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ON THE SMOOTHNESS OF THE SOLUTION FOR THE INITIAL-NEUMANN PROBLEM FOR THE HYPERBOLIC SYSTEMS IN LIPSCHITZ CYLINDERS

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Abstract. In this paper, we study the second initial boundary value problem for strongly hyperbolic systems in cylinders with Lipschitz base. We investigate the unique solvability of the problem and the smoothness with respect to time of the generanized solution.

1. Introduction

We are concerned with initial boundary value problems for non-stationary systems in cylinders with non-smooth base. These problems with Dirichlet boundary condition in the cylinders with base containing conical points have been investigated in [5,7,9], where some important results on the unique existence, smoothess and asymptotical representation of the solution for the problems in Sobolev spaces were given. The initial-Neumann problem for hyperbolic equations and systems in the cylinders with base containing conical points was described in [2,8]. In [2] coefficients of the systems are indepedent of the time variable and in [8] the problem has been dealt with for the second order equations. The initial boundary value problems for parabolic equations in the cylinders with base containing conical points were established in [6,10]. Such problems for Schrödinger systems have been studied in Sobolev spaces with weights [3,4]. In the present paper, we consider the second initial problem for the hyperbolic systems with coefficients depending on both spatial and time variables in the Lipschitz cylinders. We study the existence, uniqueness and smoothness with respect to time of the generanized solution for these problems.

The paper is organized as follows. In the second section we define the initial-Neumann problem for the hyperbolic systems in Lipschitz cylinders. In third section we study the solvability of the problem. In the fourth setion we consider regularity with respect to time of the generanized solution. The last section is intended to a problem of mathematical physics.

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2. Formulation of the Problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with the Lischitz boundary $\partial\Omega$. Denote $\Omega_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$ for each $T: \ 0 < T \leq \infty$.

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the symbol $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}$ denotes the generalized derivative of order α with respect to $x = (x_1, ..., x_n)$; $\partial^k/\partial t^k$ is the generalized derivative of order k with respect to t. Let $u = (u_1, ..., u_s)$ be a complex-valued vector function defined on Ω_T. We use notation: $D^\alpha u = (D^\alpha u_1, ..., D^\alpha u_s)$; $u_{tj} = \partial^k u/\partial t^k = (\partial^j u_1/\partial t^j, .., \partial^j u_s/\partial t^j)$.

Throughout the paper we need the following functional spaces (see [1]):

$H^m(\Omega)$ is the space consisting of all vector functions $u(x)$ defined on Ω such that

$$||u||_{H^m(\Omega)} = \left(\sum_{0 \leq |p| \leq m} \int_\Omega |D^p u|^2 dx \right)^{\frac{1}{2}} < +\infty.$$

$H^{m,k}(\Omega_T)$ is the space consisting of all vector functions $u(x, t)$ defined on Ω_T such that

$$||u||^2_{H^{m,k}(\Omega_T)} = \int_{\Omega_T} \left(\sum_{0 \leq |p| \leq m} |D^p u|^2 + \sum_{j=1}^{k} |u_{tj}|^2 \right) dxdt < +\infty.$$

Let γ be a positive number. We denote the following spaces:

$H^{m,k}(\gamma, \Omega_T)$ is the space consisting of all vector functions $u(x, t)$ defined on Ω_T such that

$$||u||^2_{H^{m,k}(\gamma,\Omega_T)} = \int_{\Omega_T} \left(\sum_{0 \leq |p| \leq m} |D^p u|^2 + \sum_{j=1}^{k} |u_{tj}|^2 \right) e^{-\gamma t} dxdt < +\infty.$$

$L_2(\gamma, \Omega_T)$ is the space of vector functions $u(x, t)$ defined on Ω_T with the normal

$$\|u\|_{L_2(\gamma,\Omega_T)} = \Big(\int_{\Omega_T} |u|^2 e^{-\gamma t} dxdt.$$

Now we introduce the differential operator

$$L(x, t, D) = \sum_{|p|,|q| \leq m} D^p a_{pq} D^q,$$

where $a_{pq} = a_{pq}(x, t)$ are the $s \times s$ matrices with the bounded complex-valued components $a^{ij}_{pq}(x, t)$ in $\overline{\Omega}_T$. Assume that $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$ $(|p|, |q| \leq m)$ and there exists a positive constant α_0 such that

$$\sum_{|p|=|q|=m} a_{pq} \eta_q \overline{\eta_p} \geq \alpha_0 \sum_{|p|=m} |\eta_p|^2 \tag{2.1}$$

for all vectors with complex components $\eta_p \in \mathbb{C}^s$ and $(x,t) \in \overline{\Omega_T}$.

Denote

$$B(u,v)(t) = \sum_{|p|,|q|\le m} (-1)^{|p|} \int_\Omega a_{pq}(.,t) D^q u(.,t) \overline{D^p v}(.,t) dx.$$

Lemma 2.1. *Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$ and assume that the condition (2.1) is fulfilled. Then there exist two constants $\mu_0 > 0$ and $\lambda_0 \ge 0$ such that the following inequality*

$$(-1)^m B(u,u)(t) \ge \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L_2(\Omega)},$$

holds for all function $u \in H^{m,1}(\gamma, \Omega_T),\ \gamma > 0$.

