (1 - \alpha_0)\|t_0 - u\| \\ &\leq \alpha_0 \|x_0 - u\| + (1 - \alpha_0)\|x_0 - u\| \\ &= \|x_0 - u\| \end{aligned}$$

which implies that (3.2) holds for $n = 0$. Suppose that (3.2) holds for $n \geq 1$. Then we have $\|x_n - u\| \leq \|x_0 - u\|$. This together with (3.1) implies that

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n x_0 + (1 - \alpha_n) S t_n - u\| \\ &= \|\alpha_n (x_0 - u) + (1 - \alpha_n)(S t_n - u)\| \\ &\leq \alpha_n \|x_0 - u\| + (1 - \alpha_n)\|S t_n - u\| \\ &\leq \alpha_n \|x_0 - u\| + (1 - \alpha_n)\|t_n - u\| \\ &\leq \alpha_n \|x_0 - u\| + (1 - \alpha_n)\|x_n - u\| \\ &\leq \alpha_n \|x_0 - u\| + (1 - \alpha_n)\|x_0 - u\| \\ &= \|x_0 - u\|. \end{aligned}$$

This shows that (3.2) holds for $n+1$. Therefore (3.2) holds for all $n \geq 0$; i.e., $\{x_n\}$ is bounded. So it follows from (3.1) that $\|t_n - u\| \leq \|x_0 - u\|$ $\forall n \geq 0$, i.e., $\{t_n\}$ is also bounded.

Step 2. $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Indeed from $(*)$ and (3.1) we get

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\alpha_n x_0 + (1-\alpha_n)St_n - u\|^2 \\
&= \|\alpha_n(x_0 - u) + (1-\alpha_n)(St_n - u)\|^2 \\
&\le \alpha_n\|x_0 - u\|^2 + (1-\alpha_n)\|St_n - u\|^2 \\
&\le \alpha_n\|x_0 - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\
&\le \alpha_n\|x_0 - u\|^2 + (1-\alpha_n)(\|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2) \\
&\le \alpha_n\|x_0 - u\|^2 + \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta\|x_n - y_n\|^2 &\le (1-\lambda_n^2 k^2)\|x_n - y_n\|^2 \\
&\le \alpha_n\|x_0 - u\|^2 + \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\le \alpha_n\|x_0 - u\|^2 + (\|x_n - u\| - \|x_{n+1} - u\|)(\|x_n - u\| + \|x_{n+1} - u\|).
\end{aligned} \tag{3.3}$$

Since $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, we have

$$|\|x_n - u\| - \|x_{n+1} - u\|| \le \|x_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty.$$

Thus combining with (3.3), the boundedness of $\{x_n\}$ and $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Step 3. $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Indeed, observe that

$$\begin{aligned}
\|y_n - t_n\| &= \|P_C(x_n - \lambda_n Ax_n) - P_C(x_n - \lambda_n Ay_n)\| \\
&\le \lambda_n\|Ax_n - Ay_n\| \\
&\le \lambda_n k\|x_n - y_n\| \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\|Sy_n - x_{n+1}\| &\le \|Sy_n - St_n\| + \|St_n - x_{n+1}\| \\
&\le \|y_n - t_n\| + \alpha_n\|St_n - x_0\| \\
&\le \|y_n - t_n\| + \alpha_n[\|St_n - u\| + \|x_0 - u\|] \\
&\le \|y_n - t_n\| + \alpha_n[\|t_n - u\| + \|x_0 - u\|] \\
&\le \|y_n - t_n\| + 2\alpha_n\|x_0 - u\| \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.5}$$

and

$$\|Sx_n - St_n\| \le \|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Consequently, from (3.4)-(3.6), we can infer that

$$\begin{aligned}\|Sx_n - x_n\| &= \|Sx_n - St_n + St_n - Sy_n + Sy_n - x_{n+1} + x_{n+1} - x_n\| \\ &\le \|Sx_n - St_n\| + \|t_n - y_n\| + \|Sy_n - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Step 4. $\limsup_{n\to\infty}\langle x_0 - u^*, x_n - u^*\rangle \le 0$ where $u^* = P_{F(S)\cap\Omega}(x_0)$. Indeed we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ so that

$$\limsup_{n\to\infty}\langle x_0 - u^*, x_n - u^*\rangle = \lim_{i\to\infty}\langle x_0 - u^*, x_{n_i} - u^*\rangle. \tag{3.7}$$