Proof. We begin by condition (2.1) with $\eta_p = u \in H^{m,1}(\gamma, \Omega_T)$. We have

$$\sum_{|p|=|q|=m} \int_\Omega a_{pq}(.,t) D^q u(.,t) \overline{D^p u}(.,t) dx \ge \alpha_0 \sum_{|p|=m} \|D^p u(.,t)\|^2_{L_2(\Omega)}.$$

From this and Cauchy inequality its follows that

$$\alpha_0 \sum_{|p|=m} \|D^p u(.,t)\|^2_{L_2(\Omega)}$$
$$\le (-1)^m B(u,u)(t) + \epsilon \sum_{|p|=m} \|D^p u(.,t)\|^2_{L_2(\Omega)} + C(\epsilon) \|u(.,t)\|^2_{H^{m-1}(\Omega},$$

where $0 < \epsilon < \alpha_0$, and $C(\epsilon) = \text{const} > 0$ depend on ϵ. Therefore, we obtain

$$\|u(.,t)\|^2_{H^m(\Omega)} \le C_1 \Big((-1)^m B(u,u)(t) + \|u(.,t)\|^2_{H^{m-1}(\Omega)} \Big),\ C_1 = \text{const} > 0.$$

Applying interpolation inequality for the domain Ω with Lipschitz boundary $\partial\Omega$ (see [11]), we get

$$\|u(.,t)\|^2_{H^m(\Omega)} \le C_2 \Big((-1)^m B(u,u)(t) + \|u(.,t)\|^2_{L_2(\Omega)} \Big),\ C_2 = \text{const} > 0.$$

Lemma is completely proved.

Denote by $N_j(x, t.D),\ j = 0, 1, ..., m-1$, the boundary operators:

$$N_j(x, t.D) = \sum_{|\alpha| \le 2m-1-j} b_{j\alpha}(x,t) D^\alpha,$$

where $b_{j\alpha}(x,t),\ j = 0, 1, ...m$, are the $s \times s$ matrices with the bounded complex-valued components on S_T. Moreover, suppose $N_j(x,t,D),\ j = 0, 1, ..., m-1$, satisfy the formula:

$$\int\limits_{\Omega} L(.,t,D)u\overline{v}dx = B(u,v)(t) + \sum_{j=0}^{m-1}\int\limits_{\partial\Omega} N_j(.,t,D)u\frac{\partial^j\overline{v}}{\partial\nu^j}ds$$

with all $u, v \in C^\infty(\overline{\Omega})$ and almost all $t \in (0,T)$, where ν is the unit exterior normal to S_T.

We consider the following problem in the cylinder Ω_T:

$$(-1)^{m-1}L(x,t,D)u - u_{tt} = f(x,t), \tag{2.2}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \tag{2.3}$$

$$N_j(x,t,D)u|_{S_T} = 0,\ j = 0,...,m-1. \tag{2.4}$$

The function $u(x,t)$ is called the generalized solution in the space $H^{m,1}(\gamma,\Omega_T)$ of problem (2.2)-(2.4) iff $u(x,t) \in H^{m,1}(\gamma,\Omega_T)$, $u(x,0) = 0$ and for each $\tau,\ 0 < \tau < T$, the equality

$$(-1)^{m-1}\int_0^\tau B(u,\eta)(t)dt + \int\limits_{\Omega_\tau} u_t\overline{\eta}_t dxdt = \int\limits_{\Omega_\tau} f\overline{\eta}dxdt \tag{2.5}$$

holds for all $\eta(x,t) \in H^{m,1}(\gamma,\Omega_T)$ satisfying $\eta(x,t) = 0$ with $t \in [\tau,T)$.

3. The Unique Solvability

In this section we study the uniqueness and existence theorems of the generalized solution of problem (2.2)-(2.4) in the space $H^{m,1}(\gamma,Q_T)$.

Donote $a_{pqt^k} = \partial^k a_{pq}/\partial t^k$ and putting

$$B_{t^k}(u,v)(t) = \sum_{|p|,|q|\le m}(-1)^{|p|}\int\limits_{\Omega} a_{pqt^k}(.,t)D^q u(.,t)\overline{D^p v}(.,t)dx,$$

we have $B(u,v)(t) = B_{t^0}(u,v)(t)$. Using integrating by parts and hypothesis $a_{pq} = (-1)^{|p|+|q|}a^*_{qp}$ $(|p|,|q|) \le m)$ in $\overline{\Omega}_T$ we have the following formula.

$$2\mathrm{Re}\int_0^\tau B_{t^k}(v,v_t)(t)dt = B_{t^k}(v,v)(\tau) - B_{t^k}(v,v)(0) - \int_0^\tau B_{t^{k+1}}(v,v)(t)dt \tag{3.1}$$

for all $\tau \in (0,T)$.

Donote by m^* the number of multiindexes which have order not excceding m. Using Cauchy inequality we have the following lemma.

Lemma 3.1. *If* $|a^{ij}_{pqt^k}| \le \mu$ *with* $(x,t) \in \overline{\Omega}_T,\ 1 \le i,j \le s, \le |p|,|q| \le m$, $\mu = \mathrm{const} > 0$*, then*

$$|B_{t^k}(v,v)(t)| \le m^*\mu\|v(.,t)\|^2_{H^m(\Omega)}.$$

Theorem 3.2. *Suppose that coefficients of the operator L(x,t,D) satisfy condition (2.1) and* $|a^{ij}_{pqt^k}(x,t)| \leq \mu$ *with* $(x,t) \in \overline{\Omega}_T,\ 1 \leq i,j \leq s,\ \leq |p|,|q| \leq m,\ k \leq 1$, $\mu = \text{const} > 0$. *Then for every* $\gamma > 0$ *problem (2.2)-(2.4) has at most generalized solution in* $H^{m,1}(\gamma, \Omega_\tau)$.

Proof. Assume problem (2.2)-(2.4) has two generalized $u_1, u_2 \in H^{m,1}(\gamma, \Omega_T)$ for a $\gamma > 0$. Put $u = u_1 - u_2$ and define the function $\eta(x,t) = \int_b^t u(x,\tau)d\tau$ with $0 \leq t \leq b$ and $\eta(x,t) = 0$ with $t > b$. Substituting $u = \eta_t$ into (2.5) and adding obtained equality to its complex conjugate, we obtain

$$(-1)^{m-1}2\text{Re}\int_0^b B(\eta_t,\eta)(t)dt + 2\text{Re}\int_{\Omega_b} \eta_{tt}\overline{\eta_t}dxdt = 0.$$

From formula (3.1) with $v = \eta,\ k = 1$ and integrating by parts the second term of this equality it follows that

$$(-1)^m B(\eta,\eta)(0) + (-1)^m \int_0^b B_t(\eta,\eta)(t)dt + \|\eta_t(.,b)\|^2_{L_2(\Omega)} = 0$$

From this equality and Lemmas 2.1, 3.1 we have

$$\begin{aligned} &\|\eta_t(.,b)\|^2_{L_2(\Omega)} + \mu_0\|\eta(.,0)\|^2_{H^m(\Omega)} \\ &\leq m^*\mu\int_0^b \|\eta(.,t)\|^2_{H^m(\Omega)}dt + \lambda_0\|\eta(.,0)\|^2_{L_2(\Omega)}. \end{aligned} \tag{3.2}$$

Putting $v_p(x,t) = \int_t^0 D^p u(x,s)ds$, we have

$$\begin{aligned} D^p\eta(x,t) &= \int_b^t D^p u(x,s)ds = v_p(x,b) - v(x,t), \\ \|\eta(.,0)\|^2_{H^m(\Omega)} &= \sum_{|p|=0}^m \|v_p(.,b)\|^2_{L_2(\Omega)} \end{aligned} \tag{3.3}$$

and

$$\int_0^b \|\eta(.,t)\|^2_{H^m(\Omega)}dt \leq b\sum_{|p|=0}^m \|v_p(.,b)\|^2_{L_2(\Omega)} + \int_0^b \sum_{|p|=0}^m \|v_p(.,t)\|^2_{L_2(\Omega)}dt. \tag{3.4}$$

On the other hand,

$$\begin{aligned} &\lambda_0\|\eta(.,0)\|^2_{L_2(\Omega)} = 2\lambda_0\text{Re}\int_b^0\int_\Omega \eta_t(x,t)\overline{\eta(x,t)}dxdt \\ &\leq \int_0^b \|\eta_t(.,t)\|^2_{L_2(\Omega)}dt + 2\lambda_0^2 b\|v_0(.,b)\|^2_{L_2(\Omega)} + 2\lambda_0^2\int_0^b \|v_0(.,t)\|^2_{L_2(\Omega)}dt. \end{aligned} \tag{3.5}$$

By Cauchy inequality from (3.2),(3.3),(3.4) and (3.5) we obtain

$$\|\eta_t(.,b)\|^2_{L_2(\Omega)} + (\mu_0 - bC_1)\sum_{|p|=0}^{m}\|v_p(.,b)\|^2_{L_2(\Omega)}$$
$$\le C_2\int_0^b \Big(\|\eta_t(x,t)\|^2_{L_2(\Omega)} + \sum_{|p|=0}^{m}\|v_p(x,t)\|^2_{L_2(\Omega)}\Big)dt,$$

where $C_{1,2} = \text{const} > 0$. Putting $J(b) = \|\eta_t(x,t)\|^2_{L_2(\Omega)} + \sum_{|p|=0}^{m}\|v_p(x,t)\|^2_{L_2(\Omega)}$, we obtain $J(b) \le C\int_0^b J(t)dt$ $(C = \text{const} > 0)$ for almost $b \in [0, \frac{\mu_0}{2C}]$. Hence, from Growall-Bellman inequality we must have $J(b) = 0$ for almost $b \in [0, \frac{\mu_0}{2C}]$. Consequently, $u(x,b) = 0$ for almost $b \in [0, \frac{\mu_0}{2C}]$. Using the similar argument as the above, we can prove that $u(x,b) = 0$ for a.e $b \in [0,T]$. Since T is arbitrary, we obtain $u_1(x,t) = u_2(x,t)$. The theorem is proved.

Denote
$$\gamma_0 = \frac{\mu m^* + \sqrt{(\mu m^*)^2 + 4\lambda_0^2\mu_0}}{2\mu_0}.$$

Theorem 3.3. *Suppose coefficients of the operator $L(x,t,D)$ satisty hypotheses of the Theorem 3.2 and $f \in L_2(\gamma_0,\Omega_T)$. Then for each $\gamma > \gamma_0$ problem (2.2)-(2.4) has the generalized solution in $H^{m,1}(\gamma,\Omega_T)$ and the inequality*
$$\|u\|^2_{H^{m,1}(\gamma,\Omega_T)} \le C\|f\|^2_{L_2(\gamma_0,\Omega_T)}$$
holds, where $C = \text{const} > 0$ is independent of u and f.

Proof. We will prove the existence of the generalized solution by Galerkin approximating method. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is the system of functions in $H^m(\Omega)$ such that its linear closure is just $H^m(\Omega)$ and it is orthonormal in $L_2(\Omega)$. Put $u^N(x,t) = \sum_{k=1}^{N} c_k^N(t)\varphi_k(x)$, where $c_k^N(t)$, $k = 1,...,N$, are the solution of the system of the ordinary differential equations of second order:

$$(3.6)\qquad (-1)^{m-1}B(u^N,\varphi_l)(t) - \int_\Omega u_{tt}^N(.,t)\overline{\varphi_l}dx = \int_\Omega f(.,t)\overline{\varphi_l}dx,\ l = 1,...,N,$$

with the initial conditions

$$(3.7)\qquad c_k^N(0) = 0,\quad \frac{d}{dt}c_k^N(0) = 0\ k = 1,...,N.$$

Assume that τ is a positive number: $\tau < T$. Let us multiply (3.6) by $\overline{dc_k^N(t)}/dt$ and take the sum with respect to l from 1 to N. Then we integrate the equality obtained with respect to t from 0 to τ and add this equality to its complex conjugate. We obtain

$$(-1)^{m-1}2\text{Re}\int_0^\tau B(u^N,u_t^N)(t)dt - \int_{\Omega_\tau}\frac{\partial}{\partial t}(u_t^N\overline{u_t^N})dxdt = 2\text{Re}\int_{\Omega_\tau} f\overline{u_t^N}dxdt.$$