Without loss of generality, we may further assume that $\{x_{n_i}\}$ converges weakly to $\tilde{u}$ for some $\tilde{u} \in H$. Hence (3.7) reduces to

$$\limsup_{n\to\infty}\langle x_0 - u^*, x_n - u^*\rangle = \langle x_0 - u^*, \tilde{u} - u^*\rangle. \tag{3.8}$$

In order to prove $\langle x_0 - u^*, \tilde{u} - u^*\rangle \le 0$, it suffices to show that $\tilde{u} \in F(S)\cap\Omega$. Note that by Lemma 2.2 and Step 3, we have $\tilde{u} \in F(S)$. Now we show $\tilde{u} \in \Omega$. Since from (3.4) and (3.6) we obtain $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we have $t_{n_i} \rightharpoonup \tilde{u}$ and $y_{n_i} \rightharpoonup \tilde{u}$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see [5]. Let$(v, w) \in G(T)$. Then we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. Therefore we have $\langle v - u, w - Av\rangle \ge 0$ for all $u \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n Ay_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - v\rangle \ge 0$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n\rangle \ge 0.$$

Therefore according to the fact that $w - Av \in N_C v$ and $t_n \in C$, we have

$$\begin{aligned}\langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle \\ &\quad -\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Thus we get $\langle v - \tilde{u}, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $\tilde{u} \in T^{-1}0$ and hence $\tilde{u} \in \Omega$. This shows that $\tilde{u} \in F(S) \cap \Omega$. Therefore by the property of the metric projection, we derive $\langle x_0 - u^*, \tilde{u} - u^*\rangle \leq 0$.

Step 5. $x_n \to u^*$ and $y_n \to u^*$ where $u^* = P_{F(S)\cap\Omega}(x_0)$. Indeed combining Lemma 2.3 with (3.1), we get

(3.9)
$$\begin{aligned}\|x_{n+1} - u^*\|^2 &= \|(1-\alpha_n)(St_n - u^*) + \alpha_n(x_0 - u^*)\|^2 \\ &\leq (1-\alpha_n)^2\|St_n - u^*\|^2 + 2\alpha_n(1-\alpha_n)\langle x_0 - u^*, x_{n+1} - u^*\rangle \\ &\leq (1-\alpha_n)\|t_n - u^*\|^2 + 2\alpha_n\langle x_0 - u^*, x_{n+1} - u^*\rangle \\ &\leq (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n\beta_n,\end{aligned}$$

where $\beta_n = 2\langle x_0 - u^*, x_{n+1} - u^*\rangle$. Thus an application of Lemma 2.1 combined with Step 4 yields that $\|x_n - u^*\| \to 0$ as $n \to \infty$. Since $x_n - y_n \to 0$, we have $y_n \to u^*$. ■

4. Applications

As in Nadezhkina and Takahashi [10], we give two applications of Theorem 3.1.

Theorem 4.1. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone, k-Lipschitz continuous mapping and $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n Ax_n, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)S(x_n - \lambda_n Ay_n) \ \forall n \geq 0 \end{cases}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$;

(b) $\{\alpha_n\} \subset (0,1),\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0.$

Then the sequence $\{x_n\}$ *converges strongly to* $P_{F(S)\cap A^{-1}0}(x_0)$ *provided*

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Proof. We have $A^{-1}0 = \Omega$ and $P_H = I$. By Theorem 3.1, we obtain the desired result. ∎

Remark 4.1. See Yamada [9] and Xu and Kim [6] for the case when $A : H \to H$ is a strongly monotone and Lipschitz continuous mapping on a real Hilbert space H and $S : H \to H$ is a nonexpansive mapping.

Theorem 4.2. *Let* H *be a real Hilbert space. Let* $A : H \to H$ *be a monotone,* k*-Lipschitz continuous mapping and* $B : H \to 2^H$ *be a maximal monotone mapping such that* $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. *Let* J_r^B *be the resolvent of* B *for each* $r > 0$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) J_r^B (x_n - \lambda_n A y_n)\ \forall n \geq 0 \end{cases}$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the conditions:*

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$;

(b) $\{\alpha_n\} \subset (0,1),\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0.$

Then the sequence $\{x_n\}$ *converges strongly to* $P_{A^{-1}0\cap B^{-1}0}(x_0)$ *provided*

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Proof. We have $A^{-1}0 = \Omega$ and $F(J_r^B) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. ∎

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