From this equality and integrating by parts with condition (3.7) we obtain

$$(-1)^m 2\mathrm{Re}\int_0^\tau B(u^N, u_t^N)(t)dt + \|u_t^N(.,\tau)\|^2_{L_2(\Omega)} = -2\mathrm{Re}\int\limits_{\Omega_\tau} f\overline{u_t^N}dxdt. \tag{3.8}$$

Applying formula (3.1) with condition (3.7) and noting that

$$\lambda_0\|u^N(\cdot,\tau)\|^2_{L_2(\Omega)} = 2\lambda_0\mathrm{Re}\int\limits_{\Omega_\tau} u_t^N\overline{u^N}dxdt,$$

we can rewrite (3.8) as follows

$$\begin{aligned}&\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + (-1)^m B(u^N, u^N)(\tau) + \lambda_0\|u^N(.,\tau)\|^2_{L_2(\Omega)}\\ &= (-1)^m\mathrm{Re}\int_0^\tau B_t(u^N, u^N)(t)dt + 2\lambda_0\mathrm{Re}\int\limits_{\Omega_\tau} u^N\overline{u_t^N}dxdt - 2\mathrm{Re}\int\limits_{\Omega_\tau} f\overline{u_t^N}dxdt.\end{aligned}$$

Using Lemmas 2.1, 3.1 and Cauchy inequality, from this equality we have

$$\begin{aligned}&\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \mu_0\|u^N(.,\tau)\|^2_{H^m(\Omega)}\\ &\le (\mu m^* + \varepsilon)\int_0^\tau \|u^N(.,t)\|^2_{H^m(\Omega)}dt + \Big(\frac{\lambda_0^2}{\varepsilon} + \delta\Big)\int_0^\tau \|u_t^N(.,t)\|^2_{L_2(\Omega)}dt\\ &+\frac{1}{\delta}\int_0^\tau \|f(.,t)\|^2_{L_2(\Omega)}dt\\ &= \Big(\frac{\lambda_0^2}{\varepsilon} + \delta\Big)\int_0^\tau \Big(\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \frac{(\mu m^* + \varepsilon)\varepsilon}{\lambda_0^2 + \delta\varepsilon}\|u^N(.,t)\|^2_{H^m(\Omega)}\Big)dt\\ &+\frac{1}{\delta}\int_0^\tau \|f(.,t)\|^2_{L_2(\Omega)}dt.\end{aligned} \tag{3.9}$$

where $\varepsilon,\ \delta = \mathrm{const} > 0$ arbitrary. Consider the following equation:

$$\frac{(\mu m^* + \varepsilon)\varepsilon}{\lambda_0^2 + \delta\varepsilon} = \mu_0. \tag{3.10}$$

From this equation we obtain

$$\delta = \frac{(\mu m^* + \varepsilon)\varepsilon - \lambda_0^2\mu_0}{\varepsilon\mu_0}. \tag{3.11}$$

Denote

$$\varepsilon_0 = \frac{-\mu m^* + \sqrt{(\mu m^*)^2 + 4\lambda_0^2\mu_0}}{2} \ge 0.$$

From (3.11) it follows that $\delta > 0$ with $\varepsilon > \varepsilon_0$ and

$$\frac{\lambda_0^2}{\varepsilon} + \delta = \frac{\mu m^* + \varepsilon}{\mu_0} > \frac{\mu m^* + \varepsilon_0}{\mu_0} = \frac{\mu m^* + \sqrt{(\mu m^*)^2 + 4\lambda_0^2\mu_0}}{2\mu_0} = \gamma_0.$$

Therefore, can rewrite (3.9) as follows

$$
\begin{aligned}
&\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \mu_0\|u^N(.,\tau)\|^2_{H^m(\Omega)} \\
&\le \frac{\mu m^* + \varepsilon}{\mu_0}\int_0^\tau \Big(\|u_t^N(.,t)\|^2_{L_2(\Omega)} + \mu_0\|u^N(.,t)\|^2_{H^m(\Omega)}\Big)dt \qquad (3.12)\\
&+ C(\varepsilon)\int_0^\tau \|f(.,t)\|^2_{L_2(\Omega)}dt
\end{aligned}
$$

with $\varepsilon > \varepsilon_0$, where $C(\varepsilon) = \text{const} > 0$ depends on ε.

Put

$$J_0^N(t) = \|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \mu_0\|u^N(.,\tau)\|^2_{H^m(\Omega)}.$$

Then for all $\gamma > \gamma_0$ there exists $\varepsilon > \varepsilon_0$, such that

$$\gamma > \gamma_0(\varepsilon) = \frac{\mu m^* + \varepsilon}{\mu_0} > \gamma_0.$$

From this fact and (3.12) we obtain

$$J_0^N(\tau) \le \gamma_0(\varepsilon)\int_0^\tau J_0^N(t)dt + C(\varepsilon)\int_0^\tau \|f(.,t)\|^2_{L_2(\Omega)}dt. \qquad (3.13)$$

Applying Gronwall-Bellman inequality to (3.13), we obtain

$$
\begin{aligned}
J_0(\tau) &\le C(\varepsilon)e^{\gamma_0(\varepsilon)\tau}\int_0^\tau e^{-\gamma_0(\varepsilon)t}\|f(.,t)\|^2_{L_2(\Omega)}dt\\
&\le C(\varepsilon)e^{\gamma_0(\varepsilon)\tau}\int_0^\tau e^{-\gamma_0 t}\|f(.,t)\|^2_{L_2(\Omega)}dt.
\end{aligned}
$$

Hence, we have

$$\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \mu_0\|u^N(.,\tau)\|^2_{H^m(\Omega)} \le C(\varepsilon)e^{\gamma_0(\varepsilon)\tau}\|f\|^2_{L_2(\gamma_0,\Omega_T)},$$

where $C_0(\varepsilon) = \text{const} > 0$ depends on ε. From this inequality we have

$$\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \|u^N(.,\tau)\|^2_{H^m(\Omega)} \le C_0(\varepsilon)e^{\gamma_0(\varepsilon)\tau}\|f\|^2_{L_2(\gamma_0,\Omega_T)}. \qquad (3.14)$$

Multiplying both sides of (3.14) by $e^{-\gamma\tau}$ and integrating with respect to τ from 0 to T, we arrive at

$$
\begin{aligned}
&\int_0^T \Big(\|u_t^N(.,\tau)\|^2_{L_2(\Omega)} + \mu_0\|u^N(.,\tau)\|^2_{H^m(\Omega)}\Big)e^{-\gamma\tau}d\tau \qquad (3.15)\\
&\le C_0(\varepsilon)\|f\|^2_{L_2(\gamma_0,\Omega_T)}\int_0^T e^{(\gamma(\varepsilon)-\gamma)\tau}d\tau.
\end{aligned}
$$

Because $\gamma_0(\varepsilon) - \gamma < 0$, so the integral $\int_0^T e^{(\gamma_0(\varepsilon)-\gamma)\tau} d\tau$ is converged. Moreover, ε depends on γ. Therefore, from (3.15) we obtain

$$\|u^N\|^2_{H^{m,1}(\gamma,\Omega_T)} \leq C(\gamma)\|f\|^2_{L_2(\gamma_0,\Omega_T)},$$

where $C(\gamma) = \text{const} > 0$ depends on γ. From this, we have the existence of a subsequence of $\{u^N\}$ weakly converging to $u \in H^{m,1}(\gamma, \Omega_T)$ and $u(x, 0) = 0$ in Ω. It is easy to verify that u is a generalized solution of the problem (2.2)-(2.4). Moreover, we have

$$\|u\|^2_{H^{m,1}(\gamma,\Omega_T)} \leq C\|f\|^2_{L_2(\gamma_0,\Omega_T)},$$

where $C = \text{const} > 0$ is independent of u and f. The theorem is proved.

Remark. If $\lambda_0 = 0$, then we do not need to estimate the term $\lambda_0\|u^N(.,\tau)\|^2_{L_2(\Omega)}$ in the proof of Theorem 3.3. Then $\varepsilon_0 = 0$ and $\gamma_0 = \dfrac{\mu m^*}{\mu_0}$.

4. The Smoothness with Respest to the Time Variable

In this section we will derive the smoothness with respect to the time of the generalized solution of problem (2.2)-(2.4). We prove that the smoothness in the time of the generalized solution depends on only the coefficients and the right side of the systems. For simplicity of presentation, we only consider the case $\lambda_0 = 0$. The case $\lambda_0 > 0$ is considered similarly.

Put

$$\gamma_h = \frac{(2h+1)m^*\mu}{\mu_0}$$

for all nonnegative integers h. We have the following theorem.

Theorem 4.1. *Let $u(x,t)$ be the generalized solution of problem (2.2)-(2.4) in Theorem 3.3. Moreover, assume that coefficients of the operator L(x,t,D) satisfy condition (2.1) and hypotheses of the Lemma 3.1 with all $k \leq h+1$ and $f_{t^k} \in L_2(\gamma_k, \Omega_T)$ $(k \leq h)$; $f_{t^k}(x,0) = 0,\ (k \leq h-1)$. Then for every $\gamma > \gamma_h$ the function $u(x,t)$ has generalized derivatives with respect to t up to order h in the space $H^{m,1}(\gamma, \Omega_T)$ and the following estimate holds*

$$\|u_{t^h}\|^2_{H^{m,1}(\gamma,\Omega_T)} \leq C\sum_{k=0}^{h} \|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)},$$

where $C = \text{const} > 0$ independent of u and f.

Proof. We use notations in the proof of Theorem 3.3. Since (3.6) is a linear ordinary differential system with initial condition (3.7), applying hypothesis of the

Theorem we have $d^{h+1}\overline{c_k^N(t)}/dt^{h+1} \in L^2(0,\tau)$ for $0 < \tau < T$. Let us differentiate (3.6) h times with respect to t. Then multiply it by $d^{h+1}\overline{c_l^N(t)}/dt^{h+1}$ and take the sum with respect to l from 1 to N. Next integrate the obtained equality with respect to t from 0 to τ and add this equality to its complex conjugate. We have

$$\begin{aligned}(4.1)\qquad &(-1)^{m-1}2\mathrm{Re}\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int\limits_{\Omega_\tau}(a_{pq}D^q u^N)_{t^h}\overline{D^p u^N_{t^{h+1}}}dxdt\\ &-2\mathrm{Re}\int\limits_{\Omega_\tau}u^N_{t^{h+2}}\overline{u^N_{t^{h+1}}}dxdt = 2\mathrm{Re}\int\limits_{\Omega_T}f_{t^h}\overline{u^N_{t^{h+1}}}dxdt.\end{aligned}$$

Put

$$\begin{aligned}(I) &= 2\mathrm{Re}\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int\limits_{\Omega_\tau}(a_{pq}D^q u^N)_{t^h}\overline{D^p u^N_{t^{h+1}}}dxdt \text{ and}\\ (II) &= 2\mathrm{Re}\int\limits_{\Omega_\tau}u^N_{t^{h+2}}\overline{u^N_{t^{h+1}}}dxdt.\end{aligned}$$

Using integrating by parts, condition (3.7) and hypotheses $f_{t^k}(x,0) = 0, 0 \le k \le h$, we obtain

$$(4.2)\qquad (II) = \|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)}$$

Denote $\binom{h}{k} = h!/k!(h-k)!$. We have

$$\begin{aligned}(I) &= 2\mathrm{Re}\sum_{k=0}^{h}\binom{h}{k}\int\limits_{\Omega_\tau}a_{pqt^k}D^q u^N_{t^{h-k}}D^p u^N_{t^{h+1}}dxdt\\ &= \sum_{k=0}^{h}\binom{h}{k}2\mathrm{Re}\int_0^\tau B_{t^k}(u^N_{t^{h-k}}, u^N_{t^{h+1}})(t)dt.\end{aligned}$$

On the other hand, from hypothesis $f_{t^k}(x,0) = 0, 0 \le k \le h$, and condition (3.7) it follows that $D^p u^N_{t^k} = 0, 0 \le k \le s, 1 \le |p| \le m$. Therefore, using formula (3.1) with $v = u^N_{t^h}$ and integrating by parts we obtain

$$
\begin{aligned}
(I) &= B(u^N_{t^h}, u^N_{t^h})(\tau) - \int_0^\tau B_t(u^N_{t^h}, u^N_{t^h})(t)dt \\
&\quad + \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^k}(u^N_{t^{h-k}}, u^N_{t^{h+1}})(t)dt \\
&= B(u^N_{t^h}, u^N_{t^h})(\tau) - \int_0^\tau B_t(u^N_{t^h}, u^N_{t^h})(t)dt \\
&\quad + \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} B_{t^k}(u^N_{t^{h-k}}, u^N_{t^h})(\tau) \\
&\quad - \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^{k+1}}(u^N_{t^{h-k}}, u^N_{t^h})(t)dt \\
&\quad - \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^k}(u^N_{t^{h-k+1}}, u^N_{t^h})(t)dt \\
&= B(u^N_{t^h}, u^N_{t^h})(\tau) - (2h+1) \int_0^\tau B_t(u^N_{t^h}, u^N_{t^h})(t)dt \\
&\quad + \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} B_{t^k}(u^N_{t^{h-k}}, u^N_{t^h})(\tau) \\
&\quad - \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^{k+1}}(u^N_{t^{h-k}}, u^N_{t^h})(t)dt \\
&\quad - \sum_{k=2}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^k}(u^N_{t^{h-k+1}}, u^N_{t^h})(t)dt
\end{aligned} \tag{4.3}
$$

From (4.2) and (4.3) we rewrite (4.1) as follows

$$
\begin{aligned}
&\|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)} + (-1)^m B(u^N_{t^h}, u^N_{t^h})(\tau) \\
&= -(-1)^{m-1}(2h+1) \int_0^\tau B_t(u^N_{t^h}, u^N_{t^h})(t)dt \\
&\quad + (-1)^{m-1} \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} B_{t^k}(u^N_{t^{h-k}}, u^N_{t^h})(\tau) \\
&\quad - (-1)^{m-1} \sum_{k=1}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^{k+1}}(u^N_{t^{h-k}}, u^N_{t^h})(t)dt \\
&\quad - (-1)^{m-1} \sum_{k=2}^{h} \binom{h}{k} 2\mathrm{Re} \int_0^\tau B_{t^k}(u^N_{t^{h-k+1}}, u^N_{t^h})(t)dt - 2\mathrm{Re} \int\limits_{\Omega_T} f_{t^h} \overline{u^N_{t^{h+1}}} dxdt.
\end{aligned} \tag{4.4}
$$

Applying Lemmas 2.1, 3.1 and Cauchy inequality to (4.4) we have

$$\begin{aligned}
&\|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)} + \mu_0\|u^N_{t^h}(.,\tau)\|^2_{H^m(\Omega)}\\
&\leq (2h+1)\mu m^* \int_0^\tau \|u^N_{t^h}\|^2_{H^m(\Omega)}dt\\
&+\varepsilon\|u^N(.,\tau)_{t^h}\|^2_{H^m(\Omega)} + C_{h,1}(\varepsilon)\sum_{k=0}^{h-1}\|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)}\\
&\varepsilon\int_0^\tau \|u^N(.,\tau)_{t^h}\|^2_{H^m(\Omega)}dt + C_{h,2}(\varepsilon)\sum_{k=0}^{h-1}\int_0^\tau \|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)}dt\\
&+\delta\int_0^\tau \|u^N_{t^{h+1}}\|^2_{L_2(\Omega)}dt + \frac{1}{\delta}\int\limits_{\Omega_\tau}|f_{t^h}|^2dxdt
\end{aligned}$$

where $0 < \varepsilon < \mu_0$, and $C_{h,1}, C_{h,2}(\varepsilon) = \text{const} > 0$ depend on ε. From this we obtain

$$\begin{aligned}
&\|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^h}(.,\tau)\|^2_{H^m(\Omega)}\\
&\leq ((2h+1)\mu m^* + \varepsilon)\int_0^\tau \|u^N_{t^h}\|^2_{H^m(\Omega)}dt + \delta\int_0^\tau \|u^N_{t^{h+1}}\|^2_{L_2(\Omega)}dt\\
&+C_{h,1}(\varepsilon)\sum_{k=0}^{h-1}\|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)} + C_{h,2}(\varepsilon)\sum_{k=0}^{h-1}\int_0^\tau \|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)}dt\\
&+\frac{1}{\delta}\int\limits_{\Omega_\tau}|f_{t^h}|^2dxdt\\
&= \delta\int_0^\tau \Big(\|u^N_{t^{h+1}}\|^2_{L_2(\Omega)} + \frac{((2h+1)\mu m^*+\varepsilon)}{\delta}\|u^N_{t^h}\|^2_{H^m(\Omega)}\Big)dt\\
&+C_{h,1}(\varepsilon)\sum_{k=0}^{h-1}\|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)} + C_{h,2}(\varepsilon)\sum_{k=0}^{h-1}\int_0^\tau \|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)}dt\\
&+\frac{1}{\delta}\int\limits_{\Omega_\tau}|f_{t^h}|^2dxdt
\end{aligned}\tag{4.5}$$

Putting

$$\frac{((2h+1)\mu m^*+\varepsilon)}{\delta} = \mu_0 - \varepsilon,$$

we have

$$\delta = \frac{((2h+1)\mu m^* + \varepsilon)}{\mu_0 - \varepsilon} \equiv \gamma_h(\varepsilon).$$

From this fact and (4.5) we obtain

$$\begin{aligned}
&\|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)} + (\mu_0 - \varepsilon)\|u^N_{t^h}(.,\tau)\|^2_{H^m(\Omega)} \\
&\leq \gamma_h(\varepsilon)\int_0^\tau \Big(\|u^N_{t^{h+1}}\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^h}\|^2_{H^m(\Omega)}\Big)dt \\
&\quad + C_{h,1}(\varepsilon)\sum_{k=0}^{h-1}\|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)} + C_{h,2}(\varepsilon)\sum_{k=0}^{h-1}\int_0^\tau \|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)}dt \\
&\quad + C_{h,3}(\varepsilon)\int_0^\tau \|f_{t^h}\|^2_{L_2(\Omega)}dt.
\end{aligned} \tag{4.6}$$

We now use induction to prove:

$$\|u^N_{t^s}(.,\tau)\|^2_{H^m(\Omega)} \leq C_s(\varepsilon)e^{\gamma_s(\varepsilon)\tau}\sum_{k=0}^{s}\|f_{t^k}(.,\tau)\|^2_{L_2(\gamma_k,\Omega_T)} \tag{4.7}$$

with for all $\gamma > \gamma_s,\ 0 < \varepsilon < \mu_0$. From (3.14) it follows that (4.7) holds for $s = 0$. Assume that (4.7) holds for $s-1$. From (4.6) with $h = s$, we have

$$\begin{aligned}
&\|u_{t^{s+1}}(.,\tau)\|^2_{L_2(\Omega)} + (\mu_0 - \varepsilon)\|u^N_{t^s}(.,\tau)\|^2_{H^m(\Omega)} \\
&\leq \gamma_s(\varepsilon)\int_0^\tau \Big(\|u^N_{t^{s+1}}\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^s}\|^2_{H^m(\Omega)}\Big)dt \\
&\quad + C_{s,1}(\varepsilon)\sum_{k=0}^{s-1}\|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)} + C_{s,2}(\varepsilon)\sum_{k=0}^{s-1}\int_0^\tau \|u^N_{t^k}(.,\tau)\|^2_{H^m(\Omega)}dt \\
&\quad + C_{s,3}(\varepsilon)\int_0^\tau \|f_{t^s}\|^2_{L_2(\Omega)}dt
\end{aligned}$$

with $0 < \varepsilon < \mu_0$. Therefore, by the induction hypothesis and $\gamma_k \leq \gamma_{s-1}$ $(k \leq s-1)$, from this inequality we obtain

$$
\begin{aligned}
&\|u_{t^{s+1}}(.,\tau)\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^s}(.,\tau)\|^2_{H^m(\Omega)}\\
&\le \gamma_s(\varepsilon)\int_0^\tau \Big(\|u^N_{t^{s+1}}\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^s}\|^2_{H^m(\Omega)}\Big)dt\\
&\quad + C_{s,1}(\varepsilon)\sum_{k=0}^{s-1} C_k(\varepsilon)e^{\gamma_k(\varepsilon)\tau}\sum_{j=0}^{k}\|f_{t^j}(.,\tau)\|^2_{L_2(\gamma_j,\Omega_T)}\\
&\quad + C_{s,2}(\varepsilon)\sum_{k=0}^{s-1}\int_0^\tau \Big(C_k(\varepsilon)e^{\gamma_k(\varepsilon)t}\sum_{j=0}^{k}\|f_{t^j}(.,\tau)\|^2_{L_2(\gamma_j,\Omega_T)}\Big)dt\\
&\quad + C_{s,3}(\varepsilon)\int_0^\tau \|f_{t^s}\|^2_{L_2(\Omega)}dt \qquad (4.8)\\
&\le \gamma_s(\varepsilon)\int_0^\tau \Big(\|u^N_{t^{s+1}}\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^s}\|^2_{H^m(\Omega)}\Big)dt\\
&\quad + C_1(\varepsilon)e^{\gamma_{s-1}(\varepsilon)\tau}\sum_{j=0}^{s-1}\|f_{t^j}(.,\tau)\|^2_{L_2(\gamma_j,\Omega_T)}\\
&\quad + C_2(\varepsilon)\int_0^\tau \Big(e^{\gamma_{s-1}(\varepsilon)t}\sum_{j=0}^{s-1}\|f_{t^j}(.,\tau)\|^2_{L_2(\gamma_j,\Omega_T)}\Big)dt\\
&\quad + C_{s,3}(\varepsilon)\int_0^\tau \|f_{t^s}\|^2_{L_2(\Omega)}dt.
\end{aligned}
$$

Denote

$$
\begin{aligned}
J_s(\tau) &= \|u_{t^{s+1}}(.,\tau)\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^s}(.,\tau)\|^2_{H^m(\Omega)},\\
\phi(\tau) &= C_1(\varepsilon)e^{\gamma_{s-1}(\varepsilon)\tau}\sum_{j=0}^{s-1}\|f_{t^j}\|^2_{L_2(\gamma_j,\Omega_T)}\\
&\quad + C_2(\varepsilon)\int_0^\tau\Big(e^{\gamma_{s-1}(\varepsilon)t}\sum_{j=0}^{s-1}\|f_{t^j}\|^2_{L_2(\gamma_j,\Omega_T)}\Big)dt + C_{s,3}(\varepsilon)\int_0^\tau \|f_{t^s}(.,t)\|^2_{L_2(\Omega)}dt.
\end{aligned}
$$

From this fact and (4.8) it follows that

$$J^N_s(\tau) \le \gamma_s(\varepsilon)\int_0^\tau J^N_s(t)dt + \phi(\tau).$$

From this relation and the Gronwal-Bellman inequality, we obtain

$$J_s(\tau) \le e^{\gamma_s(\varepsilon)\tau}\int_0^\tau e^{-\gamma_s(\varepsilon)t}\phi'(t)dt.$$

Therefore, we have

$$
\begin{aligned}
&\|u^N_{t^{s+1}}(.,\tau)\|^2_{L^2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^s}(.,\tau)\|^2_{H^m(\Omega)} \\
&\le C(\varepsilon)e^{\gamma_s(\varepsilon)\tau}\int_0^\tau e^{-\gamma_s(\varepsilon)t}\Big(e^{\gamma_{s-1}(\varepsilon)}\sum_{j=0}^{s-1}\|f_{t^j}\|^2_{L_2(\gamma_j,\Omega_T)} + \|f_{t^s}(.,t)\|^2_{L_2(\Omega)}\Big)dt \qquad (4.9)\\
&\le C_{s,0}(\varepsilon)e^{\gamma_s(\varepsilon)\tau}\sum_{j=0}^{s}\|f_{t^j}\|^2_{L_2(\gamma_j,\Omega_T)}.
\end{aligned}
$$

From this we obtain (4.7).

Now we return to inequality (4.6). By inequality (4.7) for all $s \le h-1$ and similar arguments as proof inequality (4.9) we obtain

$$
\|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)} + (\mu_0-\varepsilon)\|u^N_{t^h}(.,\tau)\|^2_{H^m(\Omega)} \le C_{h,0}(\varepsilon)e^{\gamma_h(\varepsilon)\tau}\sum_{k=0}^{h}\|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)}.
$$

Hence,

$$
\|u_{t^{h+1}}(.,\tau)\|^2_{L_2(\Omega)} + \|u^N_{t^h}(.,\tau)\|^2_{H^m(\Omega)} \le C_h(\varepsilon)e^{\gamma_h(\varepsilon)\tau}\sum_{k=0}^{h}\|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)}. \qquad (4.10)
$$

Let γ be a positive number: $\gamma > \gamma_h$. Then there exists $\varepsilon:\ 0<\varepsilon<\mu_0$, such that

$$
\gamma > \gamma_h(\varepsilon) = \frac{(2h+1)\mu m^* + \varepsilon}{\mu_0-\varepsilon} > \gamma_h.
$$

Multiplying up $e^{-2\gamma\tau}$ to both sides of this inequality and integrating obtained inequality with respect to t from 0 to T, we have the following result:

$$
\begin{aligned}
&\int_0^T \Big(\|u^N_t(.,\tau)\|^2_{L_2(\Omega)} + \|u^N(.,\tau)\|^2_{H^m(\Omega)}\Big)e^{-\gamma\tau}d\tau \\
&\le C_h(\varepsilon)\sum_{k=0}^{s}\|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)}\int_0^T e^{(\gamma_h(\varepsilon)-\gamma)\tau}d\tau. \qquad (4.11)
\end{aligned}
$$

Because $\gamma_h(\varepsilon)-\gamma<0$, so the integral $\int_0^T e^{(\gamma_h(\varepsilon)-\gamma)\tau}d\tau$ is converged. Moreover, ε depends on γ. Therefore, from (4.11) we obtain

$$
\|u^N_{t^h}\|^2_{H^{m,1}(\gamma,\Omega_T)} \le C(\gamma)\sum_{k=0}^{h}\|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)},
$$

where $C(\gamma) = \text{const} > 0$ depends on γ.

Since $\{u^N_{t^h}\}$ is bounded in $H^{m,1}(\gamma,\Omega_T)$, we can choose a subsequence which converges weakly to a function u_{t^h} in $H^{m,1}(\gamma,\Omega_T)$. It is easy to check that u_{t^h} is the generalized derivative of order h with respect to t of u. Moveover, we get

$$
\|u_{t^h}\|^2_{H^{m,1}(\gamma,\Omega_T)} \le C\sum_{k=0}^{h}\|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)},
$$

where $C = \text{const} > 0$ independent of u and f. The Theorem 4.1 is proved completely.

5. An Example

In this section we apply the previous results to the Cauchy-Neumann problem for the wave equation. We consider the following problem:

$$\triangle u - u_{tt} = f(x,t),\ (x,t) \in \Omega_T, \tag{5.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0,\ x \in \Omega, \tag{5.2}$$

$$\frac{\partial u}{\partial \nu}\Big|_{S_T} = 0, \tag{5.3}$$

where $\triangle$ is the Laplace operator, ν is the unit exterior normal to S_T.

For problem (5.1)-(5.3) we have $\lambda_0 = 0,\ \mu_0 = 1,\ \mu = 1$ and $m^* = 2$. Therefor, $\gamma_h = 2(2h+1)$. From this fact and Theorem 4.1 we obtain the following result.

Theorem 5.1. *Suppose that* $f_{t^k} \in L_2(2(2h+1), \Omega_T)),\ 0 \le k \le h$, $f_{t^k}(x,0) = 0,\ 0 \le k \le h-1$. *Then for every* $\gamma > 2(2h+1)$ *problem (5.1)-(5.3) has the unique generalized solution* $u(x,t) \in H^{1,1}(\gamma, \Omega_T)$. *Moreover, the function* $u(x,t)$ *has derivatives with respect to* t *up to order* h *belonging to the space* $H^{1,1}(\gamma, \Omega_T)$ *and*

$$\|u_{t^h}\|^2_{H^{1,1}(\gamma,\Omega_T)} \le C \sum_{k=0}^{h} \|f_{t^k}\|^2_{L_2(\gamma_k,\Omega_T)},$$

where $\mathrm{C} = \text{const} > 0$ *is independent of* u *and* f.

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